Elements of ∞-Category Theory

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We would like to dedicate this book to our supervisors: J.M.E. Hyland and J.P. May.

Peter —

My debt to you is evident in everything that I write. I found it astounding that, after each of three or four consecutive close-readings of my thesis in which you suggested complete structural revisions, your advice remained entirely correct every time. You taught me to derive immense pleasure from each comprehensive reappraisal that is demanded whenever deepened mathematical understanding allows a more fundamental narrative to emerge.

By that standard, this text isn't ready yet. Nevertheless, in honor of your 80th birthday, I wanted to dedicate an early draft to you. I'm hoping you'll like the final version.

— Emily

Contents

Preface

Mathematical objects of a certain sophistication are frequently accompanied by higher homotopical structures in the sense that the maps between them might be connected by homotopies, which might then be connected by higher homotopies, which might then be connected by even higher homotopies ad infinitum. In such contexts, the natural habitat for these mathematical objects is not an ordinary 1-category but instead an ∞*-category* or, more precisely, an $(\infty, 1)$*-category*, with the index "1" referring to the fact that its morphisms above the lowest dimension 1, the homotopies just discussed, are invertible.

Here the homotopies defining the higher morphisms of an ∞-category are to be regarded as *data* rather than as mere witnesses to an equivalence relation borne by the 1-dimensional morphisms. This has the consequence that all of the categorical structures in an ∞-category are weak. Even at the base level of 1-morphisms, composition is not necessarily uniquely defined but instead witnessed by a 2-morphism and associative up to a 3-morphism whose boundary data involves specified 2-morphism witnesses. Thus, diagrams valued in an ∞-category cannot be said to *commute* on the nose but are instead interpreted as *homotopy coherent*.

A fundamental challenge in defining ∞-categories has to do with giving a precise mathematical meaning of this notion of a weak composition law, not just for the 1-morphisms but also for the morphisms in higher dimensions. This is achieved through a variety of *models* of $(\infty, 1)$-categories, which are Bourbaki-style mathematical structures that encode infinite-dimensional categories with a weak composition law in which all morphisms above dimension 1 are weakly invertible. In order of appearance, these include *simplicial categories*, *quasi-categories* (nee. *weak Kan complexes*), *relative categories*, *Segal categories*, *complete Segal spaces*, and *1-complicial sets* (nee. *saturated 1-trivial weak complicial sets*), each of which comes with an associated array of naturally-occurring examples. The proliferation of models of $(\infty, 1)$-categories begs the question of how they might be compared. In the first decades of the 21st century, Bergner, Joyal–Tierney, Verity, Lurie, and Barwick–Kan built various bridges that prove that each of the models listed above "has the same homotopy theory" in the sense of defining the fibrant objects in Quillen equivalent model categories.[1]

In parallel with the development of models of $(\infty, 1)$-categories and the construction of comparisons between them, Joyal pioneered and Lurie and many others extended a wildly successful project to extend basic category theory from ordinary 1-categories to $(\infty, 1)$-categories modeled as quasi-categories in such a way that the new quasi-categorical notions restrict along the standard embedding $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ to the standard 1-categorical notions. A natural question is then: does this work extend to other models of $(\infty, 1)$-categories? And to what extent are basic ∞-categorical notions invariant under change of model? For instance, $(\infty, 1)$-categories of manifolds are most naturally constructed as complete Segal spaces, so Kazhdan–Varshavsky **[51]** and Rasekh **[73, 72]** have recently endeavored to redevelop some of the category theory of quasi-categories using complete Segal space instead so as

[1]A recent book by Bergner surveys all but the last of these models and their interrelationships **[12]**. For a more whirlwind tour, see **[21]**.

to have direct access to constructions and definitions that had previously been introduced only in the quasi-categorical model.

