OPERADS AS ANALYTIC MONADS

NOTES BY EMILY RIEHL

ABSTRACT. The aim is to study the paper "∞-operads as analytic monads" by David Gepner, Rune Haugseng, and Joachim Kock.

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0. DOES KNOWLEDGE OF 1-CATEGORY THEORY PROVIDE MORALLY SUFFICIENT GROUNDS UPON WHICH TO FAKE KNOWLEDGE OF ∞-CATEGORY THEORY? — EMILY RIEHL

This talk will proffer an ethical tactic for engaging with ∞-categories as a non-expert. It will start by explaining exactly what an ∞-category is from the point of view of much of the literature that works with them. Along the way, it will also illustrate the similarities and differences between 1-categories and ∞-categories by giving an in-depth discussion of one of the equivalences between ∞-categories that is used without comment in [GHK].

Q. What is an ∞-category?

A 1-category has a collection of objects and then a collection of morphisms between any ordered pair of objects, with an associative and unital binary composition operation defined whenever co/domains align. An equivalence of 1-categories need not induce a bijection on objects, so in some sense “the set of objects” in a category is not really well-defined when working with categorical constructions up to equivalence. An equivalence does define a local bijection on collections of morphisms with fixed co/domains and a global bijection on isomorphism classes of objects.

An ∞-category has a collection of objects and then a space of morphisms between any ordered pair objects, with a weakly associate and weakly unital composition operation defined up to a contractible space of choices whenever co/domains align. The homotopy category of an ∞-category is a 1-category obtained by replacing each mapping space with its set of path
components. An equivalence of infinity-categories induces a local equivalence of mapping spaces and a global bijection on isomorphism classes of objects in the homotopy category. So neither the set of objects in an infinity-category nor the space of morphisms between two objects is well-defined — though the homotopy types of mapping spaces are preserved by equivalences.

This sketch of the idea of an infinity-category as a “category weakly enriched in spaces” can be made precise by various models, to which we now turn.

Q. What are the “shortcomings afflicting topological categories when viewed as a model for infinity-categories”?

We’ve just declared that an infinity-category is a category weakly enriched in spaces. Somewhat surprisingly, and certainly non-obviously, every infinity-category can be modeled by a category strictly enriched in space, i.e., by either a topological category or by a simplicial category. This result can be interpreted as some sort of coherence theorem for infinity-categories.

However, it is not the case that any functor between infinity-categories can be represented by a topologically-enriched functor between the topological categories that represent its domain and codomain: infinity-functors correspond to “homotopy coherent” functors, with the strictly enriched functors being a special case.

And more generally one would like to access the space or even better the infinity-category of infinity-functors between a fixed pair of infinity-categories. This can be constructed for topologically enriched categories but not natural with respect to composition of infinity-functors. This is related to properties of Bergner’s model structure for simplicial categories \([\text{Ber}]\), which is not cartesian closed and in which relatively few objects are both fibrant and cofibrant.

Q. What is a “better-behaved model for infinity-categories”?

The most popular model of infinity-categories, because it is the simplest to get up and running, is the aforementioned weak Kan complexes, now called quasi-categories.

**defn.** A quasi-category is a “simplicial set with composition”: a simplicial set \(A\) in which every inner horn can be filled to a simplex.

\[
\Lambda^k[n] \rightarrow A \\
\downarrow \\
\Delta[n]
\]

\(n \geq 2, 0 < k < n\)

Note that this is weaker than the lifting property which characterizes Kan complexes. In particular, Kan complexes are examples of quasi-categories, as infinity-groupoids are instances of infinity-categories.

The vertices of a quasi-category represent its objects, as an infinity-category, and the edges represent its 1-arrows. By extending along the horn inclusion \(\Lambda^1[2] \hookrightarrow \Delta[2]\), any composable pair of arrows admits a composite, with each composition relation witnessed by a 2-simplex; degenerate 2-simplices can be used to witness the identity axioms. The higher horn filling conditions imply that this operation is associative and homotopically unique: the fibers of the right-hand vertical map

\[
\bullet \rightarrow A^\Delta[2] \\
\downarrow \\
\Delta[0] \rightarrow A^\Delta[2]
\]

\(^{\text{¹}}\text{One of the earliest examples of an infinity-category was the “weak Kan complex” of homotopy coherent functors studied by Boardman and Vogt [BV].}\)
are contractible Kan complexes: hence any composable pair of arrows \((g, f)\) has a unique composite, up to homotopy.

