In these notes, we will often be working in the topos of sheaves (of sets) on a topological space X. Therefore, we recall the forcing conditions for propositions in the internal logic in $\mathbf{Sh}(X)$.

We read

$$U \vDash \varphi$$

as "φ holds on U" for U an open subset of X. Then, for $\mathcal{F}$ a sheaf on X,

$$\begin{aligned}
U \vDash (f = g) : \mathcal{F} &\iff f|_U = g|_U \in \mathcal{F}(U) \\
U \vDash \varphi \wedge \psi &\iff U \vDash \varphi \text{ and } U \vDash \psi \\
U \vDash \varphi \vee \psi &\iff \text{there is an open cover } U = \textstyle\bigcup_i U_i \text{ s.t. for all } i,\ U_i \vDash \varphi \text{ or } U_i \vDash \psi \\
U \vDash \varphi \Rightarrow \psi &\iff \text{for all open } V \subseteq U,\ V \vDash \varphi \text{ implies } V \vDash \psi \\
U \vDash \forall f : \mathcal{F}.\, \varphi(f) &\iff \text{for all open } V \subseteq U \text{ and sections } f \in \mathcal{F}(V),\ V \vDash \varphi(f) \\
U \vDash \exists f : \mathcal{F}.\, \varphi(f) &\iff \text{there is an open cover } U = \textstyle\bigcup_i U_i \text{ s.t. for all } i, \text{ there is a } f_i \in \mathcal{F}(U_i) \text{ with } U_i \vDash \varphi(f_i)
\end{aligned}$$

There are two main properties of the internal logic that will be important for us:

- **Locality:** Suppose $U = \bigcup_i U_i$ is an open cover. Then $U \vDash \varphi$ if and only if for all i, $U_i \vDash \varphi$.
- **Soundness:** Suppose that ψ may be proved from φ by an *intuitionistic*[1] argument, then if $U \vDash \varphi$, it follows that $U \vDash \psi$.

1 A Sheaf of Rings is a Ring of Sheaves, and other Tales from Algebra on the Inside

In this section, we will gently introduce ourselves to the internal logic by looking at sheaves of rings and modules from the internal point of view. We will find that internally, a sheaf of rings is a ring, and a sheaf of modules is a module. We will then see how some common properties of modules look internally.

Definition 1. A sheaf of rings is a sheaf (of sets) $\mathcal{F}$ on a space X such that for each open $U \subseteq X$, $\mathcal{F}(U)$ has the structure of a ring and for each open $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a ring homomorphism.

Lemma. In the internal logic of $\mathbf{Sh}(X)$, a sheaf of rings is precisely a ring. Furthermore, a map of sheaves is a homomorphism of sheaves of rings if and only if it is a homomorphism internally.

Definition 2. Let $\mathcal{O}_X$ be a sheaf of rings on X. A sheaf of $\mathcal{O}_X$-modules is a sheaf (of sets) $\mathcal{M}$ such that for each open $U \subseteq X$, $\mathcal{M}(U)$ is a $\mathcal{O}_X(U)$-module, and for each $V \subseteq U$, the restriction maps $\mathcal{M}(U) \to \mathcal{M}(V)$ are homomorphisms in the sense that thay commute with addition and the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(U) \times \mathcal{M}(U) & \longrightarrow & \mathcal{M}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(V) \times \mathcal{M}(V) & \longrightarrow & \mathcal{M}(V)
\end{array}$$

Lemma. In the internal logic of $\mathbf{Sh}(X)$, a sheaf of $\mathcal{O}_X$-modules is just a module of the ring $\mathcal{O}_X$. Furthermore, a map of sheaves is a homomorphism of $\mathcal{O}_X$-modules if and only if internally it is a homomorphism of modules.

Example 1. Let M be a smooth manifold. Then $\mathcal{C}^\infty$ is a sheaf of rings on M, sending each open $U \subseteq M$ to the ring $\mathcal{C}^\infty(U)$ of real-valued smooth functions on U.

Let $\mathcal{T}^*$ be the functor sending an open $U \subseteq M$ to the set of 1-forms $\mathcal{T}^*(U)$ on U (local sections of the cotangent bundle of M over U). Then $\mathcal{T}^*$ is and $\mathcal{C}^\infty$-module.

