Overview of
∞-Operads as Analytic Monads

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Operads have been around for quite some time, back to May in the 70s Originally meant to capture some of the computational complexities of different associative and unital algebraic operations, Operads have found a home in a multitude of algebraic contexts, such as commutative ring theory, lie theory etc. Classicaly, Operads are defined as a sequence of O(n) of "n-ary" operations, sets which are meant to act on some category. These Operads are symmetric sequences, i.e. Functors $O \in \mathfrak{C}^{Fin}$ where Fin is the disconnected category of finite sets and bijections. Any symmetric sequence gives rise to an endofunctor

$$T(X) = \coprod (O(n) \times_{\Sigma_n} X^n)$$

which the authors call an "analytic functor" due to it's resemblance to a power series.

An operad is a symmetric sequence with a notion of composition (of the n-ary operations) which is associative and unital. When O is an operad, the associated endofunctor takes the form of a monad, a special endofunctor with a unit and a multiplication.

In both settings, we can consider algebras over operads, defined as maps $O(n) \times A^n \to A$ and algebras over a monad, maps $T(A) \to A$. In the case where the target category $\mathfrak{C} = \mathfrak{S}et$ the algebras of an operad and it's associated monad can be recovered from one another

However, when we are working with space, this equivalence fails. The point of the presented paper is to re-establish operads in the context of higher category theory in order to recover this algebra equivalence. This is achieved by defining analytic functors over slice categories of spaces, and restating the equivalence as a monadic adjunction.

$$Fr : AnEnd(I) \leftrightarrows AnMnd(I) : U$$

This is built off the work of Weber, who described 1-Operads as 2-Polnomial Monads.

The strategy is basically defined by the order of the sections:

In section 2 we define a polynomial functor. The main result of section one is a characterization of polynomial functors

Theorem 2.2.3)
1)F is poly
2)F is accessible and preserves weakly contractible colimits
3) F has a local right adjoint.

in terms of weakly contractible limits, which will be used to define analytic functors. We establish a two equivalent categories

$$POLY \cong POLYFUN$$

of polynomial functors and the relevant morphisms, the cartesian natural transformations, that will be used as the context for the rest of the theorems

In section 3 we specify even further to analytic functors and prove a characterization of them as polynomial functors with a middle map having finite fibers. This class of polynomial functors is classified by an space iFin, and we will see that analytic functors can be rebranded as functors with a cartesian morphism over a specific analytic functor E, who's components are iFin.

$$\begin{array}{ccccccc} I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & J \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ * & \longleftarrow & iFin_* & \longrightarrow & iFin & \longrightarrow & * \end{array}$$

We finish this section with a description of a specific set of analytic functors, the dendroidal trees, and find an equivelance between the category of analytic endofunctors and segal presheaves on trees.

$$AnFun \cong \mathscr{P}_{seg}(\Omega_{int})$$

Section 4 begins with a bar-cobar adjunction

$$Map_{alg_P((C)}(\Omega A, C) \cong Map_{coalg_P((C)}(A, BC)$$

for algebras and co-algebras on ∞-endofunctors, which is used to define the notion of a free monad on an endofunctor via another (monadic) adjuncton)

$$Fr : \mathfrak{C} \leftrightarrows alg_P(\mathfrak{C})$$

This adjunction induces a free monad $\bar{P}$ and we show that the algebras of a free monad are equivalent to the algebras of the endofunctors P.

$$alg_P(\mathfrak{C}) \cong alg_{\bar{P}}(\mathfrak{C})$$

And when restricted to the analytic endofunctors, and (free) analytic monads we get the higher analog of the algebraic equivalence we are looking for, namely a monadic adjunction

$$Fr : AnEnd(I) \leftrightarrows AnMnd(I) : U$$

And the monadicity of the adjunction implies the equivalence of categories:

$$AnMnd(I) \cong alg_{U \circ Fr}(AnEnd(I))$$

The only thing left to do is to actually describe describe ∞-operads as analytic monads, which is precisely the goal of the 5th section. In which we show that the category of analytic monads is equivalent to the category of dendroidal segal spaces

$$AnMnd \cong \mathcal{P}_{seg}(\Omega)$$

(which are an accepted model of ∞-operads i.e. are equivalent to symmetric sequences of topological spaces as defined in Lurie HA)

Alright so that's the game plan let's get started.

Disclosure

First and formost, we are working in some etherial category of spaces which they denote

$$\mathcal{S} = \text{"Category of spaces"}$$

I assume this is some "nice" category of topological spaces, or maybe ∞-groupoids. The real necessity is that it's an LCcC (i.e. with an internal hom) who's slices form ∞-Topoi

Second, there is a category of ∞-categories which follows Luries definition as some "scaled nerve of the simplicial category of fibrant marked simplicial sets $Cat_\infty = N^{sc}(\Delta^{+,o})$

Section 2: Polynomial Functors

Recall in an LCcC we have the dependent sum and product defined as

$$f_! \dashv f^* : S/J \to S/I \dashv f_*$$

Definition.

