A model-independent theory of $\infty$-categories

joint with Dominic Verity

Joint International Meeting of the AMS and the CMS
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We develop the theory of $\infty$-categories from first principles in a “model-independent” fashion, that is, using a common axiomatic framework that is satisfied by a variety of models. Our “synthetic” definitions and proofs may be interpreted simultaneously in many models of $\infty$-categories, in contrast with “analytic” results proven using the combinatorics of a particular model. Nevertheless, we prove that both “synthetic” and “analytic” theorems transfer across specified “change of model” functors to establish the same results for other equivalent models.
Plan

**Goal:** develop model-independent foundations of $\infty$-category theory

1. What are model-independent foundations?

2. $\infty$-cosmoi of $\infty$-categories

3. A taste of the formal category theory of $\infty$-categories

4. The proof of model-independence of $\infty$-category theory
What are model-independent foundations?
The motivation for $\infty$-categories

Mere 1-categories are insufficient habitats for those mathematical objects that have higher-dimensional transformations encoding the “higher homotopical information” needed for a good theory of derived functors.

A better setting is given by $\infty$-categories, which have spaces rather than sets of morphisms, satisfying a weak composition law.

Thus, we want to extend 1-category theory (e.g., adjunctions, limits and colimits, universal properties, Kan extensions) to $\infty$-category theory.

First problem: it is hard to say exactly what an $\infty$-category is.
The idea of an $\infty$-category

$\infty$-categories are the nickname that Lurie gave to $(\infty, 1)$-categories, which are categories weakly enriched over homotopy types.

The schematic idea is that an $\infty$-category should have

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

But this definition is tricky to make precise.
Models of $\infty$-categories

- Topological categories and relative categories are the simplest to define but do not have enough maps between them.
  - Quasi-categories (nee. weak Kan complexes),
  - Rezk spaces (nee. complete Segal spaces),
  - Segal categories,
  - (saturated 1-trivial weak) 1-complicial sets

Each have enough maps and also an internal hom, and in fact any of these categories can be enriched over any of the others.

Summary: the meaning of the notion of $\infty$-category is made precise by several models, connected by “change-of-model” functors.
The analytic vs synthetic theory of $\infty$-categories

Q: How might you develop the category theory of $\infty$-categories?

Two strategies:

• work analytically to give categorical definitions and prove theorems using the combinatorics of one model
  
  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\text{Cat}$; Kazhdan-Varshavsky, Rasekh in $\text{Rezk}$; Simpson in $\text{Segal}$)

• work synthetically to give categorical definitions and prove theorems in all four models $q\text{Cat}$, $\text{Rezk}$, $\text{Segal}$, $1\text{-Comp}$ at once

Our method: introduce an $\infty$-cosmos to axiomatize the common features of the categories $q\text{Cat}$, $\text{Rezk}$, $\text{Segal}$, $1\text{-Comp}$ of $\infty$-categories.
∞-cosmoi of ∞-categories
∞-cosmoi of ∞-categories

Idea: An ∞-cosmos is an “(∞, 2)-category with (∞, 2)-categorical limits” whose objects we call ∞-categories.

An ∞-cosmos is a category that

- is enriched over quasi-categories, i.e., functors \( f: A \to B \) between ∞-categories define the points of a quasi-category \( \text{Fun}(A, B) \),
- has a class of isofibrations \( E \twoheadrightarrow B \) with familiar closure properties,
- and has flexibly-weighted limits of diagrams of ∞-categories and isofibrations that satisfy strict simplicial universal properties.

Theorem. \( \text{qCat}, \text{Rezk}, \text{Segal}, \) and \( 1\text{-Comp} \) define ∞-cosmoi, and so do certain models of \((\infty, n)\)-categories for \( 0 \leq n \leq \infty \), fibered versions of all of the above, and many more things besides.

Henceforth ∞-category and ∞-functor are technical terms that mean the objects and morphisms of some ∞-cosmos.
The homotopy 2-category

The homotopy 2-category of an $\infty$-cosmos is a strict 2-category whose:

- objects are the $\infty$-categories $A, B$ in the $\infty$-cosmos
- 1-cells are the $\infty$-functors $f: A \to B$ in the $\infty$-cosmos
- 2-cells we call $\infty$-natural transformations $A \xrightarrow{\gamma} B$ which are defined to be homotopy classes of 1-simplices in $\text{Fun}(A, B)$

**Prop (R-Verity). Equivalences in the homotopy 2-category**

$$A \xrightarrow{f} B \xleftarrow{g} A$$

$$A \xrightarrow{1_A} A \xleftarrow{gf} B$$

$$B \xrightarrow{1_B} B \xleftarrow{fg} B$$

coincide with equivalences in the $\infty$-cosmos.

Thus, non-evil 2-categorical definitions are “homotopically correct.”
A taste of the formal category theory of $\infty$-categories
Adjunctions between $\infty$-categories

An adjunction between $\infty$-categories is an adjunction in the homotopy 2-category, consisting of:

- $\infty$-categories $A$ and $B$
- $\infty$-functors $u : A \to B$, $f : B \to A$
- $\infty$-natural transformations $\eta : \text{id}_B \Rightarrow uf$ and $\epsilon : fu \Rightarrow \text{id}_A$

satisfying the triangle equalities

Write $f \dashv u$ to indicate that $f$ is the left adjoint and $u$ is the right adjoint.
The 2-category theory of adjunctions

Since an adjunction between $\infty$-categories is just an adjunction in the homotopy 2-category, all 2-categorical theorems about adjunctions become theorems about adjunctions between $\infty$-categories.