So, internally, $\mathcal{C}^\infty$ is a ring and $\mathcal{T}^*$ is a module of that ring. Furthermore, it can be shown that $\mathcal{T}^*$ is, internally, $\mathcal{C}^\infty \otimes_{\mathbb{R}} \Omega_{\mathcal{C}^\infty/\mathbb{R}}$ where $\Omega_{\mathcal{C}^\infty/\mathbb{R}}$ is the K ahler module of derivations of $\mathcal{C}^\infty$ over the sheaf of locally constant real functions $\mathbb{R}$ (which is, internally, the set of real numbers as a quotient of Cauchy sequences).

[1] This means, effectively, not using the Law of Excluded Middle, or the Axiom of Choice, or a host of other (weaker) "Choice Principles" or "Omniscience Principles".

Example 2. Let X be a scheme over a base scheme S, and let $\mathcal{O}_X$ be its structure sheaf. Then $\mathcal{O}_X$ is a sheaf of rings.

Lemma (Principle of Unique Choice)**.** Let $\varphi : A \times B \to \Omega$ be a proposition concerning $a : A$ and $b : B$. Suppose that $X \vDash \forall a : A. \exists! b : B. \varphi(a, b)$. Then there is exactly one morphism $\epsilon : A \to B$ of sheaves such that $X \vDash \forall a : A. \varphi(a, \epsilon(a))$.

Lemma (Unique Existence implies Global Existence)**.** Suppose that $X \vDash \exists! a : A. \varphi(a)$. Then for any open $U \subseteq X$, there is exactly one $a \in A(U)$ such that $U \vDash \varphi(a)$.

Corollary 1. Let $f : A \to B$ be a morphism of sheaves. Then f is an isomorphism if and only if $X \vDash \forall a : A. \exists! b : B. f(a) = b$.

Remark 1. Note that we have assumed $f : A \to B$ exists *globally* in the above propositition. If instead we asked for A and B to be isomorphic in the internal logic (i.e. $X \vDash \exists f : A \cong B$), this would only show that there is an open cover of X such that A and B are isomorphic on each open of this cover – that is, that they are *locally* isomorphic.

Theorem 2. Let $\mathcal{O}_X$ be a sheaf of rings. Then the category $\mathbf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules is an abelian category.

1. It has a zero object and biproducts.
2. Every map has a kernel and a cokernel.
3. For every $f : M \to N$, the induced map $\mathbf{coim}(f) \to \mathbf{im}(f)$ is an isomorphism.

Proof. Proved in exactly the way that one proves that the category of modules of a ring is an abelian category, in the internal logic. The universal properties mean the same thing because we can test that a map is an isomorphism internally.

One can also prove Grothendieck's other axioms the same way. □

Theorem 3 (B4.1)**.** Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{M}$ is locally finitely free if and only if it is finitely free from the internal perspective, that is

$$X \vDash \exists n : \mathbb{N}\,. \exists g_1, \dots, g_n : \mathcal{M}. \forall m : \mathcal{M}. \exists! c_1, \dots, c_n : \mathcal{O}_X\,. m = \sum_i c_i g_i.$$

Proof. Internally, begin finitely free is equivalent to $\exists n : \mathbb{N}\,. \mathcal{M} \cong \mathcal{O}_X^n$. The forcing condition shows that this means that there exists an open cover $X = \bigcup_i U_i$ for which $\mathcal{M}(U_i)$ is finitely free for all i. □

Using the internal logic, we can now quickly and easily prove these exercises from Vakil's *Rising Sea*.

Lemma. We solve the following exercises from Vakil's *Rising Sea*.

- 13.1.B : Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of rank m and n respectively. Show that $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf of rank nm.
- 13.1.F : If $\mathcal{E}$ is a locally free sheaf of finite rank, and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, show that $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \cong \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G})$.
- 13.7.B : Suppose that $\mathcal{F}$ is a finite rank locally free sheaf and $\mathcal{G}$ is a $\mathcal{O}_X$-module. Give an isomorphism $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{F}^\vee \otimes \mathcal{G}$.