Arbitrary maps of simplicial sets \(A \to B\) preserve objects, arrows, and composition, and hence are regarded as \(\infty\)-functors between \(\infty\)-categories. The nerve of a 1-category \(J\) always defines a quasi-category, since the nerve embedding is fully faithful, it is increasingly common to use the same notation for the 1-category and the corresponding \(\infty\)-category. By examination, the data involved in a simplicial map \(J \to A\) assembles into a “homotopy coherent diagram of shape \(J\) in \(A\).” If \(A\) is a quasi-category and \(J\) is any simplicial set, then the internal hom \(\text{Fun}(J, A) := A^J\) is again a quasi-category, defining a natural model for the \(\infty\)-category of \(\infty\)-functors from \(J\) to \(A\).

Q. How should one approach \(\infty\)-categories assuming only a background in 1-categories?

In an ideal world, where every mathematician had an unlimited amount of unencumbered free time and could pause any conversation or research project to go read all the relevant literature, we would quote a theorem that we didn’t know how to prove. But in the real world, it’s desirable to figure out a way to interact with new technology without necessarily understanding how everything works under the hood.

In this vein, many learners, users, or experts in abstract homotopy theory are wondering to what extent their familiarity with 1-category theory can serve as a proxy for understanding of \(\infty\)-category theory. In the interest of welcoming as many people as possible into the conversation, I think it’s important to avoid having too high a cost of admission into this new direction the field is taking.

I’d advocate approaching \(\infty\)-categories with a mix of confidence (that the theorems and constructions that you know and love from 1-category theory likely extend fully faithfully to \(\infty\)-categories) and humility (that if you don’t happen to know the details of a particular extension, it’s likely the case that someone has had or will have to work quite hard to nail them down). In particular, it does not help the field advance if you write a paper asserting that some \(\infty\)-categorical fact is analogous to the corresponding 1-categorical fact if you have no idea how one would prove the \(\infty\)-categorical version of the theorem you’d like to use.

In talks, the ethical standard is somewhat different, because when trying to tell a coherent story in a constrained amount of time it’s often advisable to suppress certain details. Here I see no issue with arguing by analogy with 1-categories — provided that in doing so you do no harm. In this context, harm is caused by intimidating members of the audience into thinking that they’re the only ones who don’t understand what’s going on, for instance by parroting stuff that you don’t understand either. If you tend to be under-confident when sketching mathematical proofs then it’s likely that your use of language when discussing \(\infty\)-categories will naturally be reassuring. But if you tend to be over-confident, you should take care to make sure you don’t inadvertently put people off.

Q. How does \(\infty\)-categorical methodology differ from 1-categorical methodology?

Part of what confuses readers familiar with 1-categories when reading papers that use \(\infty\)-categories is that the arguments that appear seem less rigorous, or at least less explicit. Some of this is because it is considerably more difficult to give “full details” in a new area whose foundations haven’t been fully streamlined and sublimated, but some of this is due to a genuine methodological difference in working model-independently with \(\infty\)-categories vs working with a theory of 1-categories ultimately grounded in set theory, as we now explain.

\footnote{It is convenient to consider quasi-category-valued diagrams indexed by an arbitrary simplicial set, like it can be convenient to consider category-valued diagrams indexed by a directed graph.}

\footnote{I.e., try not to become one of what I’ve heard referred to as “infinity blah blah blah” people.}
Some definitions and constructions in 1-category theory are “evil,” failing to be invariant under equivalences of categories. To be justifiably considered as an aspect of \(\infty\)-category theory (rather than “quasi-category theory” or “complete Segal space theory”) a construction must be invariant under equivalence of \(\infty\)-categories because the various “change-of-model functors” only respect equivalence classes of \(\infty\)-categories. So, for instance, any particular \(\infty\)-category may be introduced by specifying a member of the correct equivalence class, in any model, as we shall now do.