**Prop.** Adjunctions compose:

\[
\begin{array}{c}
C \xleftarrow{f'} \mathllap{\perp} B \xrightarrow{f} A \xrightleftharpoons{\sim} C \xleftarrow{f f'} \mathllap{\perp} A
\end{array}
\]

**Prop.** Adjoint to a given functor $u : A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \simeq f'$.

**Prop.** Any equivalence can be promoted to an adjoint equivalence: if $u : A \xrightarrow{\sim} B$ then $u$ is left and right adjoint to its equivalence inverse.
Limits and colimits in an $\infty$-category

An $\infty$-category $A$ has

- a terminal element iff $A \xrightarrow{t} 1$

- limits of shape $J$ iff $A \xrightarrow{\text{lim}} A^J$ or equivalently iff the limit cone $\xrightarrow{\epsilon} \Delta$ is an absolute right lifting

- a limit of a diagram $d$ iff $\xrightarrow{d} \text{lim} d \xrightarrow{\epsilon} \Delta$ is an absolute right lifting.

Prop. Right adjoints preserve limits and left adjoints preserve colimits — and the proof is the usual one!
Universal properties of adjunctions, limits, and colimits

Any ∞-category $A$ has an ∞-category of arrows $A^2$, pulling back to define the comma ∞-category:

$$
\begin{align*}
\text{Hom}_A(f, g) & \longrightarrow A^2 \\
\downarrow \ (\text{cod, dom}) & \downarrow \\
C \times B & \longrightarrow A \times A \\
\downarrow \ (\text{cod, dom}) & \\
A^2 & \end{align*}
$$

Prop. $A \perp_u B$ if and only if $\text{Hom}_A(f, A) \simeq_{A \times B} \text{Hom}_B(B, u)$.

Prop. If $f \dashv u$ with unit $\eta$ and counit $\epsilon$ then

- $\eta b$ is initial in $\text{Hom}_B(b, u)$ and $\epsilon a$ is terminal in $\text{Hom}_A(f, a)$.

Prop. $d : 1 \rightarrow A^J$ has a limit $\ell$ iff $\text{Hom}_A(A, \ell) \simeq_A \text{Hom}_{A^J}(\Delta, d)$.

Prop. $d : 1 \rightarrow A^J$ has a limit iff $\text{Hom}_{A^J}(\Delta, d)$ has a terminal element $\epsilon$. 
The proof of model-independence of $\infty$-category theory
Cosmological biequivalences and change-of-model

A cosmological biequivalence $F: \mathcal{K} \rightarrow \mathcal{L}$ between $\infty$-cosmoi is

- a cosmological functor: a simplicial functor that preserves the isofibrations and the simplicial limits

that is additionally

- surjective on objects up to equivalence: if $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ with $FA \cong C \in \mathcal{L}$

- a local equivalence: $\text{Fun}(A, B) \sim \text{Fun}(FA, FB) \in q\text{Cat}$

Prop. A cosmological biequivalence induces bijections on:

- equivalence classes of $\infty$-categories
- isomorphism classes of parallel $\infty$-functors
- 2-cells with corresponding boundary
- fibered equivalence classes of modules such as $\text{Hom}_A(f, g)$ respecting representability, e.g., $\text{Hom}_{A J}(\Delta, d) \cong_A \text{Hom}_A(A, \ell)$
Model-independence

\[
\begin{array}{ccc}
\text{Rezk} & \xrightarrow{} & \text{Segal} \\
\downarrow & & \downarrow \\
\text{1-Comp} & \xleftrightarrow{} & \text{qCat}
\end{array}
\]

\[\sim\]

Model-Independence Theorem. Cosmological biequivalences preserve, reflect, and create all $\infty$-categorical properties and structures.

- The existence of an adjoint to a given functor.
- The existence of a limit for a given diagram.
- The property of a given functor defining a cartesian fibration.
- The existence of a pointwise Kan extension.

Analytically-proven theorems also transfer along biequivalences:
- Universal properties in an $(\infty, 1)$-category are determined objectwise.
Summary

- In the past, the theory of $\infty$-categories has been developed analytically, in a particular model.
- A large part of that theory can be developed simultaneously in many models by working synthetically with $\infty$-categories as objects in an $\infty$-cosmos.
- The axioms of an $\infty$-cosmos are chosen to simplify proofs by allowing us to work strictly up to isomorphism insofar as possible.
- Much of this development in fact takes place in a strict 2-category of $\infty$-categories, $\infty$-functors, and $\infty$-natural transformations using the methods of formal category theory.
- Both analytically- and synthetically-proven results about $\infty$-categories transfer across “change-of-model” functors called biequivalences.
For more on the model-independent theory of ∞-categories see:

Emily Riehl and Dominic Verity

- mini-course lecture notes:
  ∞-Category Theory from Scratch
  arXiv:1608.05314

- draft book in progress:
  ∞-Categories for the Working Mathematician
  www.math.jhu.edu/~eriehl/ICWM.pdf

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