Categorifying cardinal arithmetic
Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers $a$, $b$, and $c$. 

- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?
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Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers $a$, $b$, and $c$ by taking a tour of some deep ideas from category theory.
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• Step 1: categorification
• Step 2: the Yoneda lemma
• Step 3: representability
• Step 4: the proof
• Epilogue: what was the point of that?
Step 1: categorification
The idea of categorification

The first step is to understand the equation

\[ a \times (b + c) = (a \times b) + (a \times c) \]

as expressing some deeper truth about mathematical structures.
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Q: What is the deeper meaning of the equation

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Q: What is the role of the natural numbers \( a, b, \) and \( c \)?
Q: What is the role of the natural numbers $a$, $b$, and $c$?

A: Natural numbers define the cardinalities, or sizes, of finite sets. Natural numbers $a$, $b$, and $c$ encode the sizes of finite sets $A$, $B$, and $C$.

\[ a \equiv |A|, \quad b \equiv |B|, \quad c \equiv |C|. \]
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$$a := |A|, \quad b := |B|, \quad c := |C|.$$
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Q: What is true of $A$ and $B$ if $a = b$?

A: $a = b$ if and only if $A$ and $B$ are isomorphic, which means there exist functions $f: A \to B$ and $g: B \to A$ that are inverses in the sense that $g \circ f = \text{id}$ and $f \circ g = \text{id}$. In this case, we write $A \cong B$. 

Eugenia Cheng: "All equations are lies." Categorification: the truth behind $a = b$ is $A \cong B$. 
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For $a := |A|$ and $b := |B|$, the equation $a = b$ asserts the existence of an isomorphism $A \cong B$. 
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\[ a \times (b + c) = (a \times b) + (a \times c) \]?
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The story so far:

- The natural numbers \( a, b, \) and \( c \) encode the sizes of finite sets \( A, B, \) and \( C \):
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The story so far:

- The natural numbers \(a\), \(b\), and \(c\) encode the sizes of finite sets \(A\), \(B\), and \(C\):
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- The equation “\(\cong\)” asserts the existence of an isomorphism “\(\cong\)”.

Q: What is the deeper meaning of the symbols “\(+\)” and “\(\times\)”?
Q: If $b := |B|$ and $c := |C|$ what set has $b + c$ elements?
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A: The disjoint union $B + C$ is a set with $b + c$ elements.

$B = \left\{ \begin{array}{c} \# \\ b \\ q \end{array}\right\}, \quad C = \left\{ \begin{array}{c} \spadesuit \\ \heartsuit \\ \diamondsuit \\ \clubsuit \end{array}\right\}, \quad B + C = \left\{ \begin{array}{c} \# \\ b \\ \spadesuit \\ \heartsuit \\ \diamondsuit \\ \clubsuit \end{array}\right\}$
Categorifying +

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\[
B = \left\{ \begin{array}{c}
\#
\end{array} \right\}, \quad
C = \left\{ \begin{array}{c}
\spadesuit, \heartsuit, \diamondsuit, \clubsuit
\end{array} \right\}, \quad
B + C = \left\{ \begin{array}{c}
\#, b, \spadesuit, \heartsuit, \diamondsuit, \clubsuit
\end{array} \right\}
\]

\[
b + c := |B + C|
\]
Q: If $a := |A|$ and $b := |B|$ what set has $a \times b$ elements?
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A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \{ \ast \, \ast \}, \quad B = \{ \begin{array}{c} \# \\ \flat \\ \natural \end{array} \}, \quad A \times B = \{ (\ast, \#), (\ast, \#), (\ast, \flat), (\ast, \natural), (\ast, \flat), (\ast, \natural) \}$$
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\( a \times b := |A \times B| \)
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In summary:

