Algebraic model structures

Emily Riehl

University of Chicago

http://www.math.uchicago.edu/~eriehl

3 June, 2011

CMS Summer Meeting

University of Alberta

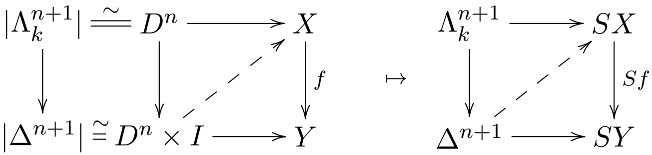
Outline

What is meant by "algebraic"?

Theorem (R.)

$| - | \colon$ **sSet** $\rightleftarrows$ **Top** $\colon S$ is an algebraic Quillen adjunction.

- all cofibrations in **sSet** are cellular, filtered by attaching stages
- images under $| - |$ not just cofibrations but cellular—here, relative cell complexes—with a specified algebraic structure—here, a cellular decomposition
- algebraic Serre fibrations are equipped with chosen lifted homotopies;



- images under S are algebraic Kan fibrations with chosen horn fillers

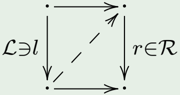
Weak factorization system perspective on model categories

Definition (Quillen)

A model structure on a homotopical category $(\mathcal{M}, \mathcal{W})$ consists of a class of cofibrations $\mathcal{C}$ and a class of fibrations $\mathcal{F}$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems (wfs).

Definition

A (functorial) weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$:

- $L, R : \mathcal{M}^{\mathbf{2}} \rightrightarrows \mathcal{M}^{\mathbf{2}}$ such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ and $f = Rf \cdot Lf$
- $\mathcal{L} \boxslash \mathcal{R}$: 

Furthermore $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$

Wfs perspective on model categories, continued

Definition

A (functorial) weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$:

- $L, R : \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$ such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ and $f = Rf \cdot Lf$
- $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$

$\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$ account for

- the extent to which the model category structure is overdetermined
- the closure properties of the classes of (trivial) cofibrations and (trivial) fibrations
- that Quillen adjunctions can be detected from the left or right adjoints alone

Furthermore, the small object argument produces functorial wfs; the model structure context is beside the point.

Algebraic model structures

Definition (Quillen)

A model structure on a homotopical category $(\mathcal{M}, \mathcal{W})$ consists of a class of cofibrations $\mathcal{C}$ and a class of fibrations $\mathcal{F}$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are wfs.

Definition (R.)

An algebraic model structure on a homotopical category $(\mathcal{M}, \mathcal{W})$ consists of a pair of algebraic weak factorization systems $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ on $\mathcal{M}$ together with a morphism

$$\xi \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$$

called the comparison map such that the underlying wfs of $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ give the trivial cofibrations, fibrations, cofibrations, and trivial fibrations of a model structure on $\mathcal{M}$, with weak equivalences $\mathcal{W}$.

Algebraic weak factorization systems

Definition (Grandis-Tholen)

An algebraic weak factorization system (awfs) $(\mathbb{L}, \mathbb{R})$ on $\mathcal{M}$ consists of a comonad $\mathbb{L}$ and a monad $\mathbb{R}$ on $\mathcal{M}^{\mathbf{2}}$ arising from a functorial factorization such that the canonical map $LR \Rightarrow RL$ is a distributive law.

Theorem (Garner)

If $\mathcal{M}$ permits the small object argument, then any small category $\mathcal{J}$ over $\mathcal{M}^{\mathbf{2}}$ generates an awfs $(\mathbb{L}, \mathbb{R})$ such that

- there is a functor $\mathcal{J} \to \mathbb{L}\text{-}\mathbf{coalg}$ over $\mathcal{M}^{\mathbf{2}}$ universal among morphisms of awfs
- $\mathbb{R}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$

Corollary (R.)

Any cofibrantly generated model category admits an algebraic model structure.

Algebraic model structures and cellularity

Definition (R.)

An algebraic model structure on $(\mathcal{M}, \mathcal{W})$ consists of a pair of algebraic weak factorization systems $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ on $\mathcal{M}$ and a morphism $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ such that the underlying wfs define a model structure.

