The synthetic approach to $\infty$-category theory

joint with Dominic Verity and Michael Shulman
The idea of an $\infty$-category

An $\infty$-category — a category weakly enriched over $\infty$-groupoids — should have:

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

But this definition is tricky to make precise in classical foundations.
Models of $\infty$-categories

The notion of $\infty$-category is made precise by several models:

- topological categories and relative categories are the simplest to define but do not have enough maps between them
- quasi-categories (nee. weak Kan complexes), Rezk spaces (nee. complete Segal spaces), Segal categories, and (saturated 1-trivial weak) 1-complicial sets

are cartesian closed, and in fact any of these categories can be enriched over any of the others.
The analytic vs synthetic theory of $\infty$-categories

Q: How might you develop the category theory of $\infty$-categories?

Strategies:

• work analytically to give categorical definitions and prove theorems using the combinatorics of one model
  
  (eg., Joyal, Lurie, Gepner-Haugse, Cisinski in $q\text{Cat}$; Kazhdan-Varshavsky, Rasekh in $\text{Rezk}$; Simpson in $\text{Segal}$)

• work synthetically to give categorical definitions and prove theorems in all four models $q\text{Cat}$, $\text{Rezk}$, $\text{Segal}$, $\text{1-Comp}$ at once
  
  (R-Verity: an $\infty$-cosmos axiomatizes the common features of the categories $q\text{Cat}$, $\text{Rezk}$, $\text{Segal}$, $\text{1-Comp}$ of $\infty$-categories)

• work synthetically in a simplicial type theory augmenting HoTT to prove theorems in $\text{Rezk}$
  
  (R-Shulman: an $\infty$-category is a type with unique binary composites in which isomorphism is equivalent to identity)
Plan

0. The analytic theory of $\infty$-categories
   
   “$\infty$-category theory for experts”

1. The synthetic theory of $\infty$-categories (in an $\infty$-cosmos)
   
   “$\infty$-category theory for graduate students”

2. The synthetic theory of $\infty$-categories (in homotopy type theory)
   
   “$\infty$-category theory for undergraduates”
The synthetic theory of $\infty$-categories (in an $\infty$-cosmos)
An $\infty$-cosmos axiomatizes the structures needed to “develop $\infty$-category theory.”

**not-the-defn.** An $\infty$-cosmos is a cartesian closed category $\mathcal{K}$ that has

- certain (flexible weighted enriched) limits
- an adjunction

$$
\mathcal{K} \overset{\text{ho}}{\rightleftarrows} \text{Cat}
$$

**Theorem.** $q\text{Cat}$, Rezk, Segal, 1-Comp define biequivalent $\infty$-cosmoi.

Henceforth $\infty$-category and $\infty$-functor are technical terms that refer to the objects and morphisms of some $\infty$-cosmos.
The homotopy 2-category

The homotopy 2-category of an $\infty$-cosmos is a strict 2-category whose:

• objects are the $\infty$-categories $A$, $B$ in the $\infty$-cosmos

• 1-cells are the $\infty$-functors $f: A \to B$ in the $\infty$-cosmos

• 2-cells, called $\infty$-natural transformations $\Downarrow \gamma$, are defined to be the arrows in the homotopy category $\text{ho}(B^A)$

Key fact: equivalences in the homotopy 2-category coincide with equivalences in the $\infty$-cosmos.

Thus, non-evil 2-categorical definitions are “homotopically correct.”
Adjunctions between $\infty$-categories

defn. An adjunction between $\infty$-categories is an adjunction in the homotopy 2-category, consisting of:

- $\infty$-categories $A$ and $B$
- $\infty$-functors $u : A \to B$, $f : B \to A$
- $\infty$-natural transformations $\eta : B \to B$ and $\varepsilon : A \to A$

satisfying the triangle equalities

Write $f \dashv u$ to indicate that $f$ is the left adjoint and $u$ is the right adjoint.
The 2-category theory of adjunctions

Since an adjunction between $\infty$-categories is just an adjunction in the homotopy 2-category, all 2-categorical theorems about adjunctions become theorems about adjunctions between $\infty$-categories.

