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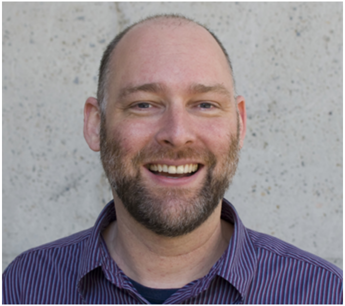
A model-independent theory of ∞-categories

joint with Dominic Verity

Joint International Meeting of the AMS and the CMS

Dominic Verity



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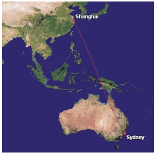

Abstract

We develop the theory of ∞-categories from first principles in a "model-independent" fashion, that is, using a common axiomatic framework that is satisfied by a variety of models. Our "synthetic" definitions and proofs may be interpreted simultaneously in many models of ∞-categories, in contrast with "analytic" results proven using the combinatorics of a particular model. Nevertheless, we prove that both "synthetic" and "analytic" theorems transfer across specified "change of model" functors to establish the same results for other equivalent models.

Plan

Goal: develop model-independent foundations of ∞-category theory

1. What are model-independent foundations?

2. ∞-cosmoi of ∞-categories

3. A taste of the formal category theory of ∞-categories

4. The proof of model-independence of ∞-category theory

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What are model-independent foundations?

The motivation for ∞-categories

Mere 1-categories are insufficient habitats for those mathematical objects that have higher-dimensional transformations encoding the "higher homotopical information" needed for a good theory of derived functors.

A better setting is given by ∞-categories, which have spaces rather than sets of morphisms, satisfying a weak composition law.

$\rightsquigarrow$ Thus, we want to extend 1-category theory (e.g., adjunctions, limits and colimits, universal properties, Kan extensions) to ∞-category theory.

First problem: it is hard to say exactly what an ∞-category is.

The idea of an ∞-category

∞-categories are the nickname that Lurie gave to $(\infty, 1)$-categories, which are categories weakly enriched over homotopy types.

The schematic idea is that an ∞-category should have

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

But this definition is tricky to make precise.

Models of ∞-categories

$$\begin{tikzcd}
 & \mathcal{R}ezk \arrow[r] \arrow[dr, shift left] & \mathcal{S}egal \arrow[d, shift left] & \\
\mathcal{R}el\mathcal{C}at \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}op\text{-}\mathcal{C}at \arrow[dl] \\
 & 1\text{-}\mathcal{C}omp \arrow[r, shift left] & q\mathcal{C}at \arrow[l, dashed, shift left] \arrow[ul, shift left] \arrow[u, shift left] &
\end{tikzcd}$$

- topological categories and relative categories are the simplest to define but do not have enough maps between them
- $\left\{\begin{array}{l}\text{quasi-categories (nee. weak Kan complexes),}\\ \text{Rezk spaces (nee. complete Segal spaces),}\\ \text{Segal categories, and}\\ \text{(saturated 1-trivial weak) 1-complicial sets}\end{array}\right.$
  each have enough maps and also an internal hom, and in fact any of these categories can be enriched over any of the others

Summary: the meaning of the notion of ∞-category is made precise by several models, connected by "change-of-model" functors.

The analytic vs synthetic theory of ∞-categories

Q: How might you develop the category theory of ∞-categories?

Two strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in all four models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

Our method: introduce an ∞-cosmos to axiomatize the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories.

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∞-cosmoi of ∞-categories

∞-cosmoi of ∞-categories

Idea: An ∞-cosmos is an "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits" whose objects we call ∞-categories.

An ∞-cosmos is a category that

- is enriched over quasi-categories, i.e., functors $f\colon A \to B$ between ∞-categories define the points of a quasi-category $\mathsf{Fun}(A, B)$,
- has a class of isofibrations $E \twoheadrightarrow B$ with familiar closure properties,
- and has flexibly-weighted limits of diagrams of ∞-categories and isofibrations that satisfy strict simplicial universal properties.

Theorem. $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, and 1-$\mathcal{C}omp$ define ∞-cosmoi, and so do certain models of (∞, n)-categories for $0 \leq n \leq \infty$, fibered versions of all of the above, and many more things besides.

Henceforth ∞-category and ∞-functor are technical terms that mean the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
- 1-cells are the ∞-functors $f\colon A \to B$ in the ∞-cosmos
- 2-cells we call ∞-natural transformations $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A,B)$

Prop (R-Verity). Equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "1_B"] \arrow[r, bend right, "fg"'] \arrow[r, phantom, "\Downarrow\cong"] & B \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

Thus, non-evil 2-categorical definitions are "homotopically correct."

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A taste of the formal category theory of ∞-categories

Adjunctions between ∞-categories

An adjunction between ∞-categories is an adjunction in the homotopy 2-category, consisting of:

- ∞-categories A and B
- ∞-functors $u\colon A \to B$, $f\colon B \to A$
- ∞-natural transformations $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_A$

satisfying the triangle equalities

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & B \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow\eta"] & {} \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

Write $f \dashv u$ to indicate that f is the left adjoint and u is the right adjoint.

