The formal theory of homotopy coherent monads

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Slogan: “It’s all in the weights!”
Plan

1. Homotopy coherent adjunctions
2. Homotopy coherent monads and the monadic adjunction
3. Codescent in the Eilenberg-Moore quasi-category
4. Monadicity theorem
Quasi-categories

A **quasi-category** is a simplicial set $A$ in which any inner horn

$$\Lambda^{n,k} \rightarrow A \quad 0 < k < n$$

has a filler.

The **homotopy category** $hA$ has

- objects = vertices
- morphisms = homotopy classes of 1-simplices

Via the adjunction

$$\text{Cat} \leftrightarrow_{h} \text{qCat}$$

quasi-category theory extends category theory.
Adjunctions of quasi-categories

$q\text{Cat}_2 := \text{the 2-category of quasi-categories}$, consisting of:

- quasi-categories $A, B$
- functors (maps of simplicial sets) $g : A \to B$
- natural transformations (homotopy classes of 1-simplices)

An adjunction of quasi-categories is an adjunction in $q\text{Cat}_2$.

\[ A \xleftarrow{f} \xrightarrow{\perp} B \quad \quad \eta : \text{id}_B \Rightarrow uf \quad \quad \epsilon : fu \Rightarrow \text{id}_A \]
Some theorems and examples

**Theorems.**

- $f \dashv u$ induces adjunctions $f^X \dashv u^X$ and $C^u \dashv C^f$ for any simplicial set $X$ and quasi-category $C$.
- Any equivalence can be promoted to an adjoint equivalence.
- Right adjoints preserve limits.
- $f : B \to A$ has a left adjoint iff $f \downarrow a$ has a terminal object for each $a \in A$.

**Examples.**

- ordinary adjunctions, topological adjunctions
- simplicial Quillen adjunctions
- colim $\dashv$ const $\dashv$ lim
- loops–suspension
A coherence question

$q\text{Cat}_{\infty} :=$ the simplicial category of quasi-categories.

Given $A \xleftarrow{u} \xrightarrow{\bot} B$ in $q\text{Cat}_2$, what adjunction data exists in $q\text{Cat}_{\infty}$?

- $\id_B \xrightarrow{\eta} uf$ in $B^B$
- $fu \xrightarrow{\epsilon} \id_A$ in $A^A$
- $\eta u u f u \xrightarrow{\alpha} u \epsilon$ in $B^A$
- $f u f \xrightarrow{\beta} \epsilon f$ in $A^B$
- filling $\Lambda^{3,1} \to B^B$

But do there exist fillers with the same bottom face?
The free adjunction

\[ \mathbf{Adj} : \text{the free adjunction}, \text{ a 2-category with} \]

- objects + and −
- \( \mathbf{Adj}(+, +) = \mathbf{Adj}(−, −)^{\text{op}} : = \Delta_+ \)
- \( \mathbf{Adj}(−, +) = \mathbf{Adj}(+, −)^{\text{op}} : = \Delta_\infty \)

**Theorem (Schanuel-Street).** 2-functors \( \mathbf{Adj} \to q\text{Cat}_2 \) correspond to adjunctions in \( q\text{Cat}_2 \).

\[
\begin{array}{c}
\text{id} \xrightarrow{\eta} u f \xleftarrow{u \epsilon} u f u f \xrightarrow{u f \eta} u f u f u f \xleftarrow{u f u \epsilon} u f u f u f u f \cdots \\
\text{u} \xleftarrow{u \epsilon} u f u \xrightarrow{u f \eta} u f u f \xrightarrow{u f u \epsilon} u f u f u f \xleftarrow{u f u f \epsilon} u f u f u f u f \cdots
\end{array}
\]
The free homotopy coherent adjunction

**Conjecture.** The free homotopy coherent adjunction is $\text{Adj}$, regarded as a simplicial category under $2\text{-Cat} \hookrightarrow \text{sSet-Cat}$.

$n$-arrows are **strictly undulating squiggles** on $n + 1$ lines

**Proposition.** $\text{Adj}$ is a simplicial computad (i.e., cofibrant).
**Theorem.** Any adjunction $\text{Adj} \to q\text{Cat}_2$ lifts to a homotopy coherent adjunction $\text{Adj} \to q\text{Cat}_\infty$.

**Theorem.** Such extensions are homotopically unique: the spaces of extensions are contractible Kan complexes.
Homotopy coherent monads

\( \text{Mnd} := \text{full subcategory of } \text{Adj} \text{ on } +. \)

**Definition.** A **homotopy coherent monad** is a simplicial functor \( T: \text{Mnd} \to \text{qCat}_\infty \), i.e.,

- \( + \mapsto B \in \text{qCat}_\infty \)
- \( \Delta_+ \overset{t}{\to} B^B =: \text{the monad resolution} \)

\[
\begin{array}{cccc}
\text{id}_B & \overset{\eta}{\longrightarrow} & t & \overset{\mu}{\leftarrow} & t^2 \\
& & \overset{t\eta}{\longrightarrow} & & \end{array}
\]

and higher data, e.g.,

\[
\begin{array}{ccc}
\eta_t & \overset{t^2}{\sim} & \mu \\
\end{array}
\]

\[
\begin{array}{cc}
t & \overset{\sim}{\longrightarrow} & t \\
\end{array}
\]
Weighted limits

Fix a simplicial functor $T$, a diagram of shape $A$.