Q. What is \(\mathcal{S}\)?

By convention \(\mathcal{S}\) denotes “the” \(\infty\)-category of spaces, well-defined up to equivalence. This mostly naturally arises as a topologically enriched category, or for technical reasons, by considering the full Kan-complex-enriched subcategory of small Kan complexes in the category of simplicial sets. The homotopy coherent nerve of this then defines a large quasi-category which we denote by \(\mathcal{S}\). Objects of \(\mathcal{S}\) are Kan complexes and morphisms are simplicial functors. Higher simplices represent homotopy coherent diagrams of Kan complexes.

Q. For \(f : I \to J\) in \(\mathcal{S}\) how is \(\mathcal{S}/\!\!/J\) equivalent to \(\text{Fun}(J, \mathcal{S})\)?

The equivalence \(\mathcal{S}/\!\!/J \cong \text{Fun}(J, \mathcal{S})\) is very special to the \(\infty\)-category \(\mathcal{S}\) as we shall explain. It is most easily described when the \(\infty\)-category of spaces \(\mathcal{S}\) is modeled as a quasi-category, defined as the homotopy coherent nerve of the category of Kan complexes. An element \(J \in \mathcal{S}\) is then a small Kan complex.

Since \(J\) and \(\mathcal{S}\) are both simplicial sets (with \(J\) being small and \(\mathcal{S}\) being large), the quasi-category \(\text{Fun}(J, \mathcal{S}) := \mathcal{S}/\!\!/J\) may be defined as above: objects are simplicial maps \(J \to \mathcal{S}\), morphisms are simplicial natural transformations \(J \times \Delta[1] \to \mathcal{S}\), and higher simplices are diagrams \(J \times \Delta[n] \to \mathcal{S}\).

The quasi-category \(\mathcal{S}/\!\!/J\) is defined by Joyal’s slice construction [Joy], which can be implemented for any vertex in any simplicial set. Objects in \(\mathcal{S}/\!\!/J\) are edges in \(\mathcal{S}\) with codomain \(J\). In general, \(n\)-simplices in \(\mathcal{S}/\!\!/J\) are \(n + 1\)-simplices in \(\mathcal{S}\) with final vertex \(J\). Since \(\mathcal{S}\) is a homotopy coherent nerve, this can be unpacked further: objects in \(\mathcal{S}/\!\!/J\) are maps of Kan complexes \(X \to J\), while an arrow is comprised of a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\Delta[1] & \xrightarrow{\sim} & J
\end{array}
\]

given by a trio of simplicial maps forming the boundary of a simplicial natural transformation \(X \times \Delta[1] \to J\).

It’s not obvious from these explicit descriptions how these quasi-categories are equivalent. The relevant 1-categorical theorem is:

**Theorem.** For any 1-category \(J\), there is an equivalence of categories

\[
\text{Fun}(J, \text{Set}) \cong \text{DLCat}_{J/} \subseteq \text{Cat}_{J/}
\]

between the category of functors \(J \to \text{Set}\) and the category of discrete left fibrations over \(J\), a full subcategory of \(\text{Cat}_{J/}\).
A discrete left fibration is a functor \( p : E \to J \) with the unique right lifting property

\[
\begin{array}{ccc}
1 & \longrightarrow & E \\
\downarrow & & \downarrow \uparrow \\
2 & \longrightarrow & J
\end{array}
\]

The corresponding maps of quasi-categories are called left fibrations by Joyal [Joy], which are maps of simplicial sets \( p : E \to J \) characterized by the right lifting property

\[
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & E \\
\downarrow & & \downarrow \uparrow \\
\Delta[n] & \longrightarrow & J
\end{array} 
\]

which in the case \( \Lambda^0[1] \hookrightarrow \Delta[1] \) represents the inclusion of the domain of an arrow. The higher left horn fillers make up for the lack of uniqueness of lifts.

By repeated attaching fillers for left horns to \( X \), any map of Kan complexes \( X \to J \) can be replaced by a left fibration \( \bar{X} \to J \) related via an equivalence \( X \simeq \bar{X} \) over \( J \). Thus \( S_J \) is equivalent to the full subcategory spanned by the left fibrations over \( J \). Since all of \( \infty \)-category is equivalence invariant, there is no need to replace \( S_J \) by this subcategory. Instead, we are free to consider \( S_J \) itself as the \( \infty \)-category of left fibrations over \( J \).

