Metric Spaces Worksheet 5

Topology I

With our understanding of metric spaces and sequences cemented, we’ll turn to examine a notion which is supported by every metric space, and in some ways subsumes the concepts we have seen so far.

Definition 1 (open ball). Let \((X,d)\) be a metric space, \(x \in X\) a point and \(r \in (0, \infty)\) a non-negative real number. The open ball of radius \(r\) centred on \(x\), written \(B_r(x)\), is the subset \(B_r(x) \equiv \{ y \in X \mid d(x, y) < r \} \subseteq X\).

We now calculate open balls in Euclidean metric spaces. To describe open balls in the Euclidean line, we need the notion of an open interval in \(\mathbb{R}\). For any \(a, b \in \mathbb{R}\), with \(a < b\), let

\((a, b) \equiv \{ z \in \mathbb{R} \mid a < z < b \}\).

Example 2 (open balls in Euclidean spaces)

1. In the Euclidean metric space \(\mathbb{R}\), the open ball \(B_r(x) = \{ y \in \mathbb{R} \mid |x - y| < r \}\) is the open interval \((x - r, x + r)\). Conversely, every open interval \((a, b)\) for \(a < b \in \mathbb{R}\), is an open ball of some radius \(r = \frac{b - a}{2}\) centred about the midpoint \(\frac{a + b}{2}\).

2. In the Euclidean metric space \(\mathbb{R}^2\), the open ball

\[ B_r((u_1, u_2)) = \{ (v_1, v_2) \in \mathbb{R}^2 \mid (u_1 - v_1)^2 + (u_2 - v_2)^2 < r^2 \} \]

is comprised of all points inside the circle of radius \(r\) centred at the point \((u_1, u_2)\). This explains the name “open ball” given to the sets \(B_r(x)\) in general metric spaces.

Question 3. What are the possible open balls in a discrete metric space \((X,d)\)?

Complete the proof here
Definition 4 (open set). A subset $U \subseteq X$ in a metric space $(X,d)$ is open if for every $u \in U$ there exists an $\varepsilon \in (0,\infty)$ such that $B_{\varepsilon}(u) \subseteq U$.

Example 5 (an open set in the Euclidean space $\mathbb{R}$)

For any $a \in \mathbb{R}$, the open ray $(a,\infty) \equiv \{x \in \mathbb{R} \mid a < x\}$ is an open set.

To see this we must prove, for every point $u \in (a,\infty)$, that there exists some $\varepsilon \in (0,\infty)$ such that $B_{\varepsilon}(u) \subseteq (a,\infty)$. To that end, consider a point $u \in (a,\infty)$. We know that $u - a > 0$, so we may choose $\varepsilon$ to be any real number so that $0 < \varepsilon < u - a$. (For sake of concreteness, we might pick $\varepsilon = \frac{u - a}{2}$, but it’s also not necessary to specify a concrete value of $\varepsilon$.)

Now if $x \in B_{\varepsilon}(u)$, then by example 2 item 1, $u - \varepsilon < x < u + \varepsilon$. Since $u - \varepsilon > a$ we conclude that $x > a$ so $x \in (a,\infty)$. Since we’ve shown that $\forall x \in B_{\varepsilon}(u)$, $x \in (a,\infty)$ this demonstrates that $B_{\varepsilon}(u) \subseteq (a,\infty)$ as required. Thus $(a,\infty)$ is an open set.

Non-example 6 (sets which are not open in the Euclidean space $\mathbb{R}$)

1. The set $\{0\} \subseteq \mathbb{R}$ is not open because there is no $\varepsilon$ small enough so that $B_{\varepsilon}(0) \subset \{0\}$.

2. For any $a \in \mathbb{R}$, the closed ray $[a,\infty) \equiv \{x \in \mathbb{R} \mid a \leq x\}$ is not an open set. The argument given in example 5 proves that for every $u \in [a,\infty)$ if $a \neq u$ then there exists $\varepsilon \in (0,\infty)$ so that $B_{\varepsilon}(u) \subseteq [a,\infty)$. However, there is no open ball that contains the point $a$ and is contained within $[a,\infty)$.

To see this, take $\varepsilon \in (0,\infty)$. Then by example 2 item 1 the point $a - \frac{\varepsilon}{2} \in B_{\varepsilon}(a)$. But since $a - \frac{\varepsilon}{2} < a$, $a - \frac{\varepsilon}{2} \notin [a,\infty)$. Thus $B_{\varepsilon}(a) \nsubseteq [a,\infty)$.

Question 7. What are the open sets in a discrete metric space $(X,d)$?

Complete the proof here
As we might hope, the subsets we were calling open balls are indeed open.

**Proposition 8** (open balls are open sets). Let \( (X,d) \) be a metric space, \( x \in X \) be a point, \( r \in (0,\infty) \) be a non-negative real number. The subset \( B_r(x) \subseteq X \) is open.

**Corollary 9** (open intervals are open sets). In the Euclidean metric space \( \mathbb{R} \), all open intervals \( (a,b) \) are open.
It turns out that open sets can be combined in certain ways and the result is always again an open set.

**Theorem 10** (open set laws). *In a metric space* $(X, d)$,

1. $X$ and $\emptyset$ are open sets.
2. If $\mathcal{F}$ is a family of open sets in $X$ then $\bigcup_{U \in \mathcal{F}} U$ is open.
3. If $U, V \subseteq X$ are open sets then $U \cap V$ is open.

*Complete the proof here*
Surprise 11 (intersection of opens is not generally open)

In the Euclidean metric space $\mathbb{R}$, the subset $I \equiv \bigcap_{n \in \mathbb{N}} \left( 0, \frac{n+2}{n+1} \right) \subseteq \mathbb{R}$ is not open.

Compute $I$ and prove this fact.

Complete the proof here