Math 301: Introduction to Proofs
Problem Set 6
due: October 16, 2019

Emily Riehl

Exercise 1. Let F_n be the n-th Fibonacci number, defined recursively by $F_0 \coloneqq 1, F_1 \coloneqq 1, F_{n+2} \coloneqq F_n + F_{n+1}$. Prove that

(i) For all n, $\sum_{k=0}^{n} F_k = F_{n+2} - 1$.
(ii) For all n, $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$.
(iii) For all n, $\sum_{k=0}^{n} F_{2k+1} = F_{2n+2} - 1$.
(iv) Determine the formula for the sum $\sum_{k=0}^{n} F_{2k}$ that works for all n and then prove it.

Exercise 2. For each of (i)-(iv) of Exercise 1, determine whether your proof required weak or strong induction. If it is not clear from the way those proofs were written, rewrite them here.

Exercise 3. Prove for all $n \in \mathbb{N}$ that the size of the set

$$\left\{ X \in P(1, \dots, n) \mid i, j \in X \implies j \neq i + 1 \right\}$$

of subsets of $[n]$ that do not contain any consecutive elements is F_{n+1}, the $(n+1)$th Fibonacci number.

Exercise 4. With only 3 cent coins and 5 cent coins it is impossible to measure out coins whose total value is exactly 7 cents. Prove, however, that it is possible to measure out coins whose total value is exactly n cents, for each $n \geq 8$.

Exercise 5. Let A and B be finite sets and assume A and B are isomorphic, i.e., that there exists a bijection between A and B. Prove that $|A| = |B|$.[1]

Exercise 6. Let X be a finite set. Prove that every subset $S \subset X$ is finite.

Exercise 7. Let X and Y be sets. If X or Y is finite prove that $X \cap Y$ is finite. Is the converse statement true?

Exercise 8. Let X and Y be finite sets.

(i) Prove that $X \cup Y$ is finite.
(ii) Look up the definition of the **disjoint union** $X \sqcup Y$ of X and Y and state it in your own words.
(iii) Prove that $|X| + |Y| = |X \sqcup Y|$.
(iv) Find a bijection
$$(X \cup Y) \sqcup (X \cap Y) \to X \sqcup Y.$$
(v) Prove that, for any finite sets X and Y,
$$|X \cup Y| = |X| + |Y| + |X \cap Y|.$$
This is called the **inclusion-exclusion formula**.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
Email address: `eriehl@math.jhu.edu`

[1] If this was proven in class, write up the proof again, in your own words.