∞-Categories for the Working Mathematician

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Preface

In this text we develop the theory of ∞-categories from first principles in a model-independent fashion using a common axiomatic framework that is satisfied by a variety of models. In contrast with prior "analytic" treatments of the theory of ∞-categories — in which the central categorical notions are defined in reference to the combinatorics of a particular model — our approach is "synthetic," proceeding from definitions that can be interpreted simultaneously in many models to which our proofs then apply. While synthetic, our work is not schematic or hand-wavy, with the details of how to make things fully precise left to "the experts" and turtles all the way down.[1] Rather, we prove our theorems starting from a short list of clearly-enumerated axioms, and our conclusions are valid in any model of ∞-categories satisfying these axioms.

The synthetic theory is developed in any *∞-cosmos*, which axiomatizes the universe in which ∞-categories live as objects. So that our theorems are natural to state, we recast *∞-category* as a technical term, to mean an object in some (typically fixed) ∞-cosmos. Several models of $(\infty, 1)$-categories[2] are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories[3] as well as fibered versions of all of the above. This usage is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional category, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

Much of the development of the theory of ∞-categories takes place not in the full ∞-cosmos but in a quotient that we call the *homotopy 2-category*.[4] The homotopy 2-category is a strict 2-category — like the 2-category of categories, functors, and natural transformations — and in this way the foundational proofs in the theory of ∞-categories closely resemble the classical foundations of ordinary category theory except that the universal properties that characterize, e.g. when a functor between ∞-categories defines a cartesian fibration, are slightly weaker than in the familiar case. In Part I, we define and develop the notions of equivalence and adjunction between ∞-categories, limits and colimits in ∞-categories, cartesian and cocartesian fibrations and their discrete variants, and prove an external version of the Yoneda lemma all from the comfort of the homotopy 2-category. In Part **??**, we turn our attention to homotopy coherent structures present in the full ∞-cosmos to define

[1] A less rigorous "model-independent" presentation of ∞-category theory might confront a problem of infinite regress, since infinite-dimensional categories are themselves the objects of an ambient infinite-dimensional category, and in developing the theory of the former one is tempted to use the theory of the latter. We avoid this problem by using a very concrete model for the ambient $(\infty, 2)$-category of ∞-categories that arises frequently in practice is designed to facilitate relatively simple proofs.

[2] Quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets (naturally marked quasi-categories) all define the ∞-categories in an ∞-cosmos.

[3] Θ_n-spaces, iterated complete Segal spaces, and n-complicial sets also define the ∞-categories in an ∞-cosmos, as do (nee. weak) complicial sets, a model for (∞, ∞)-categories. We hope to add other models of (∞, n)-categories to this list.

[4] An ∞-cosmos is something like a category of fibrant objects in an enriched model category and the homotopy 2-category is a categorification of its homotopy category.

and study homotopy coherent adjunctions and monads borne by ∞-categories as a mechanism for universal algebra.

What's missing from this basic account of the category theory of ∞-categories is a satisfactory treatment of the "hom" bifunctor associated to an ∞-category, which is the prototypical example of what we call a *module*. In Part **??**, we develop the calculus of modules between ∞-categories and apply this to define and study pointwise Kan extensions. This will give us an opportunity to repackage universal properties proven in Part I as parts of the "formal category theory" of ∞-categories.

This work is all "model-agnostic" in the sense of being blind to details about the specifications of any particular ∞-cosmos. In Part **??** we prove that the category theory of ∞-categories is also "model-independent" in a precise sense: all categorical notions are preserved, reflected, and created by any "change-of-model" functor that defines what we call a *biequivalence*. This model-independence theorem is stronger than our axiomatic framework might initially suggest in that it also allows us to transfer theorems proven using "analytic" techniques to all biequivalent ∞-cosmoi. For instance, the four ∞-cosmoi whose objects model $(\infty, 1)$-categories are all biequivalent. It follows that the analytically-proven theorems about quasi-categories[5] from **[13]** transfer to complete Segal spaces, and vice versa.

[5]Importantly, the synthetic theory developed in the ∞-cosmos of quasi-categories is fully compatible the analytic theory developed by Joyal, Lurie, and many others. This is the subject of Appendix E.

Part I

Basic ∞-category theory

CHAPTER 1

∞-Cosmoi and their homotopy 2-categories

1.1. Quasi-categories

Before introducing an axiomatic framework that will allow us to develop ∞-category theory in general, we first consider one model in particular: namely, *quasi-categories*, which were first analyzed by Joyal in [**8**] and [**9**] and in several unpublished draft book manuscripts.

1.1.1. Notation (the simplex category). Let $\mathbf{\Delta}$ denote the **simplex category** of finite non-empty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps. These include in particular the

$$\text{elementary face operators} \qquad [n-1] \overset{\delta^i}{\rightarrowtail} [n] \qquad 0 \le i \le n \qquad \text{and the}$$

$$\text{elementary degeneracy operators} \qquad [n+1] \overset{\sigma^i}{\twoheadrightarrow} [n] \qquad 0 \le i \le n$$

whose images respectively omit and double up on the element $i \in [n]$. Every morphism in $\mathbf{\Delta}$ factors uniquely as an epimorphism followed by a monomorphism; these epimorphisms, the **degeneracy operators**, decompose as composites of elementary degeneracy operators, while the monomorphisms, the **face operators**, decompose as composites of elementary face operators.

The category of **simplicial sets** is the category $\mathcal{SS}et := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ of presheaves on the simplex category. We write $\Delta[n]$ for the **standard n-simplex**, the simplicial set represented by $[n] \in \mathbf{\Delta}$, and $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$ for its **boundary sphere** and **k-horn** respectively.

Given a simplicial set X, it is conventional to write X_n for the set of **n-simplices**, defined by evaluating at $[n] \in \mathbf{\Delta}$. By the Yoneda lemma, each n-simplex $x \in X_n$ corresponds to a map of simplicial sets $x \colon \Delta[n] \to X$. Accordingly, we write $x \cdot \delta^i$ for the ith face of the n-simplex, an $(n-1)$-simplex classified by the composite map

$$\Delta[n-1] \xrightarrow{\delta^i} \Delta[n] \xrightarrow{x} X.$$

Geometrically, $x \cdot \delta^i$ is the "face opposite the vertex i" in the n-simplex x.

Since the morphisms of $\mathbf{\Delta}$ are generated by the elementary face and degeneracy operators, the data of a simplicial set[1] X is often presented by a diagram

$$\cdots \qquad X_3 \;\substack{\xrightarrow{\delta_3} \\ \xleftarrow{\sigma_2} \\ \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}}\; X_2 \;\substack{\xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}}\; X_1 \;\substack{\xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}}\; X_0,$$

identifying the set of n-simplices for each $[n] \in \mathbf{\Delta}$ as well as the (contravariant) actions of the elementary operators, conventionally denoted using subscripts.

[1]This presentation is also used for more general simplicial *objects* valued in any category.

1.1.2. Definition (quasi-category). A **quasi-category** is a simplicial set A in which any **inner horn** can be extended to a simplex, solving the displayed lifting problem:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & A \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array} \qquad n \geq 2,\ 0 < k < n. \tag{1.1.3}$$

Quasi-categories were first introduced by Boardman and Vogt [4] under the name "weak Kan complexes," a **Kan complex** being a simplicial set admitting extensions as in (1.1.3) along all horn inclusions. Since any topological space can be encoded as a Kan complex,[2] in this way spaces provide examples of quasi-categories.

Categories also provide examples of quasi-categories via the nerve construction.

1.1.4. Definition (nerve). The category $\mathcal{C}at$ of strict 1-categories embeds fully faithfully into the category of simplicial sets via the **nerve** functor. An n-simplex in the nerve of a category C is a sequence of n composable arrows in C, or equally a functor $[n] \to \mathsf{C}$ from the ordinal category $\mathbb{n}+\mathbb{1} := [n]$.

The nerve of a category C is 2-coskeletal as a simplicial set, meaning that every sphere $\partial\Delta[n] \to C$ with $n \geq 3$ is filled uniquely by an n-simplex in C (see Definition C.2.1). This is because the simplices in dimension 3 and above witness the associativity of the composition of the path of composable arrows found along their **spine**, the 1-skeletal simplicial subset formed by the edges connecting adjacent vertices.

1.1.5. Proposition (nerves are quasi-categories). *Nerves of categories are quasi-categories.*

Proof. Via the isomorphism $\mathsf{C} \cong \mathrm{cosk}_2\, \mathsf{C}$ and the adjunction $\mathrm{sk}_2 \dashv \mathrm{cosk}_2$ of C.2.1, the required lifting problem displayed below-left transposes to the one displayed below-right

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathsf{C} \cong \mathrm{cosk}_2\, \mathsf{C} \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ccc} \mathrm{sk}_2\, \Lambda^k[n] & \longrightarrow & \mathsf{C} \\ \big\downarrow & \nearrow & \\ \mathrm{sk}_2\, \Delta[n] & & \end{array}$$

For $n \geq 4$, the inclusion $\mathrm{sk}_2\, \Lambda^k[n] \hookrightarrow \mathrm{sk}_2\, \Delta[n]$ is an isomorphism, in which case the lifting problems on the right admit (unique) solutions. So it remains only to solve the lifting problems on the left in the cases $n = 2$ and $n = 3$.

To that end consider

$$\begin{array}{ccc} \Lambda^1[2] & \longrightarrow & \mathsf{C} \\ \big\downarrow & \nearrow & \\ \Delta[2] & & \end{array} \qquad \begin{array}{ccc} \Lambda^1[3] & \longrightarrow & \mathsf{C} \\ \big\downarrow & \nearrow & \\ \Delta[3] & & \end{array} \qquad \begin{array}{ccc} \Lambda^2[3] & \longrightarrow & \mathsf{C} \\ \big\downarrow & \nearrow & \\ \Delta[3] & & \end{array}$$

An inner horn $\Lambda^1[2] \to \mathsf{C}$ defines a composable pair of arrows in C; an extension to a 2-simplex exists precisely because any composable pair of arrows admits a (unique) composite.

[2]The total singular complex construction defines a functor from topological spaces to simplicial sets that is an equivalence on their respective homotopy categories — weak homotopy types of spaces correspond to homotopy equivalence classes of Kan complexes.

An inner horn $\Lambda^1[3] \to C$ specifies the data of three composable arrows in C, as displayed in the diagram below, together with the composites gf, hg, and $(hg)f$.

$$\begin{array}{ccccc} & & c_1 & & \\ & \overset{f}{\nearrow} & & \overset{hg}{\searrow} & \\ c_0 & \longrightarrow & \Big\downarrow{\scriptstyle g} \; \xrightarrow{(hg)f} & & c_3 \\ & \underset{gf}{\searrow} & & \underset{h}{\nearrow} & \\ & & c_2 & & \end{array}$$

Because composition is associative, the arrow $(hg)f$ is also the composite of gf followed by h, which proves that the 2-simplex opposite the vertex c_1 is present in C; by 2-coskeletality, the 3-simplex filling this boundary sphere is also present in C. The filler for a horn $\Lambda^2[3] \to C$ is constructed similarly. ☐

In fact, as suggested by the proof of Proposition 1.1.5, any quasi-category in which inner horns admit *unique* fillers is isomorphic to the nerve of a strict 1-category; see Exercise 1.1.iii.

We decline to introduce explicit notation for the nerve functor, preferring instead to identify strict 1-categories with their nerves. As we shall discover the theory of strict 1-categories extends to ∞-categories modeled as quasi-categories in such a way that the restriction of each ∞-categorical concept along the nerve embedding recovers the corresponding strict 1-categorical concept.

1.1.6. Definition (homotopy relation on 1-simplices). A parallel pair of 1-simplices f, g in a simplicial set X are **homotopic** if there exists a 2-simplex of either of the following forms

$$\begin{array}{ccc} & x_1 & \\ {\scriptstyle f}\nearrow & & \searrow\!\!\searrow \\ x_0 & \xrightarrow[g]{} & x_1 \end{array} \qquad\qquad \begin{array}{ccc} & x_0 & \\ \nearrow\!\!\nearrow & & \searrow{\scriptstyle f} \\ x_0 & \xrightarrow[g]{} & x_1 \end{array} \tag{1.1.7}$$

or if f and g are in the same equivalence class generated by this relation.

In a quasi-category, the relation witnessed by any of the types of 2-simplex on display in (1.1.7) is an equivalence relation and these equivalence relations coincide:

1.1.8. Lemma (homotopic 1-simplices in a quasi-category). *Parallel 1-simplices f and g in a quasi-category are homotopic if and only if there exists a 2-simplex of any or equivalently all of the forms displayed in* (1.1.7).

Proof. Exercise 1.1.i. ☐

1.1.9. Definition (the homotopy category). By 1-truncating, any simplicial set X has an underlying reflexive directed graph

$$X_1 \begin{array}{c} \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow[\delta_0]{} \end{array} X_0,$$

the 0-simplices of X defining the "objects" and the 1-simplices defining the "arrows," by convention pointing from their 0th vertex (the face opposite 1) to their 1st vertex (the face opposite 0). The **free category** on this reflexive directed graph has X_0 as its object set, degenerate 1-simplices serving as identity morphisms, and non-identity morphisms defined to be finite directed paths of non-degenerate 1-simplices. The **homotopy category** $\mathrm{h}X$ of X is the quotient of the free category on its underlying reflexive directed graph by the congruence[3] generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{array}{ccc} & x_1 & \\ {}^{f}\nearrow & & \searrow^{g} \\ x_0 & \xrightarrow[h]{} & x_2 \end{array}$$

This implies in particular that homotopic 1-simplices represent the same arrow in the homotopy category.

1.1.10. Proposition. *The nerve embedding admits a left adjoint, namely the functor which sends a simplicial set to its homotopy category:*

$$\mathcal{C}at \underset{\hookrightarrow}{\overset{\mathsf{h}}{\underset{\perp}{\leftarrow}}} s\mathcal{S}et$$

Proof. Using the description of $\mathsf{h}X$ as a quotient of the free category on the underlying reflexive directed graph of X, we argue that the data of a functor $\mathsf{h}X \to C$ can be extended uniquely to a simplicial map $X \to C$. Presented as a quotient in this way, the functor $\mathsf{h}X \to C$ defines a map from the 1-skeleton of X into C, and since every 2-simplex in X witnesses a composite in $\mathsf{h}X$, this map extends to the 2-skeleton. Now C is 2-coskeletal, so via the adjunction $\mathsf{sk}_2 \dashv \mathsf{cosk}_2$ of Definition C.2.1, this map from the 2-truncation of X into C extends uniquely to a simplicial map $X \to C$. ☐

The homotopy category of a quasi-category admits a simplified description.

1.1.11. Lemma (the homotopy category of a quasi-category). *If A is a quasi-category then its* ***homotopy category*** *$\mathsf{h}A$ has*

- *the set of 0-simplices A_0 as its objects*
- *the set of homotopy classes of 1-simplices A_1 as its arrows*
- *a composition relation $h = g \circ f$ in $\mathsf{h}A$ if and only if, for any choices of 1-simplices representing these arrows, there exists a 2-simplex with boundary*

$$\begin{array}{ccc} & a_1 & \\ {}^{f}\nearrow & & \searrow^{g} \\ a_0 & \xrightarrow[h]{} & a_2 \end{array}$$

Proof. Exercise 1.1.ii. ☐

1.1.12. Definition (isomorphisms in a quasi-category). A 1-simplex in a quasi-category is an **isomorphism** just when it represents an isomorphism in the homotopy category. By Lemma 1.1.11 this means that $f \colon a \to b$ is an isomorphism if and only if there exist a 1-simplex $f^{-1} \colon b \to a$ together with a pair of 2-simplices

$$\begin{array}{ccc} & b & \\ {}^{f}\nearrow & & \searrow^{f^{-1}} \\ a & = & a \end{array} \qquad \begin{array}{ccc} & a & \\ {}^{f^{-1}}\nearrow & & \searrow^{f} \\ b & = & b \end{array}$$

[3] A relation on parallel pairs of arrows of a strict 1-category is a **congruence** if it is an equivalence relation that is closed under pre- and post-composition: if $f \sim g$ then $hfk \sim hgk$.

The properties of the isomorphisms in a quasi-category are most easily proved by arguing in a slightly different category where simplicial sets have the additional structure of a "marking" on a specified subset of the 1-simplices subject to the condition that all degenerate 1-simplices are marked; maps of these so-called **marked simplicial sets** must then preserve the markings. Because these objects will seldom appear outside of the proofs of certain combinatorial lemmas about the isomorphisms in quasi-categories, we save the details for Appendix C.

Let us now motivate the first of several results proven using marked techniques. Quasi-categories are defined to have extensions along all *inner horns*. But if in an outer horn $\Lambda^0[2] \to A$ or $\Lambda^2[2] \to A$, the initial or final edges, respectively, are isomorphisms, then intuitively a filler should exist

$$\begin{array}{ccccc} & & a_1 & & \\ & \overset{f}{\nearrow} & & \overset{hf^{-1}}{\searrow} & \\ a_0 & & \xrightarrow[h]{\quad} & & a_2 \end{array} \qquad \begin{array}{ccccc} & & a_1 & & \\ & \overset{g^{-1}h}{\nearrow} & & \overset{g}{\searrow} & \\ a_0 & & \xrightarrow[h]{\quad} & & a_2 \end{array}$$

and similarly for the higher-dimensional outer horns.

1.1.13. Proposition (special outer horn lifting).

(i) Let A be a quasi-category. Then any outer horns

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{g} & A \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array} \qquad \begin{array}{ccc} \Lambda^n[n] & \xrightarrow{h} & A \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.

(ii) Let A and B be quasi-categories and $f\colon A \to B$ a map that lifts against the inner horn inclusions. Then any outer horns

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{g} & A \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta[n] & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} \Lambda^n[n] & \xrightarrow{h} & A \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta[n] & \longrightarrow & B \end{array}$$

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.

The proof of Proposition 1.1.13 requires clever combinatorics, due to Joyal, and is deferred to Appendix C.[4] Here, we enjoy its myriad consequences. Immediately:

1.1.14. Corollary. *A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.*

Proof. If the homotopy category of a quasi-category is a groupoid, then all of its 1-simplices are isomorphisms, and Proposition 1.1.13 then implies that all inner and outer horns have fillers. Thus, the quasi-category is a Kan complex. Conversely, in a Kan complex, all outer horns can be filled and in particular fillers for the horns $\Lambda^0[2]$ and $\Lambda^2[2]$ can be used to construct left and right inverses for any 1-simplex of the form displayed in Definition 1.1.12.[5] □

[4]The second statement subsumes the first, but the first is typically used to prove the second.

[5]In a quasi-category, any left and right inverses to a common 1-simplex are homotopic, but as Corollary 1.1.15 proves, any isomorphism in fact has a single two-sided inverse.

A quasi-category contains a canonical **maximal sub Kan complex**, the simplicial subset spanned by those 1-simplices that are isomorphisms. Just as the arrows in a quasi-category A are represented by simplicial maps $\mathbb{2} \to A$ whose domain is the nerve of the free-living arrow, the isomorphisms in a quasi-category are represented by diagrams $\mathbb{I} \to A$ whose domain is the free-living isomorphism:

1.1.15. Corollary. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a* ***homotopy coherent isomorphism***

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & A \\ \big\downarrow & \nearrow & \\ \mathbb{I} & & \end{array}$$

Proof. If f is an isomorphism, the map $f\colon \mathbb{2} \to A$ lands in the maximal sub Kan complex contained in A. The postulated extension also lands in this maximal sub Kan complex because the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ can be expressed as a sequential composite of outer horn inclusions; see Exercise 1.1.iv. □

The category of simplicial sets, like any category of presheaves, is cartesian closed. By the Yoneda lemma and the defining adjunction, an n-simplex in the exponential Y^X corresponds to a simplicial map $X \times \Delta[n] \to Y$, and its faces and degeneracies are computed by precomposing in the simplex variable. Our aim is now to show that the quasi-categories define an exponential ideal in the simplicially enriched category of simplicial sets: if X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. We will deduce this as a corollary of the "relative" version of this result involving a class of maps called isofibrations that we now introduce.

1.1.16. Definition (isofibrations between quasi-categories). A simplicial map $f\colon A \to B$ is a **isofibration** if it lifts against the inner horn inclusions, as displayed below left, and also against the inclusion of either vertex into the free-standing isomorphism $\mathbb{I}$.

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & A \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta[n] & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{1} & \longrightarrow & A \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathbb{I} & \longrightarrow & B \end{array}$$

To notationally distinguish the isofibrations, we depict them as arrows "$\twoheadrightarrow$" with two heads.

1.1.17. Observation.

(i) For any simplicial set X, the unique map $X \to *$ whose codomain is the terminal simplicial set is an isofibration if and only if X is a quasi-category.

(ii) Any class of maps characterized by a right lifting property is automatically closed under composition, product, pullback, retract, and limits of towers; see Lemma C.1.1.

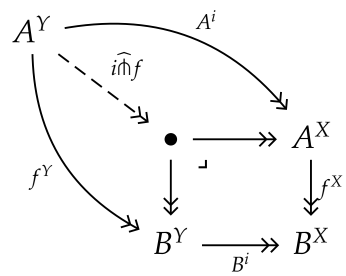
(iii) Combining (i) and (ii), if $A \twoheadrightarrow B$ is an isofibration, and B is a quasi-category, then so is A.

(iv) The isofibrations generalize the eponymous categorical notion. The nerve of any functor $f\colon A \to B$ between categories defines a map of simplicial sets that lifts against the inner horn inclusions. This map then defines an isofibration if and only if given any isomorphism in B and specified object in A lifting either its domain or codomain, there exists an isomorphism in A with that domain or codomain lifting the isomorphism in B.

We typically only deploy the term "isofibration" for a map between quasi-categories because our usage of this class of maps intentionally parallels the classical categorical case.

Much harder to establish is the stability of the class of isofibrations under forming "Leibniz exponentials" as displayed in (1.1.19). The proof of this result is given in Proposition **??** in Appendix C.

1.1.18. Proposition. *If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map*



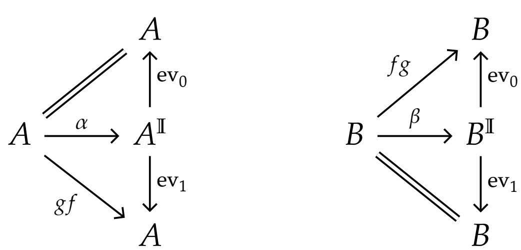
(1.1.19)

is again an isofibration.[6]

1.1.20. Corollary. *If X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. Moreover, a 1-simplex in A^X is an isomorphism if and only if its components at each vertex of X are isomorphisms in A.*

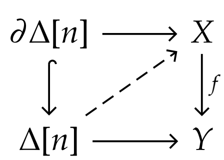
Proof. The first statement is a special case of Proposition 1.1.18; see Exercise 1.1.vi. The second statement is proven similarly by arguing with marked simplicial sets. See Lemma **??**. □

1.1.21. Definition (equivalences of quasi-categories). A map $f\colon A \to B$ between quasi-categories is an **equivalence** if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g\colon B \to A$ and



We write "$\xrightarrow{\sim}$" to decorate equivalences and $A \simeq B$ to indicate the presence of an equivalence $A \xrightarrow{\sim} B$.

1.1.22. Definition. A map $f\colon X \to Y$ between simplicial sets is a **trivial fibration** if it admits lifts against the boundary inclusions for all simplices



$n \geq 0$

(1.1.23)

We write "$\overset{\sim}{\twoheadrightarrow}$" to decorate trivial fibrations.

1.1.24. Remark. The simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ "cellularly generate" the monomorphisms of simplicial sets — see Definition C.1.2 and Lemma C.2.2. Hence the dual of Lemma C.1.1 implies that trivial fibrations lift against any monomorphism between simplicial sets. In particular, applying this to the map $\emptyset \to Y$, it follows that any trivial fibration $X \overset{\sim}{\twoheadrightarrow} Y$ is a split epimorphism.

[6] Degenerate cases of this result, taking $X = \emptyset$ or $B = 1$, imply that the other six maps in this diagram are also isofibrations; see Exercise 1.1.vi.

The notation "$\overset{\sim}{\twoheadrightarrow}$" is suggestive: the trivial fibrations between quasi-categories are exactly those maps that are both isofibrations and equivalences. This can be proven by a relatively standard although rather technical argument in simplicial homotopy theory, given as Proposition **??** in Appendix C.

1.1.25. Proposition. *For a map $f\colon A \to B$ between quasi-categories the following are equivalent:*

(i) f is at trivial fibration
(ii) f is both an isofibration and an equivalence
*(iii) f is a **split fiber homotopy equivalence**: an isofibration admitting a section s that is also an equivalence inverse via a homotopy from sf to 1_A that composes with f to the constant homotopy from f to f.*

As a class characterized by a right lifting property, the trivial fibrations are also closed under composition, product, pullback, limits of towers, and contain the isomorphisms. The stability of these maps under Leibniz exponentiation will be verified along with Proposition 1.1.18 in Proposition **??**.

1.1.26. Proposition. *If $i\colon X \to Y$ is a monomorphism and $f\colon A \to B$ is an isofibration, then if either f is a trivial fibration or if i is in the class cellularly generated*[7] *by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ then the induced Leibniz exponential map*

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

a trivial fibration.

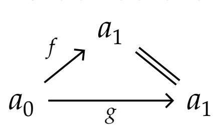
1.1.27. Digression (the Joyal model structure). The category of simplicial set bears a model structure (see Appendix C) whose fibrant objects are exactly the quasi-categories; all objects are cofibrant. The fibrations, weak equivalences, and trivial fibrations between fibrant objects are precisely the classes of isofibrations, equivalences, and trivial fibrations defined above. Proposition 1.1.25 proves that the trivial fibrations are the intersection of the classes of fibrations and weak equivalences. Propositions 1.1.18 and 1.1.26 reflect the fact that the Joyal model structure is a *closed monoidal model category* with respect to the cartesian closed structure on the category of simplicial sets.

We have declined to elaborate on the Joyal model structure for quasi-categories alluded to in Digression 1.1.27 because the only aspects of it that we will need are those described above. The results proven here suffice to show that the category of quasi-categories defines an ∞-*cosmos*, a concept to which we now turn.

Exercises.

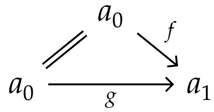
1.1.i. Exercise. Consider the set of 1-simplices in a quasi-category with initial vertex a_0 and final vertex a_1.

(i) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary



is an equivalence relation.

(ii) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary



is an equivalence relation.

[7] See Definition C.1.2.

(iii) Prove that the equivalence relations defined by (i) and (ii) are the same.

This proves Lemma 1.1.8.

1.1.ii. Exercise. Consider the free category on the reflexive directed graph

$$A_1 \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \xleftarrow{\sigma_0} A_0,$$

underlying a quasi-category A.

(i) Consider the relation that identifies a pair of sequences of composable 1-simplices with common source and common target if and only if there exists a simplex of A in which the sequences of 1-simplices define two paths from its initial vertex to its final vertex. Sketch a proof that this relation is an equivalence relation and prove that it is a congruence.[8]

(ii) Consider the congruence relation generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{array}{ccc} & a_1 & \\ {}^{f}\nearrow & & \searrow^{g} \\ a_0 & \xrightarrow[h]{} & a_2 \end{array}$$

and prove that this coincides with the relation considered in (i).

