THE FORMAL THEORY OF ADJUNCTIONS, MONADS, ALGEBRAS, AND DESCENT

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This was a slide talk. See the bottom for the slides.

Joint with Dominic Verity of Macquarie.

The goal here is to simplify the foundations of algebraic topology. This will contain new results but also new proofs of old results (e.g. Beck's Monadicity Theorem).

We begin with an $(\infty, 2)$-category, i.e. a simplicially enriched category where the hom-spaces are quasi-categories. A special case is a strict 2-category. We have an adjunction incl:2-Cat $\leftrightarrows$ $(\infty, 2)$-cat:htpy

The big theorem is that any adjunction in a homotopy 2-category extends to a homotopy coherent adjunction in the $(\infty, 2)$-category.

The new approach is via weighted limits. That's how you get a context free proof which works in both settings.

1. Pre-Talk

Consider the 2-category CAT. Everything we say here is strict. An adjunction is a pair of categories A, B, a pair of functors U, F, a unit $1 \to UF$, and a counit $FU \to 1$ satisfying certain commutative diagrams.

Equivalently, you have Bar Resolutions $id_B \to UF \overset{\rightrightarrows}{\leftarrow} UFUF \Rrightarrow UFUFUF \dots$ where we have abbreviated by $\Rrightarrow$ the map with 3 arrows going forward and 2 coming back. This data is the same as the data of a diagram $\Delta_+ \to \text{Hom}(B, B)$. Similarly you have $U \leftrightarrows UFU \Rrightarrow UFUFU \Rrightarrow \dots$ where again we leave the reader to fill in the collections of arrows which the symbol $\Rrightarrow$ is standing in for. This is the data of a diagram $\Delta_\infty \to \text{Hom}(A, B)$.

Equivalently this data is a 2-functor Adj $\to$ CAT taking $+ \mapsto B$ and $- \mapsto A$, and in which $\text{Hom}(+, +) = \Delta_+ = \text{Hom}(-, -)^{op}$ and $\text{Hom}(-, +) = \Delta_\infty = \text{Hom}(+, -)^{op}$. Here Adj is the free adjunction.

To get at monads consider the full subcategory Mnd in Adj on the object +. Then a monad is a 2-functor Mnd $\to$ Cat taking $+ \mapsto B$ and with $\Delta_+ \to \text{Hom}(B, B)$.

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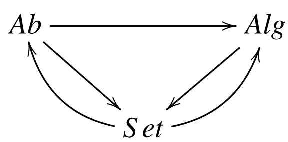
Examples:

(1) Set $\leftrightarrows$ Ab given by free-forgetful

(2) $\text{Mod}_R \leftrightarrows \text{Mod}_S$ given by induction and restriction along a ring homomorphism $R \to S$

An algebra is then given by a set X and a map $\beta : \mathbb{Z}[X] \to X$ which gives a Bar-resolution $X \leftrightarrows \mathbb{Z}[X] \rightrightarrows \mathbb{Z}[\mathbb{Z}[X]] \rightrightarrows \ldots$, i.e. a diagram $\Delta_\infty \to Set$.

We may define a full subcategory Alg of Set^{Δ_∞} as a weighted limit. So we immediately have



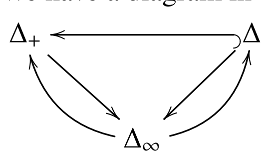
where both pairs are adjunctions.

Lastly, we turn to *descent*. Descent data is a coalgebra structure for the induced comonad on the category Alg.

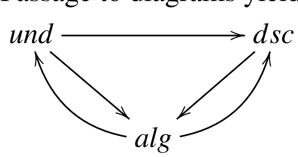
Example: For the $R \to S$ adjunction above, descent data is given by an R-module X and $X \leftrightarrows S \otimes_R X \rightrightarrows S \otimes_R S \otimes_R X \rightrightarrows \ldots$. Once again this is equivalent to a functor $\Delta \to Mod_R$. We define a full subcategory Dsc in Mod_R^Δ via weak limits, as we did previously for Alg.

Cats with Monad	Weights for
Δ_+	underlying obj
Δ_∞	obj of algebras
Δ	descent obj

We have a diagram in Cat^{Mnd}:



Passage to diagrams yields



2. Adjunctions and Monads

Let qCat denote the category of quasi-categories. Cat embeds in qCat via the nerve functor. The left adjoint ho works via passage to homotopy category. Both these functors preserve finite products, so they pass to an adjunction on things enriched in Cat and qCat (in particular, to 2-Cat $\leftrightarrows$ $(\infty, 2)$-CAT).