By a theorem of Joyal [Joy], the fibers of a left fibration \( p : X \to J \) are Kan complexes. Thus \( p \) may be regarded as a \( J \)-indexed family of Kan complexes. Moreover, the fact that \( p \) is a left fibration implies that these fibers vary covariantly functorially in morphisms in the \( \infty \)-category \( J \). In this way, a left fibration over \( J \) morally corresponds to a \( \infty \)-functor \( J \to \mathcal{S} \).

The \( \infty \)-groupoid variant of the straightening-unstraightening theorem of Lurie [Lur] then establishes an equivalence

\[ S_J \simeq \text{Fun}(J, \mathcal{S}) \]

which can be understood in a very explicit way. Cisinski [Cis] proves that there is a universal left fibration \( u : \mathcal{S}_* \to \mathcal{S} \) using techniques similar to those used to establish universal fibrations in presheaf models of homotopy type theory. Here \( \mathcal{S}_* \) is the \( \infty \)-category of pointed spaces and \( u \) is the evident forgetful functor. From right to left, the “unstraightening” of an \( \infty \)-functor \( F : J \to \mathcal{S} \) is given by forming the pullback

\[
\begin{array}{ccc}
\int F & \longrightarrow & \mathcal{S}_* \\
\downarrow & & \downarrow \nu \\
J & \overset{f}{\longrightarrow} & \mathcal{S}
\end{array}
\]

of quasi-categories. See [RV] for more details about how this mapping on objects is extended to an \( \infty \)-functor.

Finally, note that the construction of the unstraightening of a functor \( J \to \mathcal{S} \) is natural in the indexing category, with the unstraightening of a composite diagram \( Ff : I \to J \to \mathcal{S} \) formed by pulling back \( \int F \to J \) along \( f \). Thus, the equivalence \( S_J \simeq \text{Fun}(J, \mathcal{S}) \) identifies the pullback and pre-composition functors \( f^* \) and \(- \circ f\) appearing in the middle of the triple of adjoints. An adjoint to an \( \infty \)-functor is well-defined up to natural isomorphism, so it follows that \( f_! \) is equivalent to \( \text{lan}_f \) and \( f_\ast \) is equivalent to \( \text{ran}_f \).
1. Introduction — Noah Chrein

The notion of an operad has been around for decades now, going back to May in the 70s. They are used to capture the computational combinatorics of algebraic structures in various situations. Classically, operads are defined using symmetric sequences of sets, these sequences give rise to "analytic endofunctors" which are monads when the symmetric sequence is an operad. For sets, there is an equivalence between the notion of an algebra on an operad and the algebra on its associated monad. Lifting to operads defined as sequences of spaces, this algebraic equivalence is lost in general. Gepner, Haugseng, and Kock notice this problem has to do with higher structures and so devise a definition of an \(\infty\)-operad as an analytic monad to recover an analog of the algebraic equivalence in the setting of higher category theory. Several other equivalent models of \(\infty\)-operads exist, for example as dendroidal Segal spaces by Cisinski and Moerdijk. GHK’s final result proves their analytic model of infinity operads is equivalent to the dendroidal Segal spaces. This introduction will focus on the main constructions and results of the paper that this seminar aims to study in detail during the remainder of the semester.

**defn.** A **symmetric sequence** in \(\mathcal{C}\) is a functor \(O : \mathcal{F}in \to \mathcal{C}\), where \(\mathcal{F}in\) is the \(\infty\)-category of finite sets and bijections.

From a symmetric sequence one can define an endofunctor \(T : \mathcal{C} \to \mathcal{C}\) by

\[
T(X) := \bigsqcup_{n} O(n) \times_{\Sigma_n} X^n,
\]

assuming \(\mathcal{C}\) has these limits and colimits, and when \(O\) is an operad, \(T\) is a monad. Moreover, **algebras** for the operad, objects \(V \in \mathcal{C}\) with structure maps \(O(n) \times V^n \to V\) correspond to algebras for the monad, which have structure maps \(\alpha : T(V) \to V\).

When \(\mathcal{C} = \mathbf{Set}\), we can recover the operad \(O\) from from the category of algebras for \(T\), but this isn’t true when \(\mathcal{C} = \mathbf{Spaces}\).