A Polynomial functor $P : S/J \to S/I$ can be factored as $P = t_! p_* s^*$ where

$$I \overset{s}{\leftarrow} E \overset{p}{\rightarrow} B \overset{t}{\rightarrow} J$$

Proposition.

Polies Compose

Definition.

A small category is weakly contractible if it's geometric realization $|C|$ is contractible

Theorem.

Let $F : S/I \to S/J$
1)F is poly
2)F is accessible F preserves weakly contractible limits
3)F is a local right adjoint ($F/_g : (S/I)/g \to (S/J)/(Fg)$ is a right adjoint)

The second two are a "local adjoint theorem" (following from the adjoint theorem accessible + preserves limits = right adjoint) and (1) $\iff$ (3) follows from Beck - Chevellay equivalences.

Definition.

PolyFun(I,J) = poly functors, cartesian natural transformations

The justification of this is 2-fold, first the Beck - Chevellay tranformations behave really well in the cartesian context, and this yeilds us a lemma

Lemma.

if $\eta : F \to P$ cartesian, then F is a poly

The second justification is in the general construction of polynomial functors over varying I,J.

Definition.

$sq^{colax}(Cat_\infty)^{v=radj} \Rightarrow PolyFun$

$$\begin{array}{ccc} S/I & \xrightarrow{P} & S/J \\ {\scriptstyle f^*}\uparrow & \searrow\!\!\searrow & {\scriptstyle f^*}\uparrow \\ S/I' & \xrightarrow{Q} & S/J' \end{array}$$

Of course, this sort of definition is rather formal, although really synthetic so it should be the "right one", but to work with polyfunctors we want a more diagrammatic definition

Definition.

let $\Pi = \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$ define
$POLY = Fun(\Delta^{1,op}, S) \times_S Fun(\Delta^1, S)^{cart} \times_S Fun(\Delta^1, S) \hookrightarrow Fun(\Pi, S)$
so a map in POLY looks like

$$\begin{array}{ccccccc} I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & J \\ \downarrow & & \downarrow & \ulcorner & \downarrow & & \downarrow \\ I' & \longleftarrow & E' & \longrightarrow & B' & \longrightarrow & J' \end{array}$$

and we get the theorem

Theorem.

$POLY \cong PolyFun$

by the way, this relies on the usage of cartesian natural transformations because this equivalence is proved "fiber wise" (fibers over dom and codom), we can do this because this fibration is a "cartesian fibration"

Finally, we will make use of this theorem later:

Theorem.

For $P \in PolyFun$ $Polyfun/_P$ is an ∞-topos.

Section 3: Analytic Functors

Definition.

A sifted category is a small category who's diagonal preserves limits (i.e $Lim(F \circ \Delta : K \to \mathfrak{D}) = Lim(F : K \times K \to \mathfrak{D})$
in 1-categories this is the statement that finite products commute with limits.

Definition.

An **Analytic functor** $F : S/I \to S/J$ preserves sifted colimits and weakly contractible limits.

The fact that accessible implies sifted gives us $AnFun \hookrightarrow PolyFun$. We have a better characterization of analytic functors:

Theorem.

F:$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$
analytic $\iff$ p has finite fibers

We can use this characterization to classify analytic by a single analytic functor.

Definition.

$\mathcal{F}$ A "bounded local class of morphisms" comes with a classifying map $U'_{\mathcal{F}} \to U_{\mathcal{F}}$. Such that if

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \downarrow & \ulcorner & \downarrow \\ U'_{\mathcal{F}} & \longrightarrow & U_{\mathcal{F}} \end{array} \text{, then } p \in \mathcal{F}$$

Let F: $* \leftarrow U_{\mathcal{F}} \to U'_{\mathcal{F}} \to *$ and $PolyFun_{\mathcal{F}} = \{P \in Polyfun | p \in \mathcal{F}\}$

Lemma.

$Polyfun_{\mathcal{F}} = U(Polyfun/F)$

So specifically, we find the classifier for maps with finite fibers $iFin_* \to iFin$ and let

$$E : * \leftarrow iFin_* \to iFin \to *$$

Then

Corollary.

$AnFun \cong PolyFun/E$

So in particual, AnFun is an ∞-topos, but in addition, we can write an analytic functor as:

$$\begin{array}{ccccccc} I & \longleftarrow & E & \xrightarrow{\ \ulcorner\ } & B & \longrightarrow & J \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & iFin_* & \longrightarrow & iFin & \longrightarrow & * \end{array}$$

Finally, we can actually describe analytic functors in terms of trees, this lays the groundwork for the future theorem equating analytic monads to dendriodal segal spaces.

Definition.

A tree is an analytic functor $A \stackrel{s}{\leftarrow} M \stackrel{p}{\rightarrow} N \stackrel{t}{\rightarrow} A$ satifying certain properties. The category of trees is $\Omega_{int} \hookrightarrow AnFun$

I leave this to David, but essentially A are the edges, N are the nodes and M are pairs of edges and node that say "e is going into v".