Proof. Exactly as it would be for finite rank free modules. □

Theorem 4 (B4.3)**.** Recall the following definition for a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules:

1. $\mathcal{M}$ is of *finite type* if and only if there is an open cover $X = \bigcup_i U_i$ and an exact sequence

   $$(\mathcal{O}_X|_{U_i})^n \to \mathcal{M}_{U_i} \to 0$$

   for each i.

2. $\mathcal{M}$ is of *finite presentation* if and only if there is an open cover $X = \bigcup_i U_i$ and an exact sequence

$$(\mathcal{O}_X|_{U_i})^m \to (\mathcal{O}_X|_{U_i})^n \to \mathcal{M}_{U_i} \to 0$$

for each i.

3. $\mathcal{M}$ is *coherent* if it is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{M}|_U$ is of finite type, where $U \subset X$ is any open.

Then

1. $\mathcal{M}$ is of finite type if and only if $\mathcal{M}$ is, internally, finitely generated, i.e.

$$X \vDash \exists n : \mathbb{N}\,.\, \exists g_1, \ldots, g_n.\, \forall m : \mathcal{M}.\, \exists c_1, \ldots, c_n : \mathcal{O}_X\,.\, m = \sum_i c_i g_i.$$

2. $\mathcal{M}$ is of finite presentation if and only if $\mathcal{M}$ is, internally, finitely presentable. i.e.

$$X \vDash \exists n, m : \mathbb{N}\,.\, \text{there is a short exact sequence } \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{M} \to 0.$$

3. $\mathcal{M}$ is coherent if and only if $\mathcal{M}$ is coherent in the internal sense, that is

$$X \vDash \mathcal{M} \text{ is finitely generated} \wedge \forall n : \mathbb{N}\,.\, \forall \varphi : \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}).\, \mathsf{ker}\varphi \text{ is finitely generated.}$$

Effectively all of the exercises in Vakil's Section 13.8 can now be proved internally.

2 At a stalk, in general, and other ways of being true

In the last talk, we learned a bit about algebra on the inside. We saw that from the internal point of view, sheaves of rings and sheaves of modules look like rings and modules, and their basic theory can be developed the same way.

But sheaves of rings and sheaves of modules *aren't* rings and modules – they have an extra local structure that modules don't. How can we reconcile this in the internal point of view.

Or, more generally, how can we know when a statement that holds in the internal logic of a topos descends to sheaves in that topos?

In order to deal with issues of locality in the internal logic, we will need to add a new logical feature to our language: a *modality*.

A modality is a "way of being true" in a logic. It consists of a modal operator $\Box$ which operates on propositions. Then a proposition φ is true in the $\Box$-way if $\Box\varphi$ is true.

Definition 3. A *modal operator* (or Lawvere-Tierney topology) is a map $\Box : \Omega \to \Omega$ such that for all φ, $\psi : \Omega$,

1. $\varphi \Rightarrow \Box\varphi$.
2. $\Box\Box\varphi \Rightarrow \Box\varphi$,
3. $\Box(\varphi \wedge \psi) \Rightarrow \Box\varphi \wedge \Box\psi$.

We denote by $\mathbf{Sh}_\Box(X)$ the topos of sheaves associated to the modal operator $\Box$. We done by $a_\Box$ the associated sheaf functor.

Here are a few examples of modalities in a topos of sheaves on a topological space X. If $U \subseteq X$ is open, then we also denote by U the sheaf represented by U – the skyscraper sheaf. Note that $U \vDash V$ if and only if $U \subseteq V$. For $x \in X$, we let $!\,x \subseteq X$ denote the interior of the complement of $\{x\}$.

Name	$\Box\varphi$	$X \vDash \Box\varphi$	$\mathbf{Sh}_\Box(X)$	$a_\Box$
Open_U (for open $U \subseteq X$)	$U \Rightarrow \varphi$	$U \vDash \varphi$	$\mathbf{Sh}(U)$	Restriction
Closed_A (for closed $A \subseteq X$)	$A^c \vee \varphi$	$\exists$ open $U \supseteq A$. $U \vDash \varphi$	$\mathbf{Sh}(A)$	Restriction
Generally	$\neg\neg\varphi$	$\exists$ a dense open U. $U \vDash \varphi$	$\mathbf{Sh}_{\neg\neg}(X)$	$\cdots$
at_x (for $x \in X$)	$(\varphi \Rightarrow {!\,x}) \Rightarrow {!\,x}$	$\exists$ open $U \ni x$. $U \vDash \varphi$	**Set**	stalk_x

The open and closed modalities associated to open and closed sets in X give us a way of talking about properties that are local to these sets. The last two are more interesting: they let us talk about what is happening "in general" and what is happening "at a point".

