Math 301: Introduction to Proofs
Problem Set 2
due: September 18, 2019

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Exercise 1. The **unique existential quantifier** $\exists!$ is defined so that $\exists! x \in X, p(x)$ is shorthand for

$$(\exists x \in X, p(x)) \wedge (\forall a \in X, \forall b \in X, ((p(a) \wedge p(b)) \implies a = b)).$$

Prove the following:

$$\forall a, b \in \mathbb{R}, \neg(a = 0) \implies (\exists! x \in \mathbb{R}, ax + b = 0).$$

Exercise 2. Define a logical formula p by:

$$\forall x \in X, \exists y \in X, (x < y) \wedge (\forall z \in X, \neg((x < z) \wedge (z < y))).$$

Write out $\neg p$ as a maximally negated logical formula (with no "$\neg$" symbols). Prove that p is true when $X = \mathbb{Z}$ and prove that p is false when $X = \mathbb{Q}$.

Exercise 3. Find a purely symbolic logical formula that is equivalent to the following statement, and then prove it: "No matter which integer you choose, there will be an integer greater than it."

Exercise 4. Find a statement in plain English, involving no variables at all, that is equivalent to the logical formula

$$\forall a \in \mathbb{Q}, \forall b \in \mathbb{Q}, (a < b \implies (\exists c \in \mathbb{Q}, a < c < b)).$$

Then prove this statement, using the structure of the logical formula as a guide.

Exercise 5. How many lines will there be in a truth table for a propositional formula involving n distinct propositional variables?

Exercise 6. Prove using truth tables that $p \implies q$ is not logically equivalent to $q \implies p$. Give an example of propositions p and q such that $p \implies q$ is true but $q \implies p$ is false.

Exercise 7. Which of the following formulas are equivalent?

(i) $p \implies (q \implies r)$
(ii) $q \implies (p \implies r)$
(iii) $(p \implies q) \wedge (p \implies r)$
(iv) $(p \wedge q) \implies r$
(v) $p \implies (q \wedge r)$

Exercise 8.

(i) Analyze the logical form of the following statement: "If it is raining, then it is windy and the sun is not shining."
(ii) Now analyze the logical form of the following statements and in each case determine whether the statement is logically equivalent to the first statement or to its converse.
   (a) It is windy and not sunny only if it is raining.
   (b) Rain is a sufficient condition for wind with no sunshine.
   (c) Rain is a necessary condition for wind with no sunshine.
   (d) It's not raining, if either the sun is shining or it's not windy.
   (e) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.
   (f) Either it is windy only if it is raining, or it is not sunny only if it is raining.

Exercise 9. In this exercise we'll learn about Pierce's law, a curiosity of classical logic.

(i) Apply a DeMorgan law and a Double Negation law to reduce the expression $\neg(\neg p \vee q) \vee p$.
(ii) Using truth tables, show that your reduced expression is logically equivalent to p. Where did q go?

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