For practical, aesthetic, and moral reasons, the ultimate desire of practitioners is to work "model independently," meaning that theorems proven with any of the models of $(\infty, 1)$-categories would apply to them all, with the technical details inherent to any particular model never entering the discussion. Since all models of $(\infty, 1)$-categories "have the same homotopy theory" the general consensus is that the choice of model should not matter greatly, but one obstacle to proving results of this kind is that, to a large extent, precise versions of the categorical definitions that have been established for quasi-categories had not been given for the other models. In cases where comparable definitions do exist in different models, an ad-hoc heuristic proof of model-invariance of the categorical notion in question can typically be supplied, with details to be filled in by experts fluent in the combinatorics of each model, but it would be more reassuring to have a systematic method of comparing the category theory of $(\infty, 1)$-categories in different models via arguments that are somewhat closer to the ground.

Aims of this text

In this text we develop the theory of ∞-categories from first principles in a model-independent fashion using a common axiomatic framework that is satisfied by a variety of models. In contrast with prior "analytic" treatments of the theory of ∞-categories — in which the central categorical notions are defined in reference to the coordinates of a particular model — our approach is "synthetic," proceeding from definitions that can be interpreted simultaneously in many models to which our proofs then apply. While synthetic, our work is not schematic or hand-wavy, with the details of how to make things fully precise left to "the experts" and turtles all the way down.[2] Rather, we prove our theorems starting from a short list of clearly-enumerated axioms, and our conclusions are valid in any model of ∞-categories satisfying these axioms.

The synthetic theory is developed in any *∞-cosmos*, which axiomatizes the universe in which ∞-categories live as objects. So that our theorem statements suggest their natural interpretation, we recast *∞-category* as a technical term, to mean an object in some (typically fixed) ∞-cosmos. Several models of $(\infty, 1)$-categories[3] are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories[4] as well as fibered versions of all of the above. Thus each of these objects are ∞-categories in our sense and our theorems apply to all of them.[5] This usage is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional categories, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

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[2]A less rigorous "model-independent" presentation of ∞-category theory might confront a problem of infinite regress, since infinite-dimensional categories are themselves the objects of an ambient infinite-dimensional category, and in developing the theory of the former one is tempted to use the theory of the latter. We avoid this problem by using a very concrete model for the ambient $(\infty, 2)$-category of ∞-categories that arises frequently in practice and is designed to facilitate relatively simple proofs. While the theory of $(\infty, 2)$-categories remains in its infancy, we are content to cut the Gordian knot in this way.

[3]Quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets (naturally marked quasi-categories) all define the ∞-categories in an ∞-cosmos.

[4]Θ_n-spaces, iterated complete Segal spaces, and n-complicial sets also define the ∞-categories in an ∞-cosmos, as do (nee. weak) complicial sets, a model for (∞, ∞)-categories. We hope to add other models of (∞, n)-categories to this list.

[5]This may seem like sorcery but in some sense is really just the Yoneda lemma. To a close approximation, an ∞-cosmos is a "category of fibrant objects enriched over quasi-categories." When the Joyal–Lurie theory of quasi-categories is expressed in a sufficiently categorical way, it extends to encompass analogous results for the corresponding "representably defined" notions in a general ∞-cosmos.

Much of the development of the theory of ∞-categories takes place not in the full ∞-cosmos but in a quotient that we call the *homotopy 2-category*, the name chosen because an ∞-cosmos is something like a category of fibrant objects in an enriched model category and the homotopy 2-category is then a categorification of its homotopy category. The homotopy 2-category is a strict 2-category — like the 2-category of categories, functors, and natural transformations[6] — and in this way the foundational proofs in the theory of ∞-categories closely resemble the classical foundations of ordinary category theory except that the universal properties that characterize, e.g. when a functor between ∞-categories defines a cartesian fibration, are slightly weaker than in the familiar case of strict 1-categories.

In Part I, we define and develop the notions of equivalence and adjunction between ∞-categories, limits and colimits in ∞-categories, cartesian and cocartesian fibrations and their discrete variants, and prove an external version of the Yoneda lemma all from the comfort of the homotopy 2-category. In Part II, we turn our attention to homotopy coherent structures present in the full ∞-cosmos to define and study homotopy coherent adjunctions and monads borne by ∞-categories as a mechanism for universal algebra. We also give a more careful account of the full class of limit constructions present in any ∞-cosmos and use this analysis to develop further examples of ∞-cosmoi, whose objects are pointed ∞-categories, or stable ∞-categories, or cartesian or cocartesian fibrations in a given ∞-cosmos.