To justify the name of "generally" for the $\neg\neg$-modality, we have the following lemma.

Lemma (B6.16)**.** Let X be a topological space and $\xi \in X$ be a *generic point* in the sense that the closure of $\{\xi\}$ is all of X. Then the modal operator at_ξ equals $\neg\neg$ – a proposition holds at a generic point if and only if it holds generally.

Proof. By definition, $U \vDash {!\,\xi}$ if and only if $\xi \notin U$. But since ξ is generic in X, X is the intersection of all closed sets containing ξ or equivalently the complement of the union of all open sets *not* containing ξ. So if $\xi \in U$, then $U = \emptyset$.

Therefore,

$$\mathrm{at}_\xi\varphi := (\varphi \Rightarrow {!\,\xi}) \Rightarrow {!\,\xi} \equiv (\varphi \Rightarrow \emptyset) \Rightarrow \emptyset =: \neg\neg\varphi.$$

$\Box$

That $X \vDash \neg\neg\varphi$ means that $U \vDash \varphi$ for some dense open set (or that φ holds at some generic point) gives a topological interpretation to what is otherwise a quirk of intuitionistic logic. One might wonder whether one can use classical reasoning to prove a statement generally (if not everywhere). We can in fact use the law of excluded middle to prove a double negated statement intuitionistically, since

$$(\varphi \Rightarrow \neg\neg\psi) \iff (\neg\neg\varphi \Rightarrow \neg\neg\psi)$$

we have that, in particular,

$$(\varphi \vee \neg\varphi \Rightarrow \neg\neg\psi) \iff (\neg\neg(\varphi \vee \neg\varphi) \Rightarrow \neg\neg\psi) \iff \neg\neg\psi$$

since $\neg\neg(\varphi \vee \neg\varphi)$ is an intuitionistic tautology. Therefore, one may assume $\varphi \vee \neg\varphi$ when trying to prove a double negated statement, and so (more topologically) one may analyze by cases when trying to show that a statement holds generally.

In 1933, G odel gave a "double negation translation" of classical logic into intuitionistic logic. Namely, from a formula φ one produces a formula $\varphi^{\neg\neg}$ such that $\varphi^{\neg\neg}$ is intuitionistically provable if and only if φ is classically provable. We can use this same translation for any modality.

Definition 4. Let $\Box$ be a modality. We define the $\Box$-translation $\varphi^\Box$ of a proposition φ to be the result of putting $\Box$ outside of every connective and atomic formula in φ.

For example, $(\exists n : \mathbb{N}\,.\forall f : \mathcal{O}_X\,.\, f \text{ is invertible} \vee 1 - f \text{ is invertible})^\Box$ is

$$\Box\exists n : \mathbb{N}\,.\Box\forall f : \mathcal{O}_X\,.\Box(\Box\exists g.\,\Box(fg = 1) \vee \Box\exists g.\Box((1-f)g = 1).$$

This can be made somewhat simpler – one doesn't need to put $\Box$ in front of $\top$, $\wedge$, and $\forall$ – but we won't go into it in detail.

Lemma (B6.23)**.** If φ implies ψ intuitionistically, then $\varphi^\Box$ implies $\psi^\Box$ intuitionistically.

The reason for introducing the $\Box$-translation is that it gives a way of talking about $\mathbf{Sh}_\Box(X)$ inside of $\mathbf{Sh}(X)$ thanks to the following nice theorem due to Ingo Blechschmidt.

Theorem 5 (B6.31)**.** Let X be a topological space and $\Box$ be a modal operator in $\mathbf{Sh}(X)$. Then for any formulat φ in $\mathbf{Sh}(X)$,

$$\mathbf{Sh}_\Box(X) \vDash \varphi \iff \mathbf{Sh}(X) \vDash \varphi^\Box$$

where on the left, the formula is pulled back to $\mathbf{Sh}_\Box(X)$.