Prop. Adjunctions compose:

$\begin{array}{ccc}
C & \xrightleftharpoons{f'} & B \\
\downarrow{u'} & & \downarrow{u} \\
A & \xrightleftharpoons{f} & \sim & A
\end{array}$

Prop. Adjoints to a given functor $u : A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u : A \xrightarrow{\sim} B$ then $u$ is left and right adjoint to its equivalence inverse.
Prop. Adjunctions compose:

\[
\begin{array}{ccc}
C & \perp & B \\
\downarrow & & \downarrow \\
& u & \\
\end{array} \quad \Rightarrow \quad 
\begin{array}{ccc}
A & \perp & C \\
\downarrow & & \downarrow \\
& u' & \\
\end{array}
\]

Proof: The composite 2-cells

\[
\begin{array}{ccc}
C & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \\
\end{array}
\]

\[
\begin{array}{ccc}
C & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \\
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\]

\[
\begin{array}{ccc}
C & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \\
\end{array}
\]

define the unit and counit of \( f f' \dashv u' u \) satisfying the triangle equalities.
Limits and colimits in an $\infty$-category

defn. An $\infty$-category $A$ has

- a terminal element iff $\begin{array}{c}1 \\ \downarrow t \end{array} \rightarrow A$ and

- all limits of shape $J$ iff $\begin{array}{c}A^J \\ \downarrow \operatorname{lim} \end{array} \rightarrow A$.

Note: the counit components $\begin{array}{c} \operatorname{lim}_d \\ \downarrow \epsilon \end{array}$ define the limit cone.

Prop. Right adjoints preserve limits and left adjoints preserve colimits.

Proof: The usual one!
Universal properties of adjunctions and limits

defn. Any ∞-category $A$ has an ∞-category of arrows $A^2$, pulling back

$$\begin{array}{ccc}
\text{Hom}_A(f, g) & \longrightarrow & A^2 \\
\downarrow & & \downarrow \\
C \times B & \xrightarrow{g \times f} & A \times A
\end{array}$$

to define the comma ∞-category:

This specializes to define the mapping space $\text{Hom}_A(x, y)$ between each pair of elements $x, y : 1 \rightarrow A$.

Prop. An ∞-functor $d : J \rightarrow A$ has limit $\ell : 1 \rightarrow A$ iff

$$\text{Hom}_A(A, \ell) \simeq_{A \times B} \text{Hom}_B(B, u).$$

Prop. Mapping spaces are discrete ∞-categories, i.e., ∞-groupoids.
The synthetic theory of $\infty$-categories (in homotopy type theory)
### The Rosetta Stone for Homotopy Type Theory

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Path induction

The identity type family is freely generated by the terms \( \text{refl}_x : x =_A x \).

Path induction. If \( B(x, y, p) \) is a type family dependent on \( x, y : A \) and \( p : x =_A y \), then to prove \( B(x, y, p) \) it suffices to assume \( y \) is \( x \) and \( p \) is \( \text{refl}_x \). I.e., there is a function

\[
\text{path-ind} : \left( \prod_{x : A} B(x, x, \text{refl}_x) \right) \to \left( \prod_{x, y : A} \prod_{p : x =_A y} B(x, y, p) \right).
\]
A model for the type theory for synthetic $\infty$-categories

Set$^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$ ⊆ Reedy ⊆ Segal ⊆ Rezk

| bisimplicial sets | types | types with composition | types with composition & univalence |

Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). $\infty$-categories are modeled by Rezk spaces aka complete Segal spaces.
Shapes in the theory of the directed interval

Our types may depend on other types and also on shapes \( \Phi \subset 2^n \), polytopes embedded in a directed cube, defined in a language

\[ \top, \bot, \land, \lor, \equiv \quad \text{and} \quad 0, 1, \leq \]

satisfying intuitionistic logic and strict interval axioms.

\[ \Delta^n := \{(t_1, \ldots, t_n) : 2^n \mid t_n \leq \cdots \leq t_1\} \quad \text{e.g.} \quad \Delta^1 := 2 \]

\[ \Delta^2 := \begin{cases} (t,t) & (1,1) \\ (0,0) & (1,0) \\ (t,0) & (1,t) \end{cases} \]

\[ \partial \Delta^2 := \{(t_1, t_2) : 2^2 \mid (t_2 \leq t_1) \land ((0 = t_2) \lor (t_2 = t_1) \lor (t_1 = 1))\} \]

\[ \Lambda_1^2 := \{(t_1, t_2) : 2^2 \mid (t_2 \leq t_1) \land ((0 = t_2) \lor (t_1 = 1))\} \]
Extension types

Formation rule for extension types

\[ \Phi \subset \Psi \text{ shape} \quad A \text{ type} \quad a : \Phi \to A \]

\[ \langle \Phi \overset{a}{\to} A \rangle \] type

A term \( f : \langle \Phi \overset{a}{\to} A \rangle \) defines

\[ f : \Psi \to A \] so that \( f(t) \equiv a(t) \) for \( t : \Phi \).