A **weight** is a simplicial functor $W : A \to \text{sSet}$.

The **weighted limit** $\{W, T\}$ represents the simplicial set of cones of shape $W$ over $T$.

Key facts:
- The limit weighted by $\text{hom}_a$ evaluates at $a$.
- The weighted limit bifunctor is cocontinuous in the weights.

**Upshot:** Weights built by gluing representables will define cones of the expected shape.
**Proposition.** $\mathcal{q}\text{Cat}_\infty$ has all limits weighted by projective cofibrant simplicial functors.

\[ \begin{array}{ccc}
\mathbf{Mnd} & \xrightarrow{W_+} & \mathbf{sSet} \\
\downarrow & & \downarrow \text{hom}_+ \\
\text{Adj} & \xrightarrow{\text{hom}_-} & \mathbf{Mnd} \\
\uparrow & & \uparrow \text{hom}_- \\
\mathbf{Mnd} & \xrightarrow{W_-} & \\
\end{array} \]

\textbf{Adj} a simplicial computad $\Rightarrow W_+$ and $W_-$ projective cofibrant.
Fix a homotopy coherent monad $T : \text{Mnd} \to \text{qCat}_\infty$

- $\{W_+, T\} = B$
- $\{W_-, T\} =: B[t]$, the **Eilenberg-Moore quasi-category**

By definition

$$B[t] = \text{eq} \left( B^{\Delta_\infty} \Rightarrow B^{\Delta_+ \times \Delta_\infty} \right)$$

so a vertex is a map $\Delta_\infty \to B$ of the form:

```
    b \xrightarrow{\eta} tb \xleftarrow{\mu} t^2b \xrightarrow{t\eta} t^3b \cdots
```

and higher data, e.g.,

```
\eta \xrightarrow{tb} \beta
```

and

```
b \sim \beta
```
The monadic homotopy coherent adjunction

...is all in the weights!

\[
\begin{array}{ccccccc}
\mathbf{Adj}^{\text{op}} & \xrightarrow{\text{hom}} & \mathbf{sSet}^{\mathbf{Adj}} & \xrightarrow{\text{res}} & \mathbf{sSet}^{\mathbf{Mnd}} & \xrightarrow{\{-,T\}} & \mathbf{qCat}_{\infty}^{\text{op}} \\
- & \mapsto & \text{hom}_- & \mapsto & W_- & \mapsto & B[t] \\
f \downarrow \quad u & \mapsto & \downarrow & \\
+ & \mapsto & \text{hom}_+ & \mapsto & W_+ & \mapsto & B \\
\end{array}
\]

**Proposition.** If \( V \hookrightarrow W \) is identity-on-0-cells, then \( \{W, T\} \to \{V, T\} \) is conservative. E.g., \( W_+ \to W_- \).

**Corollary.** The monadic forgetful functor \( u^t : B[t] \to B \) is conservative.
Suppose \((b, \beta)\) is an algebra for a monad \(t\) on a category \(B\).

**Fact.** There is a canonical colimit diagram in \(B[t]\)

\[
\begin{array}{c}
\cdots \ t^3b \xrightarrow{t\beta} t^2b \xrightarrow{t\eta} tb \xrightarrow{\beta} b \\
\xleftarrow{tt\eta} \xleftarrow{tt\mu} \xleftarrow{t\eta} \xleftarrow{\mu} \xleftarrow{\eta}
\end{array}
\]

which is a \(u^t\)-split reflexive coequalizer diagram, and preserved by \(u^t\).

\[(b, \beta) \leadsto \text{a } u^t\text{-split (augmented) simplicial object}\]
**Theorem.** Every vertex in $B[t]$ is the colimit of a canonical $u^t$-split simplicial object that is preserved by $u^t$.

