Cellularity, composition, and morphisms of algebraic weak factorization systems

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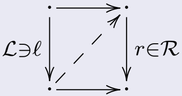
Algebraic weak factorization systems

A weak factorization system $(\mathcal{L}, \mathcal{R})$

- has left and right classes $\mathcal{L}$ and $\mathcal{R}$ of maps s.t.



An algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$

- has a comonad $\mathbb{L}$ and monad $\mathbb{R}$ arising from a functorial factorization
- coalgebras are left maps; algebras are right maps
- (co)algebra structures witness membership and solve lifting problems

Examples

- (monos,epis) in **Set**
- (injective with projective cokernel, surjective) in $\textbf{Mod}_R$

Cellularity

Motivating example

- There is an algebraic weak factorization system on $\mathbf{Top}$ whose coalgebras for the comonad are relative cell complexes.
- Hence, we call the maps admitting a coalgebra structure cellular.
- Not all cofibrations (elements of the left class of the weak factorization system) are cellular: cellularity is a condition!
- Generic cofibrations are retracts of relative cell complexes, equivalently, coalgebras for the pointed endofunctor of the comonad.

Composition

Composing coalgebras in $\mathbf{Top}$

- A coalgebra structure for a relative cell complex $i \colon A \to B$ is a cellular decomposition:

$$A = A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow B$$

  each object obtained by attaching cells.
- Cellular cofibrations can be composed: the composite of two relative cell complexes is one again.
- Furthermore, the coalgebra structures are composable: the composite is equipped with a canonical cellular decomposition.

In general

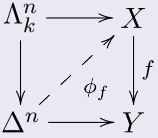
- Coalgebras for the comonad of an algebraic weak factorization system can be composed and the composition is functorial.

Composition, continued

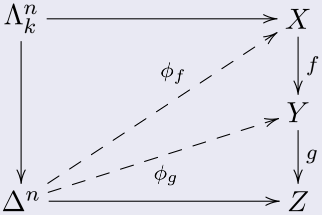
Composing algebras in $\mathbf{sSet}$

- Kan fibrations admit algebra structures for the monad of an algebraic weak factorization system.

- An algebra structure is a choice of fillers for all horns



- Algebra structures are composable: Define ϕ_{gf} by

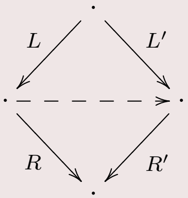


Morphisms of algebraic weak factorization systems

Preliminary definition.

A morphism between two algebraic weak factorization systems is

- a natural transformation comparing their functorial factorizations



- that induces functors $\mathbb{L}\textbf{-coalg} \to \mathbb{L}'\textbf{-coalg}, \mathbb{R}'\textbf{-alg} \to \mathbb{R}\textbf{-alg}$; i.e., defines a colax morphism of comonads and a lax morphism of monads

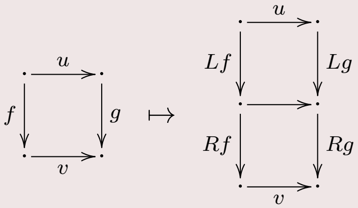
We will define morphisms between algebraic weak factorization systems on different categories lifting (two-variable) adjunctions.

Weak factorization systems

Definition

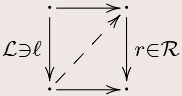
A weak factorization system (wfs) $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$:

- (factorization) there exists a functorial factorization $\mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$:



with $Lf \in \mathcal{L}, Rf \in \mathcal{R}$.

- (lifting) $\mathcal{L} \boxslash \mathcal{R}$:



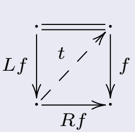
- (closure) furthermore $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$

Algebraic left and right maps

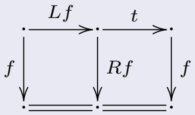
Left maps are coalgebras and right maps are algebras, resp., for the pointed endofunctors $L, R \colon \mathcal{M}^{\mathbf{2}} \rightrightarrows \mathcal{M}^{\mathbf{2}}$ with $\epsilon \colon L \Rightarrow 1$, $\eta \colon 1 \Rightarrow R$.

