Exercise 1. For each of the following functions determine whether they are injective, surjective, and bijective and construct a left, right, or two-sided inverse whenever these exist.

(i) The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.
(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
(iii) The function $f : \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q}$ defined by $(a, b) \mapsto \frac{a}{b}$; here $\mathbb{Z}_{>0}$ denotes the set of positive integers.
(iv) The function $\pi_B : A \times B \to B$ defined by $\pi_B(a, b) = b$.
(v) The function $\pi : A \to A/\sim$ associated to an equivalence relation $\sim$ on $A$ defined by $\pi(a) = [a]_\sim$.
(vi) The function from the set of subsets of $\mathbb{N}$ to the set of countably-infinite binary sequences (0, 1, 0, 0, 1, ... ) that sends a subset $S \in \mathbb{N}$ to the sequence that has a 1 in the $n$th coordinate if and only if $n \in S$.
(vii) Writing $10 = \{0, \ldots, 9\}$ and $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, the function $f : 10^{\mathbb{N}} \to [0, 1]$ that sends a sequence of decimal digits $(x_n)_{n \in \mathbb{N}}$ to the real number $0.x_1x_2x_3\ldots$.

Exercise 2. Write $B^A$ for the set of functions from $A$ to $B$.

(i) Express the cardinality of $B^A$ in terms of the cardinalities of $A$ and $B$, assuming these are finite sets.
(ii) Express the cardinality of the powerset $2^A$ of $A$ in terms of the cardinality of $A$, assuming that $A$ is a finite set.
(iii) Explain why (ii) is a special case of (i).

Exercise 3. How many functions are there from a set of $n$ elements to itself? How many bijections are there between a set with $n$ elements and itself?

Exercise 4.

(i) Let $f : A \to B$ be a function that has a left inverse $g : B \to A$ and also a right inverse $h : B \to A$. Prove that $h = g$.
(ii) Prove that $f : A \to B$ is a bijection if and only if $f$ is an isomorphism without using (i).

Exercise 5.

(i) For any function $f : A \to B$ define an explicit isomorphism between $A$ and the graph $\Gamma_f \subset A \times B$.
(ii) Define a natural function $\Gamma_f \to B$. Is it necessarily injective? Is it necessarily surjective?

Exercise 6.

(i) Define an isomorphism between any set $A$ and the set $A^1$ of functions from a singleton set to $A$.
(ii) Argue that a function $f : A \to B$ is uniquely determined by its collection of composites with functions $1 \to A$. 

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Exercise 7. Explain in your own words why all sets with three elements are isomorphic and speculate why I don’t care what we call the elements of a 3-element set.

Exercise 8. Suppose \( p: A \to B \) is a surjective function. Explain how the fibers of \( p \) define an equivalence relation on \( A \) and prove that \( B \) is isomorphic to the set of equivalence classes for this equivalence relation.