HW 9

14.5

(3) $\zeta_5 + \zeta_5^{-1}$ is a root of $x^2 + x - 1$ since

$$(\zeta_5 + \zeta_5^{-1})^2 + (\zeta_5 + \zeta_5^{-1}) - 1$$
$$= (\zeta_5^2 + 2 + \zeta_5^{-2}) + (\zeta_5 + \zeta_5^{-1}) - 1$$
$$= (\zeta_2^2 + 2 + \zeta_5^3) + (\zeta_5 + \zeta_5^4) - 1$$
$$= 1 + \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = 0$$

ζ_5 is a root of $x^2 - \alpha x + 1$

By the quadratic formula, $\zeta_5 + \zeta_5^{-1}$ is either $\frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$. We know $\zeta_5 + \zeta_5^{-1}$ is a positive real number (draw a picture) so its $\frac{-1+\sqrt{5}}{2}$. Again by the quadratic formula, ζ_5 (which has positive imaginary part) is $\frac{\sqrt{5}-1}{4} + i\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)$.

(8) a) $[K_n : \mathbb{Q}] = \varphi(2^{n+2}) = 2^{n+1}$

$[K_n : K_n^+] = 2$ by page 601

$[K_n^+ : \mathbb{Q}] = 2^n$ by the above equalities combined

$[K_{n+1}^+ : K_n^+] = 2$ since $[K_{n+1}^+ : \mathbb{Q}] = 2^{n+1}$ and $[K_n^+ : \mathbb{Q}] = 2^n$, $K_n^+ \subseteq K_{n+1}^+$ since $\mathbb{Q}(\zeta_{2^{n+2}}) \subseteq \mathbb{Q}(\zeta_{2^{n+3}})$ and K_n^+ is the maximal real subfield of $\mathbb{Q}(\zeta_{2^{n+2}})$

b) $x^2 - \alpha_n x + 1$, just like in #3

c) Clearly $\alpha_{n+1}^2 = 2 + \alpha_n$. The formula follows from this and

$\alpha_0 = \zeta_4 + \zeta_4^{-1} = i - i = 0$

(10) Suppose $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\zeta_n)$. $\mathbb{Q}(\zeta_n)$ is Galois $/\mathbb{Q}$ so the Galois closure $L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ of $\mathbb{Q}(\sqrt[3]{2})$ is contained in $\mathbb{Q}(\zeta_n)$. We've seen that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$. But S_3 can't be a quotient of the abelian group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, so we get a contradiction to the Fund. Thm. of Galois theory.

$\boxed{14.7}$

(3) Suppose $F(\sqrt{\alpha}) = F(\sqrt{\beta})$. Then $\sqrt{\beta} \in F(\sqrt{\alpha})$ so can be written as $\sqrt{\beta} = c_1 + c_2\sqrt{\alpha}$ with $c_1, c_2 \in F$. Squaring both sides, we get $\beta = (c_1^2 + c_2^2\alpha) + 2c_1c_2\sqrt{\alpha}$. Since $\beta \in F$, we must have $2c_1c_2 = 0$. Since char $F \neq 2$, $2 \neq 0$, so $c_1c_2 = 0$ and thus $c_1 = 0$ or $c_2 = 0$. Assuming $\sqrt{\beta} \notin F$, we can't have $c_2 = 0$, so $c_1 = 0$. Thus $\sqrt{\beta} = c_2\sqrt{\alpha}$. So $\beta = c_2^2\alpha$. Conversely, suppose $\beta = c^2\alpha$ for some $c \in F$. Then $\sqrt{\beta} = \pm c\sqrt{\alpha}$, so $F(\sqrt{\alpha}) = F(\sqrt{\beta})$.

For the last part, use $F = \mathbb{Q}(\sqrt{2})$, $\alpha = 1-\sqrt{2}$, and $\beta = -1$. If $\alpha = c^2\beta$, then $c^2 = \sqrt{2}-1$. If $c = a+b\sqrt{2}$, then $c^2 = (a^2+2b^2) + 2ab\sqrt{2} \neq \sqrt{2}-1$ since $2ab$ is even. So $\mathbb{Q}(\sqrt{1-\sqrt{2}}) \neq \mathbb{Q}(i,\sqrt{2})$.