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Categorifying cardinal arithmetic

MAA MathFest, August 4, 2018

Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c.

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- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof

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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?

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Step 1: categorification

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Categorification: the truth behind $a = b$ is $A \cong B$.

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Q: What is the deeper meaning of the symbols "$+$" and "$\times$"?

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A: The disjoint union $B + C$ is a set with $b + c$ elements.

$$B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\}, \qquad C = \left\{ \begin{array}{cc} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{array} \right\}, \qquad B + C = \left\{ \begin{array}{ccccccc} \sharp & & \flat & & \spadesuit & & \heartsuit \\ & \natural & & \diamondsuit & & \clubsuit & \end{array} \right\}$$

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A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

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A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

Summary of Step 1

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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Step 2: the Yoneda lemma

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- for all sets X, the sets of functions

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$$
\begin{array}{ccc}
\mathsf{Fun}(A, A) & \cong & \mathsf{Fun}(B, A) \\
\Uparrow & & \Uparrow \\
\mathrm{id}_A & \longmapsto & \psi
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{Fun}(A, B) & \cong & \mathsf{Fun}(B, B) \\
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\phi & \longleftarrow\!\shortmid & \mathrm{id}_B
\end{array}
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to define functions $\psi\colon B \to A$ and $\phi\colon A \to B$.

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to define functions $\psi\colon B \to A$ and $\phi\colon A \to B$. In fact, they're inverses! $\square$

Exercise: Use the "naturality" of $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ to prove that $\psi \circ \phi = \mathrm{id}_A$ and $\phi \circ \psi = \mathrm{id}_B$.

Summary of Steps 1 and 2

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism

$$\mathrm{Fun}(A \times (B + C), X) \cong \mathrm{Fun}((A \times B) + (A \times C), X).$$

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Step 3: representability

The universal property of the disjoint union

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By "pairing"

$$\begin{array}{ccc} \mathsf{Fun}(B+C,X) & \cong & \mathsf{Fun}(B,X)\times\mathsf{Fun}(C,X) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \leftrightsquigarrow & (f_B,f_C) \end{array}$$

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A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b) \colon A \to X$ sending a to $f(a, b)$.

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A universal property of the cartesian product

Q: For sets A, B, and X, what is $\mathsf{Fun}(A \times B, X)$?

Q: What is needed to define a function $f \colon A \times B \to X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b) \colon A \to X$ sending a to $f(a, b)$. So, altogether we need to define a function $f \colon B \to \mathsf{Fun}(A, X)$.

By "currying"

$$\begin{array}{ccc} \mathsf{Fun}(A \times B, X) & \cong & \mathsf{Fun}(B, \mathsf{Fun}(A, X)) \\ \Cup & & \Cup \\ f \colon A \times B \to X & \leftrightsquigarrow & f \colon B \to \mathsf{Fun}(A, X) \end{array}$$

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Step 4: the proof

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Epilogue: what was the point of that?

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Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta\times\gamma} = (\alpha^{\gamma})^{\beta}.$$

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- For abelian groups A, B, C,
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Thank you!