Metric Spaces Worksheet 7

Topology III

Now we are ready to address the elephant in the room. There is indeed a relationship between the closed sets and the open sets in a metric space. In order to address, however, we must first establish a useful theorem about closed sets.

**Theorem 1** (points outside a closed set are separated from that closed set). Let $(X,d)$ be a metric space, $G \subseteq X$ be a closed set, and $x \in X \setminus G$ be a point outside $G$. There exists an $\varepsilon \in (0,\infty)$ such that $B_\varepsilon(x) \cap G = \emptyset$.

**Hint 2.** To prove this aim for a contradiction,

1. Suppose this wasn’t true, and understand what that means.

2. Argue that for every $n \in \mathbb{N}$, under this assumption there must be at least one point in $B_{\frac{1}{n+1}}(x) \cap G$.

3. By appealing to the $\mathfrak{A}$ Axiom of Choice $\mathfrak{A}$, define a sequence $a_n$ by requiring that each $a_n \in B_{\frac{1}{n+1}}(x) \cap G$. (Essentially, you may assume there is such a sequence by invoking this plot device.)

4. Prove that this sequence converges.

*Complete the proof here*

(continued on next page)
Proof continued
Theorem 3 (open iff complement is closed). In a metric space \((X, d)\), a subset \(U \subseteq X\) is open iff its complement \(U^c\) is closed, where \(U^c \equiv X \setminus U\).

To prove theorem 3, we can break up this statement into two parts.

Proposition 4 (complements of open sets are closed). In a metric space \((X, d)\), if \(U \subseteq X\) is open then its complement \(U^c\) is closed.

Hint 5. Look back at your proof of theorem 6 of Worksheet 6, and try to figure out how to present a similar argument in the setting of a general metric space.

Complete the proof here
**Proposition 6** (complements of closed sets are open). *In a metric space* $(X,d)$, if $G \subseteq X$ is closed then its complement $G^c$ is open.

**Hint 7.** The following proof skeleton may be useful:

1. Given a closed set $G$, to show that $G^c$ we must choose a point $x \in G^c$ and find an $\varepsilon \in (0,\infty)$ for which $B_\varepsilon(x) \subseteq G^c$.

2. Argue that $B_\varepsilon(x) \subseteq G^c \iff B_\varepsilon(x) \cap G = \emptyset$.

3. Apply a previous result.

*Complete the proof here*
We are now in a position to prove theorem 3 by combining the proofs of propositions 4 and 6.

Complete the proof of theorem 3

Now that we understand the relationship between closed sets and open sets, it might seem natural to ask whether there are sets which are both open and closed. Whence we arrive at the following amusing terminology.

**Definition 8 (clopen set).** A subset $S \subseteq X$ of a metric space $(X,d)$ is said to be **clopen** if it is both open and closed.

**Example 9 (singletons are clopen in a discrete space)**

Let $(X,d)$ be a discrete metric space, and $x \in X$ a point. The singleton set $\{x\}$ is clopen.

**Hint 10.** In proving that a set $C$ is clopen, we may prove any one of:

1. $C$ is both open and closed, directly
2. $C$ is open and $C^c$ is open
3. $C$ is closed and $C^c$ is closed
4. $C^c$ is both open and closed, directly

Only context and experience can aid us in determining which route is likely to be easier.

Complete the proof here
**Question 11.** Can you find a metric space in which every subset is clopen? If so, describe it mathematically. If not, prove that such a space cannot exist.

*Complete your answer here*
Review 12 (open sets in the Euclidean space \( \mathbb{R} \)). Determine whether each of the following subsets of the Euclidean metric space \( \mathbb{R} \) are open or closed or both or neither.

1. The interval \((a, b) = \{x \in \mathbb{R} \mid a < x < b\}\) for fixed \(a < b \in \mathbb{R}\).

2. The interval \([a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}\) for fixed \(a < b \in \mathbb{R}\).

3. The interval \([a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}\) for fixed \(a < b \in \mathbb{R}\).

4. The interval \((a, \infty) = \{x \in \mathbb{R} \mid a < x\}\) for fixed \(a \in \mathbb{R}\).

5. The interval \([a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}\) for fixed \(a \in \mathbb{R}\).

6. The point \(\{0\} \in \mathbb{R}\).

7. The set \(\mathbb{Z} \in \mathbb{R}\).

8. The set \(\mathbb{Q} \in \mathbb{R}\).

Complete the review here