- Natural numbers define cardinalities: there are sets $A$, $B$, and $C$ so that $a \equiv |A|$, $b \equiv |B|$, and $c \equiv |C|$.
- The equation $a = b$ encodes an isomorphism $A \cong B$.
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Q: What is the deeper meaning of the equation $a \times (b + c) = (a \times b) + (a \times c)$?
A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic! $A \times (B + C) \cong (A \times B) + (A \times C)$.
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\[
\begin{align*}
&\{(\ast, \#)\} \quad \{\ast, \#\} \\
&\{(\ast, \flat)\} \quad \{\ast, \flat\} \\
&\{\ast, \spadesuit\} \quad \{\ast, \spadesuit\} \\
&\{(\ast, \diamondsuit)\} \quad \{\ast, \diamondsuit\} \\
&\{(\ast, \clubsuit)\} \quad \{\ast, \clubsuit\}
\end{align*}
\]

\[\cong\]

\[
\begin{align*}
&\{(\ast, \#)\} \quad \{(\ast, \flat)\} \quad \{(\ast, \spadesuit)\} \quad \{(\ast, \diamondsuit)\} \quad \{(\ast, \clubsuit)\} \\
&\{(\ast, \#)\} \quad \{(\ast, \diamondsuit)\} \quad \{(\ast, \spadesuit)\} \quad \{(\ast, \clubsuit)\} \quad \{(\ast, \diamondsuit)\}
\end{align*}
\]

\[A \times (B + C) \cong (A \times B) + (A \times C)\]

By categorification:

Step 1 summary: To prove \( a \times (b + c) = (a \times b) + (a \times c) \)
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\[
\begin{align*}
\{(\ast, \#), (\ast, b), (\ast, \flat), (\ast, \spadesuit), (\ast, \heartsuit), (\ast, \diamondsuit), (\ast, \clubsuit)\} & \cong \{(\ast, \#, \ast, b, \ast, \flat, \ast, \spadesuit, \ast, \heartsuit, \ast, \diamondsuit, \ast, \clubsuit)\} \\
& \cong \{(\ast, \#, \ast, b, \ast, \spadesuit, \ast, \heartsuit, \ast, \diamondsuit, \ast, \clubsuit)\}
\end{align*}
\]

By categorification:

Step 1 summary: To prove \( a \times (b + c) = (a \times b) + (a \times c) \)

\( \Rightarrow \) we'll instead show that \( A \times (B + C) \cong (A \times B) + (A \times C). \)
Step 2: the Yoneda lemma
The Yoneda lemma. Two sets $A$ and $B$ are isomorphic if and only if
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- for all sets $X$, the sets of functions

$$\text{Fun}(A, X) := \{h: A \to X\} \quad \text{and} \quad \text{Fun}(B, X) := \{k: B \to X\}$$

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The Yoneda lemma. Two sets $A$ and $B$ are isomorphic if and only if

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are isomorphic and moreover

- the isomorphisms $\text{Fun}(A, X) \cong \text{Fun}(B, X)$ are “natural” in the sense of commuting with composition with any function $\ell : X \to Y$. 
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Proof of the Yoneda lemma

The Yoneda lemma. $A$ and $B$ are isomorphic if and only if for any $X$ the sets of functions $\text{Fun}(A, X)$ and $\text{Fun}(B, X)$ are "naturally" isomorphic.
Proof of the Yoneda lemma

The Yoneda lemma. \( A \) and \( B \) are isomorphic if and only if for any \( X \) the sets of functions \( \text{Fun}(A, X) \) and \( \text{Fun}(B, X) \) are “naturally” isomorphic.

Proof (\( \Leftarrow \)): 

\[
\text{id}_A \circ g = \text{Fun}(A, A) \xrightarrow{\text{bijection}} \text{Fun}(B, A) \xrightarrow{\text{bijection}} \text{Fun}(A, B) \xrightarrow{\text{bijection}} \text{id}_B \circ f \in \text{Fun}(B, B) \\
\Rightarrow f \circ g = \text{id}_A \\
\Rightarrow g \circ f = \text{id}_B
\]
The Yoneda lemma. $A$ and $B$ are isomorphic if and only if for any $X$ the sets of functions $\text{Fun}(A, X)$ and $\text{Fun}(B, X)$ are “naturally” isomorphic.

Proof ($\Leftarrow$): Suppose $\text{Fun}(A, X) \cong \text{Fun}(B, X)$ for all $X$. 
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The Yoneda lemma. \( A \) and \( B \) are isomorphic if and only if for any \( X \) the sets of functions \( \text{Fun}(A, X) \) and \( \text{Fun}(B, X) \) are “naturally” isomorphic.