Dictionary

- algebraic cofibration $\equiv$ $\mathbb{C}$-coalgebra; objects in $\mathbb{C}$-$\mathbf{coalg}$
- cellular cofibration $\equiv$ arrow in $\mathcal{M}$ admitting a $\mathbb{C}$-coalgebra structure
- cofibration $\equiv$ retract of a cellular cofibration
- algebraic fibration $\equiv$ $\mathbb{F}$-algebra; objects in $\mathbb{F}$-$\mathbf{alg}$

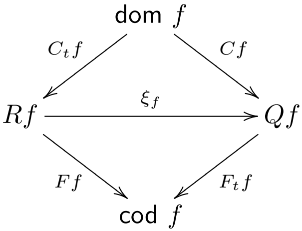
Lemma (R.)

In a cofibrantly generated algebraic model structure all fibrations and trivial fibrations admit algebra structures.

The comparison map

The comparison map $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$

- consists of natural arrows ξ_f satisfying



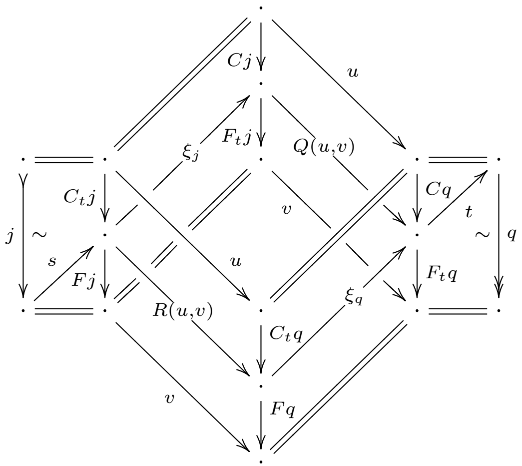
- induces functors

$$\xi_*\colon \mathbb{C}_t\text{-}\mathbf{coalg} \to \mathbb{C}\text{-}\mathbf{coalg} \quad \text{and} \quad \xi^*\colon \mathbb{F}_t\text{-}\mathbf{alg} \to \mathbb{F}\text{-}\mathbf{alg},$$

which map algebraic trivial cofibrations (algebraic trivial fibrations) to algebraic cofibrations (algebraic fibrations).

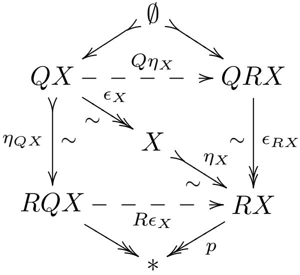
Naturality of the comparison map

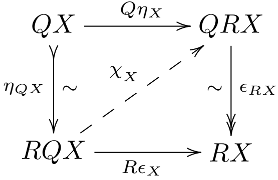
Both ways of lifting an algebraic trivial cofibration $(j, s) \in \mathbb{C}_t$-**coalg** against an algebraic trivial fibration $(q, t) \in \mathbb{F}_t$-**alg** are the same!



Algebraic fibrant-cofibrant objects

Any algebraic model structure induces a fibrant replacement monad $\mathbb{R}$ and a cofibrant replacement comonad $\mathbb{Q}$ on $\mathcal{M}$ together with $\chi : RQ \Rightarrow QR$.





Theorem (R.)

The comonad Q lifts to $\mathbb{R}$-$\mathbf{alg}$ the category of algebraic fibrant objects and the monad R lifts to $\mathbb{Q}$-$\mathbf{coalg}$. Their (co)algebras are isomorphic and define a category of algebraic bifibrant objects.

Passing algebraic model structures across an adjunction

Many ordinary model structures are constructed by lifting a cofibrantly generated model structure along an adjunction. In this setting, much more is true algebraically:

Theorem (R.)

Let $\mathcal{M}$ have an algebraic model structure generated by $\mathcal{J}$ and $\mathcal{I}$, let $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ be an adjunction, and suppose $\mathcal{K}$ permits the small object argument. If

($\star$) S maps $T\mathcal{J}$-cellular arrows into weak equivalences

then $T\mathcal{J}$ and $T\mathcal{I}$ generate an algebraic model structure on $\mathcal{K}$ with weak equivalences created by S.

Theorem (R.)