(iii) In the congruence relations of (i) and (ii), prove that every sequence of composable 1-simplices in A is equivalent to a single 1-simplex. Conclude that every morphism in the quotient of the free category by this congruence relation is represented by a 1-simplex in A.

(iv) Prove that for any triple of 1-simplices f, g, h in A, $h = g \circ f$ in the quotient category if and only if there exists a 2-simplex with boundary

$$\begin{array}{ccc} & a_1 & \\ {}^{f}\nearrow & & \searrow^{g} \\ a_0 & \xrightarrow[h]{} & a_2 \end{array}$$

This proves Lemma 1.1.11.

1.1.iii. Exercise. Show that any quasi-category in which inner horns admit unique fillers is isomorphic to the nerve of its homotopy category.

1.1.iv. Exercise.

(i) Prove that $\mathbb{I}$ contains exactly two non-degenerate simplices in each dimension.

(ii) Inductively build $\mathbb{I}$ from $\mathbb{2}$ by expressing the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ as a sequential composite of pushouts of outer horn inclusions[9] $\Lambda^0[n] \hookrightarrow \Delta[n]$, one in each dimension starting with $n = 2$.[10]

---

[8]Given a congruence relation on the hom-sets of a strict 1-category, the quotient category can be formed by quotienting each hom-set; see [**14**, §II.8].

[9]By duality — the **opposite** of a simplicial set X is the simplicial set obtained by reindexing along the involution $(-)^{op} : \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering in each ordinal — the outer horn inclusions $\Lambda^n[n] \hookrightarrow \Delta[n]$ can be used instead.

[10]This decomposition of the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ reveals which data can always be extended to a homotopy coherent isomorphism: for instance, the 1- and 2-simplices of Definition 1.1.12 together with a single 3-simplex that has these as its outer faces with its inner faces degenerate.

1.1.v. Exercise. Prove the relative version of Corollary 1.1.15: for any isofibration $p\colon A \twoheadrightarrow B$ between quasi-categories and any isomorphism $f\colon \mathbb{2} \to A$ any homotopy coherent isomorphism in B extending pf lifts to a homotopy coherent isomorphism in A extending f.

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & A \\ \big\downarrow & \nearrow & \big\downarrow p \\ \mathbb{I} & \longrightarrow & B \end{array}$$

1.1.vi. Exercise. Specialize Proposition 1.1.18 to prove the following:

(i) If A is a quasi-category and X is a simplicial set then A^X is a quasi-category.
(ii) If A is a quasi-category and $X \hookrightarrow Y$ is a monomorphism then $A^Y \twoheadrightarrow A^X$ is an isofibration.
(iii) If $A \twoheadrightarrow B$ is an isofibration and X is a simplicial set then $A^X \twoheadrightarrow B^X$ is an isofibration.

1.1.vii. Exercise.

(i) Prove that the equivalences defined in Definition 1.1.21 are closed under retracts.
(ii) Prove that the equivalences defined in Definition 1.1.21 satisfy the 2-of-3 property.

1.2. ∞-Cosmoi

In §1.1, we presented "analytic" proofs of a few of the basic facts about quasi-categories. The category theory of quasi-categories can be developed in a similar style, but we aim instead to develop the "synthetic" theory of infinite-dimensional categories, so that our results will apply to many models at once. To achieve this, our strategy is not to axiomatize what these infinite-dimensional categories *are*, but rather axiomatize the "universe" in which they *live*.

The following definition abstracts the properties of the quasi-categories and the classes of isofibrations, equivalences, and trivial fibrations introduced in §1.1. Firstly, the category of quasi-categories and simplicial maps is *enriched* over the category of simplicial sets — the set of morphisms from A to B coincides with the set of vertices of the simplicial set B^A — and moreover these hom-spaces are all quasi-categories. Secondly, a number of limit constructions that can be defined in the underlying 1-category of quasi-categories and simplicial maps satisfy universal properties relative to this simplicial enrichment. And finally, the classes of isofibrations, equivalences, and trivial fibrations satisfy properties that are familiar from abstract homotopy theory.

As will be explained in Digression 1.2.9, there are a variety of models of infinite-dimensional categories for which the category of "∞-categories," as we will call them, and "functors" between them is enriched over quasi-categories and admits classes of isofibrations, equivalences, and trivial fibrations satisfying analogous properties. This motivates the following axiomatization:

1.2.1. Definition (∞-cosmoi). An **∞-cosmos** $\mathcal{K}$ is a category enriched over quasi-categories,[11] meaning that it has

- objects A, B, that we call **∞-categories**, and
- its morphisms define the vertices of **functor-spaces** $\mathsf{Fun}(A,B)$, which are quasi-categories,

that is also equipped with a specified class of maps that we call **isofibrations** and denote by "$\twoheadrightarrow$." From these classes, we define a map $f\colon A \to B$ to be an **equivalence** if and only the induced map $f_*\colon \mathsf{Fun}(X,A) \xrightarrow{\sim} \mathsf{Fun}(X,B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathcal{K}$,

[11] This is to say $\mathcal{K}$ is a simplicially enriched category whose hom-spaces are quasi-categories; this will be unpacked in 1.2.2.

and we define f to be a **trivial fibration** just when f is both an isofibration and an equivalence; these classes are denoted by "$\xrightarrow{\sim}$" and "$\overset{\sim}{\twoheadrightarrow}$" respectively. These classes must satisfy the following three axioms:

(i) (completeness) The simplicial category $\mathcal{K}$ possesses a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.[12]

(ii) (isofibrations) The class of isofibrations contains all isomorphisms and any map whose codomain is the terminal object; is closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and has the property that if $f \colon A \twoheadrightarrow B$ is an isofibration and X is any object then $\mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

(iii) (cofibrancy) Every trivial fibration admits a section

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow{\wr} \\ B & = & B \end{array}$$

1.2.2. Digression (simplicial categories). A **simplicial category** $\mathcal{A}$ is given by categories $\mathcal{A}_n$, with a common set of objects and whose arrows are called n-**arrows**, that assemble into a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ of identity-on-objects functors

$$\cdots \qquad \mathcal{A}_3 \; \substack{\xrightarrow{\delta_3} \\ \xleftarrow{\sigma_2} \\ \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}} \; \mathcal{A}_2 \; \substack{\xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}} \; \mathcal{A}_1 \; \substack{\xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0}} \; \mathcal{A}_0, \qquad =: \mathcal{A} \tag{1.2.3}$$

The data of a simplicial category can equivalently be encoded by a **simplicially enriched category** with a set of objects and a simplicial set $\mathcal{A}(x, y)$ of morphisms between each ordered pair of objects: an n-arrow in $\mathcal{A}_n$ from x to y corresponds to an n-simplex in $\mathcal{A}(x, y)$ (see Exercise 1.2.i). Each endo-hom-space contains a distinguished identity 0-arrow (the degenerate images of which define the corresponding identity n-arrows) and composition is required to define a simplicial map

$$\mathcal{A}(y, z) \times \mathcal{A}(x, y) \xrightarrow{\circ} \mathcal{A}(x, z)$$

the single map encoding the compositions in each of the categories $\mathcal{A}_n$ and also the functoriality of the diagram (1.2.3). The composition is required to be associative and unital, in a sense expressed by the commutative diagrams

$$\begin{array}{ccc} \mathcal{A}(y,z) \times \mathcal{A}(x,y) \times \mathcal{A}(w,x) & \xrightarrow{\circ \times 1} & \mathcal{A}(x,z) \times \mathcal{A}(w,x) \\ {\scriptstyle 1\times\circ}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathcal{A}(y,z) \times \mathcal{A}(w,y) & \xrightarrow[\circ]{} & \mathcal{A}(w,z) \end{array} \qquad \begin{array}{ccc} \mathcal{A}(x,y) & \xrightarrow{\mathrm{id}_y \times 1} & \mathcal{A}(y,y) \times \mathcal{A}(x,y) \\ {\scriptstyle 1\times \mathrm{id}_x}\downarrow & \searrow{=} & \downarrow{\scriptstyle \circ} \\ \mathcal{A}(x,y) \times \mathcal{A}(x,x) & \xrightarrow[\circ]{} & \mathcal{A}(x,y) \end{array}$$

the latter making use of natural isomorphisms $\mathcal{A}(x, y) \times 1 \cong \mathcal{A}(x, y) \cong 1 \times \mathcal{A}(x, y)$ in the domain vertex.

[12]This will be elaborated in 1.2.4.

On account of the equivalence between these two presentations, the terms "simplicial category" and "simplicially-enriched category" are generally taken to be synonyms.[13] The category $\mathcal{A}_0$ of 0-arrows is the **underlying category** of the simplicial category $\mathcal{A}$, which forgets the higher dimensional simplicial structure.

In particular, the underlying category of an ∞-cosmos $\mathcal{K}$ is the category whose objects are the ∞-categories in $\mathcal{K}$ and whose morphisms are the 0-arrows in the functor spaces. In all of the examples to appear below, this recovers the expected category of ∞-categories in a particular model and functors between them.

1.2.4. Digression (simplicially enriched limits). Let $\mathcal{A}$ be a simplicial category. The **cotensor** of an object $A \in \mathcal{A}$ by a simplicial set U is characterized by an isomorphism of simplicial sets

$$\mathcal{A}(X, A^U) \cong \mathcal{A}(X, A)^U \tag{1.2.5}$$

natural in $X \in \mathcal{A}$. Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\begin{array}{ccc} \mathcal{SS}et^{\mathrm{op}} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ (U, A) & \longmapsto & A^U \end{array}$$

in a unique way making the isomorphism (1.2.5) natural in U and A as well.

The other simplicial limit notions postulated by axiom 1.2.1(i) are **conical**, which is the term used for ordinary 1-categorical limit shapes that satisfy an enriched analog of the usual universal property; see Example **??**. When these limits exist they corresponds to the usual limits in the underlying category, but the usual universal property is strengthened. Applying the covariant representable functor $\mathcal{A}(X, -)\colon \mathcal{A}_0 \to \mathcal{SS}et$ to a limit cone $(\lim_{j\in J} A_j \to A_j)_{j \in J}$ in $\mathcal{A}_0$, there is natural comparison map

$$\mathcal{A}(X, \lim_{j\in J} A_j) \to \lim_{j \in J} \mathcal{A}(X, A_j) \tag{1.2.6}$$

and we say that $\lim_{j\in J} A_j$ defines a **simplicially enriched limit** if and only if (1.2.6) is an isomorphism of simplicial sets for all $X \in \mathcal{A}$.

Considerably more details on the general theory of enriched categories can be found in [**11**].

1.2.7. Remark (flexible weighted limits in ∞-cosmoi). The axiom 1.2.1(i) implies that any ∞-cosmos $\mathcal{K}$ admits all *flexible limits* (see Corollary **??**), a much larger class of simplicially enriched "weighted" limits that will be introduced in §**??**.

Using the results of Joyal discussed in §1.1, we can easily verify:

1.2.8. Proposition. *The full subcategory $\mathcal{QC}at \subset \mathcal{SS}et$ of quasi-categories defines an ∞-cosmos with the isofibrations, equivalences, and trivial fibrations of Definitions 1.1.16, 1.1.21, and 1.1.22.*

Proof. The subcategory $\mathcal{QC}at \subset \mathcal{SS}et$ inherits its simplicial enrichment from the cartesian closed category of simplicial sets: note that for quasi-categories A and B, $\mathsf{Fun}(A, B) := B^A$ is again a quasi-category.

[13]The phrase "simplicial object in $\mathcal{C}at$" is reserved for the more general yet less common notion of a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ that is not necessarily comprised of identity-on-objects functors.

The limits postulated in 1.2.1(i) exist in the ambient category of simplicial sets.[14] The defining universal property of the simplicial cotensor is satisfied by the exponentials of simplicial sets. We now argue that the full subcategory of quasi-categories inherits all these limit notions.

Since the quasi-categories are characterized by a right lifting property, it is clear that they are closed under small products. Similarly, since the class of isofibrations is characterized by a right lifting property, Lemma C.1.1 implies that the fibrations are closed under all of the limit constructions of 1.2.1(ii) except for the last two: Leibniz closure and closure under exponentiation $(-)^X$. These last closure properties are established in Proposition 1.1.18. This completes the proof of 1.2.1(i) and 1.2.1(ii).

Remark 1.1.24 proves that trivial fibrations split once we verify that the classes of trivial fibrations and of equivalences coincide with those defined by 1.1.22 and 1.1.21. By Proposition 1.1.25 the former coincidence follows from the latter, so it remains only to show that the equivalences of 1.1.21 coincide with the **representably-defined equivalences**: those maps of quasi-categories $f: A \to B$ for which $A^X \to B^X$ is an equivalence of quasi-categories in the sense of 1.1.21. Taking $X = \Delta[0]$, we see immediately that representably-defined equivalences are equivalences, and the converse holds since the exponential $(-)^X$ preserves the data defining a simplicial homotopy. ☐

We mention a common source of ∞-cosmoi found in nature at the outside to help ground the intuition for readers familiar with Quillen's model categories, a popular framework for "abstract homotopy theory," but reassure others that model categories are not needed outside of Appendix D.

1.2.9. Digression (a source of ∞-cosmoi in nature). As explained in Appendix D, certain easily described properties of a model category imply that the full subcategory of fibrant objects defines an ∞-cosmos whose isofibrations, equivalences, and trivial fibrations are the fibrations, weak equivalences, and trivial fibrations between fibrant objects. Namely, any model category that is enriched as such over the Joyal model structure on simplicial sets and with the property that all fibrant objects are cofibrant has this property. This compatible enrichment in the Joyal model structure can be defined when the model category is cartesian closed and equipped with a right Quillen adjoint to the Joyal model structure on simplicial sets whose left adjoint preserves finite products. In this case, the right adjoint becomes the underlying quasi-category functor (see Proposition 1.3.3(ii)) and the ∞-cosmoi so-produced will then be cartesian closed (see Definition 1.2.17). The ∞-cosmoi listed in Example 1.2.18 all arise in this way.

The following results are consequences of the axioms of Definition 1.2.1. The first of these results tells us that the trivial fibrations enjoy all of the same stability properties satisfied by the isofibrations.

1.2.10. Lemma (stability of trivial fibrations). *The trivial fibrations in an ∞-cosmos define a subcategory containing the isomorphisms; are stable under product, pullback, forming inverse limits of towers; the Leibniz cotensors of any trivial fibration with a monomorphism of simplicial sets is a trivial fibration as is the Leibniz cotensor of an isofibration with a map in the class cellularly generated by the inner horn inclusions and the map* $\mathbb{1} \hookrightarrow \mathbb{I}$; *and finally, if* $E \overset{\sim}{\twoheadrightarrow} B$ *is a trivial fibration then so is* $\mathsf{Fun}(X, E) \overset{\sim}{\twoheadrightarrow} \mathsf{Fun}(X, B)$.

Proof. We prove these statements in the reverse order. By axiom 1.2.1(ii) and the definition of the trivial fibrations in an ∞-cosmos, we know that if $E \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration then is $\mathsf{Fun}(X, E) \overset{\sim}{\twoheadrightarrow} \mathsf{Fun}(X, B)$ is both an isofibration and an equivalence, and hence by Proposition 1.1.25 a trivial fibration. For stability under the remaining constructions, we know in each case that the maps in question are isofibrations in the ∞-cosmos; it remains to show only that the maps are also equivalences.

[14] Any category of presheaves is cartesian closed, complete, and cocomplete — a "cosmos" in the sense of Bénabou.

The equivalences in an ∞-cosmos are defined to be the maps that $\mathsf{Fun}(X, -)$ carries to equivalences of quasi-categories, so it suffices to verify that trivial fibrations of quasi-categories satisfy the corresponding stability properties. This is established in Proposition 1.1.26 and the fact that that class is characterized by a right lifting property. ☐

A classical construction in abstract homotopy theory proves the following:

1.2.11. Lemma (Brown factorization lemma). *Any functor $f\colon A \to B$ in an ∞-cosmos may be factored as an equivalence followed by an isofibration, where this equivalence is constructed as a section of a trivial fibration.*

$$\begin{array}{ccccc} & & Pf & & \\ & {\scriptstyle q}\swarrow\!\!\nearrow{\scriptstyle s\,\sim} & & \searrow{\scriptstyle p} & \\ A & & \xrightarrow{\quad f \quad} & & B \end{array} \tag{1.2.12}$$

Proof. The displayed factorization is constructed by the pullback of an isofibration formed by the simplicial cotensor of the inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{I}$ into the ∞-category B.

$$\begin{array}{ccccc} & \overset{\Delta}{\nearrow} & A^{\mathbb{I}} & \overset{f^{\mathbb{I}}}{\searrow} & \\ A & \overset{\sim}{\underset{s}{\dashrightarrow}} & Pf & \longrightarrow & B^{\mathbb{I}} \\ & \underset{(A,f)}{\searrow} & \downarrow{\scriptstyle (q,p)} & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ & & A \times B & \xrightarrow[f\times B]{} & B \times B \end{array}$$

Note the map q is a pullback of the trivial fibration $\mathrm{ev}_0\colon B^{\mathbb{I}} \xrightarrow{\sim} B$ and is hence a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence. ☐

1.2.13. Remark (equivalences satisfy the 2-of-6 property). Exercise 1.1.vii proves that the equivalences between quasi-categories — and hence also the equivalences in any ∞-cosmos — are closed under retracts and have the 2-of-3 property. Using this latter fact, we see that in the case where $f\colon A \to B$ is an equivalence, the map p of (1.2.12) is also a trivial fibration, and in particular has a section by axiom 1.2.1(iii). Combining these facts, a result of Blumberg and Mandell [**3**, 6.4] reproduced in Appendix C applies to prove that the equivalences in any ∞-cosmos satisfy the stronger **2-of-6 property**: for any composable triple of morphisms

$$\begin{array}{ccccc} & \overset{f}{\nearrow} & B & \overset{hg}{\underset{\sim}{\searrow}} & \\ A & & \xrightarrow{\;hgf\;}\;\downarrow{\scriptstyle g} & & D \\ & \underset{gf}{\overset{\sim}{\searrow}} & C & \underset{h}{\nearrow} & \end{array}$$

if gf and hg are equivalences then f, g, h, and hgf are too.

By a Yoneda-style argument, the "homotopy equivalence" characterization of the equivalences in the ∞-cosmos of quasi-categories extends to an analogous characterization of the equivalences in any ∞-cosmos:

1.2.14. Lemma (equivalences are homotopy equivalences). *A map $f\colon A \to B$ between ∞-categories in an ∞-cosmos $\mathcal{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g\colon B \to A$ and*

$$
\begin{array}{ccc}
 & & A \\
 & \nearrow\!\!\!\!= & \uparrow{\scriptstyle ev_0} \\
A & \xrightarrow{\alpha} & A^{\mathbb{I}} \\
 & {\scriptstyle gf}\searrow & \downarrow{\scriptstyle ev_1} \\
 & & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & B \\
 & {\scriptstyle fg}\nearrow & \uparrow{\scriptstyle ev_0} \\
B & \xrightarrow{\beta} & B^{\mathbb{I}} \\
 & \searrow\!\!\!\!= & \downarrow{\scriptstyle ev_1} \\
 & & B
\end{array}
\tag{1.2.15}
$$

in the ∞-cosmos.

Proof. By hypothesis, if $f\colon A \to B$ defines an equivalence in the ∞-cosmos $\mathcal{K}$ then the induced map on post-composition $f_*\colon \mathsf{Fun}(B,A) \xrightarrow{\sim} \mathsf{Fun}(B,B)$ is an equivalence of quasi-categories. Evaluating the equivalence inverse $\tilde{g}\colon \mathsf{Fun}(B,B) \xrightarrow{\sim} \mathsf{Fun}(B,A)$ and homotopy $\tilde{\beta}\colon \mathsf{Fun}(B,B) \to \mathsf{Fun}(B,B)^{\mathbb{I}}$ at the 0-arrow $1_B \in \mathsf{Fun}(B,B)$, we obtain a 0-arrow $g\colon B \to A$ together with an isomorphism $\mathbb{I} \to \mathsf{Fun}(B,B)$ from the composite fg to 1_B. By the defining universal property of the cotensor, this isomorphism internalizes to define the map $\beta\colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the right of (1.2.15).

Now the hypothesis that f is an equivalence also provides an equivalence of quasi-categories $f_*\colon \mathsf{Fun}(A,A) \xrightarrow{\sim} \mathsf{Fun}(A,B)$ and the map $\beta f\colon A \to B^{\mathbb{I}}$ represents an isomorphism in $\mathsf{Fun}(A,B)$ from fgf to f. Since f_* is an equivalence, we can conclude that 1_A and gf are isomorphic in the quasi-category $\mathsf{Fun}(A,A)$: such an isomorphism may be defined by applying the inverse equivalence $\tilde{h}\colon \mathsf{Fun}(A,B) \to \mathsf{Fun}(A,A)$ and composing with the components at $1_A, gf \in \mathsf{Fun}(A,A)$ of the isomorphism $\tilde{\alpha}\colon \mathsf{Fun}(A,A) \to \mathsf{Fun}(A,A)^{\mathbb{I}}$ from $1_{\mathsf{Fun}(A,A)}$ to $\tilde{h}f_*$. Now by Corollary 1.1.15 this isomorphism is represented by a map $\mathbb{I} \to \mathsf{Fun}(A,A)$ from 1_A to gf, which internalizes to a map $\alpha\colon A \to A^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the left of (1.2.15).

The converse is easy: the simplicial cotensor construction commutes with $\mathsf{Fun}(X,-)$ so homotopy equivalences are preserved and by Definition 1.1.21 homotopy equivalences of quasi-categories define equivalences of quasi-categories. □

One of the key advantages of the ∞-cosmological approaches to abstract category theory is that there a myriad varieties of "fibered" ∞-cosmoi that can be built from a given ∞-cosmos, which means that any theorem proven in this axiomatic framework specializes and generalizes to those contexts. The most basic of these derived ∞-cosmos is the ∞-cosmos of isofibrations over a fixed base[15]:

1.2.16. Proposition (sliced ∞-cosmoi). *For any ∞-cosmos $\mathcal{K}$ and any object $B \in \mathcal{K}$ there is an **∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over** B whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ with codomain B
(ii) functor-spaces, say from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$, are defined by pullback

$$
\begin{array}{ccc}
\mathsf{Fun}_B(p\colon E \twoheadrightarrow B, q\colon F \twoheadrightarrow B) & \longrightarrow & \mathsf{Fun}(E,F) \\
\big\downarrow\!\!\!\downarrow & \lrcorner & \big\downarrow\!\!\!\downarrow{\scriptstyle q_*} \\
\mathbb{1} & \xrightarrow{\quad p \quad} & \mathsf{Fun}(E,B)
\end{array}
$$

[15]Other examples of ∞-cosmoi will be introduced in Part ??, once we have developed a greater facility with the simplicial limits of axiom 1.2.1(i).

and abbreviated to $\mathsf{Fun}_B(E,F)$ *when the specified isofibrations are clear from context*

(iii) isofibrations are commutative triangles of isofibrations over B

$$\begin{array}{ccc} E & \xrightarrow{r} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

(iv) terminal object is $1\colon B \twoheadrightarrow B$ *and products are defined by the pullback along the diagonal*

$$\begin{array}{ccc} \times_i^B E_i & \longrightarrow & \prod_i E_i \\ \downarrow & \lrcorner & \downarrow \\ B & \xrightarrow{\Delta} & \prod_i B \end{array}$$

(v) pullbacks and limits of towers of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$

(vi) simplicial cotensors $p\colon E \twoheadrightarrow B$ *by* $U \in \mathcal{SS}et$ *are denoted* $U \pitchfork_B p$ *and constructed by the pullback*

$$\begin{array}{ccc} U \pitchfork_B p & \longrightarrow & E^U \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p^U} \\ B & \xrightarrow{\Delta} & B^U \end{array}$$

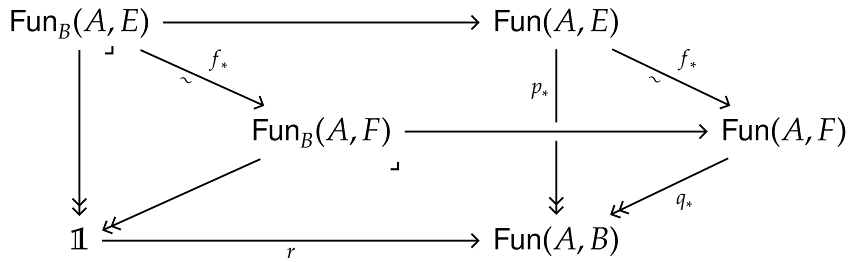
(vii) and in which a map

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

over B is an equivalence in the ∞*-cosmos* $\mathcal{K}_{/B}$ *if and only if* f *is an equivalence in* $\mathcal{K}$*.*

Proof. Note first that the functor spaces are quasi-categories since axiom 1.2.1(ii) asserts that for any isofibration $q\colon F \twoheadrightarrow B$ in $\mathcal{K}$ the map $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration of quasi-categories. Other parts of this axiom imply that that each of the limit constructions define isofibrations over B. The closure properties of the isofibrations in $\mathcal{K}_{/B}$ follow from the corresponding ones in $\mathcal{K}$. The most complicated of these is the Leibniz cotensor stability of the isofibrations in $\mathcal{K}_{/B}$, which follows from the corresponding property in $\mathcal{K}$, since for a monomorphism of simplicial sets $i\colon X \hookrightarrow Y$ and an isofibration r over B as above, the map $i \mathbin{\widehat{\pitchfork}_B} r$ is constructed by pulling back $i \mathbin{\widehat{\pitchfork}} r$ along $\Delta\colon B \to B^Y$.

The fact that the above constructions define simplicially enriched limits in a simplicially enriched slice category are standard from enriched category theory. It remains only to verify that the equivalences in the ∞-cosmos of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$. First note that if $f\colon E \to F$ defines an equivalence in $\mathcal{K}$, then for any isofibration $s\colon A \twoheadrightarrow B$ the induced equivalence on functor-spaces in $\mathcal{K}$ pulls back to define an equivalence on corresponding functor spaces in $\mathcal{K}_{/B}$.



This can be verified either by appealing to Lemmas 1.2.10 and 1.2.11 and using standard techniques from simplicial homotopy theory[16] or by appealing to Lemma 1.2.14 and using the fact that pullback along r defines a simplicial functor.

For the converse implication, we appeal to Lemma 1.2.14. If $f\colon E \to F$ is an equivalence in $\mathcal{K}_{/B}$ then it admits a homotopy inverse in $\mathcal{K}_{/B}$. The inverse equivalence $g\colon F \to E$ also defines an inverse equivalence in $\mathcal{K}$ and the required simplicial homotopies in $\mathcal{K}$ are defined by composing

$$E \xrightarrow{\alpha} \mathbb{I} \pitchfork_B E \longrightarrow E^{\mathbb{I}} \qquad F \xrightarrow{\beta} \mathbb{I} \pitchfork_B F \to F^{\mathbb{I}}$$

with the top horizontal leg of the pullback defining the cotensor in $\mathcal{K}_{/B}$. □

1.2.17. Definition (cartesian closed ∞-cosmoi). An ∞-cosmos $\mathcal{K}$ is **cartesian closed** if the product bifunctor $- \times - \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ extends to a simplicially enriched two-variable adjunction

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A)$$

in which the right adjoint $(-)^A$ preserve the class of isofibrations.