What's missing from this basic account of the category theory of ∞-categories is a satisfactory treatment of the "hom" bifunctor associated to an ∞-category, which is the prototypical example of what we call a *module*. In Part III, we develop the calculus of modules between ∞-categories and apply this to define and study pointwise Kan extensions. This will give us an opportunity to repackage universal properties proven in Part I as parts of the "formal category theory" of ∞-categories.

This work is all "model-agnostic" in the sense of being blind to details about the specifications of any particular ∞-cosmos. In Part IV we prove that the category theory of ∞-categories is also "model-independent" in a precise sense: all categorical notions are preserved, reflected, and created by any "change-of-model" functor that defines what we call a *biequivalence*. This model-independence theorem is stronger than our axiomatic framework might initially suggest in that it also allows us to transfer theorems proven using "analytic" techniques to all biequivalent ∞-cosmoi. For instance, the four ∞-cosmoi whose objects model $(\infty, 1)$-categories are all biequivalent. It follows that the analytically-proven theorems about quasi-categories from **[61]** transfer to complete Segal spaces, and vice versa. We conclude in Part V by developing more sophisticated Yoneda-type structures for ∞-cosmoi of $(\infty, 1)$-categories using a mix of synthetic and analytic methods.

The ideal reader might already have some acquaintance with enriched category theory, with 2-category theory, and with abstract homotopy theory so that the constructions and proofs with antecedents in these traditions will be familiar. Because ∞-categories are of interest to mathematicians with a wide variety of backgrounds, we review all of the material we need on each of these topics in Appendices A, B, and C respectively. Some basic facts about quasi-categories first proven by Joyal are needed to establish the corresponding features of general ∞-cosmoi in Chapter 1. We state all of these results in §1.1 but defer the proofs that require a lengthy combinatorial digression to Appendix D, where we also introduce *n-complicial sets*, a model of (∞, n)-categories for any $0 \leq n \leq \infty$. The proofs that many examples of ∞-cosmoi appear "in the wild" can be found in Appendix E, where we also present general techniques that the reader might use to find even more examples. The final appendix addresses a crucial bit of unfinished business. Importantly, the synthetic theory developed in the ∞-cosmos

[6]In fact this is another special case: there is an ∞-cosmos whose objects are ordinary categories and its homotopy 2-category is the usual category of categories, functors, and natural transformations.

of quasi-categories is fully compatible with the analytic theory developed by Joyal, Lurie, and many others. This is the subject of Appendix F.

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Part I

Basic ∞-category theory

CHAPTER 1

∞-Cosmoi and their homotopy 2-categories

1.1. Quasi-categories

Before introducing an axiomatic framework that will allow us to develop ∞-category theory in general, we first consider one model in particular: namely, *quasi-categories*, which were first analyzed by Joyal in [**48**] and [**49**] and in several unpublished draft book manuscripts.

1.1.1. Notation (the simplex category). Let $\mathbf{\Delta}$ denote the **simplex category** of finite non-empty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps. These include in particular the

$$\begin{array}{llll} \text{elementary face operators} & [n-1] \xrightarrow{\delta^i} [n] & 0 \le i \le n & \text{and the} \\ \text{elementary degeneracy operators} & [n+1] \xrightarrow{\sigma^i} [n] & 0 \le i \le n & \end{array}$$

whose images respectively omit and double up on the element $i \in [n]$. Every morphism in $\mathbf{\Delta}$ factors uniquely as an epimorphism followed by a monomorphism; these epimorphisms, the **degeneracy operators**, decompose as composites of elementary degeneracy operators, while the monomorphisms, the **face operators**, decompose as composites of elementary face operators.

The category of **simplicial sets** is the category $s\mathcal{S}et \coloneqq \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ of presheaves on the simplex category. We write $\Delta[n]$ for the **standard n-simplex**, the simplicial set represented by $[n] \in \mathbf{\Delta}$, and $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$ for its k-**horn** and **boundary sphere** respectively.