The yoga of $(\infty, 2)$-categories is that when you restrict to 2-CAT you recover the classical notions.

Examples of 2-categories: CAT, monoidal categories, accessible categories, algebras for any 2-monads

Examples of $(\infty, 2)$-categories: qCat, compete Segal spaces, Rezk objects (take a model category $\mathcal{M}$, look at Reedy model structure on simplicial objects in $\mathcal{M}$, and then do a localization to end up with something enriched over Joyal's model structure on sSet).

The set-up in this talk is totally different from Barwick–Schommer-Pries, though it is true that the model for $(\infty, 2)$-cat is equivalent to CSS, which is the universal $(\infty, 2)$-cat in Barwick–Schommer-Pries. The category theory we're presenting here is the accepted category theory for higher topos theory.

Δ_+ is the category of finite non-empty ordinals and order preserving maps.

Δ_∞ is finite non-empty ordinals and order preserving maps which preserve the top element.

Motivated by the theorem of Schanuel-Street which identifies 2-categories with functors from Adj, we define a *homotopy coherent adjunction* to be an $(\infty, 2)$-category K to be a simplicial functor Adj $\to K$.

The data of K picks out A, B; $f : A \leftrightarrows B : u$; and coherence data which is like higher simplices.

Adj has 2 objects $+, -$. It has structure maps represented pictorially in the slides by horse-shoes. Then coherence data is equivalent to n-arrows, and these are given pictorially by *strictly undulating squiggles* on $n + 1$ lines. Degenerate simplices are given by duplicating a vertical line in the squiggle. The triangle identities are used for shaking squiggles so that the only turn-arounds happen between horizontal lines. The reader is encouraged to consult the slides for images of these strictly undulating squiggles. This shaking out procedure is demonstrated as you click through the pictures in the slides.

Remark: This simplicial category is isomorphic to the hammock localization of the walking weak equivalence.

We are now ready to state our first main theorem.

Theorem 2.1. *Any adjunction in the homotopy 2-category of an $(\infty, 2)$-category extends to a homotopy coherent adjunction.*

In qCat you can start with any adjunction between model categories, replace it by a simplicial Quillen pair, and get an adjunction on qCat level. That last step is hard, but once it's done you can extend and get all higher coherence data for free by this theorem.

Theorem 2.2. *The spaces of extensions are contractible Kan complexes.*

This is a uniqueness result. Up to homotopy your decisions in the previous theorem didn't matter.

Upshot: there are lots of examples of adjunctions of $(\infty, 2)$-categories.

The proof makes use of the fact that Adj is a simplicial computad (i.e. is cellularly cofibrant in a particular model structure).

Homotopy coherent monad in an $(\infty, 2)$-category K is a simplicial functor $Mnd \to K$. As in the pre-talk, $+ \mapsto B$, we have the *monad resolution* $\Delta_+ \to \mathrm{Hom}(B, B)$, and we have higher data which we can keep track of pictorially via our graphical calculus from previous slides.

A monad in the homotopy 2-category need not lift to a homotopy coherent monad, because monads don't come with a universal property. Mostly, examples of homotopy coherent monads come from homotopy coherent adjunctions.

3. Weighted Limits

A weighted limit is a way to vary the shape of a diagram and take its limit.

weight$^{o}p\times$ diagram $\mapsto$ limit

$(Cat^A)^{op} \times K^A \to K$

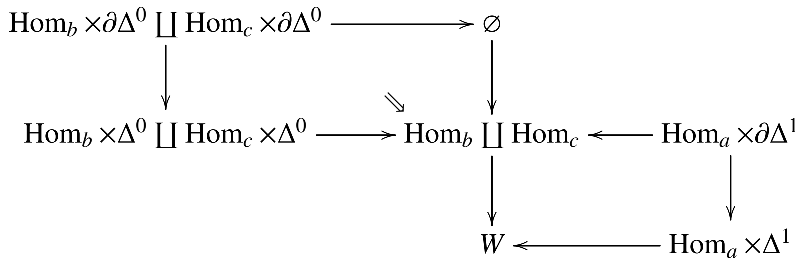
The weight is a 2-functor $A \to Cat$, but secretly it's $A \to sSet$ when you're doing $(\infty, 2)$-categories.

There is a formula for $\{W, D\}_A$ as an equalizer. In particular, it's an end.

If the weight is representable then the formula reduces so that $\{\mathrm{Hom}_a, D\} \cong D_a$. This is the Yoneda lemma.

Colimits of weights give rise to limits of weighted limits. Slogan: weighted limits can be made to order. Once you know what you want you can cook up the right weight by gluing together representables.