For instance, the ∞-cosmos of quasi-categories is cartesian closed, with the exponentials defined as (special cases of) simplicial cotensors. This is one of the reasons that we use the same notation for cotensor and for exponential.[17]

1.2.18. Example (∞-cosmoi of $(\infty,1)$-categories). The following models of $(\infty,1)$-categories define cartesian closed ∞-cosmoi:

(i) Rezk's **complete Segal spaces** define the objects of an ∞-cosmos $\mathcal{CSS}$, in which the isofibrations, equivalences, and trivial fibrations are the corresponding classes of the model structure of [**16**].[18]
(ii) The **Segal categories** defined by Dwyer, Kan, and Smith [**6**] and developed by Hirschowitz and Simpson [**7**] define the objects of an ∞-cosmos $\mathcal{S}egal$, in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure of [**15**] and [**1**].[19]

[16] In more detail: any functor between the 1-categories underlying ∞-cosmoi that preserves trivial fibrations also preserves equivalences

[17] Another reason for this convenient notational conflation will be explained in §2.3.

[18] Warning: the model category of complete Segal spaces is enriched over simplicial sets in two distinct "directions" — one enrichment makes the simplicial set of maps between two complete Segal spaces into a Kan complex that probes the "spacial" structure while another enrichment makes the simplicial set of maps into a quasi-category that probes the "categorical" structure [**10**]. It is this latter enrichment that we want.

[19] Here we reserve the term "Segal category" for those simplicial objects with a discrete set of objects that are Reedy fibrant and satisfy the Segal condition. The traditional definition does not include the Reedy fibrancy condition because it is not satisfied by the simplicial object defined as the nerve of a Kan complex enriched category. Since Kan complex enriched categories are not among our preferred models of $(\infty,1)$-categories this does not bother us.

(iii) The **1-complicial sets** of **[21]**, equivalently the "naturally marked quasi-categories" of **[13]**, define the objects of an ∞-cosmos $1\text{-}\mathcal{C}omp$ in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure from either of these sources.

Proofs of these facts can be found in Appendix D.

Appendix D also proves that certain models of (∞, n)-categories or even (∞, ∞)-categories define ∞-cosmoi.

1.2.19. Example ($\mathcal{C}at$ as an ∞-cosmos). The category $\mathcal{C}at$ of strict 1-categories defines a cartesian closed ∞-cosmos, inheriting its structure as a full subcategory $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ of the ∞-cosmos of quasi-categories via the nerve embedding, which preserves all limits and also exponentials: the nerve of the functor category B^A is the exponential of the nerves.

In the ∞-cosmos of categories, the isofibrations are the **isofibrations**: functors satisfying the displayed right lifting property:

$$\begin{array}{ccc} \mathbb{1} & \longrightarrow & A \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \mathbb{I} & \longrightarrow & B \end{array}$$

The equivalences are the equivalences of categories and the trivial fibrations are **surjective equivalences**: equivalences of categories that are also surjective on objects.

1.2.20. Definition (discrete ∞-categories). An object E in an ∞-cosmos $\mathcal{K}$ is **discrete** just when for all $A \in \mathcal{K}$ the functor-space $\mathsf{Fun}(A, E)$ is a Kan complex.

In the ∞-cosmos of quasi-categories, the discrete objects are exactly the Kan complexes: by Proposition **??** the Kan complexes also define an exponential ideal in the category of simplicial sets. Similarly, in the ∞-cosmoi of Example 1.2.18 whose ∞-categories are $(\infty, 1)$-categories in some model, the discrete objects are the ∞-groupoids.

1.2.21. Proposition (∞-cosmos of discrete objects). *The full subcategory spanned by the discrete objects in any ∞-cosmos form an ∞-cosmos.*

Proof. We first establish this result for the ∞-cosmos of quasi-categories. By Proposition 1.1.13 an isofibration between Kan complexes is a **Kan fibration**: a map with the right lifting property with respect to all horn inclusions. Conversely, all Kan fibrations define isofibrations. Since Kan complexes are closed under simplicial cotensor (which coincides with exponentiation), It follows that the full subcategory $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ is closed under all of the limit constructions of axiom 1.2.1(i). The remaining axioms 1.2.1(ii) and (iii) are inherited from the analogous properties established for quasi-categories in Proposition 1.2.8.

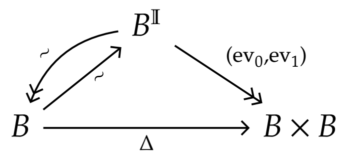
In a generic ∞-cosmos $\mathcal{K}$ we need only show that the discrete objects are closed in $\mathcal{K}$ under the limit constructions of 1.2.1(i). The definition natural isomorphism (1.2.6) characterizing these simplicial limits expresses the functor-space $\mathsf{Fun}(X, \lim_{j \in J} A_j)$ as an analogous limit of functor space $\mathsf{Fun}(X, A_j)$. If each A_j is discrete then these objects are Kan complexes and the previous paragraph then establishes that the limit is a Kan complex as well. This holds for all objects $X \in \mathcal{K}$ so it follows that $\lim_{j \in J} A_j$ is discrete as required. □

Exercises.

1.2.i. Exercise. Prove that the following are equivalent:

(i) a simplicial category, as in 1.2.2,
(ii) a category enriched over simplicial sets.

1.2.ii. Exercise. Prove that any object in an ∞-cosmos has a **path object**



constructed by cotensoring with the free-living isomorphism.

1.3. Functors of ∞-cosmoi

Certain "right adjoint type" constructions define maps between ∞-cosmoi that preserve all of the structures axiomatized in Definition 1.2.1. The simple option that such constructions define *functors* between ∞-cosmoi will streamline many proofs.

1.3.1. Definition (functor of ∞-cosmoi). A **functor of ∞-cosmoi** is a simplicial functor $F\colon \mathcal{K} \to \mathcal{L}$ that preserves the specified classes of isofibrations and all of the simplicial limits enumerated 1.2.1(i).

1.3.2. Lemma. *Any functor of ∞-cosmoi also preserves the equivalences and the trivial fibrations.*

Proof. By Lemma 1.2.14 the equivalences in an ∞-cosmos coincide with the homotopy equivalences defined relative to cotensoring with the free-living isomorphism. Since a functor of ∞-cosmoi preserves simplicial cotensors, it preserves the data displayed in (1.2.15) and hence carries equivalences to equivalences. The statement about trivial fibrations follows. ☐

1.3.3. Proposition.

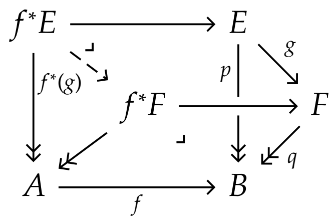
(i) For any object A in an ∞-cosmos $\mathcal{K}$, $\mathsf{Fun}(A, -)\colon \mathcal{K} \to \mathcal{QC}at$ defines a functor of ∞-cosmoi.
*(ii) Specializing, each ∞-cosmos has an **underlying quasi-category** functor*

$$(-)_0 \coloneqq \mathsf{Fun}(1, -)\colon \mathcal{K} \to \mathcal{QC}at.$$

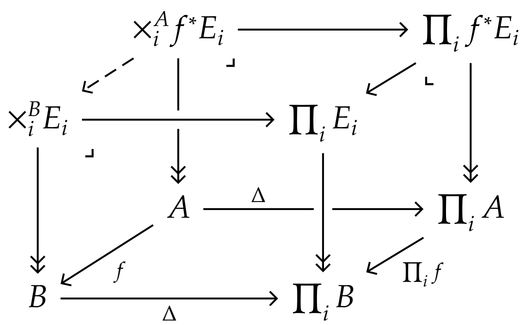
(iii) For any ∞-cosmos $\mathcal{K}$ and any simplicial set U, the simplicial cotensor defines a functor of ∞-cosmoi $(-)^U\colon \mathcal{K} \to \mathcal{K}$.
(iv) For any object A in a cartesian closed ∞-cosmos $\mathcal{K}$, exponentiation defines a functor of ∞-cosmoi $(-)^A\colon \mathcal{K} \to \mathcal{K}$.
(v) For any map $f\colon A \to B$ in an ∞-cosmos $\mathcal{K}$, pullback defines a functor of ∞-cosmoi $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$.
(vi) For any functor of ∞-cosmoi $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map on slices $F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a functor of ∞-cosmoi.

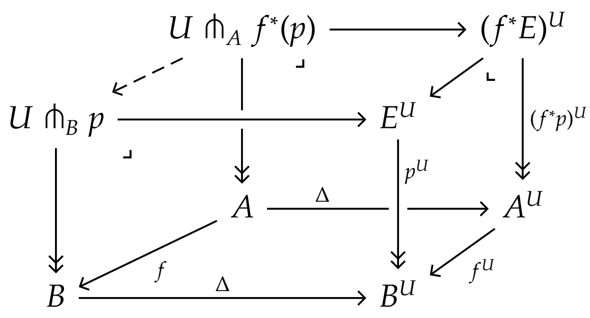
Proof. The first four of these statements are nearly immediate, the preservation of isofibrations being asserted explicitly as a hypothesis in each case and the preservation of limits following from standard categorical arguments.

For (v), pullback in an ∞-cosmos $\mathcal{K}$ is a simplicially enriched limit construction; one consequence of this is that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ defines a simplicial functor. The action of the functor f^* on a 0-arrow g in $\mathcal{K}_{/B}$ is also defined by a pullback square: since the front and back squares in the displayed diagram are pullbacks the top square is as well



Since isofibrations are stable under pullback, it follows that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves isofibrations. It remains to prove that this functor preserves the simplicial limits constructed in Proposition 1.2.16. In the case of connected limits, which are created by the forgetful functors to $\mathcal{K}$, this is clear. For products and simplicial cotensors, this follows from the commutative cubes





Since the front, back, and right faces are pullbacks, the left is as well, which is what we wanted to show.

The final statement (vi) is left as Exercise 1.3.i. ☐

1.3.4. Non-Example. The forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ is simplicial and preserves the class of isofibrations but does *not* define a functor of ∞-cosmoi, failing to preserve cotensors and products.

1.3.5. Definition (biequivalences). A functor of ∞-cosmoi defines a *biequivalence* $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ if additionally it

(i) is **essentially surjective on objects up to equivalence**: for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$ and
(ii) it defines a **local equivalence**: for all $A, B \in \mathcal{K}$, the action of F on functor quasi-categories defines an equivalence

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB).$$

1.3.6. Remark. Biequivalences of ∞-cosmoi will be studied more systematically in Part **??**, where we think of them as "change-of-model" functors. A basic fact is that any biequivalence of ∞-cosmoi not only preserves equivalences but also *creates* them: a pair of objects in an ∞-cosmos are equivalent if and only if their images in any biequivalent ∞-cosmos are equivalent (Exercise 1.3.ii). It follows that the biequivalences of ∞-cosmoi satisfy the 2-of-3 property.

1.3.7. Example (biequivalences between ∞-cosmoi of $(\infty,1)$-categories).

(i) The underlying quasi-category functors defined on the ∞-cosmoi of complete Segal spaces, Segal categories, and 1-complicial sets

$$\mathcal{CSS} \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \qquad \mathcal{S}egal \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \qquad 1\text{-}\mathcal{C}omp \xrightarrow[\sim]{(-)_0} \mathcal{QC}at$$

are all biequivalences. In the first two cases these are defined by "evaluating at the 0th row" and in the last case this is defined by "forgetting the markings."

(ii) There is also a biequivalence of ∞-cosmoi $\mathcal{QC}at \rightsquigarrow \mathcal{CSS}$ defined by Joyal and Tierney [**10**].

(iii) The functor $\mathcal{CSS} \rightsquigarrow \mathcal{S}egal$ defined by Bergner [**2**] that "discretizes" a complete Segal spaces also defines a biequivalence of ∞-cosmoi.

Proofs of these facts can be found in Appendix D.

Exercises.

1.3.i. Exercise. Prove that for any functor of ∞-cosmoi $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map $F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a functor of ∞-cosmoi.

1.3.ii. Exercise. Let $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ be a biequivalence of ∞-cosmoi and let $A, B \in \mathcal{K}$. Sketch a proof that if $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$ (and see Exercise 1.4.i).

1.4. The homotopy 2-category

Small 1-categories define the objects of a strict 2-category[20] $\mathcal{C}at$ of categories, functors, and natural transformations. Many basic categorical notions — those defined in terms of categories, functors, and natural transformations and their various composition operations — can be defined internally to the 2-category $\mathcal{C}at$. This suggests a natural avenue for generalization: reinterpreting these same definitions in an generic 2-category using its objects in place of small categories, its 1-cells in place of functors, and its 2-cells in place of natural transformations.

In Chapter 2, we will develop a non-trivial portion of the theory of ∞-categories in any fixed ∞-cosmos following exactly this outline, working internally to a strict 2-category that we refer to as the *homotopy 2-category* that we associate to any ∞-cosmos. The homotopy 2-category of an ∞-cosmos is a quotient of the full ∞-cosmos, replacing each quasi-categorical functor-space by its homotopy category. Surprisingly, this rather destructive quotienting operation preserves quite a lot of information. Indeed, essentially all of the work in Part I will take place in the homotopy 2-category of an ∞-cosmos. This said, we caution the reader against becoming overly seduced by homotopy 2-categories, for that structure is more of a technical convenience for reducing the complexity of our arguments than a fundamental notion of ∞-category theory.

The homotopy 2-category for the ∞-cosmos of quasi-categories was first introduced by Joyal in his work on the foundations of quasi-category theory.

1.4.1. Definition (homotopy 2-category). Let $\mathcal{K}$ be an ∞-cosmos. Its **homotopy 2-category** is the strict 2-category $\mathfrak{h}\mathcal{K}$ whose

- objects are the ∞-categories, i.e., the objects A, B of $\mathcal{K}$;

[20]A comprehensive introduction to strict 2-categories appears as Appendix B. Succinctly, in parallel with Digression 1.2.2, 2-categories can be understood equally as

- "two-dimensional" categories, with objects, 0-arrows (typically called **1-cells**), and 1-arrows (typically called **2-cells**)
- or as categories enriched over $\mathcal{C}at$.

- 1-cells $f\colon A \to B$ are the 0-arrows in the functor space $\mathsf{Fun}(A,B)$ of $\mathcal{K}$; and
- 2-cells $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$ are homotopy classes of 1-simplices in $\mathsf{Fun}(A,B)$.

Put another way $\mathfrak{h}\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$ and with hom-categories defined by

$$\mathsf{hFun}(A,B) := \mathsf{h}(\mathsf{Fun}(A,B)),$$

that is, as the homotopy category of the quasi-category $\mathsf{Fun}(A,B)$.

The **underlying category** of a 2-category is defined by simply forgetting its 2-cells. Note that an ∞-cosmos $\mathcal{K}$ and its homotopy 2-category $\mathfrak{h}\mathcal{K}$ share the same underlying category of ∞-categories and ∞-functors in $\mathcal{K}$.

1.4.2. Digression. The homotopy category functor $\mathsf{h}\colon \mathcal{SSet} \to \mathcal{Cat}$ preserves finite products, as of course does its right adjoint. It follows that the adjunction of Proposition 1.1.10 induces a change-of-base adjunction

$$2\text{-}\mathcal{Cat} \underset{}{\overset{\mathsf{h}_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{SSet}\text{-}\mathcal{Cat}$$

whose left and right adjoints change the enrichment by applying the homotopy category functor or the nerve functor to the hom objects of the enriched category. Here $2\text{-}\mathcal{Cat}$ and $\mathcal{SSet}\text{-}\mathcal{Cat}$ can each be understood as 2-categories — of enriched categories, enriched functors, and enriched natural transformations — and both change of base constructions define 2-functors [**5**, 6.4.3].

1.4.3. Observation (functors representing (invertible) 2-cells). By definition, 2-cells in the homotopy category $\mathfrak{h}\mathcal{K}$ are represented by maps $\mathbb{2} \to \mathsf{Fun}(A,B)$ valued in the appropriate functor space and two such maps represent the same 2-cell if and only if their images are homotopic as 1-simplices in $\mathsf{Fun}(A,B)$ in the sense defined by Lemma 1.1.8.

Now a 2-cell in a 2-category is **invertible** if and only if it defines an isomorphism in the appropriate hom-category $\mathsf{hFun}(A,B)$. By Definition 1.1.12 and Corollary 1.1.15 it follows that each invertible 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a map $\mathbb{I} \to \mathsf{Fun}(A,B)$.

1.4.4. Lemma. *Any simplicial functor $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi induces a 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ between their homotopy 2-categories.*

Proof. This follows immediately from the remarks on change of base in Digression 1.4.2 but we can also argue directly. The action of the induced 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ on objects and 1-cells is given by the corresponding action of $F\colon \mathcal{K} \to \mathcal{L}$; recall an ∞-cosmos and its homotopy 2-category have the same underlying 1-category. Each 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a 1-simplex in $\mathsf{Fun}(A,B)$ which is mapped via

$$\mathsf{Fun}(A,B) \xrightarrow{\ F\ } \mathsf{Fun}(FA,FB)$$

$$A \underset{g}{\overset{f}{\Downarrow\alpha}} B \longmapsto FA \underset{Fg}{\overset{Ff}{\Downarrow F\alpha}} FB$$

to a 1-simplex representing a 2-cell in $\mathfrak{h}\mathcal{L}$. Since the action $F\colon \mathsf{Fun}(A,B) \to \mathsf{Fun}(FA,FB)$ on functor spaces defines a morphism of simplicial sets, it preserves faces and degeneracies. In particular, homotopic 1-simplices in $\mathsf{Fun}(A,B)$ are carried to homotopic 1-simplices in $\mathsf{Fun}(FA,FB)$ so the action on 2-cells just described is well-defined. The 2-functoriality of these mappings follows from the simplicial functoriality of the original mapping. □

We now begin to relate the simplicially enriched structures of an ∞-cosmos to the 2-categorical structures in its homotopy 2-category. The first result proves that homotopy 2-categories inherit products from their ∞-cosmoi, which satisfy a 2-categorical universal property. To illustrate, recall that the terminal ∞-category $1 \in \mathcal{K}$ has the universal property $\mathsf{Fun}(X,1) \cong \mathbb{1}$ for all $X \in \mathcal{K}$. Applying the homotopy category functor we see that $1 \in \mathfrak{h}\mathcal{K}$ has the universal property $\mathsf{hFun}(X,1) \cong \mathbb{1}$ for all $X \in \mathfrak{h}\mathcal{K}$. This 2-categorical universal property has both a 1-dimensional and a 2-dimensional aspect. Since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ is a category with a single object, there exists a unique morphism $X \to 1$ in $\mathcal{K}$. And since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ has only a single identity morphism, we see that the only 2-cells in $\mathfrak{h}\mathcal{K}$ with codomain 1 are identities.

1.4.5. Proposition (cartesian (closure)).

(i) The homotopy 2-category of any ∞-cosmos has 2-categorical products.
(ii) The homotopy 2-category of a cartesian closed ∞-cosmos is cartesian closed as a 2-category.

Proof. While the functor $\mathsf{h}\colon \mathcal{SSet} \to \mathcal{C}at$ only preserves finite products, the restricted functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ preserves *all* products on account of the simplified description of the homotopy category of a quasi-category given in Lemma 1.1.11. Thus for any set I and family of ∞-categories $(A_i)_{i\in I}$ in $\mathcal{K}$, the homotopy category functor carries the isomorphism of quasi-categories displayed below left to an isomorphisms of hom-categories displayed below right

$$\mathsf{Fun}(X, \textstyle\prod_{i\in I} A_i) \xrightarrow{\cong} \textstyle\prod_{i\in I} \mathsf{Fun}(X, A_i) \qquad \overset{\mathsf{h}}{\mapsto} \qquad \mathsf{hFun}(X, \textstyle\prod_{i\in I} A_i) \xrightarrow{\cong} \textstyle\prod_{i\in I} \mathsf{hFun}(X, A_i).$$

This proves that the homotopy 2-category $\mathfrak{h}\mathcal{K}$ has products whose universal properties have both a 1- and 2-dimensional component, as described for terminal objects above.

If $\mathcal{K}$ is a cartesian closed ∞-cosmos, then for any triple of ∞-categories $A, B, C \in \mathcal{K}$ there exist exponential objects $C^A, C^B \in \mathcal{K}$ characterized by natural isomorphisms

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A).$$

Passing to homotopy categories we have natural isomorphisms

$$\mathsf{hFun}(A \times B, C) \cong \mathsf{hFun}(A, C^B) \cong \mathsf{hFun}(B, C^A),$$

which demonstrates that $\mathfrak{h}\mathcal{K}$ is cartesian closed as a 1-category: functors $A \times B \to C$ transpose to define functors $A \to C^B$ and $B \to C^A$, and 2-cells transpose similarly. □

There is a standard definition of *isomorphism* between two objects in any 1-category. Similarly, there is a standard definition of *equivalence* between two objects in any 2-category:

1.4.6. Definition (equivalence). An **equivalence** in a 2-category is given by

- a pair of objects A and B
- a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$

- a pair of invertible 2-cells

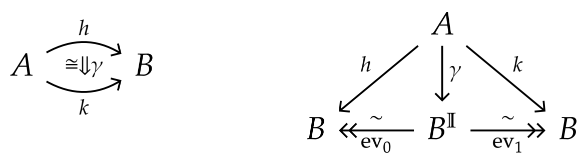
When A and B are **equivalent**, we write $A \simeq B$ and refer to the 1-cells f and g as **equivalences**, denoted by "$\xrightarrow{\sim}$."

In the case of the homotopy 2-category of an ∞-cosmos we have a competing definition of equivalence from 1.2.1: namely a 1-cell $f\colon A \xrightarrow{\sim} B$ that induces an equivalence $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ on functor-spaces — or equivalently, by Lemma 1.2.14, a homotopy equivalence defined relative to the interval $\mathbb{I}$. Crucially, these two notions of equivalence coincide:

1.4.7. Theorem (equivalences are equivalences). *A functor $f\colon A \to B$ between ∞-categories defines an equivalence in the ∞-cosmos $\mathcal{K}$ if and only if it defines an equivalence in its homotopy 2-category $\mathfrak{h}\mathcal{K}$.*

Proof. Given an equivalence $f\colon A \xrightarrow{\sim} B$ in the ∞-cosmos $\mathcal{K}$, Lemma 1.2.14 provides an inverse equivalence $g\colon B \xrightarrow{\sim} A$ and homotopies $\alpha\colon A \to A^{\mathbb{I}}$ and $\beta\colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$. By Observation 1.4.3 this data represents an equivalence in the homotopy 2-category $\mathfrak{h}\mathcal{K}$.

For the converse, first note that if parallel 1-cells $h, k\colon A \rightrightarrows B$ in the homotopy 2-category are connected by an invertible 2-cell as displayed below left, then h is an equivalence in the ∞-cosmos $\mathcal{K}$ if and only if k is:



By Observation 1.4.3, the invertible 2-cell can be represented by a map $\gamma\colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ as displayed above right; now apply the 2-of-3 property for the equivalences in $\mathcal{K}$.

Now it follows immediately from the 2-of-6 property of the equivalences in any ∞-cosmos established in Remark 1.2.13 and the fact that the equivalences contain the identities that any 2-categorical equivalence defines an equivalence in the ∞-cosmos: since gf and fg are isomorphic to identities, they must be equivalences in $\mathcal{K}$, and hence so must f and g. ☐

1.4.8. Digression (on the importance of Theorem 1.4.7). It is hard to overstate the importance of Theorem 1.4.7 to the work that follows. The categorical constructions that we will introduce for ∞-categories, ∞-functors, and ∞-natural transformations are invariant under 2-categorical equivalence in the homotopy 2-category and the universal properties we develop similarly characterize a 2-categorical equivalence class of ∞-categories. Theorem 1.4.7 then asserts that such constructions are "homotopically correct": both invariant under equivalence in the ∞-cosmos and precisely identifying equivalence classes of objects.

The equivalence invariance of the functor space in the codomain variable is axiomatic, but equivalence invariance in the domain variable is not.[21] But using 2-categorical techniques, there is now a short proof:

[21] Lemma 1.3.2 does not apply since $\mathsf{Fun}(-, B)$ is not a functor of ∞-cosmoi.

1.4.9. Corollary. *Equivalences of ∞-categories $A' \xrightarrow{\sim} A$ and $B \xrightarrow{\sim} B'$ induce an equivalence of functor spaces* $\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(A',B')$.

Proof. The simplicial functors $\mathsf{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$ and $\mathsf{Fun}(-,B)\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{QC}at$ induce 2-functors $\mathsf{Fun}(A,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ and $\mathsf{Fun}(-,B)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$, which preserve the 2-categorical equivalences of Definition 1.4.6. By Theorem 1.4.7 this is what we wanted to show. □

Similarly, there is a standard 2-categorical notion of an isofibration, defined in the statement of Proposition 1.4.10, and any isofibration in an ∞-cosmos defines an isofibration in its homotopy 2-category.[22]

1.4.10. Proposition (isofibrations define isofibrations). *Any isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$, also defines an isofibration in the homotopy 2-category $\mathfrak{h}\mathcal{K}$: given any invertible 2-cell as displayed below left abutting to B with a specified lift of one of its boundary 1-cells through p, there exists an invertible 2-cell abutting to E with this boundary 1-cell as displayed below right that whiskers with p to the original 2-cell.*

$$\begin{array}{ccc} X & \xrightarrow{e} & E \\ & \searrow^{b} \;\; \cong\Downarrow\beta & \downarrow p \\ & & B \end{array} \qquad = \qquad \begin{array}{ccc} X & \underset{\bar{e}}{\overset{e}{\rightrightarrows}}\; \cong\Downarrow\gamma & E \\ & & \downarrow p \\ & & B \end{array}$$

Proof. Put another way, the universal property of the statement says that the functor

$$p_*\colon \mathsf{hFun}(X,E) \twoheadrightarrow \mathsf{hFun}(X,B)$$

is an isofibration of categories in the sense defined in Example 1.2.19. By axiom 1.2.1(ii), since $p\colon E \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, the induced map $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ is an isofibration of quasi-categories. So it suffices to show that the functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries isofibrations of quasi-categories to isofibrations of categories.[23]

So let us now consider an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories. By Corollary 1.1.15, every isomorphism β in the homotopy category $\mathsf{h}B$ of the quasi-category B is represented by a simplicial map $\beta\colon \mathbb{I} \to B$. By Definition 1.1.16, the lifting problem

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{e} & E \\ \downarrow & \overset{\gamma}{\nearrow} & \downarrow p \\ \mathbb{I} & \xrightarrow[\beta]{} & B \end{array}$$

can be solved, and the map $\gamma\colon \mathbb{I} \to E$ so-produced represents a lift of the isomorphism from $\mathsf{h}B$ to an isomorphism in $\mathsf{h}E$ with domain e. □

1.4.11. Convention (on "isofibrations" in homotopy 2-categories). Since the converse to Proposition 1.4.10 does not hold, there is a potential ambiguity when using the term "isofibration" to refer to a map in the homotopy 2-category of an ∞-cosmos. We adopt the convention that when we declare that a map in $\mathfrak{h}\mathcal{K}$ is an isofibration we always mean this is the stronger sense of defining an isofibration in $\mathcal{K}$. This stronger condition gives us access to the 2-categorical lifting property of Proposition 1.4.10

[22] In this case, the converse does not hold, nor is it the case that a representably-defined isofibration of quasi-categories is necessarily an isofibration in the ∞-cosmos; consider the case of sliced ∞-cosmoi for instance.