Algebraic right maps

$f \in \mathcal{R}$ iff



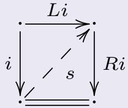
iff



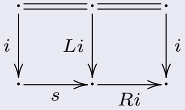
iff $f \in (R, \eta)$-alg

Algebraic left maps

$i \in \mathcal{L}$ iff



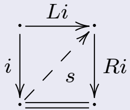
iff

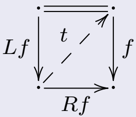


iff $i \in (L, \epsilon)$-coalg

Algebraic lifts

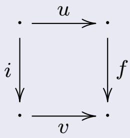
Recall

$i \in \mathcal{L}$ iff 

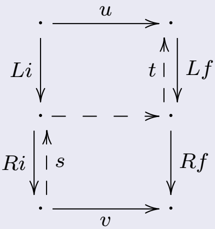
$f \in \mathcal{R}$ iff 

Constructing lifts

Given a coalgebra (i, s) and an algebra (f, t), any lifting problem



has a solution



Algebraic weak factorization systems

Definition (Grandis, Tholen)

An algebraic weak factorization system (awfs) $(\mathbb{L}, \mathbb{R})$ on a category $\mathcal{M}$:

- a comonad $\mathbb{L} = (L, \epsilon, \delta)$ and a monad $\mathbb{R} = (R, \eta, \mu)$

such that

- (L, ϵ) and (R, η) come from a functorial factorization
- the canonical map $LR \Rightarrow RL$ is a distributive law.

$\mathbb{L}$-coalgebras lift against $\mathbb{R}$-algebras—but so do (L, ϵ)-coalgebras and (R, η)-algebras. Hence the underlying wfs has

$$\mathcal{L} = \text{ retract closure of the } \mathbb{L}\text{-coalgebras}$$
$$\mathcal{R} = \text{ retract closure of the } \mathbb{R}\text{-algebras}$$

Cellularity

Cellular maps

A map in the left class of an underlying wfs of an awfs $(\mathbb{L}, \mathbb{R})$ is cellular if it admits an $\mathbb{L}$-coalgebra structure.

Examples

- In **Top**, there is an awfs such that the relative cell complexes are the cellular maps.
- In **sSet**, there is an awfs such that the left class is the monomorphisms, all of which are cellular.

Lemma (R.)

In a cofibrantly generated awfs, all right maps admit $\mathbb{R}$-algebra structures.

Cofibrantly generated algebraic weak factorization systems

Cofibrantly generated wfs

A wfs $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if there exists a set $\mathcal{J}$ such that $\mathcal{J}^{\boxslash} = \mathcal{R}$. Quillen's small object argument constructs the factorizations.

Theorem (Garner)

A small category of arrows $\mathcal{J}$ generates an awfs $(\mathbb{L}, \mathbb{R})$ such that

- there is a canonical isomorphism $\mathbb{R}\textbf{-alg} \cong \mathcal{J}^{\boxslash}$
- there exists a canonical functor $\mathcal{J} \to \mathbb{L}\textbf{-coalg}$ over $\mathcal{M}^{\mathbf{2}}$, universal among morphisms of awfs

This second universal property says

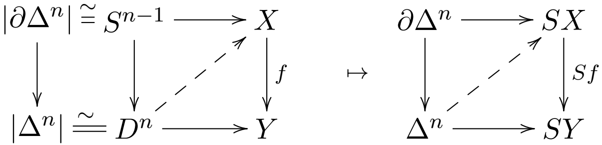
- morphisms of awfs $(\mathbb{L}, \mathbb{R}) \to (\mathbb{L}', \mathbb{R}') \leftrightsquigarrow \mathcal{J} \to \mathbb{L}'\textbf{-coalg}$
- i.e., a morphism exists iff the generators $\mathcal{J}$ are cellular for $\mathbb{L}'$.