The 2-category theory of adjunctions

Since an adjunction between ∞-categories is just an adjunction in the homotopy 2-category, all 2-categorical theorems about adjunctions become theorems about adjunctions between ∞-categories.

Prop. Adjunctions compose:

$$\begin{tikzcd} C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"] \end{tikzcd}$$

Prop. Adjoints to a given functor $u\colon A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u\colon A \xrightarrow{\sim} B$ then u is left and right adjoint to its equivalence inverse.

Limits and colimits in an ∞-category

An ∞-category A has

- a terminal element iff $\begin{tikzcd} A \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, dashed, "t"] \end{tikzcd}$

- limits of shape J iff $\begin{tikzcd} A \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & A^J \arrow[l, bend left, dashed, "\lim"] \end{tikzcd}$ or equivalently iff the limit cone

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, dashed, "\lim"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

is an absolute right lifting

- a limit of a diagram d iff $\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\epsilon"] & A^J \end{tikzcd}$ is an absolute right lifting.

Prop. Right adjoints preserve limits and left adjoints preserve colimits

— and the proof is the usual one !

Universal properties of adjunctions, limits, and colimits

Any ∞-category A has an ∞-category of arrows $A^{\mathbb{2}}$, pulling back to define the comma ∞-category:

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[r] \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\ C \times B \arrow[r, "g \times f"'] & A \times A \end{tikzcd}$$

Prop. $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ if and only if $\mathrm{Hom}_A(f, A) \simeq_{A\times B} \mathrm{Hom}_B(B, u)$.

Prop. If $f \dashv u$ with unit η and counit ϵ then

- ηb is initial in $\mathrm{Hom}_B(b, u)$ and ϵa is terminal in $\mathrm{Hom}_A(f, a)$.

Prop. $d\colon 1 \to A^J$ has a limit ℓ iff $\mathrm{Hom}_A(A, \ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta, d)$.

Prop. $d\colon 1 \to A^J$ has a limit iff $\mathrm{Hom}_{A^J}(\Delta, d)$ has a terminal element ϵ.

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The proof of model-independence of ∞-category theory

Cosmological biequivalences and change-of-model

A cosmological biequivalence $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi is

- a cosmological functor: a simplicial functor that preserves the isofibrations and the simplicial limits

that is additionally

- surjective on objects up to equivalence: if $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ with $FA \simeq C \in \mathcal{L}$
- a local equivalence: $\text{Fun}(A, B) \xrightarrow{\sim} \text{Fun}(FA, FB) \quad \in q\mathcal{C}at$

Prop. A cosmological biequivalence induces bijections on:

- equivalence classes of ∞-categories
- isomorphism classes of parallel ∞-functors
- 2-cells with corresponding boundary
- fibered equivalence classes of modules such as $\text{Hom}_A(f, g)$ respecting representability, e.g., $\text{Hom}_{A^J}(\Delta, d) \simeq_A \text{Hom}_A(A, \ell)$

Model-independence

$$\begin{tikzcd} \mathcal{R}\mathrm{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\mathrm{egal} \arrow[d, shift left] \\ 1\text{-}\mathcal{C}\mathrm{omp} \arrow[r, shift left] & \mathrm{q}\mathcal{C}\mathrm{at} \arrow[l, shift left] \arrow[u, shift left] \arrow[ul, shift left] \end{tikzcd} \qquad \leftsquigarrow$$

cosmological biequivalences between models of $(\infty, 1)$-categories

Model-Independence Theorem. Cosmological biequivalences preserve, reflect, and create all ∞-categorical properties and structures.

- The existence of an adjoint to a given functor.
- The existence of a limit for a given diagram.
- The property of a given functor defining a cartesian fibration.
- The existence of a pointwise Kan extension.

Analytically-proven theorems also transfer along biequivalences:

- Universal properties in an $(\infty, 1)$-category are determined objectwise.

Summary

- In the past, the theory of ∞-categories has been developed analytically, in a particular model.
- A large part of that theory can be developed simultaneously in many models by working synthetically with ∞-categories as objects in an ∞-cosmos.
- The axioms of an ∞-cosmos are chosen to simplify proofs by allowing us to work strictly up to isomorphism insofar as possible.
- Much of this development in fact takes place in a strict 2-category of ∞-categories, ∞-functors, and ∞-natural transformations using the methods of formal category theory.
- Both analytically- and synthetically-proven results about ∞-categories transfer across "change-of-model" functors called biequivalences.

References

For more on the model-independent theory of ∞-categories see:

Emily Riehl and Dominic Verity

- mini-course lecture notes:

  ∞-Category Theory from Scratch
  arXiv:1608.05314

- draft book in progress:

  ∞-Categories for the Working Mathematician
  www.math.jhu.edu/∼eriehl/ICWM.pdf

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