The simplicial type theory allows us to prove equivalences between extension types along composites or products of shape inclusions.
Hom types

The hom type for $A$ depends on two terms in $A$:

$$x, y : A \vdash \text{Hom}_A(x, y)$$

$$\text{Hom}_A(x, y) := \left\langle \begin{array}{c} \partial \Delta^1 \downarrow \quad [x, y] \\ \Delta^1 \end{array} \rightarrow A \right\rangle \text{ type}$$

A term $f : \text{Hom}_A(x, y)$ defines an arrow in $A$ from $x$ to $y$.

In the $\infty$-cosmos $\text{Rezk}$:

- $\text{Hom}_A(x, y)$ recovers the mapping space from $x$ to $y$ and
- $\sum_{x, y : A} \text{Hom}_A(x, y)$ recovers the $\infty$-category of arrows $A^2$. 
A type $A$ is Segal iff every composable pair of arrows has a unique composite, i.e., for every $f : \text{Hom}_A(x, y)$ and $g : \text{Hom}_A(y, z)$ the type $\langle \Lambda^2 \rightarrow A \rangle$ is contractible.

Semantically, a Reedy fibrant bisimplicial set $A$ is Segal if and only if $A^{\Delta^2} \rightarrow A^{\Lambda^1}$ has contractible fibers.

By contractibility, $\langle \Lambda^2 \rightarrow A \rangle$ has a unique inhabitant. Write $g \circ f : \text{Hom}_A(x, z)$ for its inner face, the composite of $f$ and $g$. 

Segal types $\equiv$ types with binary composition
Identity arrows

For any $x : A$, the constant function defines a term

$$id_x := \lambda t.x : \text{Hom}_A(x, x) := \langle \begin{array}{c} \partial \Delta^1 \downarrow \Delta^1 \end{array} \xrightarrow{[x,x]} A \rangle,$$

which we denote by $id_x$ and call the identity arrow.

For any $f : \text{Hom}_A(x, y)$ in a Segal type $A$, the term

$$\lambda (s, t). f(t) : \langle \begin{array}{c} \Lambda^2_1 \downarrow \Delta^2 \end{array} \xrightarrow{[id_x, f]} A \rangle$$

witnesses the unit axiom $f = f \circ id_x$. 
Associativity of composition

Let $A$ be a Segal type with arrows

\[ f : \text{Hom}_A(x, y), \quad g : \text{Hom}_A(y, z), \quad h : \text{Hom}_A(z, w). \]

Prop. \[ h \circ (g \circ f) = (h \circ g) \circ f. \]
Proof: Consider the composable arrows in the Segal type $\Delta^1 \to A$:

Composing defines a term in the type $\Delta^2 \to (\Delta^1 \to A)$ which yields a term $\ell : \text{Hom}_A(x, w)$ so that $\ell = h \circ (g \circ f)$ and $\ell = (h \circ g) \circ f$. 
An arrow $f : \text{Hom}_A(x, y)$ in a Segal type is an isomorphism if it has a two-sided inverse $g : \text{Hom}_A(y, x)$. However, the type

$$
\sum_{g : \text{Hom}_A(y, x)} \left( (g \circ f = \text{id}_x) \times (f \circ g = \text{id}_y) \right)
$$

has higher-dimensional structure and is not a proposition. Instead define

$$
isiso(f) := \left( \sum_{g : \text{Hom}_A(y, x)} g \circ f = \text{id}_x \right) \times \left( \sum_{h : \text{Hom}_A(y, x)} f \circ h = \text{id}_y \right).
$$

For $x, y : A$, the type of isomorphisms from $x$ to $y$ is:

$$
x \cong_A y := \sum_{f : \text{Hom}_A(x, y)} \text{isiso}(f).
$$
By path induction, to define a map

\[
\text{path-to-iso} : (x =_A y) \to (x \cong_A y)
\]

for all \(x, y : A\) it suffices to define

\[
\text{path-to-iso}(\text{refl}_x) := \text{id}_x.
\]

A Segal type \(A\) is Rezk iff every isomorphism is an identity, i.e., iff the map

\[
\text{path-to-iso} : \prod_{x, y : A} (x =_A y) \to (x \cong_A y)
\]

is an equivalence.
Discrete types $\equiv \infty$-groupoids

Similarly by path induction define

$$\text{path-to-arr}: \ (x =_A y) \to \text{Hom}_A(x, y)$$

for all $x, y : A$ by $\text{path-to-arr}(\text{refl}_x) := \text{id}_x$.

A type $A$ is discrete iff every arrow is an identity, i.e., iff $\text{path-to-arr}$ is an equivalence.

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms.