A sample theorem

Theorem (R.)

$|-|\colon \mathbf{sSet} \rightleftarrows \mathbf{Top}\colon S$ is an adjunction of awfs.

- left class in $\mathbf{sSet}$ are the monomorphisms, all uniquely cellular
- map via $|-|$ to relative cell complexes with a specified coalgebra structure, here a cellular (in fact CW-) decomposition
- right class in $\mathbf{Top}$ are the algebraic trivial fibrations, equipped with chosen lifted contractions



- map via S to algebraic trivial fibrations with chosen sphere fillers

Toward adjunctions of awfs

Adjunctions interact well with ordinary wfs:

Given $F\colon \mathcal{K} \rightleftarrows \mathcal{M} \colon U$ and wfs on $\mathcal{K}$ and $\mathcal{M}$

- F preserves the left class iff U preserves the right class

in $\mathcal{M}$

in $\mathcal{K}$

In an adjunction of awfs, want:

- a lift of U to a functor between the categories of algebras
- a lift of F to a functor between the categories of coalgebras
- the lifts to somehow determine each other

One way to make this precise uses the theory of mates. Alternatively . . .

Awfs encoded as double categories

Lemma (Garner)

An awfs $(\mathbb{L}, \mathbb{R})$ gives rise to and can be recovered from either of two double categories $\mathbb{C}\mathbf{oalg}(\mathbb{L})$ or $\mathbb{A}\mathbf{lg}(\mathbb{R})$.

$$\mathbb{A}\mathbf{lg}(\mathbb{R}) : \qquad \mathbb{R}\text{-}\mathbf{alg} \times_{\mathcal{M}} \mathbb{R}\text{-}\mathbf{alg} \xrightarrow{\circ} \mathbb{R}\text{-}\mathbf{alg} \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{M} \quad (i : \mathcal{M} \to \mathbb{R}\text{-}\mathbf{alg})$$

- objects and horizontal 1-cells are the objects and morphisms of $\mathcal{M}$
- vertical 1-cells and squares the the objects and morphisms of $\mathbb{R}$-$\mathbf{alg}$

There is a forgetful double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{S}\mathbf{q}(\mathcal{M})$.

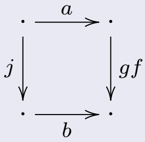
Vertical composition of awfs algebras and coalgebras

The essential point is that there is a canonical vertical composition law for algebras functorial with respect to $\mathbb{R}$-algebra morphisms:

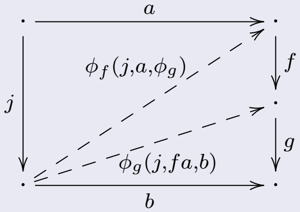
Example: $(\mathbb{L}, \mathbb{R})$ generated by $\mathcal{J}$

Algebra structures for $f, g \in \mathbb{R}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ are lifting functions ϕ_f, ϕ_g against all $j \in \mathcal{J}$.

Define ϕ_{gf} by solving



via



This composition law encodes the comultiplication for $\mathbb{L}$ (and dually).

Adjunctions of algebraic weak factorization systems

Lemma/Definition

Given an adjunction $F\colon \mathcal{K} \rightleftarrows \mathcal{M}\colon U$ together with awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$ and $(\mathbb{L}', \mathbb{R}')$ on $\mathcal{M}$, the following data are equivalent and define an adjunction of awfs $(F, U)\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{L}', \mathbb{R}')$.