[23] Alternately, argue directly using Observation 1.4.3.

but also to the many homotopical properties axiomatized in Definition 1.2.1, which guarantee that the strictly defined limits of 1.2.1(i) are automatically equivalence invariant constructions.

The 1- and 2-cells in the homotopy 2-category from the terminal ∞-category $1 \in \mathcal{K}$ to a generic ∞-category $A \in \mathcal{K}$ define the objects and morphisms in the *homotopy category* of A.

1.4.12. Definition (homotopy category of an ∞-category). The **homotopy category** of an ∞-category A in an ∞-cosmos $\mathcal{K}$ is defined to be the homotopy category of its underlying quasi-category, that is:

$$\mathsf{h}A := \mathsf{hFun}(1, A) := \mathsf{h}(\mathit{Fun}(1, A)).$$

As we shall discover, homotopy categories generally bear "derived" analogues of structures present at the level of ∞-categories. See the remark after the statement Proposition 2.1.7 for an early example of this.

Exercises.

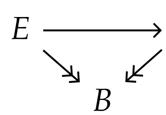
1.4.i. Exercise. Let $F \colon \mathcal{K} \rightsquigarrow \mathcal{L}$ be a biequivalence of ∞-cosmoi and let $A, B \in \mathcal{K}$. Prove that $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$ and ruminate on why this exercise is considerably easier than Exercise 1.3.ii).

1.4.ii. Exercise.

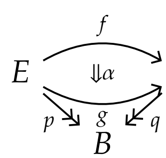
(i) What is the homotopy 2-category of the ∞-cosmos $\mathcal{C}at$ of strict 1-categories?
(ii) Prove that the nerve defines a 2-functor $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{Q}\mathcal{C}at$ that is *locally fully faithful*.

1.4.iii. Exercise. Let B be an ∞-category in the ∞-cosmos $\mathcal{K}$ and let $\mathfrak{h}\mathcal{K}_{/B}$ denote the 2-category whose

- objects are isofibrations $E \twoheadrightarrow B$ in $\mathcal{K}$ with codomain B
- 1-cells are 1-cells in $\mathfrak{h}\mathcal{K}$ over B



- 2-cells are 2-cells in $\mathfrak{h}\mathcal{K}$ over B



in the sense that $q\alpha = \mathrm{id}_p$.

Argue that the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos has the same underlying 1-category but different 2-cells. How do these compare with the 2-cells of $\mathfrak{h}\mathcal{K}_{/B}$?[24]

[24] A more systematic comparison will be given in Proposition 3.5.3.

CHAPTER 2

Adjunctions, limits, and colimits I

Heuristically, ∞-categories generalize ordinary strict 1-categories by adding in higher dimensional morphisms and weakening the composition law. The dream is that proofs establishing the theory of strict 1-categories similarly generalize to give proofs for ∞-categories, just by adding a prefix "∞-" everywhere. In this chapter, we make this dream a reality — at least for a library of basic propositions concerning equivalences, adjunctions, limits, and colimits and the relationships between these notions.

Recall that categories, functors, and natural transformations assemble into a 2-category $\mathcal{C}at$. Similarly, the ∞-categories, ∞-functors, and ∞-natural transformations in any ∞-cosmos assemble into a 2-category, namely the *homotopy 2-category* of the ∞-cosmos, introduced in §1.4. By Exercise 1.4.ii, $\mathcal{C}at$ can be regarded as a special case of a homotopy 2-category. In this chapter, we will use strict 2-categorical techniques to define *adjunctions* between ∞-categories and *limits* and *colimits* of diagrams valued in an ∞-category and prove that these notions interact in the expected ways. In the homotopy 2-category of categories, these recover the classical results from strict 1-category theory. As these proofs are equally valid in any homotopy 2-category, our arguments also establish the desired generalizations by simply appending the prefix "∞-."

2.1. Adjunctions and equivalences

In §1.4, we encountered the definition of an *equivalence* between a pair of objects in a 2-category. In the case where the ambient 2-category is the homotopy 2-category of an ∞-cosmos, we observed in Theorem 1.4.7 that the 2-categorical notion of equivalence precisely recaptures the notion of equivalence introduced in Definition 1.2.1 between ∞-categories in the full ∞-cosmos. In each of the examples of ∞-cosmoi we have considered, the representably-defined equivalences in the ∞-cosmos coincide with the standard notion of equivalences between ∞-categories as presented in that particular model.[1] Thus, the 2-categorical notion of equivalence is the "correct" notion of equivalence between ∞-categories.

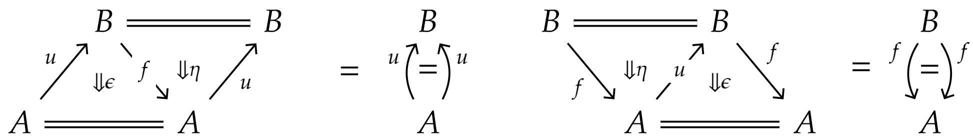
Similarly, there is a standard definition of an *adjunction* between a pair of objects in a 2-category, which, when interpreted in the homotopy 2-category of ∞-categories, functors, and natural transformations in an ∞-cosmos, will define the correct notion of adjunction between ∞-categories.

2.1.1. Definition (adjunction). An **adjunction** in between ∞-categories is comprised of:

- a pair of ∞-categories A and B,
- a pair of functors $u\colon A \to B$ and $f\colon B \to A$,
- and a pair of natural transformations $\eta\colon 1_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow 1_A$, called the **unit** and **counit** respectively,

[1] For instance, as outlined in Digression 1.2.9, the equivalences in the ∞-cosmoi of Example 1.2.18 recapture the weak equivalences between fibrant-cofibrant objects in the usual model structure.

so that the triangle equalities hold:[2]



The functor f is called the **left adjoint** and u is called the **right adjoint**, a relationship that is denoted symbolically in text by writing $f \dashv u$ or in a displayed diagram such as[3]

$$A \underset{u}{\overset{f}{\underset{\perp}{\leftrightarrows}}} B$$

2.1.2. Digression (why this is the right definition). For readers who find Definition 2.1.1 implausible — perhaps too simple to be trusted — we offer a few words of justification. Firstly, the correct notion of *adjunction* between quasi-categories is well established, though the definition appearing in [**13**, §5.2] takes a quite different form. In Appendix E, we prove that in the ∞-cosmos of quasi-categories, our definition of adjunction precisely recovers Lurie's. As explained in Part **??**, each of the models of $(\infty, 1)$-categories described in Example 1.2.18 "has the same category theory," so Definition 2.1.1 agrees with the community consensus notion of adjunction between $(\infty, 1)$-categories.

But what about those ∞-cosmoi whose objects model (∞, n)- or (∞, ∞)-categories? For instance in the ∞-cosmos of complicial sets, the adjunctions defined in the homotopy 2-category are the "pseudo-style" adjunctions. While these are not the most general adjunctions that might be considered — for instance, one could have (op)lax units and counits — they are an important class of adjunctions. One reason for the relevance of Definition 2.1.1 in all ∞-cosmoi is its formal properties vis-a-vis the related notion of equivalence, which Theorem 1.4.7 has established is morally "correct," and with the notions of limits and colimits to be introduced.

Finally, a reasonable objection is that Definition 2.1.1 appears too "low dimensional," comprised of data found entirely in the homotopy 2-category and ignoring the higher dimensional morphisms in an ∞-cosmos. This deficiency will be addressed in Chapter **??**, when we prove that any adjunction between ∞-categories extends to a *homotopy coherent adjunction*, and moreover such extensions are homotopically unique.

The definition of an adjunction given in Definition 2.1.1 is "equational" in character: stated in terms of the objects, 1-cells, and 2-cells of a 2-category and their composites. Immediately:

2.1.3. Lemma. *Adjunctions in a 2-category are preserved by any 2-functor.* □

Lemma 2.1.3 provides an easy source of examples of adjunctions between quasi-categories. The 2-functors underlying the ∞-cosmos functors of Example 1.3.7 then transfer adjunctions defined in one model of $(\infty, 1)$-categories to adjunctions defined in each of the other models.

2.1.4. Example (adjunctions between 1-categories). Via the nerve embedding $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$, any adjunction between strict 1-categories induces an adjunction between their nerves regarded as quasi-categories.

[2]The left-hand equality of pasting diagrams asserts that $u\epsilon \cdot \eta u = \mathrm{id}_u$, while the right-hand equality asserts that $\epsilon f \cdot f\eta = \mathrm{id}_f$.

[3]Some authors contort adjunction diagrams so that the left adjoint is always on the left; we instead use the turnstile symbol "$\perp$" to indicate which adjoint is the left adjoint.

2.1.5. Example (adjunctions between topological categories). The *homotopy coherent nerve* defines a 2-functor $\mathfrak{N}\colon \mathcal{K}an\text{-}\mathcal{C}at \to \mathfrak{h}\mathcal{QC}at$ from the 2-category of Kan complex enriched categories, simplicially enriched functors, and simplicial natural transformations, to the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$. In this way, topologically enriched adjunctions define adjunctions between quasi-categories.

2.1.6. Remark. Topologically enriched adjunctions are relatively rare. More prevalent are "up-to-homotopy" topologically enriched adjunctions, such as those given by Quillen adjunctions between simplicial model categories. These also define adjunctions between quasi-categories, though the proof will have to wait until Part ??.

The preservation of adjunctions by 2-functors proves:

2.1.7. Proposition. *Given any adjunction* $A \underset{u}{\overset{f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} B$ *between* ∞*-categories then:*

(i) for any ∞*-category* X,

$$\mathsf{Fun}(X,A) \underset{u_*}{\overset{f_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Fun}(X,B)$$

defines an adjunction between quasi-categories;

(ii) for any ∞*-category* X,

$$\mathsf{hFun}(X,A) \underset{u_*}{\overset{f_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{hFun}(X,B)$$

defines an adjunction between categories;

(iii) for any simplicial set U,

$$A^U \underset{u^U}{\overset{f^U}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} B^U$$

defines an adjunction between ∞*-categories;*

(iv) and if the ambient ∞*-cosmos is cartesian closed, then for any* ∞*-category* C,

$$A^C \underset{u^C}{\overset{f^C}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} B^C$$

defines an adjunction between ∞*-categories.*

For instance, taking $X = 1$ in (ii) yields a "derived" adjunction between the homotopy categories of the ∞-categories A and B.

Proof. Any adjunction $f \dashv u$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ is preserved by the 2-functors $\mathsf{Fun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, $\mathsf{hFun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, $(-)^U\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$, and $(-)^C\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$. □

2.1.8. Remark. There are contravariant versions of each of the adjunction-preservation results of Proposition 2.1.7, the first of which we explain in detail. Fixing the codomain variable of the functor-space at any ∞-category $C \in \mathcal{K}$ defines a 2-functor

$$\mathsf{Fun}(-, C)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$$

that is contravariant on 1-cells and covariant on 2-cells.[4] Similarly, the cotensor or exponential $C^{(-)}$ is contravariant on 1-cells and covariant on 2-cells.[5] Such 2-functors preserve adjunctions, but exchange left and right adjoints: for instance, given $f \dashv u$ in $\mathcal{K}$, we obtain an adjunction

$$\mathsf{Fun}(A, C) \underset{f^*}{\overset{u^*}{\leftrightarrows}} \mathsf{Fun}(B, C) \quad (\bot)$$

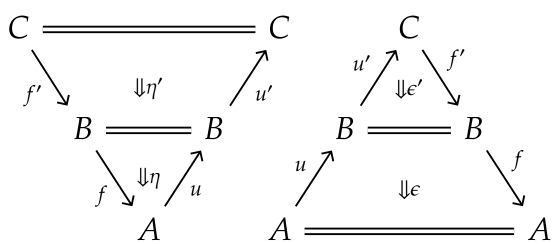
between the functor- spaces.

2.1.9. Proposition. *Adjunctions compose: given adjoint functors*

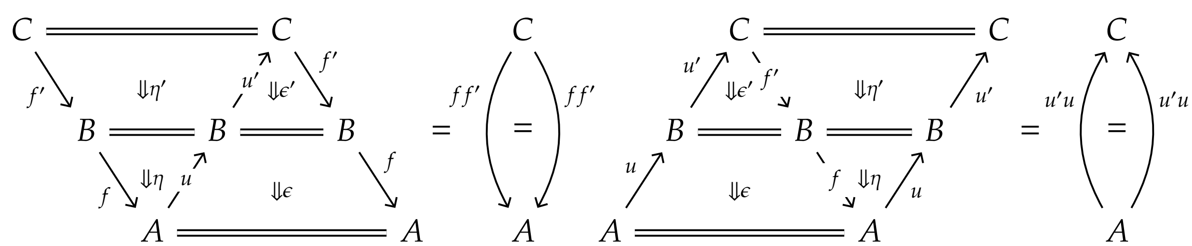
$$C \underset{u'}{\overset{f'}{\rightleftarrows}} B \underset{u}{\overset{f}{\rightleftarrows}} A \quad (\bot,\ \bot) \qquad \rightsquigarrow \qquad C \underset{u'u}{\overset{ff'}{\rightleftarrows}} A \quad (\bot)$$

the composite functors are adjoint.

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_c \Rightarrow u'f'$, and $\epsilon'\colon fu \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams



define the unit and counit of $ff' \dashv u'u$ so that the triangle equalities



hold. □

An adjoint to a given functor is unique up to natural isomorphism:

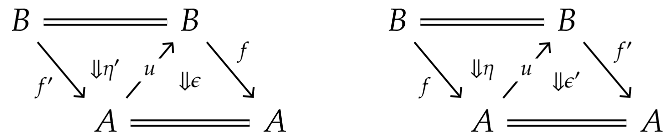
[4]On a strict 2-category, the superscript “op” is used to signal that the 1-cells should be reversed but not the 2-cells, the superscript “co” is used to signal that the 2-cells should be reversed but not the 1-cells, and the superscript “coop” is used to signal that both the 1- and 2-cells should be reversed; see Chapter B.

[5]In the case of the simplicial cotensor, the domain can safely be restricted to the homotopy 2-category of quasi-categories or can be regarded as an analogously-defined homotopy 2-category of simplicial sets.

2.1.10. Proposition (uniqueness of adjoints).

(i) If $f \dashv u$ and $f' \dashv u$, then $f \cong f'$.
(ii) Conversely, if $f \dashv u$ and $f \cong f'$ then $f' \dashv u$.

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_c \Rightarrow u'f'$, and $\epsilon'\colon fu \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams



define 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f$. The composites $f \Rightarrow f' \Rightarrow f$ and $f' \Rightarrow f \Rightarrow f$ are computed by pasting these diagrams together horizontally on one side or the other. Applying the triangle equalities for the adjunctions $f \dashv u$ and $f' \dashv u$ both composites are easily seen to be identities. Hence $f \cong f'$ as functors from B to A.

Part (ii) is left as Exercise 2.1.i. □

We will make repeated use of the following standard 2-categorical result, which says that any equivalence in a 2-category can be promoted to an equivalence that also defines an adjunction:

2.1.11. Proposition (adjoint equivalences). *Any equivalence can be promoted to an* ***adjoint equivalence*** *by modifying one of the 2-cells. That is, the invertible 2-cells in an equivalence can be chosen so as to satisfy the triangle equalities. Hence, if f and g are inverse equivalences then $f \dashv g$ and $g \dashv f$.*

Proof. Consider an equivalence comprised of functors $f\colon A \to B$ and $g\colon B \to A$ and invertible 2-cells

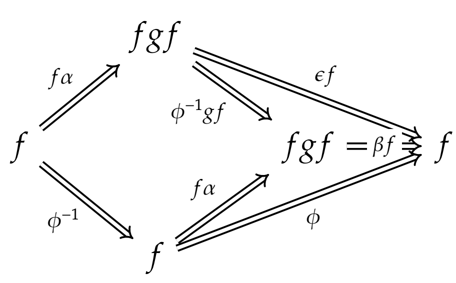
We will construct an adjunction $f \dashv g$ with unit $\eta \coloneqq \alpha$ by modifying β. The "triangle identity composite"

$$\phi \coloneqq f \xRightarrow{f\alpha} fgf \xRightarrow{\beta f} f$$

is an isomorphism, though likely not an identity. Define

$$\epsilon \coloneqq fg \xRightarrow{\phi^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B \coloneqq fg \xRightarrow{\beta^{-1}fg} fgfg \xRightarrow{f\alpha^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B$$

This "corrects" the counit so that now the composite $\epsilon f \cdot f\eta$, displayed on the top of the diagram



which agrees with the bottom composite by "naturality of whiskering," is the identity id_f.

Now by another diagram chase, the other triangle composite $u\epsilon \cdot \eta u$ is an idempotent:

$$\begin{array}{ccccc} u & \overset{\eta u}{\Longrightarrow} & ufu & \overset{u\epsilon}{\Longrightarrow} & u \\ {\scriptstyle \eta u}\Downarrow & & \Downarrow{\scriptstyle \eta u f u} & & \Downarrow{\scriptstyle \eta u} \\ ufu & \overset{uf\eta u}{\Longrightarrow} & ufufu & \overset{ufu\epsilon}{\Longrightarrow} & ufu \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle u\epsilon f u} & & \Downarrow{\scriptstyle u\epsilon} \\ & & ufu & \underset{u\epsilon}{\Longrightarrow} & u \end{array}$$

By cancelation, any idempotent isomorphism is the identity, proving that $u\epsilon \cdot \eta u = \mathrm{id}_u$. □

One use of Proposition 2.1.10 is to show that adjunctions are equivalence invariant: there exists a right adjoint $A \to B$ if and only if, for any pair of equivalent ∞-categories $A' \simeq A$ and $B' \simeq B$, there also exists a right adjoint $A' \to B'$; see Exercise 2.1.ii. As we will discover, all of ∞-category theory is equivalence invariant in this way.

2.1.12. Lemma. *For any ∞-category A, the "composition" functor*

$$A^{\mathbb{2}} \underset{A}{\times} A^{\mathbb{2}} \xrightarrow{\circ} A^{\mathbb{2}} \qquad \begin{array}{l} (-,\mathrm{id}_{\mathrm{dom}(-)}) \dashv \circ \\ \circ \dashv (\mathrm{id}_{\mathrm{cod}(-)},-) \end{array} \tag{2.1.13}$$

admits left and right adjoints, which, respectively, "extend an arrow into a composable pair" by pairing it with the identities at its domain or its codomain.

Proof. There is a dual adjunction in $\mathcal{C}at$ whose functors we describe using notation for simplicial operators introduced in 1.1.1; the full subcategory of $\mathcal{C}at$ spanned by the finite non-empty ordinals is isomorphic to $\mathbf{\Delta}$.

$$\mathbb{3} \xleftarrow{d^1} \mathbb{2} \quad (s^0 \dashv d^1 \dashv s^1) \qquad \rightsquigarrow \qquad A^{\mathbb{3}} \xrightarrow{A^{d^1}} A^{\mathbb{2}} \quad (A^{s^0} \dashv A^{d^1} \dashv A^{s^1})$$

Any ∞-category A in an ∞-cosmos $\mathcal{K}$ defines a 2-functor $A^{(-)} \colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$ carrying the adjoint triple displayed above-left to the one displayed above-right.

Now we claim there is a trivial fibration $A^{\mathbb{3}} \xrightarrow{\sim} A^{\mathbb{2}} \times_A A^{\mathbb{2}}$ constructed as follows. The pushout diagram of simplicial sets displayed below-left is carried by the simplicial cotensor $A^{(-)} \colon \mathcal{SS}et^{\mathrm{op}} \to \mathcal{K}$ to a pullback diagram displayed below-right; since the legs of the pushout square are monomorphisms, the legs of the pullback square are isofibrations

$$\begin{array}{ccc} \Lambda^1[2] & \longleftarrow & \mathbb{2} \\ \uparrow & \lrcorner & \uparrow{\scriptstyle d^1} \\ \mathbb{2} & \underset{d^0}{\longleftarrow} & \mathbb{1} \end{array} \qquad\qquad \begin{array}{ccc} A^{\Lambda^1[2]} & \twoheadrightarrow & A^{\mathbb{2}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} \\ A^{\mathbb{2}} & \underset{\mathrm{ev}_1}{\twoheadrightarrow} & A \end{array}$$

Lemma 1.2.10 tells us that the cotensor of the inner horn inclusion $\Lambda^1[2] \hookrightarrow \mathbb{3}$ with the ∞-category A defines a trivial fibration $A^{\mathbb{3}} \xrightarrow{\sim} A^{\Lambda^1[2]}$ and the pullback square above left recognizes its codomain as the desired ∞-category of "composable pairs." Any section s to $q\colon A^3 \twoheadrightarrow A^2 \times_A A^2$ can be made into an equivalence inverse. By Proposition 2.1.11, these functors are both left and right adjoints. Composing the adjunction $q \dashv s \dashv q$ with the adjunction constructed above defines the desired adjunction. ☐

Note that the adjoint functors of (2.1.13) commute with the "endpoint evaluation" functors to $A \times A$. In fact, the units and counits can similarly be fibered over $A \times A$. We will prove this in §**??**.

Exercises.

2.1.i. Exercise. Prove Proposition 2.1.10(ii).

2.1.ii. Exercise. Given an adjunction $A \underset{u}{\overset{f}{\underset{\perp}{\leftrightarrows}}} B$ and equivalences $A \simeq A'$ and $B \simeq B'$ construct an adjunction between A' and B'.

2.2. Initial and terminal elements

Employing the tactic used to define the homotopy category of A in Definition 1.4.12, we use the terminal ∞-category 1 to probe inside the ∞-category A. The objects $a \in \mathrm{h}A$ of the homotopy category of A were defined to be maps of ∞-categories $a\colon 1 \to A$, but to avoid the proliferation of the term "objects" we refer to maps $a\colon 1 \to A$ as **elements** of the ∞-category A instead.

Before introducing limits and colimits of general diagram shapes, we warm up by defining initial and terminal elements in an ∞-category A.

2.2.1. Definition (initial/terminal element). An **initial element** in an ∞-category A is a left adjoint to the unique functor $!\colon A \to 1$, as displayed below left, while a **terminal element** in an ∞-category A is a right adjoint, as displayed below right.

$$1 \underset{!}{\overset{i}{\underset{\perp}{\rightleftarrows}}} A \qquad\qquad 1 \underset{t}{\overset{!}{\underset{\perp}{\leftrightarrows}}} A$$

Let us unpack the definition of an initial element; dual remarks apply to terminal elements.

2.2.2. Lemma (the minimal data required to present an initial element). *To define an initial element in A, it suffices to specify*

- *an element $i\colon 1 \to A$ and*
- *a natural transformation*

$$\begin{array}{ccccc} & & 1 & & \\ & {}^{!}\nearrow & \Downarrow\epsilon & \searrow^{i} & \\ A & & = = = = & & A \end{array}$$

so that the component $\epsilon i\colon i \Rightarrow i$ is the identity in $\mathrm{h}A$.

Proof. Proposition 1.4.5 demonstrates that the ∞-category $1 \in \mathcal{K}$ is terminal in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. The 1-dimensional aspect of this universal property implies that i defines a section of the unique map $A \to 1$ and from the 2-dimensional aspects, we see that there exist no non-identity 2-cells with codomain 1. In particular, the unit of the adjunction $i \dashv !$ is necessarily an identity and one of the triangle equalities comes for free. The data enumerated above is what remains of Definition 2.1.1 in this setting. ☐

Put more concisely, an initial element i defines a **left adjoint right inverse** to the functor $!\colon A \to 1$. Such adjunctions are studied more systematically in §B.2. In fact, it suffices to assume that the counit component ϵi is an isomorphism, not necessarily the identity; see Lemma B.2.1.

2.2.3. Remark. Applying the 2-functor $\mathsf{Fun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ to an initial or terminal element of an ∞-category $A \in \mathcal{K}$ yields adjunctions

$$\mathbb{1} \cong \mathsf{Fun}(X,1) \underset{!}{\overset{i_*}{\rightleftarrows}} \mathsf{Fun}(X,A) \quad (\perp) \qquad\qquad \mathbb{1} \cong \mathsf{Fun}(X,1) \underset{t_*}{\overset{!}{\leftrightarrows}} \mathsf{Fun}(X,A) \quad (\perp)$$

Via the isomorphisms $\mathsf{Fun}(X,1) \cong \mathbb{1}$ that express the universal property of the terminal ∞-category 1, we see that initial or terminal elements of A define initial or terminal elements of the functor-space $\mathsf{Fun}(X,A)$, namely the composite functors

$$X \xrightarrow{!} 1 \xrightarrow{i} A \qquad \text{or} \qquad X \xrightarrow{!} 1 \xrightarrow{t} A\ .$$

In particular, initial or terminal elements are representably initial or terminal at the level of the ∞-cosmos.

This representable universal property is also captured at the level of the homotopy 2-category. The next lemma shows that the initial element $i\colon 1 \to A$ is initial among all *generalized elements* $f\colon X \to A$ in the following precise sense.

2.2.4. Lemma. *An element $i\colon 1 \to A$ is initial if and only if for all $f\colon X \to A$ there exists a unique 2-cell with boundary*

$$\begin{array}{ccccc} & & 1 & & \\ & {\scriptstyle !}\nearrow & \Downarrow\exists! & \searrow{\scriptstyle i} & \\ X & & \xrightarrow[f]{\qquad\qquad} & & A \end{array}$$

Proof. If $i\colon 1 \to A$ is initial, then the adjunction of Definition 2.2.1 is preserved by the 2-functor $\mathsf{hFun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, defining an adjunction

$$\mathbb{1} \cong \mathsf{hFun}(X,1) \underset{!}{\overset{i_*}{\rightleftarrows}} \mathsf{hFun}(X,A) \quad (\perp)$$

Via the isomorphism $\mathsf{hFun}(X,1) \cong \mathbb{1}$, this adjunction proves that the element $i!\colon X \to A$ is initial in $\mathsf{hFun}(X,A)$ and thus has the universal property of the statement.

Conversely, if $i\colon 1 \to A$ satisfies the universal property of the statement, applying this to the generic element of A (the identity map $\mathrm{id}_A\colon A \to A$) easily produces the data of Lemma 2.2.2. ☐

2.2.5. Remark. Lemma 2.2.4 says that initial elements are *representably initial* in the homotopy 2-category. Specializing the generalized elements to ordinary elements, we see that initial and terminal elements in A respectively define initial and terminal elements in the homotopy category $\mathsf{h}A$.