The main result of this paper is that \(\infty\)-operads are analytic monads, that is analytic monads are equivalent to the exist models of \(\infty\)-operads due to Lurie and due to Cisinski-Moerdijk (as dendroidal Segal spaces).

Now let’s give an overview of the sections of the paper.4

1.2. Polynomial functors. Let \(\mathcal{S}\) be the \(\infty\)-category of spaces, or of \(\infty\)-groupoids. Some key facts:

- \(\mathcal{S}\) is an \(\infty\)-topos
- \(\mathcal{S}\) is locally cartesian closed as an \(\infty\)-category, meaning for every \(f : I \to J\) in \(\mathcal{S}\) there exists an adjoint triple

\[
\begin{array}{c}
\mathcal{S}_J \\
\downarrow \\
\downarrow \\
\downarrow \\
\mathcal{S}_I
\end{array}
\]

\[
\begin{array}{c}
f^* \\
\downarrow \\
f_* \\
f^! \end{array}
\]

Let \(f : I \to J\) be a morphism in \(\mathcal{S}\).

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4The content of the talk has been reordered somewhat to optimize for note-taking.
defn. A polynomial functor $P : S/I \to S/J$ is a functor that arises as $P = t \cdot p \cdot s^*$ for some polynomial

$$
\begin{array}{ccc}
I & \xleftarrow{s} & E \\
| & | & | \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{a} & \bullet
\end{array}
\quad
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
| & | & | \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{t} & \bullet
\end{array}
\quad
\begin{array}{cc}
& J \\
& \bullet
\end{array}
$$

in $S$.

There is a geometric realization functor $|-| : \text{Cat}_\infty \to S$ from the $\infty$-category of $\infty$-categories to the $\infty$-category of spaces that freely inverts all of the morphisms in $C$. An $\infty$-category $C$ is weakly contractible if $|C|$ is weakly contractible.

**Theorem.** For a functor $F : S/I \to S/J$, the following are equivalent:

(i) $F$ is a polynomial functor

(ii) $F$ is accessible and preserves weakly contractible limits

(iii) $F$ is a local right adjoint.

**defn.** A functor $F : C \to D$ is a local right adjoint if for all $x \in C$ the functor $F_{/x} : C_{/x} \to D_{/x}$ is a right adjoint for all $x \in C$.

**Remark.** By the adjoint functor theorem, a functor between locally presentable $\infty$-categories is a left adjoint if and only if it preserves colimits and is a right adjoint if and only if it is accessible and preserves limits.

For any $I, J \in S$, there is an $\infty$-category $\text{PolyFun}(I, J)$ whose objects are polynomial functors from $I$ to $J$ and whose morphisms are cartesian natural transformations, natural transformations whose naturality squares are pullback squares.

**Lemma.** If $\eta : F \to P$ is a cartesian natural transformation whose codomain $P$ is a polynomial functor, then $F$ is a polynomial functor.

The definition of the $\infty$-category $\text{PolyFun}$ of polynomial functors with varying endpoints is somewhat complicated.

The category $\text{Poly}$ is defined as a subcategory of diagrams of shape $\bullet \xleftarrow{\bullet} \bullet \xrightarrow{\bullet} \bullet$ in $S$ containing all objects and only those morphisms whose “middle square” as below is a pullback

$$
\begin{array}{ccc}
\bullet & \xleftarrow{\bullet} & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\cdot & \xleftarrow{\cdot} & \bullet
\end{array}
$$

**Theorem.** The $\infty$-categories $\text{Poly}$ and $\text{PolyFun}$ are equivalent.

1.3. Analytic functors.

**defn.** An analytic functor $F : S/I \to S/J$ preserves weakly contractible limits and sifted colimits.

This is a strengthening of the second characterization of polynomial functors in the theorem above, since filtered colimits are sifted. Consequently:

**Corollary.** Analytic functors are polynomial functors.

**Theorem.** Analytic functors are those polynomial functor represented by polynomials

$$
\begin{array}{ccc}
I & \xleftarrow{s} & E \\
| & | & | \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{t} & \bullet
\end{array}
\quad
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
| & | & | \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{t} & \bullet
\end{array}
\quad
\begin{array}{cc}
& J \\
& \bullet
\end{array}
$$

in which $p$ has finite discrete fibers.
Proposition. Let $\mathcal{F}$ be a “bounded local class of morphisms” in $\mathcal{S}$ with classifying family $U_\mathcal{F} \to U_\mathcal{F}$, and let $F : \mathcal{S} \to \mathcal{S}$ be the polynomial functor corresponding to this map. Then the forgetful functor $\text{PolyFun}_{/F} \to \text{PolyFun}$ is fully faithful and its image is the full subcategory $\text{PolyFun}_{/\mathcal{F}}$ of polynomial functors represented by polynomials whose middle map is in the class $\mathcal{F}$.