In this case the trivial tree is

$$* \leftarrow \emptyset \rightarrow \emptyset \rightarrow *$$

And the n-Corolla is

$$n+1 \leftarrow n \rightarrow * \rightarrow n+1$$

These are the elemetary trees $\Omega_{el} \hookrightarrow \Omega_{int}$

The inclusion induces a functor on the presheaf categories $\mathscr{P}(\Omega_{el}) \hookrightarrow \mathscr{P}(\Omega_{int})$ the image of which are called the **segal presheafs**

$$\mathscr{P}(\Omega_{el}) \hookrightarrow \mathscr{P}_{seg}(\Omega_{int}) \subseteq \mathscr{P}(\Omega_{int})$$

Finally, the yoneda embedding on $AnFun$ yeilds an equivelence of categories

$$AnFun \cong P_{seg}(\Omega_{int})$$

Section 4: Lambek Algebras and Free Monads This section begins with a bar-cobar construction who's resulting adjunction will be used to construct free monads from an endofunctor and equate the algebras on either.

Definition.

For an endofunctor $P : \mathfrak{C} \to \mathfrak{C}$ A lambek P-(co)algebra is a pair (A,a), (resp. (C,c)) such that $a : PA \to A$ (resp. $c : C \to PC$).

These both have their respective categories $alg_P(\mathfrak{C})$ and $coalg_P(\mathfrak{C})$

For an endofunctor P that preserves finite limits(remember Polies do this), from a P-coalgebra $c : C \to PC$ we can consider

$$\Omega C = colim(C \to PC \to P^2C \to ...)$$

and this comes with a canonical equivalence $\Omega C \to P(\Omega C)$ with inverse $u : P(\Omega C) \to \Omega C$ and so $(\Omega C, u) \in alg_P(\mathfrak{C})$.

We get a functor $\Omega : coalg_P(\mathfrak{C}) \to alg_P(\mathfrak{C})$ and dually we can construct a functor $B : alg_P(\mathfrak{C}) \to coalg_P(\mathfrak{C})$. These are the **bar/cobar**.

Theorem.

$$Map_{alg_P(\mathfrak{C})}(\Omega C, A) \cong Map_{coalg_P(\mathfrak{C})}(C, BA)$$

We are now ready to construct the free monad on an endofunctor:

Proposition.

Let $U_P : alg_P(\mathfrak{C}) \to \mathfrak{C}$ the underlying functor then U has a left adjoint

$$Fr_P \dashv U_P$$

and this adjunction is monadic.

Now since $Fr_P \dashv U_P$ we can induce a monad on $\mathfrak{C}$, $\bar{P} = U_P \circ Fr_P$ and we get the following equivalence

Proposition.

$$alg_P(\mathfrak{C}) \cong alg_{\bar{P}}(\mathfrak{C})$$

This is essentially the algebraic equivalence we are looking for, but we need to specify to analytic endofunctors $P : S/_I \to S/_I$ and show that everything restricts properly. Namely:

In 1-Categories: If

$$F : C \to D \dashv G$$

is an adjunction $\eta : Id \to G \circ F$ the unit, and $\epsilon : F \circ G \to Id$ the counit, then the composite $T = G \circ F$ becomes a monad on C

$$(T, \eta : Id \to T, G\epsilon F : T^2 \to T)$$

Moreover, for an object of $d \in D$ the map

$$G\epsilon_d : G \circ F \circ G(d) = TG(d) \to G(d)$$

gives a functor $G^T : D \to alg_T(\mathfrak{C})$. The adjunction is called **monadic** if this functor is an equivalence

Theorem.

For $P \in AnEnd(I)$ the free monad $Fr(P) = \bar{P}$ is also analytic, and we have a monadic adjunction

$$Fr : AnEnd(I) \leftrightarrows AnMnd(I) : U$$

Interpreting as 1-categories, monadicity implies:

$$AnMnd(I) \cong alg_{U \circ Fr}(AnEnd(I))$$

i.e., that analytic monads are equivalent to free monad algebras on analytic endofunctors. This generalizes the 1-categorical:

Operads determine and are determined by the algebra on their analytic endofunctor.

That is, if we can show that analytic monads are actually models for ∞-operads:

Definition.

Let Ω be the subcategory of AnMnd spanned by free monads on the analytic endofunctor trees Ω_int, A **Dendroidal Segal Space** is a presheaf on Ω who's restriction along the inclusion $\Omega_int \to \Omega$ is a segal presheaf. Denote this presheaf subcategory $\mathscr{P}_{seg}(\Omega)$

Theorem.

$$AnMnd \cong \mathscr{P}_{seg}(\Omega)$$

The dendroidal segal spaces are an accepted model for ∞-operads, they have been shown to be equivelent to the concrete construction of ∞-operads as symmetric sequences of spaces by Lurie.