Categorifying cardinal arithmetic
Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers $a, b, \text{ and } c$. by taking a tour of some deep ideas from category theory.

- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?
Step 1: categorification
The idea of categorification

The first step is to understand the equation

\[ a \times (b + c) = (a \times b) + (a \times c) \]

as expressing some **deeper truth** about mathematical structures.

**Q:** What is the deeper meaning of the equation

\[ a \times (b + c) = (a \times b) + (a \times c) \]?

**Q:** What is the role of the natural numbers \(a, b,\) and \(c\)?
Q: What is the role of the natural numbers $a$, $b$, and $c$?

A: Natural numbers define the cardinalities, or sizes, of finite sets.

Natural numbers $a$, $b$, and $c$ encode the sizes of finite sets $A$, $B$, and $C$.

$$a := |A|, \quad b := |B|, \quad c := |C|.$$
Categorifying equality

Natural numbers $a, b, \text{ and } c$ encode the sizes of finite sets $A, B, \text{ and } C$.

$$a := |A|, \quad b := |B|, \quad c := |C|.$$ 

Q: What is true of $A$ and $B$ if $a = b$?

A: $a = b$ if and only if $A$ and $B$ are isomorphic, which means there exist functions $f: A \to B$ and $g: B \to A$ that are inverses in the sense that $g \circ f = \text{id}$ and $f \circ g = \text{id}$. In this case, we write $A \cong B$.

For $a := |A|$ and $b := |B|$, the equation $a = b$ asserts the existence of an isomorphism $A \cong B$.

Eugenia Cheng: “All equations are lies.”

Categorification: the truth behind $a = b$ is $A \cong B$. 
Categorification progress report

Q: What is the deeper meaning of the equation

\[ a \times (b + c) = (a \times b) + (a \times c) \]?

The story so far:

- The natural numbers \( a, b, \) and \( c \) encode the sizes of finite sets \( A, B, \) and \( C \):
  \[
a := |A|, \quad b := |B|, \quad c := |C|.
\]
- The equation “=” asserts the existence of an isomorphism “\( \cong \)”.

Q: What is the deeper meaning of the symbols “+” and “\( \times \)”?
Q: If $b := |B|$ and $c := |C|$ what set has $b + c$ elements?

A: The disjoint union $B + C$ is a set with $b + c$ elements.

$B = \begin{cases} \{ \# \} \end{cases}$, \quad C = \begin{cases} \{ \spadesuit, \heartsuit \} \end{cases}$, \quad B + C = \begin{cases} \{ \# \spadesuit \heartsuit \diamondsuit \clubsuit \} \end{cases}$

$b + c := |B + C|$
Q: If \( a := |A| \) and \( b := |B| \) what set has \( a \times b \) elements?

A: The cartesian product \( A \times B \) is a set with \( a \times b \) elements.

\[
A = \{ * \, \star \} , \quad B = \left\{ \begin{array}{c} \# \\ b \\ ♮ \end{array} \right\} , \quad A \times B = \left\{ \begin{array}{c} (*, \#) \\ (\star, \#) \\ (\star, b) \\ (\star, ♮) \end{array} \right\}
\]

\[
a \times b := |A \times B|
\]
Categorifying cardinal arithmetic

In summary:

- Natural numbers define cardinalities: there are sets $A$, $B$, and $C$ so that $a := |A|$, $b := |B|$, and $c := |C|$.
- The equation $a = b$ encodes an isomorphism $A \cong B$.
- The disjoint union $B + C$ is a set with $b + c$ elements.
- The cartesian product $A \times B$ is a set with $a \times b$ elements.

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)$$

A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$
Summary of Step 1

Q: What is the deeper meaning of the equation

\[ a \times (b + c) = (a \times b) + (a \times c) \]?

A: The sets \( A \times (B + C) \) and \( (A \times B) + (A \times C) \) are isomorphic!

\[
\begin{array}{ccc}
(\ast, \#) & (\ast, \#) \\
(\ast, b) & (\ast, b) \\
(\ast, \flat) & (\ast, \flat) \\
(\ast, \spadesuit) & (\ast, \spadesuit) \\
(\ast, \heartsuit) & (\ast, \heartsuit) \\
(\ast, \diamondsuit) & (\ast, \diamondsuit) \\
(\ast, \clubsuit) & (\ast, \clubsuit)
\end{array}
\cong
\begin{array}{ccc}
(\ast, \#) & (\ast, b) & (\ast, \spadesuit) & (\ast, \heartsuit) \\
(\ast, \flat) & (\ast, \diamondsuit) & (\ast, \clubsuit) \\
(\ast, \spadesuit) & (\ast, \diamondsuit) & (\ast, \clubsuit)
\end{array}
\]

\( A \times (B + C) \cong (A \times B) + (A \times C) \)

By categorification:

Step 1 summary: To prove \( a \times (b + c) = (a \times b) + (a \times c) \)
\( \Rightarrow \) we'll instead show that \( A \times (B + C) \cong (A \times B) + (A \times C) \).
Step 2: the Yoneda lemma
The Yoneda lemma. Two sets $A$ and $B$ are isomorphic if and only if

- for all sets $X$, the sets of functions

$$\text{Fun}(A, X) := \{ h : A \to X \} \quad \text{and} \quad \text{Fun}(B, X) := \{ k : B \to X \}$$

are isomorphic and moreover
- the isomorphisms $\text{Fun}(A, X) \cong \text{Fun}(B, X)$ are “natural” in the sense of commuting with composition with any function $\ell : X \to Y$. 