Proof:
**∞-categories for undergraduates**

**defn.** An ∞-groupoid is a type in which arrows are equivalent to identities:

\[ \text{path-to-arr: } (x =_A y) \to \text{Hom}_A(x, y) \text{ is an equivalence.} \]

**defn.** An ∞-category is a type

- which has unique binary composites of arrows:

\[ \begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{[f,g]} & A \\
\vee & \Downarrow & \vee \\
\Delta^2 & \end{array} \]

is contractible

- and in which isomorphisms are equivalent to identities:

\[ \text{path-to-iso: } (x =_A y) \to (x \cong_A y) \text{ is an equivalence.} \]
Covariant type families $\equiv$ categorical fibrations

A type family $x : A \vdash B(x)$ over a Segal type $A$ is covariant if for every $f : \text{Hom}_A(x, y)$ and $u : B(x)$ there is a unique lift of $f$ with domain $u$.

The codomain of the unique lift defines a term $f_*u : B(y)$.

**Prop.** For $u : B(x)$, $f : \text{Hom}_A(x, y)$, and $g : \text{Hom}_A(y, z)$,

$$g_*(f_*u) = (g \circ f)_*u \quad \text{and} \quad (\text{id}_x)_*u = u.$$

**Prop.** If $x : A \vdash B(x)$ is covariant then for each $x : A$ the fiber $B(x)$ is discrete. Thus covariant type families are fibered in $\infty$-groupoids.

**Prop.** Fix $a : A$. The type family $x : A \vdash \text{Hom}_A(a, x)$ is covariant.
The Yoneda lemma

Let $x : A \vdash B(x)$ be a covariant family over a Segal type and fix $a : A$.

Yoneda lemma. The maps

$$
ev\text{-}id : \lambda \phi. \phi(a, \text{id}_a) : \left( \prod_{x: A} \text{Hom}_A(a, x) \to B(x) \right) \to B(a)$$

and

$$\text{yon} : \lambda u. \lambda x. \lambda f. f_* u : B(a) \to \left( \prod_{x: A} \text{Hom}_A(a, x) \to B(x) \right)$$

are inverse equivalences.

Corollary. A natural isomorphism $\phi : \prod_{x: A} \text{Hom}_A(a, x) \cong \text{Hom}_A(b, x)$ induces an identity $\text{ev}\text{-}id(\phi) : b =_A a$ if the type $A$ is Rezk.
The dependent Yoneda lemma

Yoneda lemma. If $A$ is a Segal type and $B(x)$ is a covariant family dependent on $x : A$, then evaluation at $(a, \text{id}_a)$ defines an equivalence

$$\text{ev-id} : \left( \prod_{x : A} \text{Hom}_A(a, x) \rightarrow B(x) \right) \rightarrow B(a)$$

The Yoneda lemma is a “directed” version of the “transport” operation for identity types, suggesting a dependently-typed generalization analogous to the full induction principle for identity types.

Dependent Yoneda lemma. If $A$ is a Segal type and $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \text{Hom}_A(x, y)$, then evaluation at $(x, x, \text{id}_x)$ defines an equivalence

$$\text{ev-id} : \left( \prod_{x, y : A} \prod_{f : \text{Hom}_A(x, y)} B(x, y, f) \right) \rightarrow \prod_{x : A} B(x, x, \text{id}_x)$$
Dependent Yoneda is directed path induction

Slogan: the dependent Yoneda lemma is directed path induction.

Path induction. If $B(x, y, p)$ is a type family dependent on $x, y : A$ and $p : x =_A y$, then to prove $B(x, y, p)$ it suffices to assume $y$ is $x$ and $p$ is $\text{refl}_x$. I.e., there is a function

$$\text{path-ind} : \left( \prod_{x:A} B(x, x, \text{refl}_x) \right) \to \left( \prod_{x,y:A} \prod_{p:x =_A y} B(x, y, p) \right).$$

Arrow induction. If $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \text{Hom}_A(x, y)$ and $A$ is Segal, then to prove $B(x, y, f)$ it suffices to assume $y$ is $x$ and $f$ is $\text{id}_x$. I.e., there is a function

$$\text{id-ind} : \left( \prod_{x:A} B(x, x, \text{id}_x) \right) \to \left( \prod_{x,y:A} \prod_{f:\text{Hom}_A(x, y)} B(x, y, f) \right).$$
For more on the synthetic theories of $\infty$-categories, see:

Emily Riehl and Dominic Verity

- draft book in progress:
  
  *Elements of $\infty$-Category Theory*
  
  www.math.jhu.edu/~eriehl/elements.pdf

- mini-course lecture notes:
  
  $\infty$-Category Theory from Scratch
  
  arXiv:1608.05314

Emily Riehl and Michael Shulman

- *A type theory for synthetic $\infty$-categories*, Higher Structures
  

Thank you!