Corollary 6. Let $x \in X$. Then $X \vDash \varphi^{\mathrm{at}_x}$ iff φ holds at x.

This is nice, but the $\Box$-translations are somewhat complicated. We need a tool for tackling them. To handle this, we introduce the notion of *geometric logic*.

Geometric logic is the sort of logic that is well preserved by the inverse image functor of a geometric morphism.

Definition 5. A formula is *geometric* if it only uses

$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists$$

but not $\Rightarrow$, $\forall$, or $\bigwedge$ (and in particular not $\neg$, which is defined as $\Rightarrow \bot$). A *geometric sequent* is a statement of the form

$$\forall \cdots \forall \varphi \Rightarrow \psi$$

where φ and ψ are geometric formulas.

Lemma (B6.26)**.** Let φ be a geometric formula and $\Box$ a modality. Then $\varphi^{\Box} \iff \Box\varphi$.

Corollary 7 (B2.10)**.** Let $x \in X$ be a point and φ a geometric formula. Then φ holds at x if and only if there is some open neighborhood U of x such that φ holds on U.

Proof. Since φ is geometric, $\varphi^{\Box}$ is equivalent to $\Box\varphi$. But $X \vDash \varphi^{\Box}$ iff φ holds at x and $X \vDash \Box\varphi$ iff φ holds on a neighborhood of x. $\Box$

Example 3. We solve Vakil's exercise 13.7.F.

Suppose that $\mathcal{F}$ is a finitely presented sheaf on a scheme X. Show that if $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module for some $p \in X$, then $\mathcal{F}$ is locally free on an open neighborhood of p. In particular, a finitely presented sheaf is locally free if and only if all of its stalks are free.

Proof. As we noted previously, $\mathcal{F}$ is a finitely presented sheaf if and only if it is a finitely presented module internally. Since a finitely presented module is free if and only if all the m generating relations R_i are 0, being free is a finite conjunction of equalities and is therefore a geometric formula. Therefore, if it holds at p, it holds in a neighborhood of p. $\Box$

Corollary 8. A geometric sequent holds on X if and only if it holds at every point of X.

Definition 6. A ring is *local* if it is non-trivial ($1 \neq 0$) and for all x, $y : R$, if $x + y$ is invertible then either x or y is invertible.

Lemma (B3.5)**.** In the internal language of $\mathbf{Sh}(X)$ for X a scheme, $\mathcal{O}_X$ is a local ring.

Proof. Being local is a finite conjuction of geometric sequents. Therefore, it holds everywhere if and only if it holds at each point. But $\mathcal{O}_{X,p}$ is local for all $p \in X$, so $\mathcal{O}_X$ is local in the internal logic. $\Box$

Lemma (B3.2)**.** For any sheaf of rings $\mathcal{O}_X$ on a topological space X and global section $f \in \Gamma(X, \mathcal{O}_X)$, the following are equivalent:

1. f is invertible internally.
2. f is invertible in all stalks.
3. f is invertible in $\Gamma(X, \mathcal{O}_X)$.

Proof. Being invertible is a geometric formula, so the first two conditions are equivalent. The third statement also implies the other two. But note that inverses are unique, and since unique existence implies global existence, the first statement implies the third. $\Box$

Definition 7. A ring R is *reduced* if $\forall r : R.(\exists n : \mathbb{N}.s^n = 0) \Rightarrow s = 0$.

Lemma (B3.3)**.** A scheme X is reduced iff $\mathcal{O}_X$ is reduced internally.

Proof. By definition, a scheme is reduced if each stalk $\mathcal{O}_{X,p}$ is reduced. But being reduced is a geometric sequent. $\square$

Lemma (B3.7)**.** Let X be a scheme. Then

$$X \vDash \forall f : \mathcal{O}_X.\, \neg(f \text{ is invertible}) \Rightarrow f \text{ is nilpotent.}$$

Proof. By locality of the internal logic and X is covered by affine schemes, it suffices to show that if $X \vDash \neg(f \text{ is invertible})$ then $X \vDash f$ is nilpotent whenever $X = \mathrm{Spec}(A)$ is affine.

