Emily Riehl

Johns Hopkins University

Categorifying cardinal arithmetic

MAA MathFest, August 4, 2018

Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c. by taking a tour of some deep ideas from category theory.

- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?

1

Step 1: categorification

The idea of categorification

The first step is to understand the equation

$$a \times (b + c) = (a \times b) + (a \times c)$$

as expressing some deeper truth about mathematical structures.

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

Q: What is the role of the natural numbers a, b, and c?

Categorifying natural numbers

Q: What is the role of the natural numbers a, b, and c?

A: Natural numbers define the cardinalities, or sizes, of finite sets.

Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$

Categorifying equality

Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$

Q: What is true of A and B if $a = b$?

A: The sets A and B are isomorphic, in which case we write $A \cong B$, if and only if $a = b$.

For $a := |A|$ and $b := |B|$, the equation $a = b$ asserts the existence of an isomorphism $A \cong B$.

Eugenia Cheng: "All equations are lies."

Categorification: the truth behind $a = b$ is $A \cong B$.

Categorification progress report

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

The story so far:

- The natural numbers a, b, and c encode the sizes of finite sets A, B, and C:
$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$
- The equation "$=$" asserts the existence of an isomorphism "$\cong$".

Q: What is the deeper meaning of the symbols "$+$" and "$\times$"?

Categorifying +

Q: If $b := |B|$ and $c := |C|$ what set has $b + c$ elements?

A: The disjoint union $B + C$ is a set with $b + c$ elements.

$$B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad C = \left\{ \begin{matrix} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{matrix} \right\}, \qquad B + C = \left\{ \begin{matrix} \sharp & \flat & \spadesuit & \heartsuit \\ \natural & \diamondsuit & \clubsuit \end{matrix} \right\}$$

$$b + c := |B + C|$$

Categorifying $\times$

Q: If $a := |A|$ and $b := |B|$ what set has $a \times b$ elements?

A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \{ \ast \quad \star \}, \qquad B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad A \times B = \left\{ \begin{matrix} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \end{matrix} \right\}$$

$$a \times b := |A \times B|$$

Categorifying cardinal arithmetic

In summary:

- Natural numbers define cardinalities: there are sets A, B, and C so that $a := |A|$, $b := |B|$, and $c := |C|$.
- The equation $a = b$ encodes an isomorphism $A \cong B$.
- The disjoint union $B + C$ is a set with $b + c$ elements.
- The cartesian product $A \times B$ is a set with $a \times b$ elements.

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

Summary of Step I

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

A: The sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$\left\{ \begin{array}{cc} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \\ (\ast, \spadesuit) & (\star, \spadesuit) \\ (\ast, \heartsuit) & (\star, \heartsuit) \\ (\ast, \diamondsuit) & (\star, \diamondsuit) \\ (\ast, \clubsuit) & (\star, \clubsuit) \end{array} \right\} \cong \left\{ \begin{array}{c} (\ast, \sharp) \quad (\ast, \flat) \quad (\ast, \spadesuit) \quad (\ast, \heartsuit) \\ (\ast, \natural) \quad (\ast, \diamondsuit) \quad (\ast, \clubsuit) \\ (\star, \sharp) \quad (\star, \flat) \quad (\star, \spadesuit) \quad (\star, \heartsuit) \\ (\star, \natural) \quad (\star, \diamondsuit) \quad (\star, \clubsuit) \end{array} \right\}$$

$$A \times (B + C) \qquad \cong \qquad (A \times B) + (A \times C)$$

By categorification:

Step I summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

2

Step 2: the Yoneda lemma

The Yoneda lemma

The Yoneda lemma. Two sets A and B are isomorphic if and only if

- for all sets X, the sets of functions

  $$\mathsf{Fun}(A, X) := \{\phi \colon A \to X\} \quad \text{and} \quad \mathsf{Fun}(B, X) := \{\psi \colon B \to X\}$$

  are isomorphic and moreover
- the isomorphisms $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ are "natural" in the sense of commuting with composition with any function $f \colon X \to Y$.

? ? ?

Proof of the Yoneda lemma

The Yoneda lemma. A and B are isomorphic if and only if for any X the sets of functions $\mathsf{Fun}(A, X)$ and $\mathsf{Fun}(B, X)$ are "naturally" isomorphic.

