Exercise 1. Let $G$ be a group.
   (i) Prove that if $x, y \in G$ are conjugate, then $x$ and $y$ have the same order.
   (ii) Prove that the order of any conjugacy class of elements in $G$ divides the order of the group $G$.
   (iii) Let $N \triangleleft G$ be a normal subgroup. Prove that $N$ is the union of the conjugacy classes of its elements.

Exercise 2. Let $I$ denote the icosahedral group, the group of symmetries of the icosahedron (or equally, by duality of the platonic solids, of the dodecahedron). In problem set 5, you discovered that $|I| = 60$.
   (i) Calculate the orders of the elements of $I$. In particular, determine how many elements have each order and describe the resulting partition:
   
   $60$ elements $= 1$ element of order one + ...

   (ii) Calculate the conjugacy classes of elements of $I$ and describe the resulting partition of $60 = |I|$ (the class equation).
   Explain why this partition refines the partition you found in (i).

   (iii) Prove that $I$ is a simple group: that is, show that $I$ has no non-trivial normal subgroups.

   The result in (iii) is useful for identifying $I$. The group $I$ acts on the set of cubes inscribed inside the dodecahedron. Since there are five such cubes, this action defines a homomorphism $I \to S_5$. Since $I$ is a simple group, the kernel of this homomorphism must either be $I$ or $\{e\}$. Since the action is non-trivial it's the latter, and consequently $I$ may be identified with a subgroup of $S_5$ of order 60. To find this subgroup, we consider the composite homomorphism $I \to S_5 \to \mathbb{Z}/2$ where the second map is the sign homomorphism, sending even cycles to $[0]$ and odd cycles to $[1]$. If this homomorphism were surjective, the first isomorphism theorem would tell us that $\mathbb{Z}/2$ is isomorphic to a quotient group $I/N$, where $N \triangleleft I$ is a normal subgroup of order 30. But such a subgroup doesn't exist, so $I \to S_5 \to \mathbb{Z}/2$ must be the zero homomorphism. Thus $I$ is contained in the kernel of the sign homomorphism $S_5 \to \mathbb{Z}/2$, which is the alternating group $A_5$. Since $I \subset A_5$ and both groups have the same order, we conclude that $I \cong A_5$.

Exercise 3. Find the center of $D_{2n}$. [Hint: the answer depends on whether $n$ is even or odd.]

Exercise 4. Prove that the center of $S_n$ is trivial for $n \geq 3$.

Exercise 5. If $H \subset G$ is a subgroup its conjugate subgroups are the subgroups of the form
   
   $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$

   for some $g \in G$.

   (i) Prove that $gHg^{-1}$ is a subgroup of $G$.
   (ii) Define a bijective group homomorphism $H \to gHg^{-1}$.
   (iii) The group $G$ acts on the set of subgroups of $G$ by conjugation: the action of a group element $g \in G$ on a subgroup $H \subset G$ is defined by $H \mapsto gHg^{-1} \subset G$. Rephrase the condition of $H$ being a normal subgroup in terms of the orbits of this action.

Exercise 6. Prove that $S_p$, where $p$ is prime, is generated by just two permutations: the transposition $(12)$ and $(12 \ldots p)$.

Exercise 7. Find the formula for the size of the conjugacy class of a permutation of any given cycle shape in $S_n$.

Exercise 8. Prove that any normal subgroup of $S_4$ must have order 1, 4, 12, or 24.