On the directed univalence axiom
joint with Evan Cavallo and Christian Sattler

AMS Special Session on Homotopy Type Theory, Joint Mathematics Meetings
1. A type theory for synthetic \((\infty, 1)\)-categories

2. A directed univalence conjecture

3. Covariant type families

4. The covariant directed univalence axiom
A type theory for synthetic $(\infty, 1)$-categories
The bisimplicial sets model

\[
\begin{align*}
\text{Set}^{\Delta^\text{op} \times \Delta^\text{op}} & \supset \text{Reedy} \supset \text{Segal} \supset \text{Rezk} \\
bisimplicial \text{ sets} & \supset \text{types} \supset \text{types with composition} \supset \text{types with composition 
& \text{& univalence}}
\end{align*}
\]

**Theorem (Shulman).** Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

**Theorem (Rezk).** \((\infty, 1)\)-categories are modeled by Rezk spaces aka complete Segal spaces.

The bisimplicial sets model of homotopy type theory has:

- an interval type \(I\), parametrizing paths inside a general type
- a directed interval type \(2\), parametrizing arrows inside a general type
Paths and arrows

- The identity type for $A$ depends on two terms in $A$:
  \[ x, y : A \vdash x =_A y \]
  and a term $p : x =_A y$ may be thought of as a path in $A$ from $x$ to $y$.

- The hom type for $A$ depends on two terms in $A$:
  \[ x, y : A \vdash \text{hom}_A(x, y) \]
  and a term $f : \text{hom}_A(x, y)$ defines an arrow in $A$ from $x$ to $y$.

Hom types are defined as instances of extension types axiomatized in a three-layered type theory with shapes due to Shulman

\[
\text{hom}_A(x, y) := \left\langle 1 + 1 \xrightarrow{[x,y]} A \right\rangle
\]
Segal, Rezk, and discrete types

• A type $A$ is Segal if every composable pair of arrows has a unique composite: if for every $f : \text{hom}_A(x, y)$ and $g : \text{hom}_A(y, z)$

$$\langle \Lambda_1^2 \xrightarrow{[f, g]} A \rangle$$

is contractible.

• A Segal type $A$ is Rezk if every isomorphism is an identity: if

$$\text{id-to-iso} : \prod_{x, y : A} (x =_A y) \to (x \cong_A y)$$

is an equivalence.

• A type $A$ is discrete if every arrow is an identity: if

$$\text{id-to-arr} : \prod_{x, y : A} (x =_A y) \to \text{hom}_A(x, y)$$

is an equivalence.

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms — the discrete types are the $\infty$-groupoids.
A directed univalence conjecture
What are the arrows in the universe?

For small types $A, B : \mathcal{U}$, the following are equivalent:

- an arrow $F : \text{hom}_\mathcal{U}(A, B)$
- a function $F : 2 \to \mathcal{U}$ with $F(0) \equiv A$ and $F(1) \equiv B$
- a type family $t : 2 \vdash F(t)$ with $F(0) \equiv A$ and $F(1) \equiv B$

In this context the dependent function type is equivalent to the dependent sum

$$
\prod_{t : 2} F(t) \simeq \sum_{a : A} \sum_{b : B} \text{hom}_{F(2)}(a, b)
$$

of dependent hom types

$$
\text{hom}_{F(2)}(a, b) := \left\langle \begin{array}{c}
1 + 1 \\
[0, 1]
\end{array} \right\rangle,
$$

the type of arrows in $F$ from $a$ to $b$ over the generic arrow in $2$. 
A conjectural directed univalence axiom

Define

\[ \text{arr-to-span} : \text{hom}_U(A, B) \to (A \times B \to U) \]

to carry \( F \) to the span given by the dependent product

\[
\prod_2 F \simeq \sum_{a:A} \sum_{b:B} \text{hom}_{F(2)}(a, b)
\]

and its domain and codomain projections.

**Directed Univalence Conjecture.**

For all small types \( A \) and \( B \) the map

\[ \text{arr-to-span} : \text{hom}_U(A, B) \to (A \times B \to U) \]

is an equivalence.
Semantics of the directed univalence conjecture

Semantically, `arr-to-span` constructs the comma object of a cospan:

\[
\begin{array}{ccc}
A & \overset{i_0}{\rightarrow} & F & \overset{i_1}{\leftarrow} & B \\
\downarrow & \ & \downarrow & \ & \downarrow \\
1 & \overset{0}{\rightarrow} & 2 & \overset{1}{\leftarrow} & 1
\end{array}
\]

\[
\begin{array}{ccc}
\prod_2 F & \rightarrow & F^2 \\
\downarrow & \ & \downarrow \\
A \times B & \overset{i_0 \times i_1}{\rightarrow} & F \times F
\end{array}
\]

2-category theory suggests a converse construction:

\[
\begin{array}{ccc}
S & \overset{p+q}{\rightarrow} & A + B \\
\downarrow & \ & \downarrow \\
S \times 2 & \rightarrow & A \star_S B
\end{array}
\]

The image of `arr-to-span` is not all spans — only the “two-sided discrete fibrations” — the definition of which involves conditions on `A` and `B`.

\[
\Rightarrow \text{Search for a directed univalence axiom in a different universe.}
\]
3

Covariant type families
Covariant type families I

Let \( x : A \vdash B(x) \) be a type family over a Segal type \( A \). Then any arrow \( f : \text{hom}_A(x, y) \) in the base, gives rise to a span

\[
\sum_{u:B(x)} \sum_{v:B(y)} \text{hom}_B(f)(u, v)
\]

\[
\begin{array}{ccc}
B(x) & \xrightarrow{\text{dom}} & \text{hom}_B(f) \left( u, v \right) \\
\downarrow & & \downarrow \\
B(y) & \xleftarrow{\text{cod}} & B(y)
\end{array}
\]

and any 2-simplex in \( A \) witnessing \( h = g \circ f \) gives rise to a “higher span.”