An example is the family $\mathcal{F}$ of maps whose fibers are finite sets. Then by the previous theorem, $\text{AnFun} \cong \text{PolyFun}_{/\mathcal{F}} \cong \text{PolyFun}_{/E}$ where $E$ is the polynomial functor with polynomial $\ast \leftarrow \iota \text{Fin} \rightarrow \iota \text{Fin} \rightarrow \ast$

Here $\text{Fin}$ is the category of finite sets and all maps while $\iota \text{Fin}$ is its maximal subgroupoid, the category of finite sets and isomorphisms. This proves:

Theorem. If $F$ is analytic then its polynomial is a pullback

$$
\begin{array}{ccc}
I & \leftarrow & E \\
\downarrow & & \downarrow \\
\ast & \leftarrow & \iota \text{Fin} \\
\end{array}
\quad
\begin{array}{ccc}
E & \to & B \\
\downarrow & & \downarrow \\
\iota \text{Fin} & \to & \iota \text{Fin} \\
\end{array}
\quad
\begin{array}{ccc}
B & \to & I \\
\downarrow & & \downarrow \\
\ast & \leftarrow & \ast
\end{array}


Consequently

$$F(X) = \bigsqcup_n (B_n \times_{\Sigma_n} X^n)$$

where the $B_n$ are the fibers of $B \to \iota \text{Fin}$.

1.4. Initial Algebras and Free Monads. Let $P$ be an endofunctor of some $\infty$-category $\mathcal{C}$. A $P$-algebra is a pair $(A, a)$ with $a : PA \to A$. Similarly a $P$-coalgebra is a pair $(C, c)$ where $c : C \to PC$.

For any $P$-coalgebra one can define the colimit

$$\Omega C = \text{colim}( C \xrightarrow{\epsilon} PC \xrightarrow{P\epsilon} P^2C \to \cdots ).$$

It turns out that $\Omega C$ is a $P$-algebra so this defines a functor

$$\Omega : \text{Coalg}_P(\mathcal{C}) \to \text{Alg}_P(\mathcal{C}).$$

Dually one defines

$$B : \text{Alg}_P(\mathcal{C}) \to \text{Coalg}_P(\mathcal{C}),$$

and these are adjoint $\Omega \dashv B$, defining the bar-cobar adjunction.

The category of $P$-algebras has a free-forgetful adjunction inducing a monad $\bar{P}$ on $\mathcal{C}$, and this defines a functor

$$\text{Fr} : \text{End}(\mathcal{C}) \to \text{Mon}(\mathcal{C}),$$

by $\text{Fr}(P) = \bar{P}$.

When this construction applied to analytic endofunctors it gives an analytic monad, and this defines a left adjoint to the inclusion

$$\begin{array}{ccc}
\text{AnEnd}(I) & \cong & \text{AnMnd}(I) \\
\downarrow & & \downarrow \\
\uparrow & & \\
\text{Fr} & \cong & \text{Fr}
\end{array}$$

Moreover this adjunction is monadic.
Consequently, the category of analytic monads is equivalent to the category of algebras for the monad $U_{Fr}$ on the category of endofunctors. What this means in practice is that from an algebra on an analytic endofunctor we can recover our analytic monad.

1.5. **Analytic monads and $\infty$-operads.** In the final section they show that analytic monads are Segal presheaves on the category of trees. This is the Cisinski-Moerdijk notion of $\infty$-operad.

2. **Polynomial Functors — Martina Rovelli**

3. **Analytic Functors — David Myers**

4. **Initial Algebras and Free Monads — TBD**

5. **Analytic Monads and $\infty$-Operads — Daniel Fuentes-Keuthan**

References


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