Proof (\( \iff \)): Suppose \( \text{Fun}(A, X) \cong \text{Fun}(B, X) \) for all \( X \). Taking \( X = A \) and \( X = B \), we use the bijections:

\[
\begin{align*}
\text{Fun}(A, A) & \cong \text{Fun}(B, A) \\
\text{Fun}(A, B) & \cong \text{Fun}(B, B)
\end{align*}
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\[
\text{Fun}(A, A) \cong \text{Fun}(B, A) \quad \text{Fun}(A, B) \cong \text{Fun}(B, B)
\]

\[
\cup
d_{A}
\]

\[
\cup
d_{B}
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$$
\begin{align*}
\text{Fun}(A, A) & \cong \text{Fun}(B, A) \\
\text{id}_A & \quad g \\
\text{Fun}(A, B) & \cong \text{Fun}(B, B) \\
\text{Fun}(A, A) & \cong \text{Fun}(B, B)
\end{align*}
$$

$\text{id}_A, g, f, \text{id}_B$

to define functions $g: B \to A$ and $f: A \to B$. 
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\begin{align*}
\text{Fun}(A, A) & \cong \text{Fun}(B, A) \\
\text{Fun}(A, B) & \cong \text{Fun}(B, B)
\end{align*}
\]

\[
\begin{array}{ccc}
\text{id}_A & \overset{\cong}{\rightarrow} & g \\
\uparrow & & \uparrow \\
\text{Fun}(A, A) & \cong & \text{Fun}(B, A) \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
\text{id}_B & \overset{\cong}{\leftarrow} & f \\
\uparrow & & \uparrow \\
f & \overset{\cong}{\rightarrow} & \text{Fun}(A, B) \cong \text{Fun}(B, B) \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
f & \overset{\cong}{\rightarrow} & \text{id}_B = f \circ g
\end{array}
\]

to define functions \( g: B \rightarrow A \) and \( f: A \rightarrow B \). By naturality:
Proof of the Yoneda lemma

The Yoneda lemma. $A$ and $B$ are isomorphic if and only if for any $X$ the sets of functions $\text{Fun}(A, X)$ and $\text{Fun}(B, X)$ are “naturally” isomorphic.

Proof ($\Leftarrow$): Suppose $\text{Fun}(A, X) \cong \text{Fun}(B, X)$ for all $X$. Taking $X = A$ and $X = B$, we use the bijections:

$$
\begin{align*}
\text{Fun}(A, A) & \cong \text{Fun}(B, A) \\
\text{id}_A & \cong g
\end{align*}
$$

$$
\begin{align*}
\text{Fun}(A, B) & \cong \text{Fun}(B, B) \\
f & \cong \text{id}_B
\end{align*}
$$

to define functions $g: B \to A$ and $f: A \to B$. By naturality:

$$
\begin{align*}
\text{id}_A & \cong g \\
\text{Fun}(A, A) \cong \text{Fun}(B, A) \\
f \cong f \circ g
\end{align*}
$$

and similarly $g \circ f = \text{id}_A$. 

\[\square\]
Summary of Steps 1 and 2

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
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By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$

$\Rightarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$. 
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**Step 1 summary:** To prove $a \times (b + c) = (a \times b) + (a \times c)$

$\Rightarrow$ we’ll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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**Step 2 summary:** To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
Summary of Steps 1 and 2

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**Step 1 summary:** To prove \( a \times (b + c) = (a \times b) + (a \times c) \)
\( \Rightarrow \) we'll instead show that \( A \times (B + C) \cong (A \times B) + (A \times C) \).

By the Yoneda lemma:

**Step 2 summary:** To prove \( A \times (B + C) \cong (A \times B) + (A \times C) \)
\( \Rightarrow \) we'll instead define a “natural” isomorphism
\[
\text{Fun}(A \times (B + C), X) \cong \text{Fun}((A \times B) + (A \times C), X).
\]
Step 3: representability
The universal property of the disjoint union

Q: For sets $B$, $C$, and $X$, what is $\text{Fun}(B + C, X)$?
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A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f : B + C \to X$ is determined by two functions $f_B : B \to X$ and $f_C : C \to X$. By "pairing" $\text{Fun}(B + C, X) \cong \text{Fun}(B, X) \times \text{Fun}(C, X)$.
The universal property of the disjoint union

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By "pairing"

$$
\text{Fun}(B + C, X) \ni f \mapsto (f_B, f_C)
$$
A universal property of the cartesian product

Q: For sets $A$, $B$, and $X$, what is $\text{Fun}(A \times B, X)$?
A universal property of the cartesian product

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Q: What is needed to define a function $f: A \times B \to X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. 
A universal property of the cartesian product

Q: For sets $A$, $B$, and $X$, what is $\text{Fun}(A \times B, X)$?