Example: A has the shape of a pullback diagram $b \to a \leftarrow c$. Define $W \in sSet^A$ via



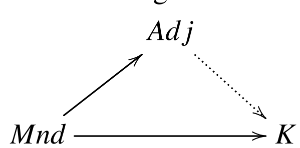
This is an example of a cellular weight, i.e. a cell complex in the projective model structure on Cat^A or $sSet^A$. An example is Bousfield-Kan homotopy limits.

Henceforth we have a completeness hypothesis that K admits cellular weighted limits. In all the examples from the pre-talk, this is satisfied.

4. Algebras and Descent Data

Given a homotopy coherent monad B, define the object of algebras alg$B \in K$ and the monadic homotopy coherent adjunction alg $B \leftrightarrows B$

Do this via right Kan extension



$B \in K^{Mnd}$ and $\Delta_\infty inCat^{Mnd}$

$algB := \{\Delta_\infty, B\}_{Mnd}$

Example $K = qCat$. A vertex in alg$B$ is a map $\Delta_\infty \to B$. So this is just a Bar resolution plus higher coherence data.

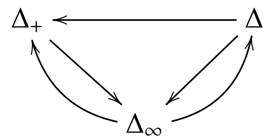
Cat^{Mnd} has a representable functor $\{-, B\}_{Mnd}$ which takes you to K^{op}. Under this functor Δ_∞ goes to $algB$ and Δ_+ goes to B. Cat^{Adj} restricts to both of these weights via Hom$_-$ and Hom$_+$, and those weights come from $-$ and $+$ on Adj^{op}. The adjunction between $-$ and $+$ pushes through to an adjunction alg $B \leftrightarrows B$.

Remark: Because Adj is a simplicial computad these weights are cellular.

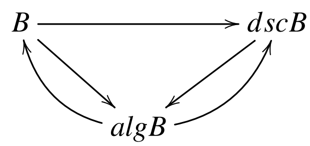
The homotopy 2-category of a 2-category doesn't admit cellular weighted limits. There are very few strict 2-limits in a homotopy 2-category.

Fix $B \in K^{Mnd}$. A descent datum is a coalgebra for the induced monad on the object of algebras. Thus, $dscB := coalg(alg(B))$.

Example: a vertex in dsc$B$ is a map $\Delta \to B$. Apply weights



and get



5. Monadicity and descent

Monadicity and descent theorems require geometric realization of simplicial objects valued in an object of an $(\infty, 2)$-category.

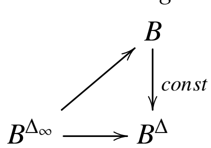
Say an object $B \in K$ admits totalizations iff there is an absolute right lifting diagram in ho$K$ for the object B^{Δ} against the map const: $B \to B^{\Delta}$. If the lift exists then it is the totalization map $B^{\Delta} \to B$. Furthermore, there is a two cell inside the resulting triangle (formed by the lifted map) which ensures that B^{Δ} is compatible with B in the way our intuition says it should be.

Note: B^{Δ} is the cotensor in the $(\infty, 2)$-category.

Note: In a similar way one can get at any homotopy 2-limit in an $(\infty$, 2-category.

Equivalently to the absolute right lifting diagram is an adjunction $B^{\Delta} \leftrightarrows B$ in $ho(K)$. So it's equivalent to a homotopy coherent adjunction.

Theorem 5.1. *In any $(\infty, 2)$-category with cotensors, the totalization of a split augmented cosimplicial object is its augmenation. Precisely, this is saying there is an absolute right lifting diagram*



This result is proven by considering the weights, i.e. the diagram $1 \leftarrow \Delta \to \Delta_{\infty}$. In particular, this is a context free proof of the result.

Theorem 5.2. *Any descent datum is the totalization of a canonical cosimplicial object of free descent data.*

Any algebra is the geometric realization of a canonical simplicial object of free algebras.

Again, both statements are formalized and proven via absolute right lifting diagrams.

Theorem 5.3. *For any homotopy coherent monad in an $(\infty, 2)$-category with cellular weighted limits, there is a canonical map $B \to dscB$ which*

(1) Admits a right adjoint if B has totalizations

(2) Is full and faithful if elements of B are totalizations of their monad resolution

(3) Is an equivalence if comonadicity is satisfied

Theorem 5.4 (Beck's Monadicity)**.** *Just as in the result above, for any homotopy coherent adjunction (f, u) there is a map $A \to algB$ which admits a left adjoint if A has geometric realizations and it provides an adjoint equivalence.*