2.2.6. Lemma. *If A has an initial element and $A \simeq A'$ then A' has an initial element and these elements are preserved up to isomorphism by the equivalences.*

PROOF. By Proposition 2.1.11, the equivalence $A \simeq A'$ can be promoted to an adjoint equivalence, which can immediately be composed with the adjunction characterizing an initial element i of A:

$$1 \underset{!}{\overset{i}{\underset{\perp}{\rightleftarrows}}} A \underset{\sim}{\overset{\sim}{\underset{\perp}{\rightleftarrows}}} A'$$

The composite adjunction provided by Proposition 2.1.9 proves that the image of i defines an initial element of A', which by construction is preserved by the equivalence $A \xrightarrow{\sim} A'$.

To see that the equivalence $A' \xrightarrow{\sim} A$ also preserves initial elements, we can use the invertible 2-cells of the equivalence to see that i is isomorphic to the image of the image of i in A'. In case the initial objects in mind are not the ones being considered here, we can appeal to the uniqueness of initial elements proven in Exercise 2.2.ii. □

Exercises.

2.2.i. EXERCISE. Prove that initial elements are preserved by left adjoints and terminal elements are preserved by right adjoints.

2.2.ii. EXERCISE. Prove that any two initial elements in an ∞-category A are isomorphic in $\mathsf{h}A$.

2.3. Limits and colimits

Our aim is now to introduce limits and colimits of diagram valued *inside* an ∞-category A in some ∞-cosmos. We will consider two varieties of diagrams:

- In a generic ∞-cosmos $\mathcal{K}$, we shall consider diagrams indexed by a simplicial set J and valued in an ∞-category A.
- In a cartesian closed ∞-cosmos $\mathcal{K}$, we shall also consider diagram indexed by an ∞-category J and valued in an ∞-category A.[6]

2.3.1. DEFINITION (diagram ∞-categories). For a simplicial set J — or possibly, in the case of a cartesian closed ∞-cosmos, an ∞-category J — and an ∞-category A, we refer to A^J as the **∞-category of J-shaped diagrams in A**. Both constructions define bifunctors

$$\begin{array}{ccc} \mathcal{SSet}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} & \qquad & \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} \\ (J, A) \longmapsto A^J & & (J, A) \longmapsto A^J \end{array}$$

In either indexing context, there is a terminal object 1 with the property that $A^1 \cong A$ for any ∞-category A. Restriction along the unique map $!\colon J \to 1$, induces the **constant diagram functor** $\Delta\colon A \to A^J$.

We are deliberately conflating the notation for ∞-categories of diagrams indexed by a simplicial set or by another ∞-category because all of the results we will prove in Part I about the former case will also apply to the latter. For economy of language, we refer only to simplicial set indexed diagrams for the remainder of this section.

[6]In Part ??, we shall discover that in the case of the ∞-cosmoi of $(\infty,1)$-categories, there is no essential difference between these notions: in $\mathcal{QCat}$ they are tautologically the same, and in all biequivalent ∞-cosmsoi the ∞-category of diagrams indexed by an ∞-category A is equivalent to the ∞-category of diagrams indexed by its underlying quasi-category, regarded as a simplicial set.

2.3.2. Definition. An ∞-category A **admits all colimits** of shape J if the constant diagram functor $\Delta\colon A \to A^J$ admits a left adjoint, while A **admits all limits** of shape J if the constant diagram functor admits a right adjoint:

$$A^J \underset{\underset{\lim}{\perp}}{\overset{\overset{\operatorname{colim}}{\perp}}{\leftarrow \Delta -}} A$$

2.3.3. Warning. Limits or colimits of set-indexed diagrams — the case where the indexing shape is a coproduct of the terminal object $\mathbb{1}$ indexed by a set J — are called **products** or **coproducts**, respectively. In this case the ∞-category of diagrams itself decomposes as a product $A^J \cong \prod_J A$. As the functor

$$\mathfrak{h}\mathcal{K} \xrightarrow{\mathsf{hFun}(1,-)} \mathcal{C}at$$

$$A \longmapsto \mathsf{h}A$$

that carries an ∞-category to its homotopy category preserves products, when J is a set there is a chain of isomorphisms

$$\mathsf{h}(A^J) \cong \mathsf{h}(\textstyle\prod_J A) \cong \prod_J \mathsf{h}A \cong (\mathsf{h}A)^J$$

Thus, in this special case the adjunctions of Definition 2.3.2 that define products or coproducts in an ∞-category descend to the adjunctions that define products or coproducts in its homotopy category.

However, this argument does *not* extend to more general limit or colimit notions, and such ∞-categorical limits or colimits are generally not limits or colimits in the homotopy category.[7] In §3.2, we shall see that the homotopy category construction fails to preserve more complicated cotensors, even in the relatively simple case of $J = \mathbb{2}$.

The problem with Definition 2.3.2 is that it is insufficiently general: many ∞-categories will have certain, but not all, limits of diagrams of a particular indexing shape. So it would be desirable to re-express Definition 2.3.2 in a form that allows us to define the limit of a single diagram $d\colon \mathbb{1} \to A^J$ or of a family of diagrams. To achieve this, we make use of the following 2-categorical notion that op-dualizes the more familiar absolute extension diagrams.

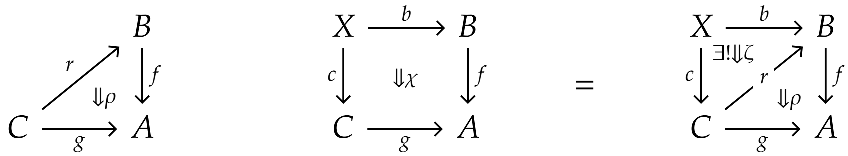
2.3.4. Definition (absolute lifting diagrams). Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in a 2-category, an **absolute left lifting** of g through f is given by a 1-cell and 2-cell as displayed below-left

$$\begin{array}{ccc} & & B \\ & \overset{\ell}{\nearrow} \;\Uparrow\lambda & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{b} & B \\ c\downarrow & \Uparrow\chi & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{b} & B \\ c\downarrow & \exists!\Uparrow\zeta \; \overset{\ell}{\nearrow} \; \Uparrow\lambda & \downarrow f \\ C & \xrightarrow[g]{} & A \end{array}$$

so that any 2-cell as displayed above-center factors uniquely through (ℓ, λ) as displayed above-right.

[7]This sort of behavior is expected in abstract homotopy theory: homotopy limits and colimits are not generally limits or colimits in the homotopy category.

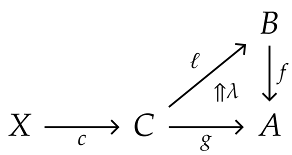
Dually, an **absolute right lifting** of g through f is given by a 1-cell and 2-cell as displayed below-left



so that any 2-cell as displayed above-center factors uniquely through (r, ρ) as displayed above-right.

The adjectives "left" and "right" refer to the handedness of the adjointness of these constructions: left and right liftings respectively define left and right adjoints to the composition functor $f_*\colon \mathsf{hFun}(C, B) \to \mathsf{hFun}(C, A)$, with the 2-cells defining the components of the unit and counit of these adjunctions, respectively, at the object g. The adjective "absolute" refers to the following stability property.

2.3.5. Lemma. *Absolute left or right lifting diagrams are stable under restriction of their domain object: if (ℓ, λ) defines an absolute left lifting of g through f, then for any $c\colon X \to C$, the restricted diagram $(\ell c, \lambda c)$ defines an absolute left lifting of gc through f.*



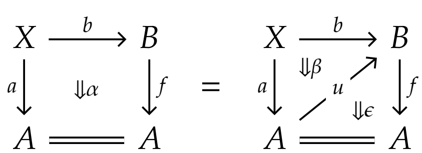
Proof. Exercise 2.3.i. □

Units and counits of adjunctions provide important examples of absolute left and right lifting diagrams respectively:

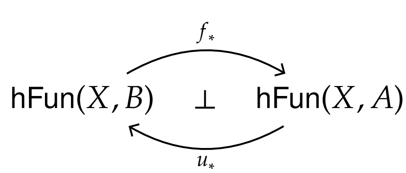
2.3.6. Lemma. *A 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ defines the unit of an adjunction $f \dashv u$ if and only if (f, η) defines an absolute left lifting diagram, displayed below-left.*

Dually a 2-cell $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ defines the counit of an adjunction if and only if (u, ϵ) defines an absolute right lifting diagram, displayed above-right.

Proof. We prove the universal property of the counit. Given a 2-cell $\alpha\colon fb \Rightarrow a$ as displayed below left



there exists a unique transpose $\beta\colon b \Rightarrow ua$ as displayed above-right across the induced adjunction



between the hom-categories of the homotopy 2-category; see Proposition 2.1.7(ii). From right to left, transposes are composed by pasting with the counit; hence the left-hand side above equals the right-hand side. The converse is left as Exercise 2.3.ii. □

In particular, the unit of the adjunction $\operatorname{colim} \dashv \Delta$ of Definition 2.3.2 defines an absolute left lifting diagram

$$\begin{array}{ccc} & & A \\ & \overset{\operatorname{colim}}{\nearrow} \ \Uparrow\eta & \downarrow \Delta \\ A^J & = & A^J \end{array}$$

By Lemma 2.3.5, this universal property is retailed upon restricting to any subobject of the ∞-category of diagrams. This motivates the following definitions:.

2.3.7. Definition. A **colimit** of a family of diagrams $d \colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute left lifting diagram

$$\begin{array}{ccc} & & A \\ & \overset{\operatorname{colim}}{\nearrow} \ \Uparrow\eta & \downarrow \Delta \\ D & \xrightarrow[d]{} & A^J \end{array}$$

comprised of a **colimit functor** $\operatorname{colim} \colon D \to A$ and a **colimit cone** $\eta \colon \Delta \operatorname{colim} \Rightarrow d$.

Dually, a **limit** of a family of diagrams $d \colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute right lifting diagram

$$\begin{array}{ccc} & & A \\ & \overset{\lim}{\nearrow} \ \Downarrow\epsilon & \downarrow \Delta \\ D & \xrightarrow[d]{} & A^J \end{array}$$

comprised of a **limit functor** $\lim \colon D \to A$ and a **limit cone** $\epsilon \colon d \Rightarrow \Delta \lim$.

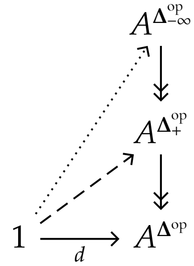
2.3.8. Remark. If A has all limits of shape J, then Lemma 2.3.5 implies that any family of diagrams $d \colon D \to A^J$ has a limit, defined by evaluating the limit functor $\lim \colon A^J \to A$ at d, i.e., by restricting $\lim$ along d. In certain ∞-cosmoi, such as $\mathcal{QC}at$, if every diagram $d \colon 1 \to A^J$ has a limit, then A has all J-indexed limits, because the quasi-category 1 generates the ∞-cosmos of quasi-categories in a suitable sense, but this result is not true for all ∞-cosmoi.

For example, a 2-categorical lemma enables general proof of a classical result from homotopy theory that computes geometric realizations of "split" simplicial objects. Before proving this, we introduce the indexing shapes involved.

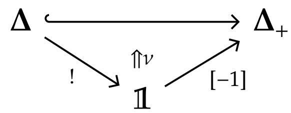
2.3.9. Definition (split augmented (co)simplicial objects). Recall $\mathbf{\Delta}$ is the simplex category of finite non-empty ordinals and order-preserving maps introduced in 1.1.1. It defines a full subcategory of a category $\mathbf{\Delta}_+$ which freely appends the empty ordinal "$[-1]$" as an initial object. This in turn defines a wide subcategory of a category $\mathbf{\Delta}_{-\infty}$, which adds an "extra" degeneracy $\sigma^{-1} \colon [n+1] \twoheadrightarrow [n]$ between each pair of consecutive ordinals, including $\sigma^{-1} \colon [0] \twoheadrightarrow [-1]$.

Diagrams indexed by $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_{-\infty}$ are respectively called **cosimplicial objects**, **coaugmented cosimplicial objects**, and **split coaugmented cosimplicial objects**, if they are covariant, and **simplicial objects**, **augmented simplicial objects**, and *split augmented simplicial objects*, if they are contravariant.

A simplicial object $d\colon 1 \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ in an ∞-category A **admits an augmentation** or **admits a splitting**, if it lifts along the restriction functors



The family of simplicial objects admitting an augmentation and splitting is then represented by the generic element $A^{\mathbf{\Delta}_{-\infty}^{\mathrm{op}}} \twoheadrightarrow A^{\mathbf{\Delta}^{\mathrm{op}}}$. The following proposition proves that for any simplicial object admitting a splitting, the augmentation defines the colimit cone; dual results apply to colimits of split cosimplicial objects. The limit and colimit cones are defined by cotensoring with the unique natural transformation



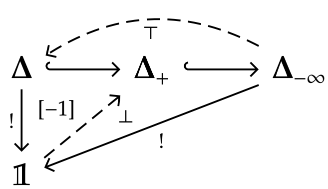
(2.3.10)

that exists because $[-1]\colon \mathbb{1} \to \mathbf{\Delta}_+$ is initial; see Lemma 2.2.4.

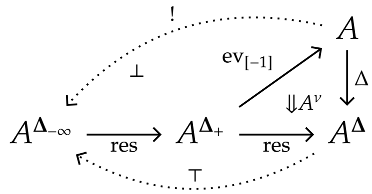
2.3.11. Proposition (geometric realizations). *Let A be any ∞-category. For every cosimplicial object in A that admits a coaugmentation and a splitting, the coaugmentation defines a limit cone. Dually, for every simplicial object in A that admits an augmentation and a splitting, the augmentation defines a colimit cone. That is, there exist absolute right and left lifting diagrams*

in which the 2-cells are obtained as restrictions of the cotensor of the 2-cell (2.3.10) *with A.*

Proof. By Example B.3.2, the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_{-\infty}$ admits a right adjoint, which can automatically be regarded as an adjunction "over" $\mathbb{1}$ since $\mathbb{1}$ is 2-terminal. The initial element $[-1] \in \mathbf{\Delta}_+ \subset \mathbf{\Delta}_{-\infty}$ defines a left adjoint to the constant functor:



with the counit of this adjunction (2.3.10) defining the colimit cone under the constant functor at the initial element. These adjunctions are preserved by the 2-functor $A^{(-)}\colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$, yielding a diagram



By Lemma B.3.1 these adjunctions witness the fact that evaluation at $[-1]$ and the 2-cell from (2.3.10) define an absolute right lifting of the canonical restriction functor $A^{\mathbb{\Delta}_{-\infty}} \to A^{\mathbb{\Delta}}$ through the constant diagram functor, as claimed. The colimit case is proven similarly by applying the composite 2-functor

$$\mathcal{C}at^{\mathrm{coop}} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}at^{\mathrm{op}} \xrightarrow{A^{(-)}} \mathfrak{h}\mathcal{K} \qquad \square$$

Exercises.

2.3.i. Exercise. Prove Lemma 2.3.5.

2.3.ii. Exercise. Re-prove the forwards implication of Lemma 2.3.6 by following your nose through a pasting diagram calculation and prove the converse similarly.

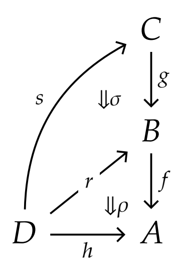
2.4. Preservation of limits and colimits

Famously, right adjoint functors preserve limits and left adjoints preserve colimits. Our aim in this section is to prove this in the ∞-categorical context and exhibit the first examples of initial and final functors, in the sense introduced in Definition 2.4.6 below.

The commutativity of right adjoints and limits is very easily established in the case where the ∞-categories in question admit *all* limits of a given shape: under these hypotheses, the limit functor is right adjoint to the constant diagram functor, which commutes with all functors between the base ∞-categories. Since the left adjoints commute, the uniqueness of adjoints (Proposition 2.1.10) implies that the right adjoints do as well. This outline gives a hint for Exercise 2.4.i.

A slightly more delicate argument is needed in the general case, involving, say the preservation of a single limit diagram without a priori assuming that any other limits exist. This follows easily from a general lemma about composition and cancelation of absolute lifting diagrams:

2.4.1. Lemma (composition and cancelation of absolute lifting diagrams). *Suppose (r, ρ) defines an absolute right lifting of h through f:*



Then (s, σ) defines an absolute right lifting of r through g if and only if $(s, \rho \cdot f\sigma)$ defines an absolute right lifting of h through fg.

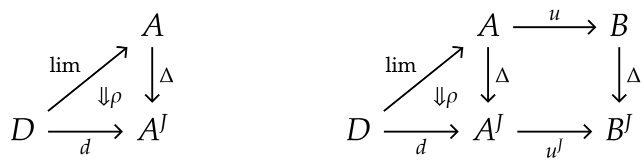
Proof. Exercise 2.4.ii. $\square$

2.4.2. Theorem (RAPL/LAPC). *Right adjoints preserve limits and left adjoints preserve colimits.*

The usual argument that right adjoints preserve limits proceeds like this: a cone over a J-shaped diagram in the image of u transposes across the adjunction $f^J \dashv u^J$ to a cone over the original diagram, which factors through the designated limit cone. This factorization transposes across the adjunction $f \dashv u$ to define the sought-for unique factorization through the image of the limit cone. The use of absolute lifting diagrams to express the universal properties of limits and colimits (Definition 2.3.7) and adjoint transposition (Lemma 2.3.6) allows us to economize on the usual proof by suppressing consideration of a generic test cone that must be shown to uniquely factor through the limit cone.

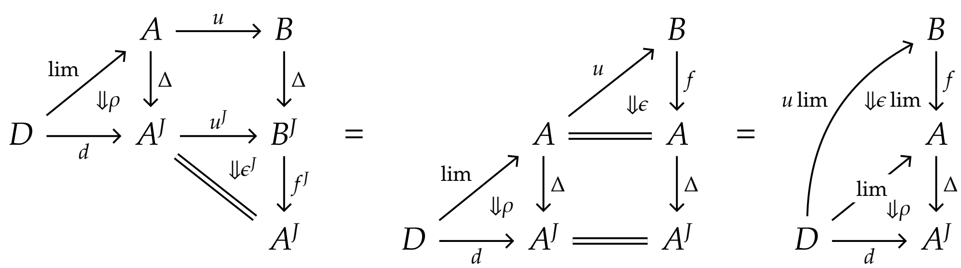
Proof. We prove that right adjoints preserve limits. By taking "co" duals the same argument demonstrates that left adjoints preserve colimits.

Suppose $u\colon A \to B$ admits a left adjoint $f\colon B \to A$ with unit $\eta\colon \mathrm{id}_B \Rightarrow uf$ and counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$. Our aim is to show that any absolute right lifting diagram as displayed below-left is carried to an absolute right lifting diagram as displayed below-right:



(2.4.3)

The cotensor $(-)^J\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ carries the adjunction $f \dashv u$ to an adjunction $f^J \dashv u^J$ with unit η^J and counit ϵ^J. In particular, by Lemma 2.3.6, (u^J, ϵ^J) defines an absolute right lifting of the identity through f^J, which is then preserved by restriction along the functor d. Thus, by Lemma 2.4.1, the diagram on the right of (2.4.3) is an absolute right lifting diagram if and only if the pasted composite displayed below-left



defines an absolute right lifting diagram. Pasting the 2-cell on the right of (2.4.3) with the counit in this way amounts to transposing the cone under $u\lim$ across the adjunction $f^J \dashv u^J$.

We'll now observe that this transposed cone factors through the limit cone $(\lim, \rho)$ in a canonical way. From the 2-functoriality of the simplicial cotensor in its exponent variable, $f^J\Delta = \Delta f$ and $\epsilon^J\Delta = \Delta\epsilon$. Hence, the pasting diagram displayed above-left equals the one displayed above-center and hence also, by naturality of whiskering, the diagram above-right.[8] This latter diagram is a pasted composite of two absolute right lifting diagrams, and is hence an absolute right lifting diagram in its own right; this universal property says that any cone over d whose summit factors through f factors uniquely through the limit cone $(\lim, \rho)$ through a map that then transposes along the adjunction $f \dashv u$. Hence all of the diagrams in the statement are absolute right lifting diagrams, including in particular the one on the right-hand side of (2.4.3). ☐

By combining Theorem 2.4.2 with Proposition 2.1.11, we have immediately that:

[8]By naturality of whiskering, $\epsilon^J d \cdot f^J u^J \rho = \rho \cdot \epsilon^J \Delta \lim$, and since $\epsilon^J\Delta = \Delta\epsilon$, this composite equals $\rho \cdot \Delta\epsilon \lim$.

2.4.4. COROLLARY. *Equivalences preserve limits and colimits.* □

We can also prove a more refined result:

2.4.5. PROPOSITION. *If $A \simeq B$, then any family of diagrams in A admitting a limit or colimit in B also admits a limit or colimit in A that is preserved by the equivalence.*

PROOF. By Proposition 2.1.11 the equivalence $B \rightsquigarrow A$ is both left and right adjoint to its equivalence inverse, preserving both limits and colimits of the composite family of diagrams $D \to A^J \rightsquigarrow B^J$. Via the invertible 2-cells of the equivalence $A^J \simeq B^J$ constructed by applying $(-)^J \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ to the equivalence $A \simeq B$, the preserved diagram $D \to A^J \rightsquigarrow B^J \rightsquigarrow A^J$ is isomorphic to the original family of diagrams $D \to A^J$. Thus, we conclude that a family of diagrams in A has a limit or colimit if and only if its image in an equivalent ∞-category B does, and such limits and colimits are preserved by the equivalence. □

The following definition makes sense between small quasi-categories or equally between arbitrary ∞-categories in a cartesian closed ∞-cosmos.

2.4.6. DEFINITION (initial and final functor). A functor $k \colon I \to J$ is **final** if J-indexed colimits exist if and only if, and in such cases coincide with, the restricted I-indexed colimits. That is, $k \colon I \to J$ is final if and only if for any ∞-category A, the square

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ A^J & \xrightarrow[A^k]{} & A^I \end{array}$$

preserves and reflects all absolute left lifting diagrams.

Dually a functor $k \colon I \to J$ is **initial** if this square preserves and reflects all absolute right lifting diagrams: or informally, if a generalized element defines a limit of a J-indexed diagram if and only if it defines a limit of the restricted I-indexed diagrams.

Historically, final functors were called "cofinal" with no obvious name for the dual notion. Our preferred terminology hinges on the following mnemonic: the inclusion of an initial element defines an initial functor, while the inclusion of a terminal (aka final) element defines a final functor. These results are special cases of a more general result we now establish, using exactly the same tactics as taken to prove Theorem 2.4.2.

2.4.7. PROPOSITION. *Left adjoints define initial functors and right adjoints define final functors.*

PROOF. If $k \dashv r$ with unit $\eta \colon \mathrm{id}_I \Rightarrow rk$ and counit $\epsilon \colon kr \Rightarrow \mathrm{id}_J$, then cotensoring into A yields an adjunction

$$A^J \underset{A^k}{\overset{A^r}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} A^I$$

with unit $A^\eta \colon \mathrm{id}_{A^I} \Rightarrow A^k A^r$ and counit $A^\epsilon \colon A^r A^k \Rightarrow \mathrm{id}_{A^J}$.

To prove that k is initial we must show that for any $(d, \lim, \rho)$ as displayed below-left,

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

the left-hand diagram is an absolute right lifting diagram if and only if the right-hand diagram is an absolute right lifting diagram.

By Lemmas 2.3.6 and 2.4.1, the right-hand diagram is an absolute right lifting diagram if and only if the pasted composite displayed below-left

$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J \arrow[r, "A^k"] \arrow[dr, equal] & A^I \arrow[d, "A^r"] \arrow[dl, phantom, "\Downarrow A^\epsilon"] \\ & & A^J \end{tikzcd} = \begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J \end{tikzcd}$$

is also an absolute right lifting diagram. On noting that $A^r\Delta = \Delta$ and $A^\epsilon\Delta = \mathrm{id}_\Delta$, the left-hand side reduces to the right-hand side, which proves the claim. ☐

Exercises.

2.4.i. Exercise. Show that any left adjoint $f\colon B \to A$ between ∞-categories admitting all J-shaped colimits preserves them in the sense that the square of functors

$$\begin{tikzcd} B^J \arrow[r, "f^J"] \arrow[d, "\mathrm{colim}"'] \arrow[dr, phantom, "\cong"] & A^J \arrow[d, "\mathrm{colim}"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

commutes up to isomorphism.

2.4.ii. Exercise. Prove Lemma 2.4.1.

2.4.iii. Exercise. Given a proof of Theorem 2.4.2 that does not appeal to Lemma 2.4.1 by directly verifying that the diagram on the right of (2.4.3) is an absolute right lifting diagram.

2.4.iv. Exercise. Use Lemma 2.4.1 to give a new proof of Proposition 2.1.9.

CHAPTER 3

Weak 2-limits in the homotopy 2-category

In Chapter 2, we introduced adjunctions between ∞-categories and limits and colimits of diagrams valued within an ∞-category through definitions that are particularly expedient for establishing the expected interrelationships. But neither 2-categorical definition clearly articulates the universal properties of these notions. Definition 2.3.7 does not obviously express the expected universal property of the limit cone: namely, that the limit cone over a diagram d defines the terminal element of the ∞-category of cones over d, yet-to-be-defined. Nor have we understood how an adjunction $f \dashv u$ induces an equivalence on as-yet-to-be-defined hom-spaces $\mathsf{Hom}_A(fb, a) \simeq \mathsf{Hom}_B(b, ua)$ for a pair of generalized elements.[1] In this section, we make use of the completeness axiom in the definition of an ∞-cosmos to exhibit a general construction that will specialize to give a definition of this ∞-category of cones and also specialize to define these hom-spaces. This construction will also permit us to represent a functor between ∞-categories *as* an ∞-category, in dual "left" or "right" fashions. Using this, we can redefine an adjunction to consist of a pair of functors $f : B \to A$ and $u : A \to B$ so that the left representation of f is equivalent to the right representation of u over $A \times B$.

Our vehicle for all of these new definitions is the *comma ∞-category* associated to a cospan

$$C \xrightarrow{g} A \xleftarrow{f} B \qquad \rightsquigarrow \qquad \begin{array}{c} \mathsf{Hom}_A(f, g) \\ \big\downarrow\!\!\downarrow {\scriptstyle (p_1, p_0)} \\ C \times B \end{array}$$

Our aim in this chapter is to develop the general theory of comma constructions from the point of view of the homotopy 2-category of an ∞-cosmos. Our first payoff for this work will occur in Chapter **??** where we study the universal properties of adjunctions, limits, and colimits in the sense of the ideas just outlined. The comma construction will also provide the essential vehicle for establishing the model-independence of the categorical notions we will introduce throughout this text.

There is a standard definition of a "comma object" that can be stated in any strict 2-category, defined as a particular weighted limit (see Example **??**). Comma ∞-categories do *not* satisfy this universal property in the homotopy 2-category, however. Instead, they satisfy a somewhat peculiar "weak" variant of the usual 2-categorical universal property that to our knowledge has not been discovered elsewhere in the categorical or homotopical literature, expressed in terms of something we call a *smothering functor*. To introduce these universal properties in a concrete rather than abstract framework, we start in §3.1 by considering smothering functors involving homotopy categories of quasi-categories. The intrepid and impatient reader may skip the entirety of §3.1 if they wish to instead first encounter these notions in their full generality.