Q: What is needed to define a function $f: A \times B \rightarrow X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b): A \rightarrow X$ sending $a$ to $f(a, b)$. 
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By “currying” $\text{Fun}(A \times B, X)$

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By “currying”

\[
\begin{align*}
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\cup \\
f: A \times B \to X & \leftrightarrow f: B \to \text{Fun}(A, X)
\end{align*}
\]
Summary of Steps 1, 2, and 3

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
Summary of Steps 1, 2, and 3

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Step 1 summary: To prove \( a \times (b + c) = (a \times b) + (a \times c) \)

\( \leadsto \) we'll instead show that \( A \times (B + C) \cong (A \times B) + (A \times C) \).
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By the Yoneda lemma:

**Step 2 summary:** To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
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By the Yoneda lemma:

**Step 2 summary:** To prove \( A \times (B + C) \cong (A \times B) + (A \times C) \)
\( \Rightarrow \) we’ll instead define a “natural” isomorphism
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**Step 3 summary:**
Summary of Steps 1, 2, and 3

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**Step 1 summary:** To prove $a \times (b + c) = (a \times b) + (a \times c)$
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By representability:

**Step 3 summary:**

- $\text{Fun}(B + C, X) \cong \text{Fun}(B, X) \times \text{Fun}(C, X)$ by “pairing”
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By representability:

Step 3 summary:

- $\text{Fun}(B + C, X) \cong \text{Fun}(B, X) \times \text{Fun}(C, X)$ by “pairing” and
- $\text{Fun}(A \times B, X) \cong \text{Fun}(B, \text{Fun}(A, X))$ by “currying.”
Step 4: the proof
The proof

**Theorem.** For any natural numbers $a$, $b$, and $c$,

$$a \times (b + c) = (a \times b) + (a \times c).$$

**Proof:**
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  \text{Fun}(A \times (B + C), X) \cong \text{Fun}(B + C, \text{Fun}(A, X)) \text{ by “currying”}
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$\square$
Epilogue: what was the point of that?
Generalization to infinite cardinals

Note we didn’t actually need the sets $A$, $B$, and $C$ to be finite.
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**Theorem.** For any cardinals $\alpha$, $\beta$, $\gamma$,

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma).$$
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**Proof:** The one we just gave.

**Exercise:** Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta + \gamma} = \alpha^\beta \times \alpha^\gamma \quad \text{and} \quad (\alpha^\beta)^\gamma = \alpha^{\beta \times \gamma} = (\alpha^\gamma)^\beta.$$
Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn’t need $A$, $B$, and $C$ to be sets at all!
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**Theorem.**

- For vector spaces $U$, $V$, $W$,
  $$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$

- For nice topological spaces $X$, $Y$, $Z$,
  $$X \times (Y \sqcup Z) = (X \times Y) \sqcup (X \times Z).$$

- For abelian groups $A$, $B$, $C$,
  $$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C).$$
Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn’t need \( A, B, \) and \( C \) to be sets at all!

**Theorem.**

- For vector spaces \( U, V, W, \)
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- For nice topological spaces \( X, Y, Z, \)
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- For abelian groups \( A, B, C, \)
  \[ A \otimes_Z (B \oplus C) \cong (A \otimes_Z B) \oplus (A \otimes_Z C). \]

**Proof:** The one we just gave.
The real point

The ideas of

• categorification (replacing equality by isomorphism),
• the Yoneda lemma (replacing isomorphism by natural isomorphism),
• representability (characterizing maps to or from an object),
• limits and colimits (like cartesian product and disjoint union),
• adjunctions (such as currying)

are all over mathematics — so keep a look out!

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