- a double functor $\mathbb{C}\mathbf{oalg}(\mathbb{L}) \to \mathbb{C}\mathbf{oalg}(\mathbb{L}')$ lifting F
- a double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}') \to \mathbb{A}\mathbf{lg}(\mathbb{R})$ lifting U
- functors $F\colon \mathbb{L}\text{-}\mathbf{coalg} \to \mathbb{L}'\text{-}\mathbf{coalg}$ and $U\colon \mathbb{R}'\text{-}\mathbf{alg} \to \mathbb{R}\text{-}\mathbf{alg}$ whose characterizing natural transformations are mates

Corollary (composition criterion)

A lifted right adjoint $U\colon \mathbb{R}'\text{-}\mathbf{alg} \to \mathbb{R}\text{-}\mathbf{alg}$ defines an adjunction of awfs iff it preserves vertical composition of algebras.

The cellularity theorem

Theorem (R.)

Given $F\colon \mathcal{K} \rightleftarrows \mathcal{M}\colon U$, an awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$ generated by $\mathcal{J}$, an awfs $(\mathbb{L}', \mathbb{R}')$ on $\mathcal{M}$,

- $F \dashv U$ is an adjunction of awfs iff $F\mathcal{J}$ is cellular, i.e., iff there exists

$$\begin{array}{ccc} \mathcal{J} & \dashrightarrow & \mathbb{L}'\text{-}\mathbf{coalg} \\ \downarrow & & \downarrow \\ \mathcal{K}^{\mathbf{2}} & \xrightarrow{F} & \mathcal{M}^{\mathbf{2}} \end{array}$$

- Furthermore, the adjunction of awfs is determined by the coalgebra structures assigned to elements of $F\mathcal{J}$.

Corollary (R.)

The functor $\mathcal{J} \to \mathbb{L}\text{-}\mathbf{coalg}$ constructed by Garner's small object argument is universal among adjunctions of awfs.

Proof of the cellularity theorem

Proof:

- $$\begin{array}{ccc} F\mathcal{J}^{\boxslash} & \xrightarrow{\text{adj}} & \mathcal{J}^{\boxslash} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{M}^{\mathbf{2}} & \xrightarrow[U]{} & \mathcal{K}^{\mathbf{2}} \end{array}$$ is a pullback in $\mathbf{CAT}$
- define $\mathbb{R}'\text{-}\mathbf{alg} \to \mathbb{R}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ to be the composite

$$\mathbb{R}'\text{-}\mathbf{alg} \xrightarrow{\text{lift}} (\mathbb{L}'\text{-}\mathbf{coalg})^{\boxslash} \xrightarrow{\text{res}} (F\mathcal{J})^{\boxslash} \xrightarrow{\text{adj}} \mathcal{J}^{\boxslash}$$

- each functor preserves vertical composition

Two-variable adjunctions and enrichment

Definition

A two-variable adjunction consists of pointwise adjoint bifunctors

$$\otimes\colon \mathcal{K} \times \mathcal{M} \to \mathcal{N} \quad \hom_\ell\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{N} \to \mathcal{M} \quad \hom_r\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{N} \to \mathcal{K}$$

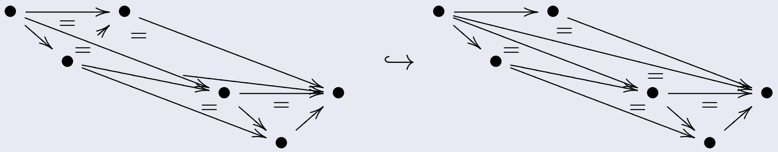
Examples

A closed monoidal structure $(\otimes, \hom_\ell, \hom_r)\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

A tensored and cotensored enriched category $(\odot, \{\}, \hom)\colon \mathcal{V} \times \mathcal{M} \to \mathcal{M}$.

Induced two-variable adjunctions

$(\hat{\otimes}, \hat{\hom}_\ell, \hat{\hom}_r)\colon \mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \to \mathcal{N}^{\mathbf{2}}$ e.g., $(\Lambda^2_1 \to \Delta^2)\hat{\otimes}(\partial\Delta^1 \to \Delta^1)$ is



Two-variable adjunctions of awfs

Definition (R.)