Given a simplicial set X, it is conventional to write X_n for the set of n-**simplices**, defined by evaluating at $[n] \in \mathbf{\Delta}$. By the Yoneda lemma, each n-simplex $x \in X_n$ corresponds to a map of simplicial sets $x\colon \Delta[n] \to X$. Accordingly, we write $x \cdot \delta^i$ for the ith face of the n-simplex, an $(n-1)$-simplex classified by the composite map

$$\Delta[n-1] \xrightarrow{\delta^i} \Delta[n] \xrightarrow{x} X.$$

Geometrically, $x \cdot \delta^i$ is the "face opposite the vertex i" in the n-simplex x.

Since the morphisms of $\mathbf{\Delta}$ are generated by the elementary face and degeneracy operators, the data of a simplicial set[1] X is often presented by a diagram

$$\cdots \qquad X_3 \begin{smallmatrix} \xrightarrow{\delta_3} \\ \xleftarrow{\sigma_2} \\ \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix} X_2 \begin{smallmatrix} \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix} X_1 \begin{smallmatrix} \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{smallmatrix} X_0,$$

identifying the set of n-simplices for each $[n] \in \mathbf{\Delta}$ as well as the (contravariant) actions of the elementary operators, conventionally denoted using subscripts.

[1]This presentation is also used for more general simplicial *objects* valued in any category.

1.1.2. Definition (quasi-category). A **quasi-category** is a simplicial set A in which any **inner horn** can be extended to a simplex, solving the displayed lifting problem:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad n \geq 2,\ 0 < k < n. \tag{1.1.3}$$

Quasi-categories were first introduced by Boardman and Vogt [**16**] under the name "weak Kan complexes," a **Kan complex** being a simplicial set admitting extensions as in (1.1.3) along all horn inclusions $n \geq 1, 0 \leq k \leq n$. Since any topological space can be encoded as a Kan complex,[2] in this way spaces provide examples of quasi-categories.

Categories also provide examples of quasi-categories via the nerve construction.

1.1.4. Definition (nerve). The category $\mathcal{C}at$ of 1-categories embeds fully faithfully into the category of simplicial sets via the **nerve** functor. An n-simplex in the nerve of a 1-category C is a sequence of n composable arrows in C, or equally a functor $[n] \to C$ from the ordinal category $\mathbb{n}+1 \coloneqq [n]$ with objects $0, \ldots, n$ and a unique arrow $i \to j$ just when $i \leq j$.

1.1.5. Remark. The nerve of a category C is 2-coskeletal as a simplicial set, meaning that every sphere $\partial\Delta[n] \to C$ with $n \geq 3$ is filled uniquely by an n-simplex in C (see Definition C.5.2). This is because the simplices in dimension 3 and above witness the associativity of the composition of the path of composable arrows found along their **spine**, the 1-skeletal simplicial subset formed by the edges connecting adjacent vertices. In fact, as suggested by the proof of the following proposition, any simplicial set in which inner horns admit *unique* fillers is isomorphic to the nerve of a 1-category; see Exercise 1.1.iii.

We decline to introduce explicit notation for the nerve functor, preferring instead to identify 1-categories with their nerves. As we shall discover the theory of 1-categories extends to ∞-categories modeled as quasi-categories in such a way that the restriction of each ∞-categorical concept along the nerve embedding recovers the corresponding 1-categorical concept. For instance, the standard simplex $\Delta[n]$ is the nerve of the ordinal category $\mathbb{n}+1$, and we frequently adopt the latter notation — writing $\mathbb{1} \coloneqq \Delta[0]$, $\mathbb{2} \coloneqq \Delta[1]$, $\mathbb{3} \coloneqq \Delta[2]$, and so on — to suggest the correct categorical intuition.