A type family \( x : A \vdash B(x) \) over a Segal type \( A \) is covariant if for every \( f : \text{hom}_A(x, y) \) and \( u : B(x) \) there is a unique lift of \( f \) with domain \( u \), i.e.:

\[
\sum_{v:B(y)} \text{hom}_B(f)(u, v)
\]

is contractible.

\( x : A \vdash B(x) \) is covariant iff for each \( f : \text{hom}_A(x, y) \) the left leg of the span from \( B(x) \) to \( B(y) \) is an equivalence — defining a covariant span.
Covariant type families II

A type family $x : A \vdash B(x)$ over a Segal type $A$ is **covariant** if for every $f : \text{hom}_A(x,y)$ and $u : B(x)$ there is a unique lift of $f$ with domain $u$.

**Prop.** If $x : A \vdash B(x)$ is covariant then for each $x : A$ the fiber $B(x)$ is discrete. Thus covariant type families are fibered in $\infty$-groupoids.

**Prop.** Fix $a : A$. The type family $x : A \vdash \text{hom}_A(a,x)$ is covariant.

The Yoneda lemma proves that the type family $x : A \vdash \text{hom}_A(a,x)$ is freely generated by the identity arrow $\text{id}_a : \text{hom}_A(a,a)$ and gives a “directed” version of the “transport” operation for identity types.
The universe of covariant fibrations

In bisimplicial sets

• type families correspond to Reedy fibrations, characterized by a right lifting property against:

\[(\partial \Delta^m \to \Delta^m) \hat{\square} (\Lambda_k^n \to \Delta^n) \quad m \geq 0, \ 0 \leq k \leq n\]

• covariant type families correspond to covariant fibrations aka left fibrations, characterized by a further right lifting property against:

\[(\Lambda_k^n \to \Delta^n) \hat{\square} (\partial \Delta^m \to \Delta^m) \quad m \geq 0, \ 0 \leq k < n.\]

The universe of covariant fibrations \(\mathcal{U}_{\text{cov}}\) is the presheaf on \(\Delta \times \Delta\) with

\[\mathcal{U}_{\text{cov}}(m, n) := \{\text{covariant fibrations over } \Delta^m \square \Delta^n\}.\]

The universal covariant fibration is defined by pullback:
The covariant directed univalence axiom
A new directed univalence axiom

- A covariant type family over \(1\) is a discrete type. Thus the terms in \(\mathcal{U}_{\text{cov}}\) are discrete types.

- A covariant type family \(t : 2 \vdash F(t)\) over \(2\) determines a pair of discrete types \(A := F(0)\) and \(B := F(1)\) together with a span

\[
\sum_{a:A} \sum_{b:B} \text{hom}_{F(2)}(a, b)
\]

whose left leg is invertible. The type of such covariant spans is equivalent to the type of functions \(A \to B\).

**Directed Univalence Axiom.** For all small discrete types \(A\) and \(B\) the map

\[
\text{arr-to-fun} : \text{hom}_{\mathcal{U}_{\text{cov}}}(A, B) \to (A \to B)
\]

is an equivalence.
Evidence supporting the directed univalence axiom

**Directed Univalence Axiom.** For all small discrete types \(A\) and \(B\) the map

\[
\text{arr-to-fun} : \hom_{\mathcal{U}_{\text{cov}}}(A, B) \rightarrow (A \rightarrow B)
\]

is an equivalence.

Sattler has sketched a verification of the Directed Univalence Axiom in bisimplicial sets:

- The canonical map \(\mathcal{U}_{\text{cov}} \rightarrow \mathcal{U}\) is a fibration; hence \(\mathcal{U}_{\text{cov}}\) is fibrant.
- The homotopy inverse to \(\text{arr-to-fun}\) is the specialization of \(\text{span-to-arr}\) to the case of covariant spans between discrete types.
- This map \(\text{cov-span-to-arr}\) automatically produces a covariant fibration over \(2\).
- The fatal flaw in the original directed univalence conjecture is avoided since discrete types are local at \(2\): \(A \simeq I \rightarrow A \simeq 2 \rightarrow A\).
A warning about the universal property of $\mathcal{U}_{\text{cov}}$

The type theoretic definition of a covariant type family can be stated in any context and the universe for covariant fibrations $\mathcal{U}_{\text{cov}}$ can be weakened to any context.

- A covariant type family $x : A \vdash B(x)$ over $A$ in the empty context defines a map $B : A \rightarrow \mathcal{U}_{\text{cov}}$ and conversely.
- But a covariant type family $x : A \vdash B(x)$ over $A$ in context $\Gamma$ will not define a map $B : \Gamma.A \rightarrow \mathcal{U}_{\text{cov}}$.
- The definition of a covariant type family over $A$ in context $\Gamma$ is covariant over arrows in $A$ fiberwise in $\Gamma$.
- Whereas a map $B : \Gamma.A \rightarrow \mathcal{U}_{\text{cov}}$ defines a type family that is covariant over arrows in the entire extended context.
A type theory for synthetic $(\infty, 1)$-categories with semantics in the bisimplicial sets model of HoTT has been developed by Riehl–Shulman but many questions about universes remain.

A directed univalence conjecture — that arrows in the universe of all types are equivalent to spans — is false in the model.

A restricted directed univalence axiom — that arrows in the universe of covariant fibrations correspond to functions between discrete types — is likely true in the model.

Much remains to be explored, so let us know if you’d like to get involved!
References

For considerably more, see:


Thank you!