The antecedent means that any open subset on which f is invertible is empty; in particular $D(f)$ is empty. Therefore, f is in every prime ideal of A and is therefore nilpotent externally and therefore internally. $\square$

Corollary 9 (B3.9)**.** Let X be a reduced scheme. Then, internally, $\mathcal{O}_X$ is a "null field" in the sense that for all $f : \mathcal{O}_X$, if f is not invertible, then $f = 0$.

With this, we can prove Vakil's "Important Hard Exercise" 13.7.K.

Example 4 (B5.8)**.** Let X be a reduced scheme and $\mathcal{F}$ be a finite type sheaf on X. Then $\mathcal{F}$ is locally finitely free on a dense open subset.[2]

Proof. Internally, we are trying to show that $\mathcal{F}$ is *not not* free. Let $g_1, \ldots, g_n$ be a generating family of $\mathcal{F}$. If $n = 0$, we're done; so we proceed by induction. As mentioned earlier, we are free to do case analysis when trying to prove a double negated statement, so either some g_i is a linear combination of the rest, or not. If not, then $\mathcal{F}$ is *not not* free. Otherwise, there is some g_i which is a linear combination of the rest, whence $g_1, \ldots, g_{i-1}, g_{i+1}, \ldots g_n$ is a smaller generating family and so by induction $\mathcal{F}$ is *not not* free. $\square$

3 A little bit of the big picture

In this talk, we'll move from the little Zariski topos to the big Zariski topos. The big Zariski topos is the setting of the "functor of points" point of view. All rings are commutative.

We will work in the "category theorist's" big Zariski topos of $\mathbf{Spec}(\mathbb{Z})$, sheaves on the opposite of the category of finitely presented rings equipped with the Zariski coverage. For a more detailed look into the different big Zariski toposes, see Blechschmidt's discussion in Chapter 15 of his thesis. What we say here is true for the big Zariski topos over any scheme.

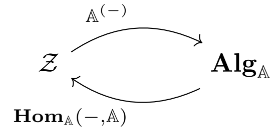
In the following, we will let $\mathcal{Z}$ denote the big Zariski topos of $\mathbf{Spec}(\mathbb{Z})$.

Definition 8. Let $\mathbb{A}$ be the forgetful functor from finitely presented rings to sets. We call $\mathbb{A}$ the *affine line*.

Note that since $\mathbb{A}(R) = R$ is a ring for each finitely presented ring R, and that $\mathbb{A}(R) \to \mathbb{A}(S)$ is a ring homomorphism for each ring homomorphism $R \to S$, $\mathbb{A}$ is a ring in $\mathcal{Z}$.

Definition 9. Let $\mathbf{Alg}_{\mathbb{A}}$ denote the category of $\mathbb{A}$-algebras – that is, commutative rings in $\mathcal{Z}$ equipped with a homomorphism from $\mathbb{A}$.

Consider the following contravariant adjunction:



One functor above sends $X \in \mathcal{Z}$ to the $\mathbb{A}$-algebra of function $X \to \mathbb{A}$. The other sends an $\mathbb{A}$-algebra R to the sheaf $\mathbf{Hom}_{\mathbb{A}}(R, \mathbb{A})$ of all $\mathbb{A}$-algebra homomorphisms from R into $\mathbb{A}$. We interpret these functors as sending a space to its algebra of functions and sending an algebra to the space of solutions to its defining equations.

[2] This is actually just the second part of the exercise; we can prove the first part using these techniques as well, but it requires a detour into the "upper naturals", the internal Dedekind-Yoneda completion of the natural numbers and externally the sheaf of upper semi-continous functions valued in the naturals.

Consider, for example, the case of $R = \mathbb{A}[x_1, \ldots, x_n]$ (which may be constructed as the functor $A \mapsto A[x_1, \ldots, x_n]$). Then by its universal property, we have $\mathbf{Hom}_{\mathbb{A}}(R, \mathbb{A}) \cong \mathbb{A}^n$. If we let R be $\mathbb{A}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, then

$$\mathbf{Hom}_{\mathbb{A}}(R, \mathbb{A}) \cong \{a : \mathbb{A}^n \mid \forall i.\, f_i(a) = 0\}$$

is the vanishing locus of the polynomials f_i. With this in mind, we will call the functor $\mathbf{Hom}_{\mathbb{A}}(-, \mathbb{A})$ the *internal spectrum* **Spec**, since it recovers a space from its algebra of functions.

