# Algebraic model structures

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# Outline

1. Flavors of the theory of algebraic model structures

2. The monoidal algebraic model structure on **sSet**

# Weak factorization systems

Any model structure has two weak factorization systems (wfs):

- (cofibrations, trivial fibrations)
- (trivial cofibrations, fibrations)

The left and right classes satisfy factorization and lifting axioms.

## Examples

- (retracts of rel. cell complexes, trivial Serre fibrations) on **Top**
- (anodyne extensions, Kan fibrations) on **sSet**
- (injections with projective cokernel, surjections) on $A$-**mod**
- (monomorphisms, epimorphisms) on **Set**

# “Algebraic” perspective

## Thinking “algebraically”

to characterize maps or objects satisfying a certain *property*, assign to each one a particular *structure* that demonstrates the property.

## Examples

- a surjective map in **Set** or $A$-**mod** admits a (set-based) section
- a relative cell complex admits a cellular decomposition
- for any Kan complex, can choose fillers for all horns

## (Co)algebras for (co)monads

For all of these examples there is a monad or a comonad whose algebras or coalgebras have exactly this form.

# Can homotopy theory be made algebraic?

## Answer: yes!

Cofibrantly generated model categories admit algebraic model structures:

- a fibrant-replacement monad and a cofibrant-replacement comonad
- fibrations and trivial fibrations are algebras for a pair of monads
- cellular cofibrations and trivial cofibrations are coalgebras
- the (co)monads define the functorial factorizations
- the (co)algebra structures give explicit solutions to lifting problems

## Algebraic weak factorization systems

In place of wfs, we use algebraic weak factorization systems (awfs):

- factorization defines a monad and a comonad on the arrow category
- maps in the right class admit pre-algebra structures
- maps in the left class admit pre-coalgebra structures

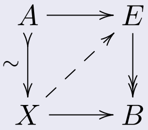
# A classical example: Hurewicz fibrations

## Consider the wfs (cofibrations and htpy equiv, fibrations) on **Top**

- A map $f\colon E \to B$ can be factored through the space of Moore paths

$$E \xrightarrow{\iota f} \Gamma_B E \xrightarrow{\pi f} B$$

- $f \mapsto \pi f$ is a monad on **Top**$/B$, or better: on $\mathbf{Top}^\mathbf{2}$
- pre-algebras are maps with path lifting functions; ie Hurewicz fibrations
- $f \mapsto \iota f$ is a comonad on $\mathbf{Top}^\mathbf{2}$; coalgebras are maps in the left class
- algebra and coalgebra structures can be used to construct lifts



# Cellularity: definition

### Cofibrantly generated awfs

An awfs is cofibrantly generated if there exists a set $\mathscr{I}$ of arrows such that the right class equals those maps that lift against $\mathscr{I}$.

### Baby example

In **Set**, $\mathscr{I} = \{\emptyset \to *\}$ generates (monomorphism, epimorphism).

### Lemma (R.)

In a cofibrantly generated awfs, all right maps admit algebra structures.

### Cellular maps

A map (in the left class) is cellular if it admits a coalgebra structure.

# Cellularity: examples

## Examples

| category | generators | cofibrations | cellular cofibrations |
|---|---|---|---|
| $A$-**mod** | $\{0 \to A\}$ | monos w/ projective cokernel | monos w/ free cokernel |
| **Top** | $\{S^{n-1} \to D^n\}$ | retracts of relative cell cxes | relative cell cxes |
| **sSet** | $\{\partial\Delta^n \to \Delta^n\}$ | monomorphisms | monomorphisms |
| **sSet** | $\{\Lambda^n_k \to \Delta^n\}$ | anodyne extensions | "anodyne cell cxes"? |

# Algebraic Quillen adjunctions by example

**Sample Theorem (R.)**

$| - | \colon \mathbf{sSet} \rightleftarrows \mathbf{Top} \colon S$ is an algebraic Quillen adjunction.

- all cofibrations in **sSet** are cellular, filtered by attaching stages
- images under $| - |$ not just cofibrations but cellular—here, relative cell complexes—with a specified algebraic structure—here, a cellular decomposition
- algebraic Serre fibrations are equipped with chosen lifted homotopies;

$$\begin{array}{ccccc} |\Lambda^{n+1}_k| \overset{\sim}{=\!=} D^n & \longrightarrow & X & & \Lambda^{n+1}_k & \longrightarrow & SX \\ \downarrow \qquad\quad \downarrow & \nearrow & \downarrow f & \mapsto & \downarrow & \nearrow & \downarrow Sf \\ |\Delta^{n+1}| \cong D^n \times I & \longrightarrow & Y & & \Delta^{n+1} & \longrightarrow & SY \end{array}$$

- images under $S$ are algebraic Kan fibrations with chosen horn fillers

# Existence of algebraic Quillen adjunctions

**In an algebraic Quillen adjunction**

the left adjoint lifts to commuting functors of coalgebras and the right adjoint lifts to commuting functors of algebras.

Modulo the usual acyclicity condition and a compatibility condition which is not the main point:

**Cellularity Theorem (R.)Cellularity & Uniqueness Theorem (R.)**

Suppose $\mathcal{M}$ has an algebraic model structure generated by $\mathscr{J}$ and $\mathscr{I}$, $\mathcal{K}$ has an algebraic model structure, and $F\colon \mathcal{M} \rightleftarrows \mathcal{K}\colon U$. Then $F \dashv U$ is an algebraic Quillen adjunction iff $F\mathscr{J}$ and $F\mathscr{I}$ are cellular. Furthermore, the coalgebra structures assigned to $F\mathscr{I}$ and $F\mathscr{J}$ determine everything.

