Math 616: Algebraic Topology
Problem Set 2[1]
due: February 25, 2016

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Exercise 1. Prove that any additive functor preserves split short exact sequences.

Exercise 2. Suppose $f\colon A \to B$ factors as an epimorphism $e\colon A \twoheadrightarrow I$ followed by a monomorphism $m\colon I \rightarrowtail B$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Prove that:

(i) $\ker f \cong \ker e$.[2]
(ii) $\mathrm{coker} f \cong \mathrm{coker} m$.
(iii) $\mathrm{im} f \cong I$.

Exercise 3. Prove that a direct summand of a projective object is projective.[3]

Exercise 4. In class we showed that given any morphism $f\colon A \to B$ and any pair of projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$, there exists a lift of f to a chain map $f_\bullet\colon P_\bullet \to Q_\bullet$ and that any two such lifts are chain homotopic. Use this to show that for any two projective resolutions $P_\bullet \to A$ and $P'_\bullet \to A$ of $A \in \mathsf{A}$ and for any additive functor $F\colon \mathsf{A} \to \mathsf{B}$, the chain complexes $FP_\bullet$ and $FP'_\bullet$ have the same homology: i.e., that $H_n FP \cong H_n FP'$ for all n.

Exercise 5. Recall that the n-th left derived functor of an right exact functor $F\colon \mathsf{A} \to \mathsf{B}$ between abelian categories is defined by

$$L_n F(A) := H_n FP_\bullet$$

for any projective resolution $P_\bullet \to A$ of A.

(i) Prove that this is well-defined.
(ii) Prove that if A is projective then $L_0 FA \cong FA$ and $L_i FA \cong 0$ for all $i \neq 0$.[4]

Exercise 6*. Suppose that $F\colon \mathsf{A} \to \mathsf{B}$ is *left* exact. Assuming that A has enough injectives,[5] the aim of this exercise is to sketch the proof of the construction of the right derived functors $R^n F\colon \mathsf{A} \to \mathsf{B}$, which define a universal *cohomological* δ-functor.[6]

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[1]Problems labelled n* are optional (fun!) challenge exercises that will not be graded.

[2]Hint: if $\ker f \rightarrowtail A$ and $\ker e \rightarrowtail A$ factor through each other they must be isomorphic. Why?

[3]This is how one shows that any direct summand of a free R-module is projective in the category Mod_R.

[4]Hint: Choose a clever projective resolution of A.

[5]An object $I \in \mathsf{A}$ is **injective** if for any map $f\colon A \to I$ and any monomorphism $m\colon A \rightarrowtail B$, the morphism f extends along m to define a morphism $\tilde{f}\colon B \to I$.

[6]My recommendation is to first try to do this on your own, consulting Weibel and your notes as necessary. Do not attempt to write down complete proofs! Just sketch or state the main ideas (e.g., "because F is additive, the image of a short exact sequence of injectives is still exact.").

(i) Explain what it means for a cohomologically graded chain complex to define an **injective resolution** of an object $A \in \mathsf{A}$.
(ii) Define the right derived functors of F and sketch a proof that they are (a) functors and (b) well-defined.
(iii) Construct a long exact sequence in B

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \xrightarrow{\delta} R^1FA \longrightarrow R^1FB \longrightarrow R^1FC \xrightarrow{\delta} R^2FA \longrightarrow \cdots$$

from a short exact sequence in A.
(iv) Calculate R^nFI where I is injective.

Exercise 7*. Under addition, the rational numbers $\mathbb{Q}$ define an injective abelian group. Think about this example and try to come up with a conjecture that describes which property of this abelian group implies its injectivity.

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