Speaking of which, we should check that $\mathbb{A}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is indeed the algebra of functions on the vanishing locus.

Theorem 10 (in B18.9)**.** Let R be a finitely presented $\mathbb{A}$-algebra. Then the canonical map

$$R \to \mathbb{A}^{\mathbf{Spec}(R)}$$

given by $r \mapsto \phi \mapsto \phi(r)$ is an isomorphism.

This wonderful theorem has many magical consequences: it implies that $\mathbb{A}$ is (in some sense) an algebraically closed field with a plenitude of nilpotents.

Corollary 11. $\mathbb{A}$ is a field in the sense that $a : \mathbb{A}$ is invertible if and only if $a \neq 0$.

Proof. Let $a : \mathbb{A}$ and suppose that $a \neq 0$. Then $\mathbb{A}/(a)$ is a finitely presented $\mathbb{A}$-algebra and so

$$\mathbb{A}/(a) \cong \mathbb{A}^{\mathbf{Spec}(\mathbb{A}/(a))}.$$

But $\mathbf{Spec}(\mathbb{A}/(a)) := \mathbf{Hom}_{\mathbb{A}}(\mathbb{A}/(a), \mathbb{A})$ is empty, because there is no homomorphism $\mathbb{A}/(a) \to \mathbb{A}$ since there is just one homomorphism $\mathbb{A} \to \mathbb{A}$ (the identity) and so for this to descend to the quotient it must send a to 0. This can't happen because $a \neq 0$.

So $\mathsf{Spec}(\mathbb{A}/(a)) = \emptyset$ and accordingly $\mathbb{A}/(a)$ is the zero ring. But then $1 \in (a)$, so that a is a unit. □

Corollary 12. Let $\sqrt{(0)}$ denote the nil-radical (ideal of nilpotents) of $\mathbb{A}$. Then

$$\sqrt{(0)} = \{a : \mathbb{A} \mid \neg(a \neq 0)\}$$

is the (internal) set of elements which are *not not* 0 in $\mathbb{A}$.

Proof. If $a \in \sqrt{(0)}$, then there is an $n : \mathbb{N}$ with $a^n = 0$. But if $a \neq 0$, then a is invertible and so a^n is also invertible and so non-zero, a contradiction. Thus, a is *not not* 0.

On the other hand, if it is not the case that $a \neq 0$, then consider $\mathbb{A}[a^{-1}] := \mathbb{A}[x]/(ax - 1)$. This is a finitely presented $\mathbb{A}$-algebra, and so

$$\mathbb{A}[a^{-1}] \cong \mathbb{A}^{\mathsf{Spec}(\mathbb{A}[a^{-1}])}.$$

But $\mathsf{Spec}(\mathbb{A}[a^{-1}]) := \mathbf{Hom}_{\mathbb{A}}(\mathbb{A}[a^{-1}], \mathbb{A})$ is the set of inverses to a; since a is not non-zero, this set is empty. Thus, $\mathbb{A}[a^{-1}] = 0$, and so by a standard argument of commutative algebra, a is nilpotent in $\mathbb{A}$. □

Corollary 13. For any $f : \mathbb{A}[x]$ which is monic of degree greater than 0, the set of roots of f is non-empty.

Proof. Note that $\mathsf{Spec}(\mathbb{A}[x]/(f)) := \mathbf{Hom}_{\mathbb{A}}(\mathbb{A}[x]/(f), \mathbb{A})$ is the set of roots of f. So, if f has no roots, then $\mathsf{Spec}(\mathbb{A}[x]/(f))$ is empty and so $\mathbb{A}[x]/(f)$ is the zero ring and f is nilpotent in $\mathbb{A}[x]$. But if f were nilpotent, then its leading coefficient would be nilpotent – contradicting our assumption that f was monic. So the set of roots is not empty. □

Note that this last corollary does not say that every monic polynomial of degree greater than 0 *has* a root, just that it *doesn't have no roots*.