???
Proof of the Yoneda lemma

The Yoneda lemma. \( A \) and \( B \) are isomorphic if and only if for any \( X \) the sets of functions \( \text{Fun}(A, X) \) and \( \text{Fun}(B, X) \) are “naturally” isomorphic.

Proof (\( \Leftarrow \)): Suppose \( \text{Fun}(A, X) \cong \text{Fun}(B, X) \) for all \( X \). Taking \( X = A \) and \( X = B \), we use the bijections:

\[
\begin{align*}
\text{Fun}(A, A) & \cong \text{Fun}(B, A) \\
\text{Fun}(A, B) & \cong \text{Fun}(B, B)
\end{align*}
\]

...to define functions \( g : B \to A \) and \( f : A \to B \). By naturality:

\[
\begin{align*}
id_A & \Rightarrow g \\
f & \Rightarrow f \circ g
\end{align*}
\]

and similarly \( g \circ f = \text{id}_A \).
Summary of Steps 1 and 2

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
\[\leadsto\] we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
\[\leadsto\] we'll instead define a “natural” isomorphism
\[
\text{Fun}(A \times (B + C), X) \cong \text{Fun}((A \times B) + (A \times C), X).
\]
Step 3: representability
Q: For sets $B$, $C$, and $X$, what is $\text{Fun}(B + C, X)$?

Q: What is needed to define a function $f: B + C \to X$?

A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f: B + C \to X$ is determined by two functions $f_B: B \to X$ and $f_C: C \to X$.

By “pairing”

$$\text{Fun}(B + C, X) \ni f \mapsto (f_B, f_C)$$
A universal property of the cartesian product

Q: For sets $A, B,$ and $X$, what is $\text{Fun}(A \times B, X)$?

Q: What is needed to define a function $f : A \times B \to X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(\_, b) : A \to X$ sending $a$ to $f(a, b)$. So, altogether we need to define a function $f : B \to \text{Fun}(A, X)$.

By “currying”

$$\text{Fun}(A \times B, X) \cong \text{Fun}(B, \text{Fun}(A, X))$$

$$\upharpoonright f : A \times B \to X \iff f : B \to \text{Fun}(A, X)$$
Summary of Steps 1, 2, and 3

By categorification:

**Step 1 summary:** To prove \(a \times (b + c) = (a \times b) + (a \times c)\)

\(\Rightarrow\) we'll instead show that \(A \times (B + C) \cong (A \times B) + (A \times C)\).

By the Yoneda lemma:

**Step 2 summary:** To prove \(A \times (B + C) \cong (A \times B) + (A \times C)\)

\(\Rightarrow\) we'll instead define a “natural” isomorphism

\[\text{Fun}(A \times (B + C),X) \cong \text{Fun}((A \times B) + (A \times C),X)\]

By representability:

**Step 3 summary:**

- \(\text{Fun}(B + C, X) \cong \text{Fun}(B, X) \times \text{Fun}(C, X)\) by “pairing” and
- \(\text{Fun}(A \times B, X) \cong \text{Fun}(B, \text{Fun}(A, X))\) by “currying.”
Step 4: the proof
Theorem. For any natural numbers $a$, $b$, and $c$,

$$a \times (b + c) = (a \times b) + (a \times c).$$

Proof: To prove $a \times (b + c) = (a \times b) + (a \times c)$:

- pick sets $A$, $B$, and $C$ so that $a := |A|$, and $b := |B|$, and $c := |C|$
- and show that $A \times (B + C) \cong (A \times B) + (A \times C)$.
- By the Yoneda lemma, this holds if and only if, “naturally,”
  $$\text{Fun}(A \times (B + C), X) \cong \text{Fun}((A \times B) + (A \times C), X).$$
- Now
  $$\text{Fun}(A \times (B + C), X) \cong \text{Fun}(B + C, \text{Fun}(A, X)) \text{ by "currying"}$$
  $$\cong \text{Fun}(B, \text{Fun}(A, X)) \times \text{Fun}(C, \text{Fun}(A, X)) \text{ by "pairing"}$$
  $$\cong \text{Fun}(A \times B, X) \times \text{Fun}(A \times C, X) \text{ by "currying"}$$
  $$\cong \text{Fun}((A \times B) + (A \times C'), X) \text{ by "pairing."}$$
Epilogue: what was the point of that?
Generalization to infinite cardinals

Note we didn’t actually need the sets $A$, $B$, and $C$ to be finite.

**Theorem.** For any cardinals $\alpha$, $\beta$, $\gamma$,
\[
\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma).
\]

**Proof:** The one we just gave.

**Exercise:** Find a similar proof for other identities of cardinal arithmetic:
\[
\alpha^{\beta+\gamma} = \alpha^\beta \times \alpha^\gamma \quad \text{and} \quad (\alpha^\beta)^\gamma = \alpha^{\beta \times \gamma} = (\alpha^\gamma)^\beta.
\]
Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn’t need $A$, $B$, and $C$ to be sets at all!

Theorem.

• For vector spaces $U$, $V$, $W$,
  \[ U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W). \]

• For nice topological spaces $X$, $Y$, $Z$,
  \[ X \times (Y \sqcup Z) = (X \times Y) \sqcup (X \times Z). \]

• For abelian groups $A$, $B$, $C$,
  \[ A \otimes_Z (B \oplus C) \cong (A \otimes_Z B) \oplus (A \otimes_Z C). \]

Proof: The one we just gave.
The real point

The ideas of

- **categorification** (replacing equality by isomorphism),
- the Yoneda lemma (replacing isomorphism by natural isomorphism),
- **representability** (characterizing maps to or from an object),
- **limits** and **colimits** (like cartesian product and disjoint union), and
- **adjunctions** (such as currying)

are all over mathematics — so keep a look out!

Thank you!