Math 601: Algebra
Problem Set 8
due: November 9, 2017

Emily Riehl

Exercise 1.
(i) Consider a short exact sequence of \( R \)-modules

\[
0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0
\]

Prove that \( M \) is Noetherian if and only if \( N \) and \( M/N \) are Noetherian.
(ii) Let \( R \) be a Noetherian ring and let \( I \subset R \) be an ideal. Prove that \( R/I \) is a Noetherian ring and explain the connection, if any, between this question and (i).

Exercise 2. Prove that the ring of continuous functions \([0, 1] \rightarrow \mathbb{R}\) is not Noetherian.

Exercise 3. Consider the subring \( \mathbb{Z}[\sqrt{-5}] := \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C} \).

(i) Express \( \mathbb{Z}[\sqrt{-5}] \) as a finite type \( \mathbb{Z} \)-algebra.
(ii) Prove that \( \mathbb{Z}[\sqrt{-5}] \) is a Noetherian integral domain.
(iii) Consider the norm function \( N: \mathbb{Z}[\sqrt{-5}] \rightarrow \mathbb{N} \) defined by \( N(a + bi\sqrt{5}) = a^2 + 5b^2 \) and describe its properties — i.e., what sort of homomorphism is it?
(iv) Find all units in \( \mathbb{Z}[\sqrt{-5}] \).
(v) Prove that \( 2, 3, 1+i\sqrt{5}, 1-i\sqrt{5} \) are all irreducible elements but none of these are prime.
(vi) Conclude that \( \mathbb{Z}[\sqrt{-5}] \) has factorizations into irreducibles but is not a UFD.

Exercise 4. Let \( R \) be a UFD and let \( I \subset R \) be a non-zero ideal. Prove that every descending chain of principal ideals containing \( I \) must stabilize.

Exercise 5. The height \( h \) of a prime ideal \( p \subset R \) is the length of the maximal chain of prime ideals

\[
0 = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_h = p.
\]

Prove that if \( R \) is a UFD then every prime ideal of height 1 is principal.

Exercise 6.
(i) Prove that any Euclidean domain \( R \) admits a Euclidean valuation \( v: R \setminus \{0\} \rightarrow \mathbb{N} \) with the property that \( v(ab) \geq v(b) \) for all non-zero \( a, b \in R \).
(ii) With respect to the valuation defined in (i) prove that associate elements have the same valuation and that units have minimum valuation. Are the converses true?

Exercise 7. A discrete valuation on a field \( \mathbb{k} \) is a surjective homomorphism of abelian groups \( v: (k^\times, \times) \rightarrow (\mathbb{Z}, +) \) so that \( v(a + b) \geq \min\{v(a), v(b)\} \) for all \( a, b \in k^\times \) so that \( a + b \in k^\times \).

(i) Prove that \( R = \{a \in k^\times \mid v(a) \geq 0\} \cup \{0\} \) is a subring of \( k \).
(ii) Prove that \( R \) is a Euclidean domain.
(iii) Rings of the form of (i) are called **discrete valuation rings**. Find an example of a discrete valuation ring.

DEPT. OF MATHEMATICS, JOHNS HOPKINS UNIV., 3400 N CHARLES ST, BALTIMORE, MD 21218
E-mail address: eriehl@math.jhu.edu