[1]A 2-categorical version of this result — exhibiting a bijection between *sets* of 2-cells — appears as Lemma 2.3.6, but in an ∞-category we'd hope for a similar equivalence of hom-spaces.

3.1. Smothering functors

Let Q be a quasi-category. Recall from Lemma 1.1.11 that its homotopy category $\mathsf{h}Q$ has

- the elements $1 \to Q$ of Q as its objects;
- the set of homotopy classes of 1-simplices of Q as its arrows, where parallel 1-simplices are homotopic just when they bound a 2-simplex with the remaining outer edge degenerate; and
- a composition relation if and only if any chosen 1-simplices representing the three arrows bound a 2-simplex.

For a strict 1-category J, it is well-known in classical homotopy theory that the homotopy category of diagrams $\mathsf{h}(Q^J)$ is not equivalent to the category $(\mathsf{h}Q)^J$ of diagrams in the homotopy category — except in very special cases, such as when J is a set (see Warning 2.3.3). The objects of $\mathsf{h}(Q^J)$ are *homotopy coherent diagrams* of shape J in Q, while the objects of $(\mathsf{h}Q)^J$ are mere *homotopy commutative diagrams*. There is, however, a canonical comparison functor

$$\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$$

defined by applying $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ to the evaluation functor $Q^J \times J \to Q$ and then transposing; a homotopy coherent diagram is in particular homotopy commutative.

Our first aim in this section is to better understand the relationship between the arrows in the homotopy category $\mathsf{h}Q$ and what we'll refer to as the **arrows of** Q, namely, the 1-simplices in the quasi-category. To study this we'll be interested in the quasi-category in which the arrows of Q live as elements, namely $Q^{\mathbb{2}}$, where $\mathbb{2} = \Delta[1]$ is the nerve of the "walking" arrow. Our notation deliberately imitates the notation commonly used for the **category of arrows**: if C is a strict 1-category, then $\mathsf{C}^{\mathbb{2}}$ is the category whose objects are arrows in C and whose morphisms are commutative squares, regarded as a morphism from the arrow displayed vertically on the left-hand side to the arrow displayed vertically on the right-hand side. This notational conflation suggests our first motivating question: how does the homotopy category of $Q^{\mathbb{2}}$ relate to the category of arrows in the homotopy category of Q?

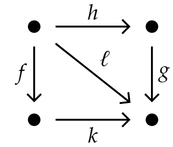
3.1.1. Lemma. *The canonical functor* $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ *is*

- *surjective on objects,*
- *full, and*
- ***conservative**, i.e., reflects invertibility of morphisms,*

but not injective on objects nor faithful.

Proof. Surjectivity on objects asserts that every arrow in the homotopy category $\mathsf{h}Q$ is represented by a 1-simplex in Q. This is the conclusion of Exercise 1.1.ii(iii) which outlines the proof of Lemma 1.1.11.

To prove fullness, consider a commutative square in $\mathsf{h}Q$ and choose arbitrary 1-simplices representing each morphism and their common composite:



By Lemma 1.1.11, every composition relation in $\mathsf{h}Q$ is witnessed by a 2-simplex in Q; choosing a pair of such 2-simplices defines a diagram $\mathbb{2} \to Q^{\mathbb{2}}$, which represents a morphism from f to g in $\mathsf{h}(Q^{\mathbb{2}})$, proving fullness.

Surjectivity on objects and fullness of the functor $\mathsf{h}(Q^{\mathbb{2}}) \to \mathsf{h}(Q)^{\mathbb{2}}$ are special properties having to do with the diagram shape $\mathbb{2}$. Conservativity is much more general as a consequence of the second statement of Corollary 1.1.20. ☐

The properties of the canonical functor $\mathsf{h}(Q^{\mathbb{2}}) \to \mathsf{h}(Q)^{\mathbb{2}}$ will reappear frequently so are worth giving a name:

3.1.2. Definition. A functor $f\colon A \to B$ between strict 1-categories is **smothering** if it is surjective on objects, full, and conservative. That is a functor is smothering if and only if it has the right lifting property with respect to the set of functors:

$$\left\{ \begin{array}{c} \emptyset \\ \big\downarrow \\ \mathbb{1} \end{array} , \begin{array}{c} \mathbb{1}+\mathbb{1} \\ \big\downarrow \\ \mathbb{2} \end{array} , \begin{array}{c} \mathbb{2} \\ \big\downarrow \\ \mathbb{I} \end{array} \right\}$$

Some elementary properties of smothering functors are established in Exercise 3.1.i. The most important of these is:

3.1.3. Lemma. *Each fibre of a smothering functor is a non-empty connected groupoid.*

Proof. Suppose $f\colon A \to B$ is smothering and consider the fiber

$$\begin{array}{ccc} A_b & \longrightarrow & A \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ \mathbb{1} & \xrightarrow{\;b\;} & B \end{array}$$

over an object b of B. By surjectivity on objects, the fiber is non-empty. Its morphisms are defined to be arrows between objects in the fiber of b that map to the identity on b. By fullness, any two objects in the fiber are connected by a morphism, indeed, by morphisms pointing in both directions. By conservativity, all the morphisms in the fiber are necessarily invertible. ☐

The argument used to prove Lemma 3.1.1 generalizes to:

3.1.4. Lemma. *If J is a strict 1-category that is free on a reflexive directed graph and Q is a quasi-category, then the canonical functor $\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$ is smothering.*

Proof. Exercise 3.1.ii. ☐

Cotensors are one of the simplicial limit constructions enumerated in axiom 1.2.1(i). Other limit constructions listed there also give rise to smothering functors.

3.1.5. Lemma. *For any pullback diagram of quasi-categories in which p is an isofibration*

$$\begin{array}{ccc} E \times_B A & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[\;f\;]{} & B \end{array}$$

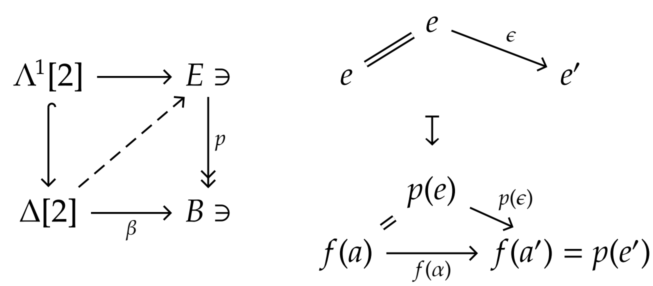
the canonical functor $\mathsf{h}(E \times_B A) \to \mathsf{h}E \times_{\mathsf{h}B} \mathsf{h}A$ is smothering.

Proof. As $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ does not preserve pullbacks, the canonical comparison functor of the statement is not an isomorphism. It is however bijective on objects since the composite functor

$$\mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at \xrightarrow{\mathsf{obj}} \mathcal{S}et$$

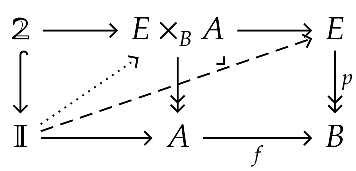
is given by evaluation on the set of vertices of each quasi-category, and this functor *does* preserve pullbacks.

For fullness, note that a morphism in $\mathsf{h}E \times_{\mathsf{h}B} \mathsf{h}A$ is represented by a pair of 1-simplices $\alpha\colon a \to a'$ and $\epsilon\colon e \to e'$ in A and E whose images in B are homotopic, a condition that implies in particular that $f(a) = p(e)$ and $f(a') = p(e')$. By Lemma 1.1.8, we can arrange this homotopy however we like, and thus we choose a 2-simplex witness β so as to define a lifting problem



Since p is an isofibration, a solution exists, defining an arrow $\tilde{\epsilon}\colon e \to e'$ in E in the same homotoyp class as ϵ so that $p(\tilde{\epsilon}) = f(\alpha)$. The pair $(\alpha, \tilde{\epsilon})$ now defines the lifted arrow in $\mathsf{h}(E \times_B A)$.

Finally, consider an arrow $\mathbb{2} \to E \times_B A$ whose image in $\mathsf{h}E \times_{\mathsf{h}B} \mathsf{h}A$ is an isomorphism, which is the case just when the projections to E and A define isomorphisms. By Corollary 1.1.15, we may choose a homotopy coherent isomorphism $\mathbb{I} \to A$ extending the given isomorphism $\mathbb{2} \to A$. This data presents us with a lifting problem



which Exercise 1.1.v tells us we can solve. This proves that $\mathsf{h}(E \times_B A) \to \mathsf{h}E \times_{\mathsf{h}B} \mathsf{h}A$ is conservative and hence also smothering. □

A similar argument proves:

3.1.6. Lemma. *For any tower of isofibrations between quasi-categories*

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathsf{h}(\lim_n E_n) \to \lim_n \mathsf{h}E_n$ *is smothering.*

Proof. Exercise 3.1.iii. □

3.1.7. LEMMA. *For any cospan between quasi-categories* $C \xrightarrow{g} A \xleftarrow{f} B$ *consider the quasi-category defined by the pullback*

$$\begin{array}{ccc} \mathsf{Hom}_A(f,g) & \longrightarrow & A^2 \\ {\scriptstyle (p_1,p_0)}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{cod},\mathrm{dom})} \\ C \times B & \xrightarrow[g\times f]{} & A \times A \end{array}$$

The canonical functor $\mathsf{hHom}_A(f,g) \to \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f,\mathsf{h}g)$ *is smothering.*

PROOF. Here, the codomain is the category defined by an analogous pullback

$$\begin{array}{ccc} \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f,\mathsf{h}g) & \longrightarrow & (\mathsf{h}A)^2 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{cod},\mathrm{dom})} \\ \mathsf{h}C \times \mathsf{h}B & \xrightarrow[\mathsf{h}g\times \mathsf{h}f]{} & \mathsf{h}A \times \mathsf{h}A \end{array}$$

in $\mathcal{C}at$ and the canonical functor factors as

$$\mathsf{hHom}_A(f,g) \to \mathsf{h}(A^2) \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B) \to (\mathsf{h}A)^2 \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B)$$

By Lemma 3.1.5 the first of these functors is smothering. By Lemma 3.1.1 the second is a pullback of a smothering functor. By Exercise 3.1.i(i) it follows that the composite functor is smothering. □

In the sections that follow, we will discover that the smothering functors just constructed express particular "weak" universal properties of arrow, pullback, and comma constructions in the homotopy 2-category of any ∞-cosmos. It is to the first of these that we now turn.

Exercises.

3.1.i. EXERCISE. Prove that:

(i) The class of smothering functors is closed under composition, retract, product, and pullback.
(ii) The class of smothering functors contains all surjective equivalences of categories.
(iii) All smothering functors are isofibrations, that is, maps that have the right lifting property with respect to $\mathbb{1} \hookrightarrow \mathbb{I}$.
(iv) Prove that if f and gf are smothering functors, then g is a smothering functor.[2]

3.1.ii. EXERCISE. Prove Lemma 3.1.4.

3.1.iii. EXERCISE. Prove Lemma 3.1.6.

3.2. ∞-categories of arrows

In this section, we replicate the discussion from the start of the previous section using an arbitrary ∞-category A in place of the quasi-category Q. The analysis of the previous section could have been developed natively in this general setting but at the cost of an extra layer of abstraction and more confusing notation — with a functor space $\mathsf{Fun}(X,A)$ replacing the quasi-category Q.

Recall an **element** of an ∞-category is defined to be a functor $1 \to A$. Tautologically, the elements of A are the vertices of the **underlying quasi-category** $\mathsf{Fun}(1,A)$ of A. In this section, we will define

[2]It suffices, in fact, to merely assume that f is surjective on objects and arrows.

and study an ∞-category $A^{\mathbb{2}}$ whose elements are the 1-simplices in the underlying quasi-category of A. We refer to $A^{\mathbb{2}}$ as the *∞-category of arrows* in A and call its elements simply *arrows* of A.

In fact, we've tacitly introduced this construction already. Recall $\mathbb{2}$ is our preferred notation for the quasi-category $\Delta[1]$, as this coincides with the nerve of the strict 1-category $\mathbb{2}$ with a single non-identity morphism $0 \to 1$.

3.2.1. DEFINITION (arrow ∞-category). Let A be an ∞-category. The **∞-category of arrows** in A is the simplicial cotensor $A^{\mathbb{2}}$ together with the canonical endpoint-evaluation isofibration

$$A^{\mathbb{2}} := A^{\Delta[1]} \xrightarrow{(p_1,p_0)} A^{\partial\Delta[1]} \cong A \times A$$

induced by the inclusion $\partial\Delta[1] \hookrightarrow \Delta[1]$. For conciseness, we write $p_0 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the domain-evaluation induced by the inclusion $0 \colon \mathbb{1} \hookrightarrow \mathbb{2}$ and write $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the codomain-evaluation induced by $1 \colon \mathbb{1} \hookrightarrow \mathbb{2}$.

As an object of the homotopy 2-category $\mathfrak{h}\mathcal{K}$, the ∞-category of arrows comes equipped with a canonical 2-cell that we now construct.

3.2.2. LEMMA. *For any ∞-category A, the ∞-category of arrows comes equipped with a canonical 2-cell*

$$A^{\mathbb{2}} \underset{p_1}{\overset{p_0}{\rtwocell{\kappa}}} A \tag{3.2.3}$$

that we refer to as the ***generic arrow*** *with codomain A.*

PROOF. The simplicial cotensor has a strict universal property described in Digression 1.2.4: namely $A^{\mathbb{2}}$ is characterized by the natural isomorphism

$$\mathsf{Fun}(X, A^{\mathbb{2}}) \cong \mathsf{Fun}(X, A)^{\mathbb{2}}. \tag{3.2.4}$$

By the Yoneda lemma, the data of the natural isomorphism (3.2.4) is encoded by its "universal element," which is defined to be the image of the identity at the representing object. Here the identity functor $\mathrm{id} \colon A^{\mathbb{2}} \to A^{\mathbb{2}}$ is mapped to an element of $\mathsf{Fun}(A^{\mathbb{2}}, A)^{\mathbb{2}}$, a 1-simplex in $\mathsf{Fun}(A^{\mathbb{2}}, A)$ which represents a 2-cell in the homotopy 2-category defining (3.2.3).

To see that its source and target must be the domain-evaluation and codomain-evaluation maps, note that the action of the simplicial cotensor $A^{(-)}$ on morphisms of simplicial sets is defined so that the isomorphism (3.2.4) is natural in the cotensor variable as well. Thus, by restricting along the endpoint inclusion $\mathbb{1}+\mathbb{1} \hookrightarrow \mathbb{2}$, we may regard the isomorphism (3.2.4) as lying over $\mathsf{Fun}(X, A \times A) \cong \mathsf{Fun}(X, A) \times \mathsf{Fun}(X, A)$. □

There is a 2-categorical limit notion that is analogous to Definition 3.2.1, which constructs, for any object A, the universal 2-cell with codomain A: namely the cotensor with the 1-category $\mathbb{2}$. Its universal property is analogous to (3.2.4) but with the hom-categories of the 2-category in place of the functor spaces. In $\mathcal{C}at$ this constructs the arrow category associated to a strict 1-category.

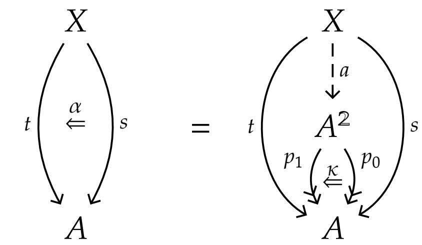
In the homotopy 2-category $\mathfrak{h}\mathcal{K}$, by the Yoneda lemma again, the data (3.2.3) encodes a natural transformation

$$\mathsf{hFun}(X, A^{\mathbb{2}}) \to \mathsf{hFun}(X, A)^{\mathbb{2}}$$

of categories but this is *not* a natural isomorphism, nor even a natural equivalence of categories but does express the arrow ∞-category as a "weak" arrow object with a universal property of the following form:

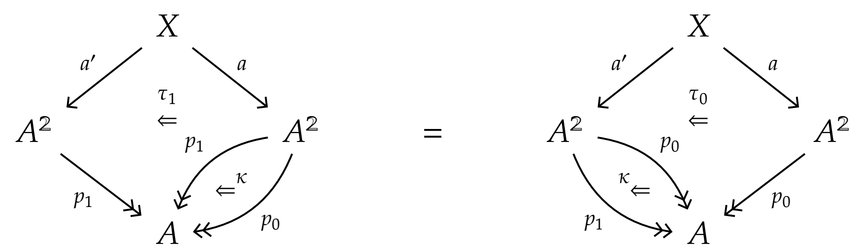
3.2.5. Proposition (the weak universal property of the arrow ∞-category). *The generic arrow* (3.2.3) *with codomain A has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction**: Given a 2-cell over A as below-left*

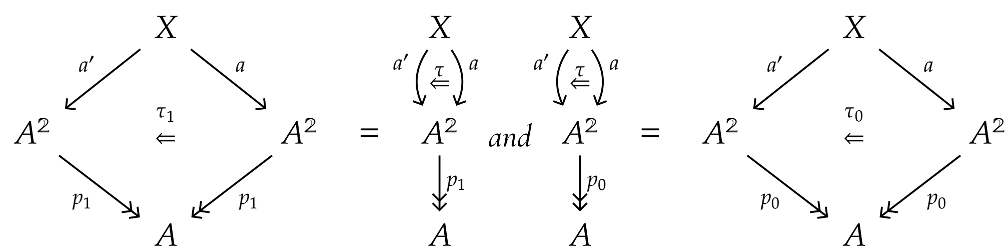


there exists a 1-cell $a\colon X \to A^2$ so that $s = p_0 a$, $t = p_1 a$, and $\alpha = \kappa a$.

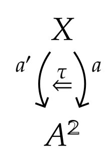
(ii) ***2-cell induction**: Given a pair of functors $a, a'\colon X \rightrightarrows A^2$ and a pair of 2-cells τ_0 and τ_1 so that*



there exists a 2-cell $\tau\colon a \Rightarrow a'$ so that



(iii) ***2-cell conservativity**: Any 2-cell*



with the property that both $p_1\tau$ and $p_0\tau$ are isomorphisms is an isomorphism.

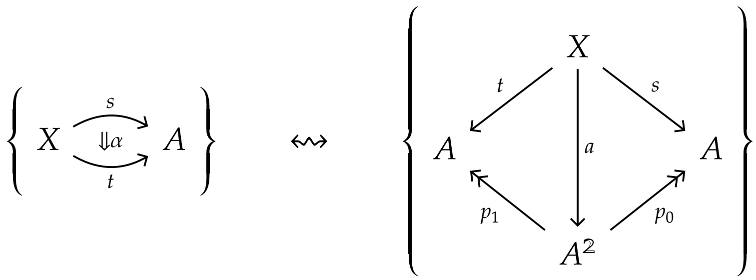
Proof. Let $Q = \mathsf{Fun}(X, A)$ and apply Lemma 3.1.1 to observe that the natural map of hom-categories

$$\begin{array}{ccc} \mathsf{hFun}(X, A^2) & \longrightarrow & \mathsf{hFun}(X, A)^2 \\ & {\scriptstyle ((p_1)_*, (p_0)_*)} \searrow \quad \swarrow {\scriptstyle (\mathrm{ev}_1, \mathrm{ev}_0)} & \\ & \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A) & \end{array}$$

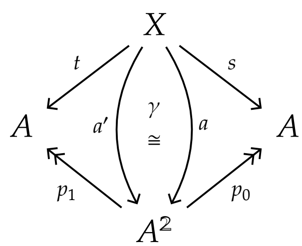
over $\mathsf{hFun}(X, A \times A) \cong \mathsf{hFun}(X, A) \times \mathfrak{h}Fun(X, A)$ is a smothering functor. Surjectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. □

Note that the functors $X \to A^2$ that represent a fixed 2-cell with domain X and codomain A are not unique. However, they are unique up to "fibered" isomorphisms that whisker with $(p_1, p_0) \colon A^2 \twoheadrightarrow A \times A$ to an identity 2-cell:

3.2.6. Proposition. *Whiskering with* (3.2.3) *induces a bijection between 2-cells with domain X and codomain A as displayed below-left*



and fibered isomorphism classes of functors $X \to A^2$ as displayed above-right, where the span isomorphisms are given by invertible 2-cells



so that $p_0\gamma = \mathrm{id}_s$ *and* $p_1\gamma = \mathrm{id}_t$.

Proof. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.2.5 are connected groupoids. The objects of these fibers are functors $X \to A^2$ and the morphisms are invertible 2-cells that whisker with $(p_1, p_0) \colon A^2 \twoheadrightarrow A \times A$ to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

Our final task is to observe that the universal property of Proposition 3.2.5 is also enjoyed by any object $(e_1, e_0) \colon E \twoheadrightarrow A \times A$ that is equivalent to the arrow ∞-category $(p_1, p_0) \colon A^2 \twoheadrightarrow A \times A$ in the slice ∞-cosmos $\mathcal{K}_{/A\times A}$. We have special terminology to allow us to concisely express the type of equivalence we have in mind.

3.2.7. Definition (fibered equivalence). A **fibered equivalence** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{array}{ccc} E & \xrightarrow{\sim} & F \\ & \searrow \!\!\!\!\searrow \quad \swarrow \!\!\!\!\swarrow & \\ & B & \end{array} \tag{3.2.8}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$.

By Proposition 1.2.16(vii), a fibered equivalence is just a map between a pair of isofibrations over a common base that defines an equivalence in the underlying ∞-cosmos: the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences. Note, however, that it does not create them: it is possible for two ∞-categories E and F to be equivalent without there existing any equivalence compatible with a pair of specified isofibration $E \twoheadrightarrow B$ and $F \twoheadrightarrow B$.

3.2.9. Remark. At this point, there is some ambiguity about the 2-categorical data that presents a fibered equivalence related to the question posed in Exercise 1.4.iii. But since Proposition 1.2.16(vii) tells us that a mere equivalence in $\mathfrak{h}\mathcal{K}$ involving a functor of the form (3.2.8) is sufficient to guarantee that this as-yet-unspecified 2-categorical data exists, we defer a careful analysis of this issue to §3.5.

3.2.10. Proposition (uniqueness of arrow ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ equipped with a fibered equivalence $e\colon E \xrightarrow{\sim} A^2$, the corresponding 2-cell*

$$E \underset{e_1}{\overset{e_0}{\underset{\Downarrow \epsilon}{\rightrightarrows}}} A$$

satisfies the weak universal property of Proposition 3.2.5. Conversely, if $(d_1, d_0)\colon D \twoheadrightarrow A \times A$ and $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ are equipped with 2-cells

$$D \underset{d_1}{\overset{d_0}{\underset{\Downarrow \delta}{\rightrightarrows}}} A \qquad \textit{and} \qquad E \underset{e_1}{\overset{e_0}{\underset{\Downarrow \epsilon}{\rightrightarrows}}} A$$

satisfying the weak universal property of Proposition 3.2.5, then D and E are fibered equivalent over $A \times A$.

Proof. We prove the first statement. By the definition equation of 1-cell induction $\epsilon = \kappa e$, where κ is the canonical 2-cell of (3.2.3). Hence, pasting with ϵ induces a functor

$$\begin{array}{ccccc} \mathsf{hFun}(X,E) & \xrightarrow{e_*} & \mathsf{hFun}(X,A^2) & \longrightarrow & \mathsf{hFun}(X,A)^2 \\ & {\scriptstyle ((p_1)_*,(p_0)_*)} \searrow & & \swarrow {\scriptstyle (\mathrm{ev}_1,\mathrm{ev}_0)} & \\ & & \mathsf{hFun}(X,A) \times \mathsf{hFun}(X,A) & & \end{array}$$

and our task is to prove that this composite functor is smothering. We see that the first functor, defined by post-composing with the equivalence $e\colon E \to A^2$, is an equivalence of categories, and the second functor is smothering. Thus, the composite is clearly full and conservative. To see that it is also surjective on objects, note first that by 1-cell induction any 2-cell

$$X \underset{t}{\overset{s}{\underset{\Downarrow \alpha}{\rightrightarrows}}} A$$

is represented by a functor $a\colon X \to A^2$ over $A \times A$. Composing with any fibered inverse equivalence e' to e yields a functor

$$\begin{array}{ccccc} X & \xrightarrow{a} & A^2 & \underset{\sim}{\xrightarrow{e'}} & E \\ & {\scriptstyle (t,s)} \searrow & \downarrow\!\!\!\downarrow {\scriptstyle (p_1,p_0)} & \swarrow {\scriptstyle (e_1,e_0)} & \\ & & A \times A & & \end{array}$$

whose image after post-composing with e is isomorphic to a over $A \times A$. Because this isomorphism is fibered (see Proposition 3.2.6), the image of ae' under the functor $\mathsf{hFun}(X,E) \to \mathsf{hFun}(X,A)^2$ returns the 2-cell α. This proves that this mapping is surjective on objects and hence defines a smothering functor as claimed.

The converse is left to Exercise 3.2.i and proven in a more general context in Proposition 3.3.11. □

3.2.11. Convention. On account of Proposition 3.2.10, we extend the appellation "∞-category of arrows" from the strict model constructed in Definition 3.2.1 to any ∞-category that is fibered equivalent to it.

Via Lemma 3.1.4, the discussion of this section extends to establish corresponding weak universal properties for the cotensors A^J of an ∞-category A with a free category J. We leave the exploration of this to the reader.

Exercises.

3.2.i. Exercise. Prove the second statement of Proposition 3.2.10.

3.3. The comma construction

The *comma ∞-category* is defined by restricting the domain and codomain of the ∞-category of arrows $A^{\mathbb{2}}$ along specified functors with codomain A.

3.3.1. Definition (comma ∞-category). Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories. The **comma ∞-category** is constructed as a pullback of the simplicial cotensor $A^{\mathbb{2}}$ along $g \times f$

$$\begin{array}{ccc} \mathrm{Hom}_A(f,g) & \xrightarrow{\phi} & A^{\mathbb{2}} \\ {\scriptstyle (p_1,p_0)}\downarrow & \lrcorner & \downarrow{\scriptstyle (p_1,p_0)} \\ C \times B & \xrightarrow[g\times f]{} & A \times A \end{array} \tag{3.3.2}$$

This construction equips the comma ∞-category with a specified isofibration $(p_1, p_0) \colon \mathrm{Hom}_A(f,g) \twoheadrightarrow C \times B$ and a canonical 2-cell

$$\begin{array}{ccccc} & & \mathrm{Hom}_A(f,g) & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_0} & \\ C & & \overset{\phi}{\Leftarrow} & & B \\ & {\scriptstyle g}\searrow & & \swarrow{\scriptstyle f} & \\ & & A & & \end{array} \tag{3.3.3}$$

in the homotopy 2-category called the **comma cone**.

