Exercise 1. Let $\phi: G \to H$ be a group homomorphism whose underlying function is a bijection. Prove that $\phi$ defines an isomorphism between the groups $G$ and $H$ by defining its inverse homomorphism.

Exercise 2. Given groups $G$ and $H$ their product $G \times H$ is the group whose:

- elements are ordered pairs $(g, h)$ with $g \in G$ and $h \in H$.
- multiplication law is defined componentwise:
  $$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2).$$

(i) What is the identity element for $G \times H$?
(ii) What is the inverse of $(g, h) \in G \times H$?
(iii) Define a pair of canonical “projection” homomorphisms $\pi_G: G \times H \to G$ and $\pi_H: G \times H \to H$.
(iv) Let $\phi: K \to G$ and $\psi: K \to H$. Define a unique homomorphism $\zeta: K \to G \times H$ so that $\phi = \pi_G \circ \zeta$ and $\psi = \pi_H \circ \zeta$. (You should check that the function $\zeta$ that you define is a homomorphism but do not need to verify that it is unique with the commutativity property.)

Exercise 3. The Klein four group\(^1\) is the group with four elements defined by the multiplication table:

$$
\begin{array}{c|cccc}
  & e & i & j & k \\
\hline
  e & e & i & j & k \\
i & i & e & j & k \\
j & j & k & e & i \\
k & k & j & i & e \\
\end{array}
$$

Define an isomorphism between the Klein four group and the product $\mathbb{Z}/2 \times \mathbb{Z}/2$ of the cyclic group with two elements\(^2\) with itself.

Exercise 4.

(i) Are there any non-zero homomorphisms $\mathbb{Z}/n \to \mathbb{Z}$ for $n \in \mathbb{N}$? If so, define one. If not, explain why not.
(ii) How many homomorphisms are there from $\mathbb{Z}$ to $\mathbb{Z}/n$?

Exercise 5. Let $p, n \in \mathbb{N}$ with $p$ prime. For which $n$ does there exist a non-zero group homomorphism $\mathbb{Z}/p \to \mathbb{Z}/n$?

Exercise 6.

(i) Prove that there are no non-zero group homomorphisms between the Klein four group and $\mathbb{Z}/7$.

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\(^1\)Problems labelled $n^*$ are optional (fun!) challenge exercises that will not be graded.
\(^2\)Note that the multiplication on $G \times H$ is defined so that the projection functions $\pi_G$ and $\pi_H$ become group homomorphisms.
\(^3\)It’s also the name of an a cappella group; google “finite simple group of order 2.”
\(^4\)Aka the “finite simple group of order 2.”
(ii) Define a non-zero homomorphism from the Klein four group to $\mathbb{Z}/4$.

**Exercise 7.** Let $S_n = \text{Aut}_\text{Set} \{1, 2, \ldots, n\}$ denote the group of permutations of an $n$-element set.

- Define an element of order $d$ in $S_n$ for any $d < n$.
- For which $n$ is $S_n$ abelian? Give a proof or supply a counterexample for each $n \geq 1$.

**Exercise 8.** Is the product of cyclic groups cyclic? If so, give a proof. If not, find a counterexample.

**Exercise 9*.** Classify the homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for any $n, m \in \mathbb{N}$. In particular, how many homomorphisms are there in the set $\text{Hom}_\text{Group}(\mathbb{Z}/n, \mathbb{Z}/m)$?

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