Exercise 1. For each of the following functions determine whether they are injective, surjective, and bijective and construct a left, right, or two-sided inverse whenever these exist.

(i) The function \( f : \mathbb{N} \to \mathbb{N} \) defined by \( f(x) = x^2 \).
(ii) The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \).
(iii) The function \( f : \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q} \) defined by \((a, b) \mapsto \frac{a}{b}\); here \( \mathbb{Z}_{>0} \) denotes the set of positive integers.
(iv) The function \( \pi_B : A \times B \to B \) defined by \( \pi_B(a, b) = b \).
(v) The function \( \pi : A \to A_{\neq} \) associated to a non-identity equivalence relation \( \sim \) on \( A \) defined by \( \pi(a) = [a]_{\sim} \).
(vi) The function \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) defined by \( f(a, b) = a + b \).
(vii) Writing \( \chi : \mathbb{P}(A) \to 2 \) in some natural way.

Exercise 2. Write \( 2 = \{\perp, \top\} \) or \( 2 = \{0, 1\} \) for the set with two elements. (Your choice which notation you want to use for its elements.) Let \( \mathbb{P}(A) \) denote the set of all subsets of a set \( A \) and let \( 2^A \) denote the set of functions from \( A \) to \( 2 \).

(i) Let \( S \subseteq A \). Define a function \( \chi_S : A \to 2 \) that is related to \( S \) in some natural way.
(ii) Use part (i) to define a natural function \( \chi : \mathbb{P}(A) \to 2^A \).
(iii) Show that the function \( \chi : \mathbb{P}(A) \to 2^A \) you’ve defined in part (ii) is a bijection.

Exercise 3. Write \( B^A \) for the set of functions from \( A \) to \( B \).

(i) Express the cardinality of \( B^A \) in terms of the cardinalities of \( A \) and \( B \), assuming these are finite sets.
(ii) Express the cardinality of the powerset \( 2^A \) of \( A \) in terms of the cardinality of \( A \), assuming that \( A \) is a finite set.
(iii) Explain why (ii) is a special case of (i).

Exercise 4. How many functions are there from a set of \( n \) elements to itself? How many bijections are there between a set with \( n \) elements and itself?

Exercise 5.

(i) Let \( f : A \to B \) be a function that has a left inverse \( g : B \to A \) and also a right inverse \( h : B \to A \). Prove that \( h = g \).

Exercise 6.

(i) For any function \( f : A \to B \) define an explicit isomorphism between \( A \) and the graph \( \Gamma_f \subseteq A \times B \).
(ii) Define a natural function \( \Gamma_f \to B \). Is it necessarily injective? Is it necessarily surjective?

Exercise 7.

(i) For any non-empty set \( A \), define an isomorphism between the set \( A \times A \) and the set \( A^2 \) of functions from the set with two elements to the set \( A \).
(ii) For any non-empty set \( A \) and positive natural number \( n \), define an isomorphism between the \( n \)-fold cartesian product \( \prod_n A := A \times \cdots \times A \) and the set \( A^n \) of functions from the set with \( n \) elements to \( A \).

Exercise 8. Explain in your own words why all sets with three elements are isomorphic and speculate why I don’t care what we call the elements of a 3-element set.

Exercise 9. Suppose \( p : A \to B \) is a surjective function. Explain how the fibers of \( p \) define an equivalence relation on \( A \) and prove that \( B \) is isomorphic to the set of equivalence classes for this equivalence relation.

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1 If the equivalence relation is defined by \( x \sim y \) iff \( x = y \), then \( \pi \) is the identity function. Please exclude this case.
2 If the function you’ve defined in part (i) is not a bijection, you might need to redefine the function \( \chi \).
3 In fact, it is for any set \( A \) and any set \( I \) (possible empty and possibly infinite) it is also the case that the product \( \prod_I A \) is isomorphic to the set of functions \( A^I \). The proof is by the same argument, but requires somewhat more complicated notation. In fact there is a sense in which the product \( \prod_I A \) of the indexed family of sets \((A)_{i\in I}\) is defined to be the set of functions \( A^I \).