**Corollary (R.)**

Whenever an algebraic model structure is lifted along an adjunction, the adjunction is canonically an algebraic Quillen adjunction.

# The monoidal model structure on **sSet**

The combinatorics necessary to prove theorems such as

**Theorem (Quillen?)**

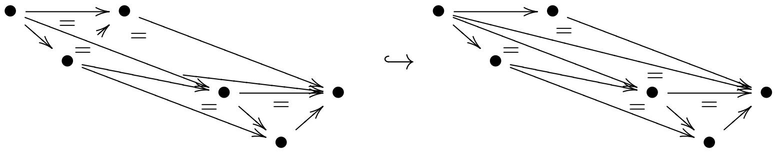
If $X$ is a Kan complex and $A$ is a simplicial set then $X^A$ is a Kan complex.

are encoded in the fact that **sSet** is a monoidal model category. Precisely:

**Equivalent Theorem (pushout-product axiom)**

The pushout-product of an anodyne extension with a monomorphism is an anodyne extension.

eg, $(\Lambda^2_1 \to \Delta^2)\hat{\times}(\partial\Delta^1 \to \Delta^1)$ is



# Two-variable adjunctions

The equivalence is because both theorems describe the interaction between wfs and a two-variable adjunction.

**Definition**

A two-variable adjunction consists of pointwise adjoint bifunctors

$$\mathcal{K} \times \mathcal{M} \xrightarrow{\times} \mathcal{N} \qquad \mathcal{K}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\hom_\ell} \mathcal{M} \qquad \mathcal{M}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\hom_r} \mathcal{K}$$

$$\mathcal{N}(k \times m, n) \cong \mathcal{M}(m, \hom_\ell(k,n)) \cong \mathcal{K}(k, \hom_r(m,n))$$

**Examples**

A closed monoidal structure $(\times, \hom_\ell, \hom_r) \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.
A tensored and cotensored enriched category $(\otimes, \{\}, \hom) \colon \mathcal{V} \times \mathcal{M} \to \mathcal{M}$.

# The monoidal model structure on **sSet**

To prove that **sSet** is a monoidal model category, suffices to show:

- $(\partial\Delta^n \to \Delta^n)\hat{\times}(\partial\Delta^m \to \Delta^m)$ is a cofibration
- $(\Lambda^n_k \to \Delta^n)\hat{\times}(\partial\Delta^m \to \Delta^m)$ is an anodyne extension
- $(\partial\Delta^m \to \Delta^m)\hat{\times}(\Lambda^n_k \to \Delta^n)$ is an anodyne extension

Analogously, though this was very hard to prove:

### Cellularity & Uniqueness Theorem (R.)

A cofibrantly generated algebraic model structure on a closed monoidal category is a monoidal algebraic model structure if and only if the pushout-products of the generating (trivial) cofibrations are cellular. The assignment of coalgebra structures to these maps completely determines the constituent algebraic Quillen two-variable adjunction.

### Corollaries (R.)

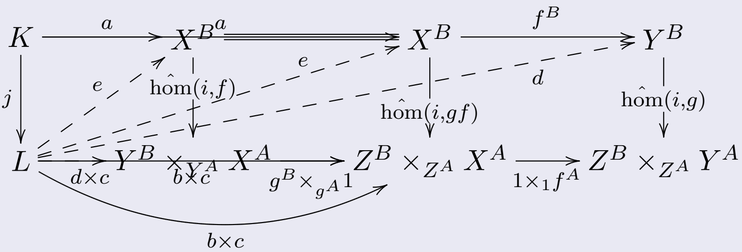
**sSet** and **Cat** are monoidal algebraic model categories

# A peek behind the curtain

The proof uses a composition criterion:

## Theorem (R.)

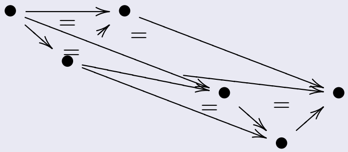
A lifted functor $\hat{\hom}(-,-)$ determines a two-variable adjunction of awfs iff, given a coalgebra $i$ and composable algebras $f, g$, the algebra $\hat{\hom}(i, gf)$ solves a lifting problem against a coalgebra $j$ as follows:




and also satisfies a dual condition in the first variable.

# A combinatorial tidbit

The monoidal algebraic model structure on **sSet** defines (a priori) two different "anodyne cell structures" for the pushout-product of two anodyne cell complexes—eg, $(\Lambda^2_1 \to \Delta^2)\hat{\times}(\Lambda^1_0 \to \Delta^1)$



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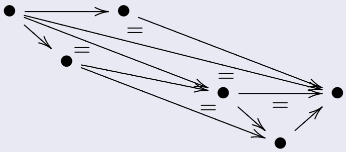

—and these *are* different:

- one fills the missing end triangle and then the "trough"
- the other fills in the top square and then the interior cylinder

Future work will explore the implications of these results for the theory of enriched model categories.

# Acknowledgments

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## For further details


- Riehl, E., Algebraic model structures, *New York J. Math* 17 (2011) 173-231.
- Riehl, E., Monoidal algebraic model structures, arXiv:1109.2883v1 [math.CT], or at `www.math.harvard.edu/~eriehl`