Proof ($\Leftarrow$): Suppose $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ for all X. Taking $X = A$ and $X = B$, we use the bijections:

$$\begin{array}{ccc} \mathsf{Fun}(A, A) & \cong & \mathsf{Fun}(B, A) \\ \Cup & & \Cup \\ \mathrm{id}_A & \longmapsto & \psi \end{array} \qquad \begin{array}{ccc} \mathsf{Fun}(A, B) & \cong & \mathsf{Fun}(B, B) \\ \Cup & & \Cup \\ \phi & \longleftarrow\!\shortmid & \mathrm{id}_B \end{array}$$

to define functions $\psi\colon B \to A$ and $\phi\colon A \to B$. In fact, they're inverses! □

Exercise: Use the "naturality" of $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ to prove that $\psi \circ \phi = \mathrm{id}_A$ and $\phi \circ \psi = \mathrm{id}_B$.

Summary of Steps 1 and 2

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism

$$\mathrm{Fun}(A \times (B + C), X) \cong \mathrm{Fun}((A \times B) + (A \times C), X).$$

3

Step 3: representability

The universal property of the disjoint union

Q: For sets B, C, and X, what is $\mathsf{Fun}(B + C, X)$?

Q: What is needed to define a function $f\colon B + C \to X$?

A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f\colon B + C \to X$ is determined by *two* functions $f_B\colon B \to X$ and $f_C\colon C \to X$.

By "pairing"

$$\begin{array}{ccc} \mathsf{Fun}(B + C, X) & \cong & \mathsf{Fun}(B, X) \times \mathsf{Fun}(C, X) \\ \uplus & & \uplus \\ f & \leftrightsquigarrow & (f_B, f_C) \end{array}$$

A universal property of the cartesian product

Q: For sets A, B, and X, what is $\mathsf{Fun}(A \times B, X)$?

Q: What is needed to define a function $f \colon A \times B \to X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b) \colon A \to X$ sending a to $f(a, b)$. So, altogether we need to define a function $f \colon B \to \mathsf{Fun}(A, X)$.

By "currying"

$$\begin{array}{ccc} \mathsf{Fun}(A \times B, X) & \cong & \mathsf{Fun}(B, \mathsf{Fun}(A, X)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f \colon A \times B \to X & \leftrightsquigarrow & f \colon B \to \mathsf{Fun}(A, X) \end{array}$$

Summary of Steps 1, 2, and 3

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism

$$\mathsf{Fun}(A \times (B + C), X) \cong \mathsf{Fun}((A \times B) + (A \times C), X).$$

By representability:

Step 3 summary:

- $\mathsf{Fun}(B + C, X) \cong \mathsf{Fun}(B, X) \times \mathsf{Fun}(C, X)$ by "pairing" and
- $\mathsf{Fun}(A \times B, X) \cong \mathsf{Fun}(B, \mathsf{Fun}(A, X))$ by "currying."

4

Step 4: the proof

The proof

Theorem. For any natural numbers a, b, and c,

$$a \times (b + c) = (a \times b) + (a \times c).$$

Proof: To prove $a \times (b + c) = (a \times b) + (a \times c)$:

- pick sets A, B, and C so that $a := |A|$, and $b := |B|$, and $c := |C|$
- and show that $A \times (B + C) \cong (A \times B) + (A \times C)$.
- By the Yoneda lemma, this holds if and only if, "naturally," $\mathrm{Fun}(A \times (B + C), X) \cong \mathrm{Fun}((A \times B) + (A \times C), X)$.
- Now

$$\begin{aligned} \mathrm{Fun}(A \times (B+C), X) &\cong \mathrm{Fun}(B + C, \mathrm{Fun}(A, X)) \text{ by "currying"} \\ &\cong \mathrm{Fun}(B, \mathrm{Fun}(A, X)) \times \mathrm{Fun}(C, \mathrm{Fun}(A, X)) \text{ by "pairing"} \\ &\cong \mathrm{Fun}(A \times B, X) \times \mathrm{Fun}(A \times C, X) \text{ by "currying"} \\ &\cong \mathrm{Fun}((A \times B) + (A \times C), X) \text{ by "pairing."} \end{aligned}$$

□

5

Epilogue: what was the point of that?

Generalization to infinite cardinals

Note we didn't actually need the sets A, B, and C to be finite.

Theorem. For any cardinals α, β, γ,

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma).$$

Proof: The one we just gave.

Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta\times\gamma} = (\alpha^{\gamma})^{\beta}.$$

Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn't need A, B, and C to be sets at all!

Theorem.

- For vector spaces U, V, W,

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$

- For nice topological spaces X, Y, Z,

$$X \times (Y \sqcup Z) = (X \times Y) \sqcup (X \times Z).$$

- For abelian groups A, B, C,

$$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C).$$

Proof: The one we just gave.

The real point

The ideas of

- categorification (replacing equality by isomorphism),
- the Yoneda lemma (replacing isomorphism by natural isomorphism),
- representability (characterizing maps to or from an object),
- limits and colimits (like cartesian product and disjoint union), and
- adjunctions (such as currying)

are all over mathematics — so keep a look out!

Thank you!