3.3.4. Example (arrow ∞-categories as comma ∞-categories). The arrow ∞-category arises as a special case of the comma construction applied to the identity span. This provides us with alternate notation for the canonical 2-cell of (3.2.3), which may be regarded as a particular instance of a comma cone.

$$\begin{array}{ccccc} & & \mathrm{Hom}_A & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_0} & \\ A & & \overset{\phi}{\Leftarrow} & & A \\ & \Vert & & \Vert & \\ & & A & & \end{array}$$

The following proposition encodes the homotopical properties of the comma construction. The proof is by a standard argument in abstract homotopy theory, which can be found in Appendix C. A hint for this proof is given in Exercise 3.3.i.

3.3.5. PROPOSITION (maps between commas). *A commutative diagram*

$$\begin{array}{ccccc} C & \xrightarrow{g} & A & \xleftarrow{f} & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \bar{C} & \xrightarrow{\bar{g}} & \bar{A} & \xleftarrow{\bar{f}} & \bar{B} \end{array}$$

induces a map between the comma ∞-categories

$$\begin{array}{ccc} \mathsf{Hom}_A(f,g) & \xrightarrow{\mathsf{Hom}_p(q,r)} & \mathsf{Hom}_{\bar{A}}(\bar{f},\bar{g}) \\ {\scriptstyle (p_1,p_0)}\downarrow & & \downarrow{\scriptstyle (p_1,p_0)} \\ C \times B & \xrightarrow{r \times q} & \bar{C} \times \bar{B} \end{array}$$

Moreover, if p, q, and r are all

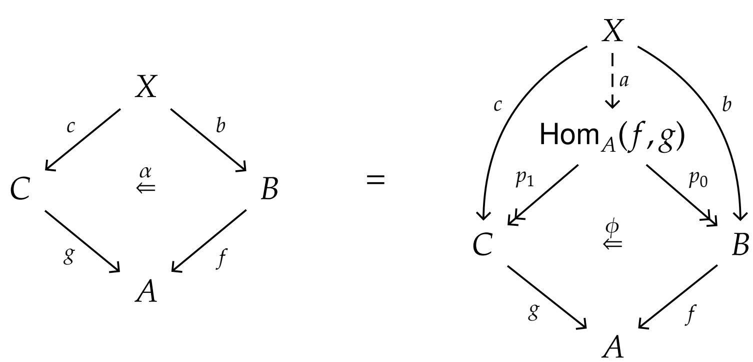
(i) *equivalences,*
(ii) *isofibrations, or*
(iii) *trivial fibrations*

then the induced map is again an equivalence, isofibration, or trivial fibration, respectively.

There is a 2-categorical limit notion that is analogous to Definition 3.3.1, which constructs the universal 2-cell inhabiting a square over a specified cospan. In $\mathcal{C}at$ the category so-constructed is referred to as a *comma category*, from when we borrow the name. As with the case of ∞-categories of arrow, comma ∞-categories do *not* satisfy this 2-universal property strictly. Instead:

3.3.6. PROPOSITION (the weak universal property of the comma ∞-category). *The canonical 2-cell* (3.3.3) *has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction**: Given a 2-cell over* $C \xrightarrow{g} A \xleftarrow{f} B$ *as below-left*



there exists a 1-cell $a\colon X \to \mathsf{Hom}_A(f,g)$ so that $b = p_0 a$, $c = p_1 a$, and $\alpha = \phi a$.

(ii) ***2-cell induction**: Given a pair of functors $a, a' \colon X \rightrightarrows \mathsf{Hom}_A(f,g)$ and a pair of 2-cells τ_0 and τ_1 so that*

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & & & & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ \mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"] \arrow[dr, "p_0"] & = & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\ & C \arrow[dr, "g"'] & \overset{\phi}{\Leftarrow} & B \arrow[dl, "f"] & C \arrow[dr, "g"'] & \overset{\phi}{\Leftarrow} & B \arrow[dl, "f"] \\ & & A & & & A & \end{tikzcd}$$

there exists a 2-cell $\tau \colon a \Rightarrow a'$ so that

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & & X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ \mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"] = & \mathsf{Hom}_A(f,g) \arrow[d, "p_1"'] \\ & C & & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] & & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ \mathsf{Hom}_A(f,g) \arrow[d, "p_0"] = & \mathsf{Hom}_A(f,g) \arrow[dr, "p_0"'] & {} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\ B & & B & \end{tikzcd}$$

(iii) ***2-cell conservativity**: Any 2-cell*

$$\begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ \mathsf{Hom}_A(f,g) \end{tikzcd}$$

with the property that both $p_1\tau$ and $p_0\tau$ are isomorphisms is an isomorphism.

Proof. The functor of ∞-cosmoi $\mathsf{Fun}(X,-) \colon \mathcal{K} \to \mathcal{QC}at$ carries the pullback (3.3.2) to a pullback

$$\begin{tikzcd} \mathsf{Fun}(X, \mathsf{Hom}_A(f,g)) \cong \mathsf{Hom}_{\mathsf{Fun}(X,A)}(\mathsf{Fun}(X,f), \mathsf{Fun}(X,g)) \arrow[r, "\phi"] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X,A)^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\ \mathsf{Fun}(X,C) \times \mathsf{Fun}(X,B) \arrow[r, "\mathsf{Fun}(X,g)\times\mathsf{Fun}(X,f)"'] & \mathsf{Fun}(X,A) \times \mathsf{Fun}(X,A) \end{tikzcd}$$

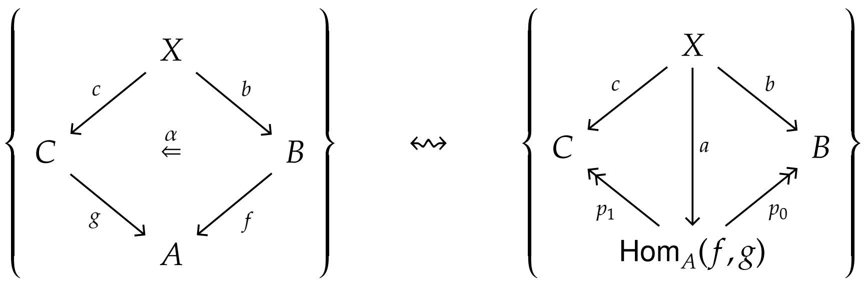
of quasi-categories. Now Lemma 3.1.7 demonstrates that the canonical 2-cell (3.3.3) induces a natural map of hom-categories

$$\begin{tikzcd} \mathsf{hFun}(X, \mathsf{Hom}_A(f,g)) \arrow[rr] \arrow[dr, "((p_1)_*,(p_0)_*)"'] & & \mathsf{Hom}_{\mathsf{hFun}(X,A)}(\mathsf{hFun}(X,f), \mathsf{hFun}(X,g)) \arrow[dl, "(\mathsf{ev}_1,\mathsf{ev}_0)"] \\ & \mathsf{hFun}(X,C) \times \mathsf{hFun}(X,B) & \end{tikzcd}$$

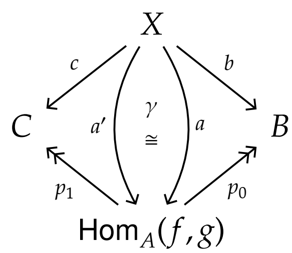
over $\mathsf{hFun}(X, C \times B) \cong \mathsf{hFun}(X,C) \times \mathsf{hFun}(X,B)$ that is a smothering functor. The properties of 1-cell induction, 2-cell induction, and 2-cell conservativity follow from surjectivity on objects, fullness, and conservativity of this smothering functor respectively. □

The 1-cells $X \to \mathsf{Hom}_A(f,g)$ that are induced by a fixed 2-cell $\alpha\colon fb \Rightarrow gc$ are unique up to fibered isomorphism over $C \times B$.

3.3.7. Proposition. *Whiskering with* (3.3.3) *induces a bijection between 2-cells as displayed below-left*



and isomorphism classes of maps of spans from C to B as displayed above-right, where the span isomorphisms are given by invertible 2-cells



so that $p_0\gamma = \mathrm{id}_b$ and $p_1\gamma = \mathrm{id}_c$.

Proof. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.3.6 are connected groupoids. The objects of these fibers are functors $X \to \mathsf{Hom}_A(f,g)$ and the morphisms are invertible 2-cells that whisker with

$$(p_1, p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$$

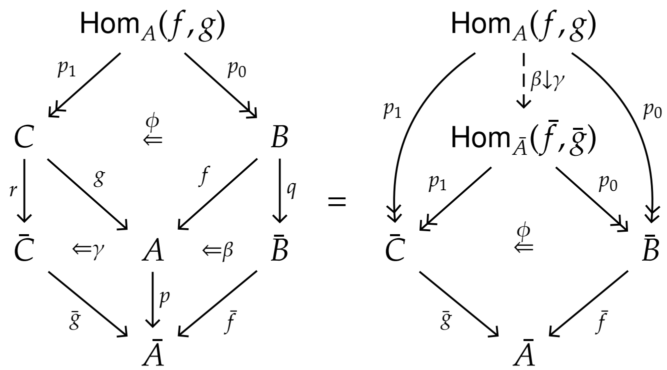
to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

The construction of the comma ∞-category is also pseudo-functorial in lax maps defined in the homotopy 2-category:

3.3.8. Observation. By 1-cell induction a diagram

$$\begin{array}{ccccc} C & \xrightarrow{g} & A & \xleftarrow{f} & B \\ {\scriptstyle r}\downarrow & \Leftarrow\gamma & \downarrow{\scriptstyle p} & \Leftarrow\beta & \downarrow{\scriptstyle q} \\ \bar{C} & \xrightarrow[\bar{g}]{} & \bar{A} & \xleftarrow[\bar{f}]{} & \bar{B} \end{array}$$

induces a map between comma ∞-categories as displayed below-right:



that is well-defined and functorial up to fibered isomorphism.

One of many uses of comma ∞-categories is to define the internal mapping spaces between two elements of an ∞-category A. This is one motivation for our notation "Hom_A."

3.3.9. Definition. For any two elements $x, y\colon 1 \rightrightarrows A$ of an ∞-category A, their **mapping space** is the comma ∞-category $\mathsf{Hom}_A(x,y)$ defined by the pullback diagram

$$\begin{array}{ccc} \mathsf{Hom}_A(x,y) & \xrightarrow{\phi} & A^2 \\ {\scriptstyle (p_1,p_0)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (p_1,p_0)} \\ 1 & \xrightarrow[(y,x)]{} & A \times A \end{array}$$

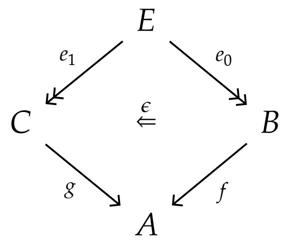
The mapping spaces in any ∞-category are discrete in the sense of Definition 1.2.20.

3.3.10. Proposition (internal mapping spaces are discrete). *For any pair of elements $x, y\colon 1 \rightrightarrows A$ of an ∞-category A, the mapping space $\mathsf{Hom}_A(x,y)$ is discrete.*

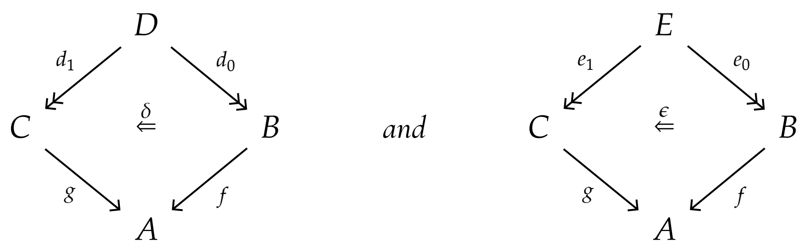
Proof. Our task is to prove that for any ∞-category X, the functor space $\mathsf{Fun}(X, \mathsf{Hom}_A(x,y))$ is a Kan complex. This is so just when $\mathsf{hFun}(X, \mathsf{Hom}_A(f,g))$ is a groupoid, i.e., when any 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible. By 2-cell conservativity, a 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible just when its whiskered composite with the isofibration $(p_1,p_0)\colon \mathsf{Hom}_A(x,y) \twoheadrightarrow 1 \times 1$ is an invertible 2-cell, but in fact this whiskered composite is an identity since 1 is terminal. ☐

As in our convention for ∞-categories of arrows, it will be convenient to weaken the meaning of "comma ∞-category" to extend this appellation to object of $\mathcal{K}_{/C\times B}$ that is fibered equivalent (see Definition 3.2.7) to the strict model $(p_1,p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ defined by 3.3.1. This is justified because such objects satisfy the weak universal property of Proposition 3.3.6 and conversely any two objects satisfying this weak universal property are equivalent over $C \times B$.

3.3.11. Proposition (uniqueness of comma ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ that is fibered equivalent to $\mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ the 2-cell*



encoded by the equivalence $E \xrightarrow{\sim} \mathsf{Hom}_A(f,g)$ satisfies the weak universal property of Proposition 3.3.6. Conversely, if $(d_1, d_0)\colon D \twoheadrightarrow C \times B$ and $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ are equipped with 2-cells

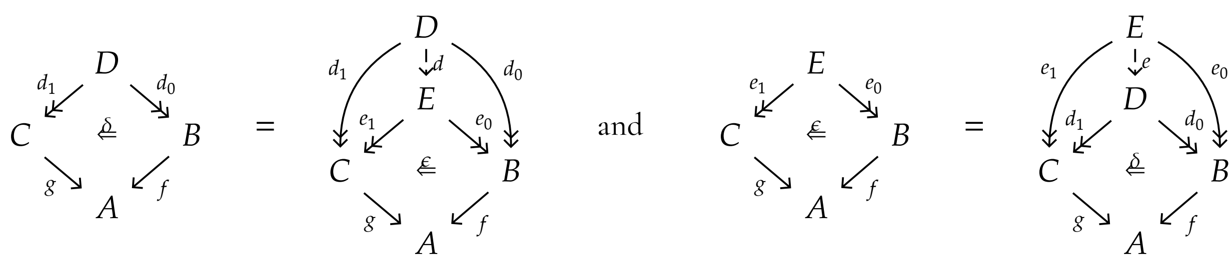


(3.3.12)

satisfying the weak universal property of Proposition 3.3.6, then D and E are fibered equivalent over $C \times B$.

Proof. The proof of the first statement proceeds exactly as in the special case of Proposition 3.2.10. We prove the converse, solving Exercise 3.2.i.

Consider a pair of 2-cells (3.3.12) satisfying the weak universal properties enumerated in Proposition 3.3.6. 1-cell induction supplies maps of spans



with the property that $\epsilon de = \epsilon$ and $\delta ed = \delta$. By Proposition 3.3.7 it follows that $de \cong \mathrm{id}_E$ over $C \times B$ and $ed \cong \mathrm{id}_D$ over $C \times B$. This defines the data of a fibered equivalence $D \simeq E$.[3] ☐

3.3.13. Convention. On account of Proposition 3.3.11, we extend the appellation "comma ∞-category" from the strict model constructed in Definition 3.3.1 to any ∞-category that is fibered equivalent to it and refer to its accompanying 2-cell as the "comma cone."

For example, in §**??** we define the ∞-category of cones over a fixed diagram as a comma ∞-category. Proposition 3.3.11 gives us the flexibility to use multiple models for this ∞-category, which will be useful in characterizing the universal properties of limits and colimits.

[3]For the reader uncomfortable with Remark 3.2.9, Proposition 3.5.3 and Lemma 3.5.4 provides a small boost to finish the proof.

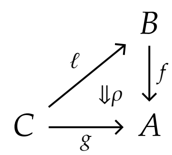
Exercises.

3.3.i. Exercise. Prove Lemma 3.3.5 by observing that the map $\mathsf{Hom}_p(q, r)$ factors as a pullback of the Leibniz cotensor of $\partial\Delta[1] \hookrightarrow \Delta[1]$ with p followed by a pullback of $r \times q$.

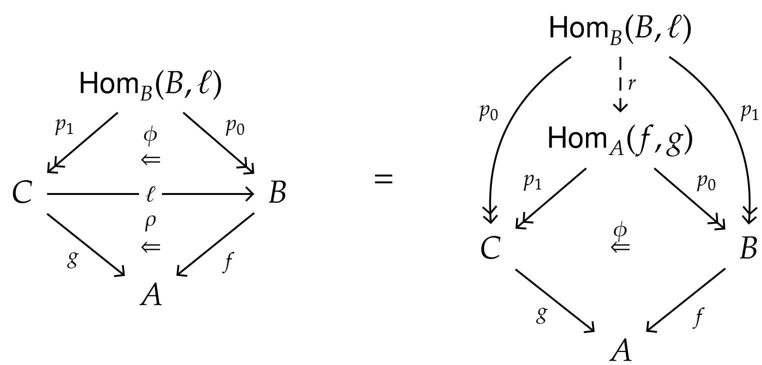
3.3.ii. Exercise. Use Proposition 3.3.7 to justify the pseudofunctoriality of the comma construction in lax morphisms described in Observation 3.3.8.

3.4. A comma characterization of absolute lifting diagrams

3.4.1. Theorem. *The data*



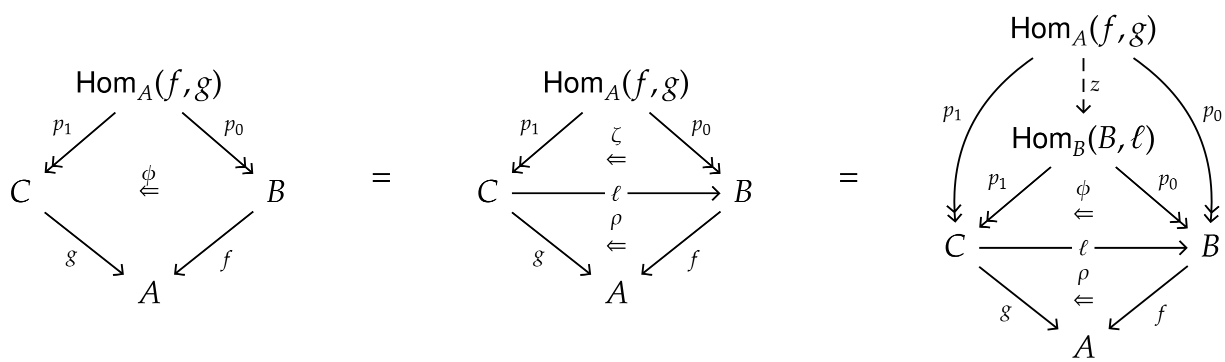
defines an absolute right lifting diagram if and only if the induced 1-cell



(3.4.2)

defines a fibered equivalence $\mathsf{Hom}_B(B, \ell) \simeq \mathsf{Hom}_A(f, g)$ *over* $C \times B$.

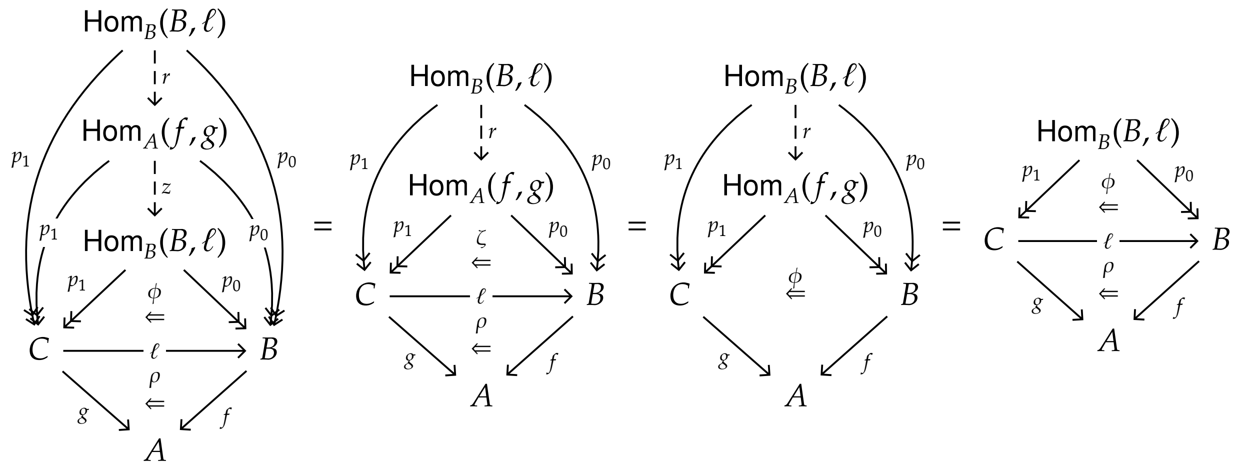
Proof. Suppose that (ℓ, ρ) defines an absolute right lifting of g through f and consider the corresponding unique factorization of the comma cone under $\mathsf{Hom}_A(f, g)$ through ρ as displayed below-center



(3.4.3)

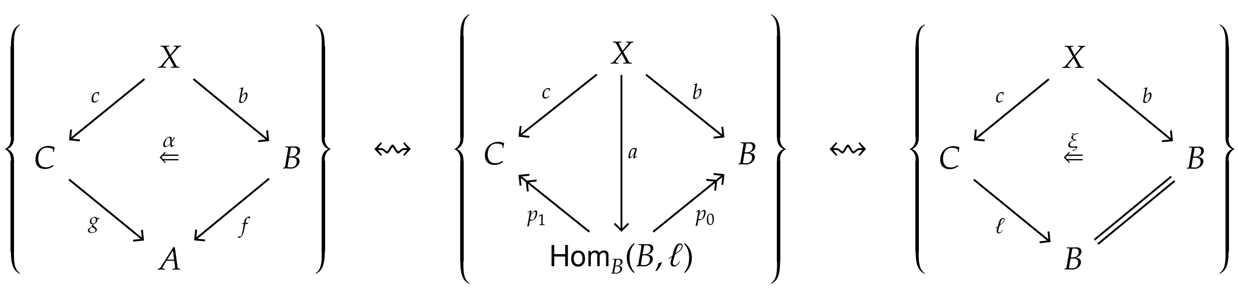
By 1-cell induction, the 2-cell ζ factors through the comma cone for $\mathsf{Hom}_B(B, \ell)$ as displayed above-right. Substituting the right-hand side of (3.4.2) into the bottom portion of the above-right diagram, we see that $rz \colon \mathsf{Hom}_A(f, g) \to \mathsf{Hom}_A(f, g)$ is a 1-cell that factors the comma cone for $\mathsf{Hom}_A(f, g)$ through itself. Applying the universal property of Proposition 3.3.7, it follows that there is a fibered isomorphism $rz \cong \mathrm{id}_{\mathsf{Hom}_A(f,g)}$ over $C \times B$.

To prove that $zr \cong \mathrm{id}_{\mathsf{Hom}_B(B,\ell)}$ it suffices to argue similarly that the comma cone for $\mathsf{Hom}_B(B,\ell)$ restricts along zr to itself. Since ρ is absolute right lifting, it suffices to verify the equality $\phi zr = \phi$ after pasting below with ρ. But now reversing the order of the equalities in (3.4.3) and (3.4.2) we have



which is exactly what we wanted to show. Thus, we see that if (ℓ, ρ) is an absolute right lifting of g through f, then the induced map (3.4.2) defines a fibered equivalence $\mathsf{Hom}_B(B,\ell) \simeq \mathsf{Hom}_A(f,g)$.

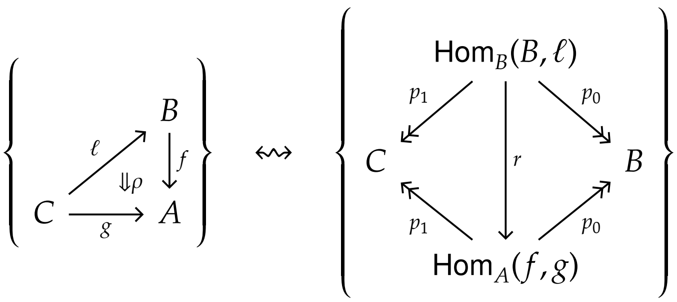
Now, conversely, suppose the 1-cell r defined by (3.4.2) is a fibered equivalence and let us argue that (ℓ, ρ) is an absolute right lifting of g through f. By Proposition 3.3.11, via this fibered equivalence the 2-cell displayed on the left-hand side of (3.4.2) inherits the weak universal property of a comma cone from $\mathsf{Hom}_A(f,g)$. So Proposition 3.3.7 supplies a bijection displayed below-left-center



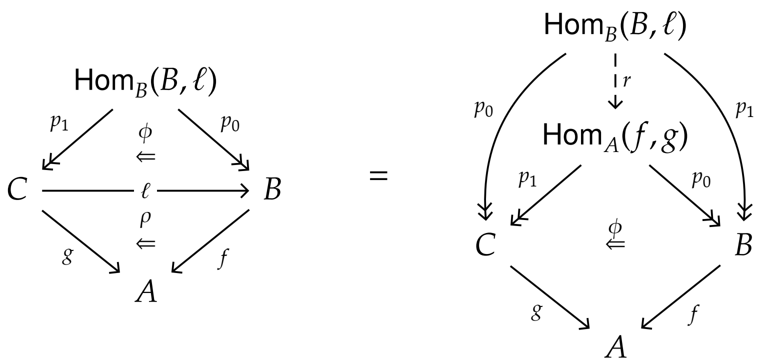
between 2-cells over the cospan and fibered isomorphism classes of maps of spans that is implemented, from center to left, by whiskering with the 2-cell on the left-hand side of (3.4.2). Proposition 3.3.7 also applies to the comma cone ϕ over $\ell\colon C \to B$ giving us a second bijection, displayed above center-right between the same fibered isomorphism classes of maps of spans and 2-cells over ℓ. This second bijection is implemented, from center to right, by pasting with the comma cone ϕ. Combining these yields a bijection between the 2-cells displayed on the left and the 2-cells displayed on the right implemented by pasting with ρ, which is precisely the universal property that characterizes absolute right lifting diagrams. ☐

Having proven Theorem 3.4.1 our immediate aim is to strengthen it to show that a fibered equivalence $\mathsf{Hom}_B(B,\ell) \simeq \mathsf{Hom}_A(f,g)$ over $C \times B$ implies that $\ell\colon C \to B$ defines an absolute right lifting of g through f without a previously specified 2-cell $\rho\colon f\ell \Rightarrow g$.

3.4.4. Theorem. *Given a trio of functors $\ell\colon C \to B$, $f\colon B \to A$, and $g\colon C \to A$ there is a bijection between 2-cells as displayed below-left and isomorphism classes of maps of spans as displayed below-right*



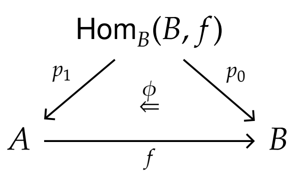
that is constructed by pasting the comma cone for $\mathsf{Hom}_B(B,\ell)$ and then applying 1-cell induction to factor through the comma cone for $\mathsf{Hom}_A(f,g)$.



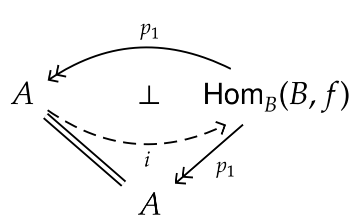
Moreover, a 2-cell $\rho\colon f\ell \Rightarrow g$ displays ℓ as an absolute right lifting of g through f if and only if the corresponding map of spans $r\colon \mathsf{Hom}_B(B,\ell) \to \mathsf{Hom}_A(f,g)$ is an equivalence.

