Math 616: Algebraic Topology
Problem Set 5
due: May 3, 2016
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Exercise 1. Let $F: A \to B$ be an additive functor between abelian categories. Its prolongation is the functor $F: \text{Ch} A \to \text{Ch} B$ that you get by just applying $F$ degreewise.

(i) Prove that the prolongation preserves chain homotopy equivalences.
(ii) Suppose further that $F$ is exact. Prove that its prolongation preserves quasi-isomorphisms.

Exercise 2. Recall:
• $S^n \in \text{Ch}_R$ is the chain complex with the $R$-module $R$ in degree $n$ and zeros elsewhere.
• $D^n \in \text{Ch}_R$ is the chain complex with the $R$-module $R$ in degrees $n$ and $n-1$, with identity differential, and zeros elsewhere.
Let $A \in \text{Ch}_R$ be a chain complex of $R$-modules.

(i) Define an injective chain map $S^{n-1} \to D^n$.
(ii) Explain precisely what data is needed to define a chain map $D^n \to A$.\(^1\)
(iii) Explain precisely what data is needed to define a chain map $S^n \to A$.

Exercise 3. Prove that a chain map $f: A \to B$ has the right lifting property against the map $0 \to D^n$

\[
\begin{array}{ccc}
0 & \to & A \\
\downarrow & & \downarrow f \\
D^n & \to & B
\end{array}
\]
if and only if $f_n: A_n \to B_n$ is a surjection at the level of underlying sets (every element of $B_n$ is in the image of $f_n$).

Exercise 4. Prove that if a chain map $f: A \to B$ has the right lifting property against the map $S^n \to D^{n+1}$

\[
\begin{array}{ccc}
S^n & \to & A \\
\downarrow & & \downarrow f \\
D^{n+1} & \to & B
\end{array}
\]
then

(i) $H_n f: H_n A \to H_n B$ is a monomorphism and
(ii) $H_{n+1} f: H_{n+1} A \to H_{n+1} B$ is an epimorphism.

Exercise 5. Consider a functor $F: M \to N$ whose domain is a model category and whose codomain is a category with a specified class of weak equivalences. Prove that if $F$ carries trivial fibrations between fibrant objects to weak equivalences then $F$ preserves all weak equivalences between fibrant objects.

\(^1\)That is, tell me that chain maps $D^n \to A$ are in (natural) bijection with elements of some set; your task is to identify this set.
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