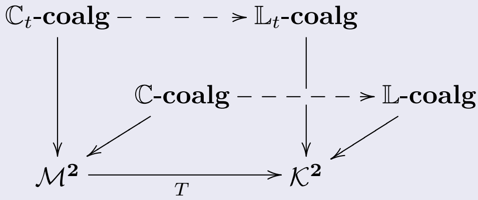
Furthermore, this adjunction is canonically an algebraic Quillen adjunction.

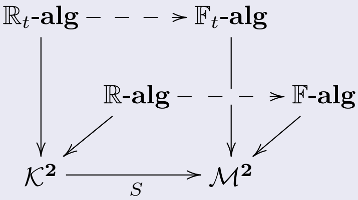
Algebraicizing Quillen adjunctions

For ordinary Quillen adjunctions $T\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathcal{K}\colon S$

$$T \text{ preserves trivial cofibrations } \leftrightsquigarrow S \text{ preserves fibrations}$$
$$T \text{ preserves cofibrations } \leftrightsquigarrow S \text{ preserves trivial fibrations}$$

For algebraic Quillen adjunctions





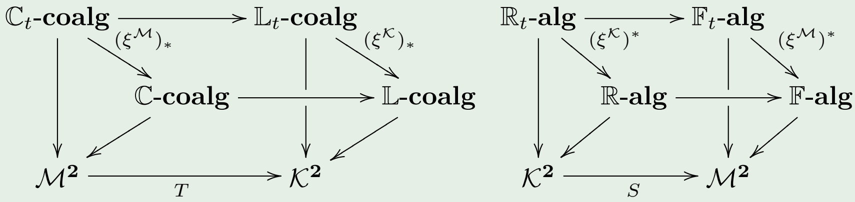
and the lifts determine each other

Algebraic Quillen adjunctions

Let $\mathcal{M}$ have an algebraic model structure $\xi^{\mathcal{M}} \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ and let $\mathcal{K}$ have an algebraic model structure $\xi^{\mathcal{K}} \colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t)$.

Definition (R.)

An adjunction $T \colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{K} \colon S$ is an algebraic Quillen adjunction if



such that

- the characterizing natural transformations are mates
- each lifted functor preserves canonical composition of (co)algebras

Adjunctions of awfs

Cellularity of cofibrations plays an essential role of identifying algebraic Quillen adjunctions in practice.

An algebraic Quillen adjunction is determined by two adjunctions of awfs

$$(T, S) \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{L}_t, \mathbb{R}) \leftrightsquigarrow \mathbb{C}_t\textbf{-coalg} \to \mathbb{L}_t\textbf{-coalg}, \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg}$$

$$(T, S) \colon (\mathbb{C}, \mathbb{F}_t) \to (\mathbb{L}, \mathbb{R}_t) \leftrightsquigarrow \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg}, \mathbb{R}_t\textbf{-alg} \to \mathbb{F}_t\textbf{-alg}$$

Cellularity & Uniqueness Theorem (R.)

Suppose $T \colon \mathcal{M} \rightleftarrows \mathcal{K} \colon S$, $\mathcal{J}$ generates an awfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$, and $(\mathbb{L}, \mathbb{R})$ is an awfs on $\mathcal{K}$. There is an adjunction of awfs $(T, S) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ iff $T\mathcal{J}$ is cellular. Furthermore, an assignment of coalgebra structures to $T\mathcal{J}$ completely determines the adjunction of awfs.

Monoidal algebraic model structures

New work introduces monoidal algebraic model structures, defining algebraic Quillen two-variable adjunctions. This is much harder, but . . .

Cellularity & Uniqueness Theorem (R.)

A cofibrantly generated algebraic model structure on a closed monoidal category is a monoidal algebraic model structure if and only if the pushout-products of the generating (trivial) cofibrations are cellular. Furthermore, an assignment of coalgebra structures to these maps completely determines the constituent algebraic Quillen two-variable adjunction.

The theory of enriched algebraic model structures is similar.

Acknowledgments

Thanks

Thanks to the organizers, Richard Garner, Martin Hyland, Peter May, Mike Shulman, and the members of the category theory seminars at Chicago, Macquarie, and Sheffield.

Further details

Further details can be found in

- Riehl, E., Algebraic model structures, *New York J. Math* 17 (2011) 173-231.
- a preprint "Monoidal algebraic model structures" available at `www.math.uchicago.edu/~eriehl`