The second clause is the statement of Theorem 3.4.1, so it remains only to prove the first. We show the claimed construction is a bijection by exhibiting its inverse, the construction of which involves a lemma that we will also make use of when we return to the study of adjunctions in Chapter **??**.

3.4.5. Lemma. *Let $f\colon A \to B$ be any functor and denote the comma cone for its right representable comma ∞-category by*

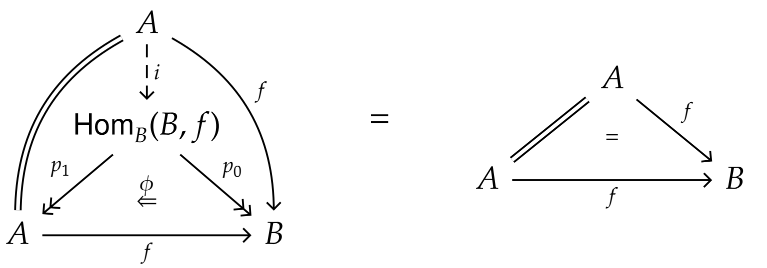


Then the codomain-projection functor $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ admits a right adjoint right inverse, defining an adjunction

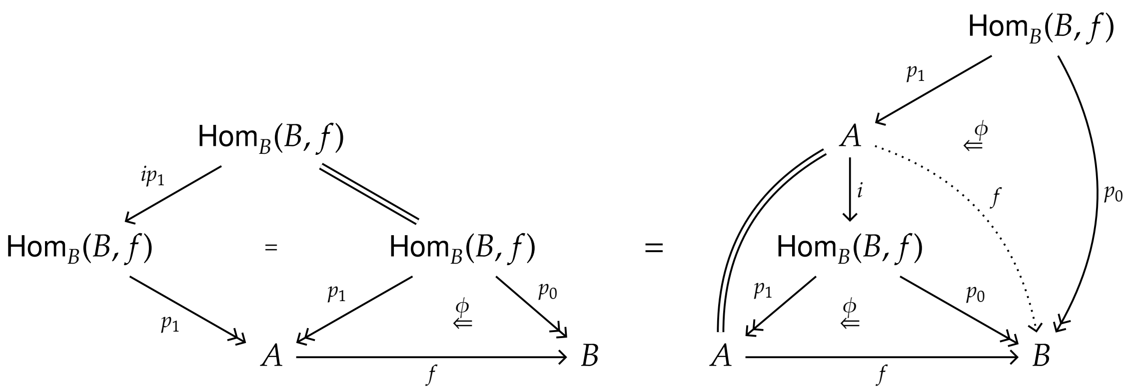


over A whose counit is an identity and whose unit $\nu\colon \mathrm{id} \Rightarrow ip_1$ satisfies the conditions $\nu i = \mathrm{id}_i$, $p_1\nu = \mathrm{id}_{p_1}$ and $p_0\nu = \phi$.

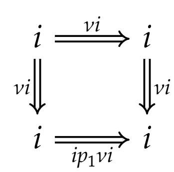
Proof. This adjunction will be constructed using the weak universal properties of the comma cone for $\mathsf{Hom}_B(B, f)$. The identity 2-cell $\mathrm{id}_f \colon f \Rightarrow f$ induces a 1-cell over the comma cone for $\mathsf{Hom}_B(B, f)$:



Note that $p_1 i = \mathrm{id}_A$, so we may take the counit to be the identity 2-cell. Since $\phi i = \mathrm{id}_f$, we have a pasting equality:



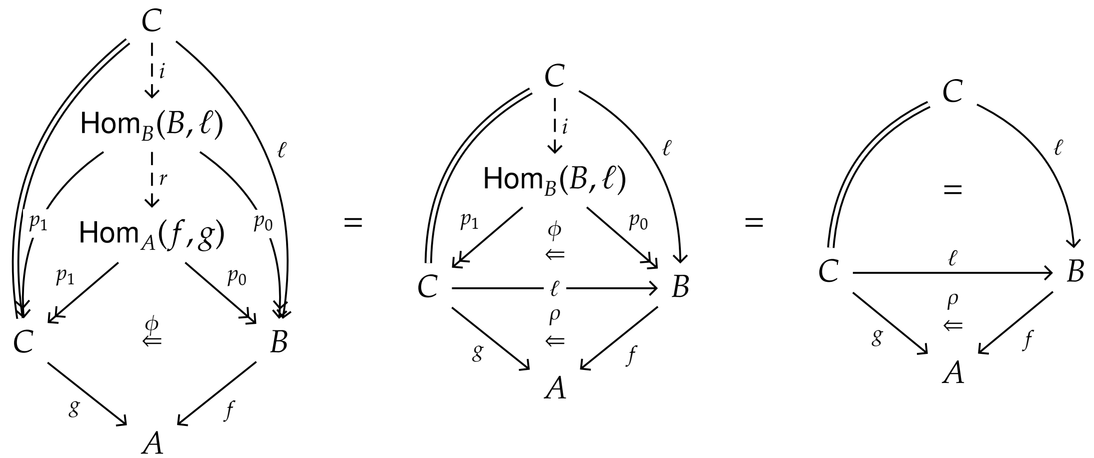
while allows us to induce a 2-cell $\nu \colon \mathrm{id} \Rightarrow ip_1$ with defining equations $p_1\nu = \mathrm{id}_{p_1}$ and $p_0\nu = \phi$. The first of these conditions ensures one triangle identity; for the other, we must verify that $\nu i = \mathrm{id}_i$. By 2-cell conservativity, νi is an isomorphism since $p_1 \nu i = \mathrm{id}_A$ and $p_0 \nu i = \phi i = \mathrm{id}_f$ are both invertible. By naturality of whiskering, we have



and the bottom edge is an identity. So $\nu i \cdot \nu i = \nu i$ and since νi is an isomorphism this implies that $\nu i = \mathrm{id}_i$ as required. □

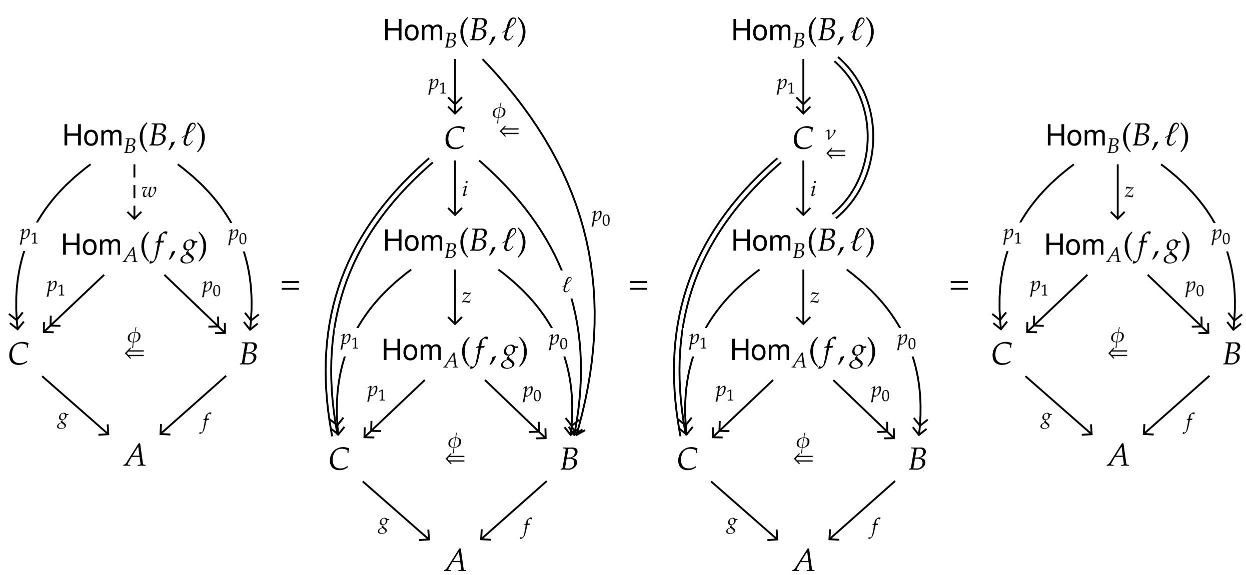
Proof of Theorem 3.4.4. The inverse to the function that takes a 2-cell $f\ell \Rightarrow g$ and produces an isomorphism class of maps $\mathsf{Hom}_B(B, \ell) \to \mathsf{Hom}_A(f, g)$ over $C \times B$ is constructed by applying Lemma 3.4.5 to the functor $\ell \colon C \to B$: give a map of spans, restrict along the right adjoint $i \colon C \to \mathsf{Hom}_B(B, \ell)$ and past with the comma cone for $\mathsf{Hom}_A(f, g)$ to define a 2-cell $f\ell \Rightarrow g$.

Starting from a 2-cell $\rho\colon f\ell \Rightarrow g$, the composite of these two functions constructs the 2-cell displayed below-left



which equals the above-center pasted composite by the definition of r from ρ, and equals the above-right composite since $\phi i = \mathrm{id}_\ell$. Thus, when a 2-cell is encoded as a map of spans, and then re-converted into a 2-cell, the original 2-cell ρ is recovered.

For the converse, starting with a map $z\colon \mathsf{Hom}_B(B,\ell) \to \mathsf{Hom}_A(f,g)$ over $C \times B$, the composite of these two functions constructs an isomorphism class of maps of spans w displayed below-left by applying 1-cell induction for the comma cone $\mathsf{Hom}_A(f,g)$ to the composite 2-cell pasted below-center-left:



Applying Lemma 3.4.5, there exists a 2-cell $\nu\colon \mathrm{id} \Rightarrow ip_1$ so that $p_0\nu = \phi$ — this gives the pasting equality above center — and $p_1\nu = \mathrm{id}$ — which gives the pasting equality above right. Proposition 3.3.7 now implies that $w \cong z$ over $C \times B$. □

Exercises.

3.4.i. EXERCISE. What does Lemma 3.4.5 tell us when applied to a functor of the form $b\colon 1 \to B$?

3.5. Sliced homotopy 2-categories and fibered equivalences

The ∞-category $A^{\mathbb{2}}$ of arrows in A together with its domain and codomain evaluation functors $(p_0, p_1): A^{\mathbb{2}} \twoheadrightarrow A \times A$ satisfies a weak universal property in the homotopy 2-category that characterizes the ∞-category up to equivalence *over* $A \times A$; see Proposition 3.2.10. Similarly the comma ∞-category is characterized up to fibered equivalence, as defined in Definition 3.2.7.

As commented upon in Remark 3.2.9 there is some ambiguity regarding the 2-categorical data required to specify a fibered equivalence, that we shall now address head-on. The issue is that, for an ∞-category B in an ∞-cosmos $\mathcal{K}$, the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos of Proposition 1.2.16 is not isomorphic to the 2-category $(\mathfrak{h}\mathcal{K})_{/B}$ of isofibrations, functors, and 2-cells over B in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of $\mathcal{K}$; see Exercise 1.4.iii.

However, there is a canonical comparison functor relating this pair of 2-categories that satisfies a property we now introduce:

3.5.1. Definition (smothering 2-functor). A 2-functor $F: \mathcal{A} \to \mathcal{B}$ is **smothering** if it is

- surjective on 0-cells;
- full on 1-cells: for any pair of objects A, A' in $\mathcal{A}$ and 1-cell $k: FA \to FA'$ in $\mathcal{B}$, there exists $f: A \to A'$ in $\mathcal{A}$ with $Ff = k$;
- full on 2-cells: for any parallel pair $f, g: A \rightrightarrows A'$ in $\mathcal{A}$ and 2-cell

$$FA \underset{Fg}{\overset{Ff}{\underset{\Downarrow\beta}{\rightrightarrows}}} FA'$$

  in $\mathcal{B}$, there exists a 2-cell $\alpha: f \Rightarrow g$ in $\mathcal{A}$ with $F\alpha = \beta$; and
- conservative on 2-cells: for any 2-cell α in $\mathcal{A}$ if $F\alpha$ is invertible in $\mathcal{B}$ then α is invertible in $\mathcal{A}$.

3.5.2. Remark. Note that smothering 2-functors are "locally smothering": surjective on objects 2-functors so that the action on hom-categories is by a smothering functor, as defined in 3.1.2.

The prototypical example of a smothering 2-functor solves Exercise 1.4.iii.

3.5.3. Proposition. *Let B be an ∞-category in an ∞-cosmos $\mathcal{K}$. There is a canonical 2-functor*

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

from the homotopy 2-category of the sliced ∞-cosmos $\mathcal{K}_{/B}$ to the 2-category of isofibrations, functors, and 2-cells over B in $\mathfrak{h}\mathcal{K}$ and this 2-functor is smothering.

This follows more-or-less immediately from Lemma 3.1.5 but we spell out the details nonetheless.

Proof. The 2-categories $\mathfrak{h}(\mathcal{K}_{/B})$ and $(\mathfrak{h}\mathcal{K})_{/B}$ have the same objects — isofibrations with codomain B — and 1-cells — functors between the "total spaces" that commute with these isofibrations to B — so the canonical mapping may be defined to act as the identity on underlying 1-categories.

By the definition of the sliced ∞-cosmos given in Proposition 1.2.16, a 2-cell between functors $f, g: E \rightrightarrows F$ from $p: E \twoheadrightarrow B$ to $q: F \twoheadrightarrow B$ is a homotopy class of 1-simplices in the quasi-category defined by the pullback of simplicial sets below-left

$$\begin{array}{ccc} \mathsf{Fun}_B(E,F) & \longrightarrow & \mathsf{Fun}(E,F) \\ \downarrow & \lrcorner & \downarrow q_* \\ \mathbb{1} & \xrightarrow[p]{} & \mathsf{Fun}(E,B) \end{array} \qquad \begin{array}{ccc} (\mathfrak{h}\mathcal{K})_{/B}(E,F) & \longrightarrow & \mathsf{hFun}(E,F) \\ \downarrow & \lrcorner & \downarrow q_* \\ \mathbb{1} & \xrightarrow[p]{} & \mathsf{hFun}(E,B) \end{array}$$

Unpacking, a 2-cell $\alpha\colon f \Rightarrow g$ is represented by a 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whiskers with q to the degenerate 1-simplex on the vertex $p \in \mathsf{Fun}(E,B)$, and two such 1-simplices represent the same 2-cell if and only if they bound a 2-simplex of the form displayed in (1.1.7) that also whiskers with q to the degenerate 2-simplex on p.

By contrast, a 2-cell in $(\mathfrak{h}\mathcal{K})_{/B}$ is a morphism in the category defined by the pullback of categories above-right. Such 2-cells are represented by 1-simplices $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whisker with q to 1-simplices in $\mathsf{Fun}(E,B)$ that are homotopic to the degenerate 1-simplex on p, and two such 1-simplices represent the same 2-cell if and only if they are homotopic in $\mathsf{Fun}(E,F)$.

The cospan in the above-right pullback is defined by applying the homotopy category functor to the cospan of in the above-left pullback, inducing a canonical map

$$\mathsf{Fun}_B(E,F) := \mathfrak{h}(\mathcal{K}_{/B})(E,F) \to (\mathfrak{h}\mathcal{K})_{/B}(E,F),$$

which is the action on homs of the canonical 2-functor $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$.

The 2-functor just constructed is bijective on 0- and 1-cells. To see that it is full on 2-cells we must show that any 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$, for which $q\alpha\colon p \to p$ is homotopic to $p \cdot \sigma^0\colon p \to p$ in $\mathsf{Fun}(E,B)$, is homotopic in $\mathsf{Fun}(E,F)$ to a 1-simplex from f to g over $p \cdot \sigma^0$. By Lemma 1.1.8, any such α defines a lifting problem

$$\begin{array}{ccccc}
\Lambda^1[2] & \longrightarrow & \mathsf{Fun}(E,F) \ni & & f \overset{f}{=\!=} \; \overset{\alpha}{\searrow} g \\
\big\downarrow & \nearrow & \big\downarrow q_* & & \big\downarrow \\
\Delta[2] & \longrightarrow & \mathsf{Fun}(E,B) \ni & & p \overset{p}{=\!=} \; \overset{q\alpha}{\searrow} p, \quad p = p
\end{array}$$

A solution exists since $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration, proving that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ is full on 2-cells.

Now suppose $\alpha\colon f \to g$ represents a 2-cell in $\mathsf{Fun}_B(E,F)$ whose image in $(\mathfrak{h}\mathcal{K})_{/B}(E,F)$ is an isomorphism. A map in a 1-category defined by a pullback is invertible if and only if its projections along the legs of the pullback cone are isomorphisms. Thus the image of α is invertible if and only $\alpha\colon f \to g$ defines an isomorphism in $\mathsf{hFun}(E,F)$, which by Definition 1.1.12 is the case if and only if $\alpha\colon f \to g$ represents an isomorphism in $\mathsf{Fun}(E,F)$. Since α is fibered over the degenerate 1-simplex at p, this presents us with a lifting problem

$$\begin{array}{ccccc}
\mathbb{2} & \xrightarrow{\alpha} & \mathsf{Fun}_B(E,F) & \longrightarrow & \mathsf{Fun}(E,F) \\
\big\downarrow & & \big\downarrow & \lrcorner & \big\downarrow q_* \\
\mathbb{I} & \longrightarrow & \mathbb{1} & \xrightarrow[p]{} & \mathsf{Fun}(E,B)
\end{array}$$

which Exercise 1.1.v tells us we can solve. This proves that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ reflects invertibility of 2-cells and hence defines a smothering 2-functor. □

3.5.4. Lemma. *Smothering 2-functors reflect equivalences: for any smothering 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ and 1-cell $f\colon A \to B$ in $\mathcal{A}$, if $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence in $\mathcal{B}$ then f is an equivalence in $\mathcal{A}$.*

Proof. By fullness on 1-cells, an equivalence inverse $g' \colon FB \xrightarrow{\sim} FA$ to Ff lifts to a 1-cell $g \colon B \to A$ in $\mathcal{A}$. By fullness on 2-cells, the isomorphisms $\mathrm{id}_{FA} \cong g' \circ Ff$ and $Ff \circ g' \cong \mathrm{id}_{FB}$ also lift to $\mathcal{A}$ and by conservativity on 2-cells these lifted 2-cells are also invertible. ☐

Applying Lemma 3.5.4 to the smothering 2-functor

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

we resolve the ambiguity about the 2-categorical data of a fibered equivalence.

3.5.5. Proposition.

(i) Any equivalence in $(\mathfrak{h}\mathcal{K})_{/B}$ lifts to an equivalence in $\mathfrak{h}(\mathcal{K}_{/B})$. That is, fibered equivalences over B may be specified by defining an opposing pair of 1-cells $f \colon E \to F$ and $g \colon F \to E$ over B together with invertible 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$ over B.

(ii) Moreover, if $f \colon E \to F$ is a map between isofibrations over B that admits an not-necessarily fibered equivalence inverse $g \colon F \to E$ with not-necessarily fibered 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$, then this data is isomorphic to a genuine fibered equivalence.

Proof. The first statement is proven by Lemma 3.5.4 and Proposition 3.5.3 The second statement asserts that the forgetful 2-functor $(\mathfrak{h}\mathcal{K})_{/B} \to \mathfrak{h}\mathcal{K}$ reflects equivalences. Exercise 3.5.i then shows that for any map between isofibrations over B that admits an equivalence inverse in the underlying 2-category, the inverse equivalence and invertible 2-cells can be lifted to also lie over B. ☐

This gives a 2-categorical proof of Proposition 1.2.16(vii), that for any ∞-category B in an ∞-cosmos $\mathcal{K}$, the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences.

Exercises.

3.5.i. Exercise. Let B be an object in a 2-category $\mathcal{C}$ and consider a map

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & \searrow \hspace{1em} \swarrow & \\ & B & \end{array}$$

between isofibrations over B. Prove that if f is an equivalence in $\mathcal{C}$ then f is also an equivalence in the slice 2-category $\mathcal{C}_{/B}$ of isofibrations over B, functors over B, and natural transformations over B.

3.5.ii. Exercise. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a smothering 2-functor. Show that any adjunction in $\mathcal{B}$ can be lifted to an adjunction in $\mathcal{A}$. Demonstrate furthermore that if we have previously specified a lift of the objects, 1-cells, and either the unit or counit of the adjunction in $\mathcal{B}$, then there is a lift of the remaining 2-cell that combines with the previously specified data to define an adjunction in $\mathcal{A}$.

Appendix

APPENDIX A

Basic concepts in enriched category theory

For now see [**11**].

APPENDIX B

Introduction to 2-category theory

B.1. 2-categories and the calculus of pasting diagrams

For now see [**12**].

B.1.1. Digression (how to read a pasting diagram). A pasting diagram in a 2-category represents a composite 2-cell, defining a morphism in one of the hom-categories between a pair of objects. To identify these objects, look at the underlying directed graph of objects and 1-cells in the pasting diagram. If the pasting diagram is well-formed, that graph should have a unique source object A and a unique target object Z. This indicates that the pasting diagram defines a 2-cell in the hom-category $\mathsf{Hom}(A, Z)$. The object A is its **source 0-cell** and the object Z is its **target 0-cell**.

The next step is to identify the source 1-cell and the target 1-cell of the pasting diagram. These should both be objects of $\mathsf{Hom}(A, Z)$, i.e., 1-cells in the 2-category from A to Z. Again if the pasting diagram is well-formed, the **source 1-cell** should be the unique composable path of 1-cells none of which occur as codomains of any 2-cells in the pasting diagram. Dually, the **target 1-cell** should be the unique composable path of 1-cells, none of which occur as domains of any 2-cells in the pasting diagram.

The final step is to represent the pasting diagram as a vertical composite of 2-cells, each of which is a map between a pair of composite 1-cells from A to Z that trace a composable path through the directed graph of the pasting diagram. Each 2-cell in the pasting diagram will label precisely one of the 2-cells of this composite. The expressions of these vertical 2-cell composites are not necessarily unique and may not necessarily pass through every possible composable path of 1-cells (though there will be some vertical composite of 2-cells that does pass through each path of 1-cells).

To start, pick any 2-cell in the pasting diagram whose 1-cell source can be found as a subsequence of the source 1-cell. Whisker it so that it defines a 2-cell from that source 1-cell to another path of composable 1-cells through the pasting diagram. Then this whiskered composite forms the first step in the sequence of composable 2-cells. Remove this part of the pasting diagram and repeat until you arrive at the target 1-cell.

B.2. Right adjoint right inverse adjunctions

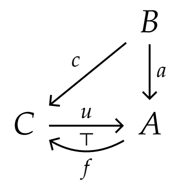
B.2.1. Lemma. *Suppose we are given a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$ and a 2-isomorphism $fu \cong \mathrm{id}_A$ in a 2-category. If there exists a 2-cell $\eta'\colon \mathrm{id}_B \Rightarrow uf$ with the property that $f\eta'$ and $\eta' u$ are 2-isomorphisms, then f is left adjoint to u. Furthermore, in the special case where u is a section of f, then f is left adjoint to u with the counit of the adjunction an identity.*

Proof. For now see [**18**, 4.1.2]. □

B.3. A bestiary of 2-categorical lemmas

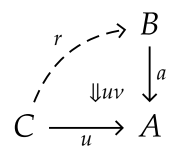
B.3.1. Lemma. *Suppose $f \dashv u$ is an adjunction under B in the sense that*

- *the triangles involving both adjoints commute*



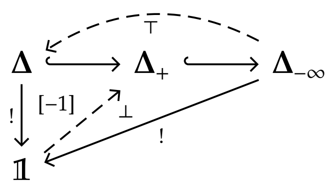
- *and* $\eta a = \mathrm{id}_a$ *and* $\epsilon c = \mathrm{id}_c$.

Then if c *admits a right adjoint* r *with counit* $\nu\colon cr \Rightarrow \mathrm{id}_C$, *then* $u\nu$ *exhibits* r *as an absolute right lifting of* u *through* a.



Proof. For now see [**18**, 5.3.2]. □

B.3.2. Example. For example, consider the diagram of adjoint functors



For now, see the proof of [**18**, 5.3.1].

APPENDIX C

Abstract homotopy theory

C.1. Lifting properties, weak factorization systems, and Leibniz closure

C.1.1. LEMMA. *Any class of maps characterized by a right lifting property is closed under composition, product, pullback, retract, and limits of towers; see Lemma C.1.1.*

PROOF. For now see [**17**, 11.1.4] and dualize. □

On account of the dual of Lemma C.1.1, any set of maps in a cocomplete category "cellularly generates" a larger class of maps with the same left lifting property.

C.1.2. DEFINITION. The class of maps **cellularly generated** by a set of maps is comprised of those maps obtained as sequential composites of pushouts of coproducts of those maps.

C.2. Simplicial sets and markings

C.2.1. DEFINITION. Write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory of the simplex category of 1.1.1 spanned by the ordinals $[0], \ldots, [n]$. Restriction and left and right Kan extension define adjunctions

$$\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \underset{\mathrm{ran}_n}{\overset{\mathrm{lan}_n}{\underset{\perp}{\overset{\perp}{\longleftarrow \mathrm{res}_n \to}}}} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}$$

inducing an idempotent comonad $\mathrm{sk}_n := \mathrm{lan}_n \circ \mathrm{res}_n$ and an idempotent monad $\mathrm{cosk}_n := \mathrm{ran}_n \circ \mathrm{res}_n$ on $\mathcal{SS}et$ that are adjoint $\mathrm{sk}_n \dashv \mathrm{cosk}_n$. The counit and unit of this comonad and monad define canonical maps

$$\mathrm{sk}_n X \overset{\epsilon}{\rightarrowtail} X \overset{\eta}{\twoheadrightarrow} \mathrm{cosk}_n X$$

relating a simplicial set X with its n**-skeleton** and n**-coskeleton**. We say X is n**-skeletal** or n**-coskeletal** if the former or latter of these maps, respectively, is an isomorphism.

The next lemma proves that the monomorphisms are cellularly generated by the simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.

C.2.2. LEMMA. *Any monomorphism of simplicial sets decomposes canonically as a sequential composite of pushouts of coproducts of the maps $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.*

PROOF. An exercise, for now. □

C.2.3. DEFINITION ((left-/right-/inner-)anodyne extensions).

- The set of **horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$ cellularly generates the **anodyne extensions**.

- The set of **left horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k < n$ cellularly generates the **left anodyne extensions**.
- The set of **right horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 < k \leq n$ cellularly generates the **right anodyne extensions**.
- The set of **inner horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 2$ and $0 < k < n$ cellularly generates the **inner anodyne extensions**.

APPENDIX D

Examples of ∞-cosmoi

For now see §IV.2 of [**19**].

APPENDIX E

Compatibility with the analytic theory of quasi-categories

The aim in this section is to prove that the synthetic theory of quasi-categories is compatible with the analytic theory pioneered by André Joyal, Jacob Lurie, and many others.

For now, see

- for limits: I.4.4.6, I.4.4.7, I.4.4.8 of [**18**]
- for adjunctions: IV.4.1.20-22 of [**19**], VIII.3.3.6 of [**20**]
- for cartesian fibrations: IV.4.1.24 of [**19**]

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