A two-variable adjunction of awfs consists of

- a two-variable adjunction $(\otimes, \hom_\ell, \hom_r) \colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$
- awfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$, $(\mathbb{L}', \mathbb{R}')$ on $\mathcal{M}$, and $(\mathbb{L}'', \mathbb{R}'')$ on $\mathcal{N}$
- lifted functors

$$
\begin{aligned}
-\hat{\otimes}- &\colon \mathbb{L}\textbf{-coalg} \times \mathbb{L}'\textbf{-coalg} \to \mathbb{L}''\textbf{-coalg} \\
\hat{\hom}_\ell(-,-) &\colon \mathbb{L}\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}''\textbf{-alg} \to \mathbb{R}'\textbf{-alg} \\
\hat{\hom}_r(-,-) &\colon \mathbb{L}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}''\textbf{-alg} \to \mathbb{R}\textbf{-alg}
\end{aligned}
$$

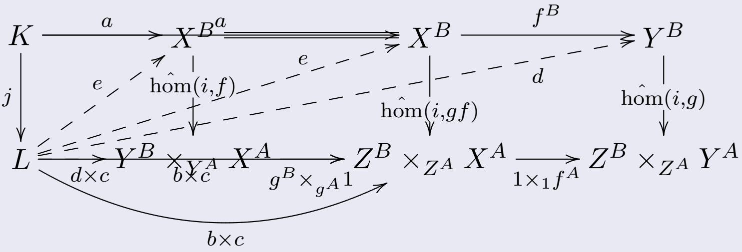
such that their characterizing natural transformations are parameterized mates.

Sadly, the lifted functors don't even preserve *composability* of (co)algebras.

The composition criterion

Theorem (R.)

A lifted functor $\hat{\hom}(-,-)\colon \mathbb{L}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}''\textbf{-alg} \to \mathbb{R}\textbf{-alg}$ determines a two-variable adjunction of awfs iff, given $i \in \mathbb{L}'\textbf{-coalg}$ and composable $f, g \in \mathbb{R}''\textbf{-alg}$, $\hat{\hom}(i, gf) \in \mathbb{R}\textbf{-alg}$ solves a lifting problem against $j \in \mathbb{L}\textbf{-coalg}$ as follows:



and also satisfies a dual condition in the first variable.

The cellularity theorem

Theorem (R.)

Given a two-variable adjunction $\otimes \colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$, awfs $(\mathbb{L}, \mathbb{R})$ and $(\mathbb{L}', \mathbb{R}')$ on $\mathcal{K}$ and $\mathcal{M}$ generated by $\mathcal{J}$ and $\mathcal{J}'$, and an awfs $(\mathbb{L}'', \mathbb{R}'')$ on $\mathcal{N}$,

- $\otimes$ is a two-variable adjunction of awfs iff $\mathcal{J} \hat{\otimes} \mathcal{J}'$ is cellular, i.e., iff there exists

$$\begin{array}{ccc} \mathcal{J} \times \mathcal{J}' & \dashrightarrow & \mathbb{L}''\textbf{-coalg} \\ \downarrow & & \downarrow \\ \mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} & \xrightarrow{\hat{\otimes}} & \mathcal{N}^{\mathbf{2}} \end{array}$$

- Furthermore, the two-variable adjunction of awfs is determined by the coalgebra structures assigned to elements of $\mathcal{J} \hat{\otimes} \mathcal{J}'$.

Sample Theorems (R.)

Quillen's model structure on $\mathbf{sSet}$ and the folk model structure on $\mathbf{Cat}$ are (cartesian) monoidal algebraic model structures.

Acknowledgments

Thanks

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Further details

Further details can be found in

- "Algebraic model structures" New York J. Math **17** (2011) 173-231
- "Monoidal algebraic model structures" a preprint available at `www.math.uchicago.edu/~eriehl`
- my Ph.D. thesis "Algebraic model structures" available at `www.math.uchicago.edu/~eriehl`