**Proof.** By cocontinuity,

\[
\begin{align*}
\Delta_+^\op \times W_+ & \longrightarrow \Delta_\infty \times W_+ \\
\downarrow & \quad \downarrow \\
\Delta_+^\op \times W_- & \longrightarrow W \\
\Delta_\infty^\op \times W & \longrightarrow B[t]^{\Delta_+^\op}
\end{align*}
\]

\[
\begin{align*}
\{W, T\} & \longrightarrow B[t]^{\Delta_+^\op} \\
\downarrow & \quad \downarrow u^t \\
B^{\Delta_\infty} & \longrightarrow B^{\Delta_+^\op}
\end{align*}
\]

\[
\begin{align*}
\{W, T\} & \longrightarrow B[t]^{\Delta_+^\op} \\
\downarrow & \quad \downarrow \text{const} \\
\{W, T\} & \longrightarrow B[t]^{\Delta_+^\op}
\end{align*}
\]

$B[t]$ in $q\text{Cat}_2$. 

\[
\begin{align*}
\{W, T\} & \longrightarrow B[t]^{\Delta_+^\op} \\
\downarrow & \quad \downarrow \text{const} \\
\{W, T\} & \longrightarrow B[t]^{\Delta_+^\op}
\end{align*}
\]
Codescent in the Eilenberg-Moore quasi-category

**Theorem.**

\[
\{W, T\} \rightarrow B[t]^{\Delta^+} \xrightarrow{\text{res}} B[t]^{\Delta^{\text{op}}}
\]

defines an absolute left lifting diagram in \(q\text{Cat}_2\) that \(u^t\) preserves.

**Proof.** Similar to:

**Theorem.**

\[
B^{\Delta^{\infty}} \xrightarrow{\text{res}} B^{\Delta^{\text{op}}}
\]

defines an absolute left lifting diagram in \(q\text{Cat}_2\) that is preserved by any functor.

**Proof.** See “The 2-category theory of quasi-categories.”
The classical monadicity theorem

Let $t$ be the monad induced by an adjunction $f \dashv u$.

**Theorem** (Beck).

- There is a comparison functor commuting with the adjunctions.

\[
\begin{array}{ccc}
A & \xleftarrow{u} & B[t] \\
\downarrow^f & & \downarrow_{f^t} \\
B & \xleftarrow{u^t} & B
\end{array}
\]

- If $A$ has $u$-split coequalizers, then $R$ has a left adjoint.
- If $u$ preserves them, then $L$ is fully faithful.
- If $u$ is conservative, then $L \dashv R$ is an adjoint equivalence.

**Goal.** Prove the analogous theorem for the homotopy coherent monad of a homotopy coherent adjunction.
Defining the comparison map

\[ \text{Adj} \xrightarrow{H} \text{qCat}_\infty \]

\[ \text{Mnd} \xrightarrow{T} \sim \quad B[t] \cong \{W_-, \text{res } H\} \cong \{\text{lan } W_-, H\} \]

Weights for the monadic adjunction, revisited.

- weight for the Eilenberg-Moore quasi-category: \( \text{lan } \text{res } \text{hom}_- \)
- weight for the monadic adjunction: \( \text{lan } \text{res } \text{hom} \)

The counit of \( \text{sSet}^{\text{Adj}} \xrightarrow{\text{res}} \text{sSet}^{\text{Mnd}} \) defines a map of weights \( \text{lan } \text{res } \text{hom} \rightarrow \text{hom} \) and hence a natural transformation

\[
\begin{array}{c}
A \xrightarrow{R} B[t] \\
\downarrow \quad \downarrow \\
\text{B} & \text{B}
\end{array}
\]

between homotopy coherent adjunctions.
The weight for $u$-split simplicial objects

Define a weight

\[
\Delta^{\text{op}} \times \text{hom}_+ \to \Delta_\infty \times \text{hom}_+ \\
\downarrow \quad \quad \downarrow \\
\Delta^{\text{op}} \times \text{hom}_- \to W'
\]

\[
\{W', H\} \to A^{\Delta^{\text{op}}} \\
\downarrow \quad \quad \downarrow u \\
B^{\Delta_\infty_{\text{res}}} \to B^{\Delta^{\text{op}}}
\]

**Definition.** The quasi-category $A$ admits colimits of $u$-split simplicial objects if there is an absolute left lifting diagram

\[
\begin{align*}
\{W', H\} & \to A^{\Delta^{\text{op}}} \\
\uparrow \quad \quad \downarrow \text{const} \\
\text{colim} & \to A \\
\end{align*}
\]

in $\text{qCat}_2$. 
The proof of the monadicity theorem

Proof.

- The obvious map $W' \to \text{lan} W_-$ induces $B[t] \to \{W', H\}$.
- If $A$ has colimits of $u$-split simplicial objects, define $L := B[t] \to \{W', H\} \text{ colim } A$.
- From the universal property of absolute left liftings, $L \dashv R$.
- If $u$ preserves these colimits, then $u^t$ carries the unit of $L \dashv R$ to an isomorphism.
- As $u^t$ is conservative, the unit is an isomorphism.
- If $u$ is conservative, it follows that the counit is also an isomorphism, and $A \simeq B[t]$ is an adjoint equivalence of quasi-categories.
Further reading

- “A weighted limits proof of monadicity” on the $n$-category café.
- “Homotopy coherent adjunctions and the formal theory of monads” — coming soon!