To begin down this path, we must first verify the assertion that has implicitly just been made:

1.1.6. Proposition (nerves are quasi-categories). *Nerves of categories are quasi-categories.*

Proof. Via the isomorphism $C \cong \mathrm{cosk}_2 C$ and the adjunction $\mathrm{sk}_2 \dashv \mathrm{cosk}_2$ of C.5.2, the required lifting problem displayed below-left transposes to the one displayed below-right

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & C \cong \mathrm{cosk}_2 C \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \mathrm{sk}_2 \Lambda^k[n] \arrow[r] \arrow[d, hook] & C \\ \mathrm{sk}_2 \Delta[n] \arrow[ur, dashed] \end{tikzcd}$$

[2]The total singular complex construction defines a functor from topological spaces to simplicial sets that is an equivalence on their respective homotopy categories — weak homotopy types of spaces correspond to homotopy equivalence classes of Kan complexes.

For $n \geq 4$, the inclusion $\mathrm{sk}_2\,\Lambda^k[n] \hookrightarrow \mathrm{sk}_2\,\Delta[n]$ is an isomorphism, in which case the lifting problems on the right admit (unique) solutions. So it remains only to solve the lifting problems on the left in the cases $n = 2$ and $n = 3$.

To that end consider

$$
\begin{tikzcd}
\Lambda^1[2] \arrow[r] \arrow[d, hook] & C \\
\Delta[2] \arrow[ur, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^1[3] \arrow[r] \arrow[d, hook] & C \\
\Delta[3] \arrow[ur, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2[3] \arrow[r] \arrow[d, hook] & C \\
\Delta[3] \arrow[ur, dashed]
\end{tikzcd}
$$

An inner horn $\Lambda^1[2] \to C$ defines a composable pair of arrows in C; an extension to a 2-simplex exists precisely because any composable pair of arrows admits a (unique) composite.

An inner horn $\Lambda^1[3] \to C$ specifies the data of three composable arrows in C, as displayed in the diagram below, together with the composites gf, hg, and $(hg)f$.

$$
\begin{tikzcd}
& c_1 \arrow[dr, "hg"] \arrow[dd, "g" near end] & \\
c_0 \arrow[ur, "f"] \arrow[rr, "(hg)f" near end] \arrow[dr, "gf"'] & & c_3 \\
& c_2 \arrow[ur, "h"'] &
\end{tikzcd}
$$

Because composition is associative, the arrow $(hg)f$ is also the composite of gf followed by h, which proves that the 2-simplex opposite the vertex c_1 is present in C; by 2-coskeletality, the 3-simplex filling this boundary sphere is also present in C. The filler for a horn $\Lambda^2[3] \to C$ is constructed similarly. $\square$

1.1.7. Definition (homotopy relation on 1-simplices). A parallel pair of 1-simplices f, g in a simplicial set X are **homotopic** if there exists a 2-simplex of either of the following forms

$$
\begin{tikzcd}
& x_1 \arrow[dr, equal] & \\
x_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & x_1
\end{tikzcd}
\qquad
\begin{tikzcd}
& x_0 \arrow[dr, "f"] & \\
x_0 \arrow[ur, equal] \arrow[rr, "g"'] & & x_1
\end{tikzcd}
\tag{1.1.8}
$$

or if f and g are in the same equivalence class generated by this relation.

In a quasi-category, the relation witnessed by any of the types of 2-simplex on display in (1.1.8) is an equivalence relation and these equivalence relations coincide:

1.1.9. Lemma (homotopic 1-simplices in a quasi-category). *Parallel 1-simplices f and g in a quasi-category are homotopic if and only if there exists a 2-simplex of any or equivalently all of the forms displayed in* (1.1.8).

Proof. Exercise 1.1.i. $\square$

1.1.10. Definition (the homotopy category). By 1-truncating, any simplicial set X has an underlying reflexive directed graph

$$
\begin{tikzcd}
X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0, \arrow[l, "\sigma_0" description]
\end{tikzcd}
$$

the 0-simplices of X defining the "objects" and the 1-simplices defining the "arrows," by convention pointing from their 0th vertex (the face opposite 1) to their 1st vertex (the face opposite 0). The **free**

category on this reflexive directed graph has X_0 as its object set, degenerate 1-simplices serving as identity morphisms, and non-identity morphisms defined to be finite directed paths of non-degenerate 1-simplices. The **homotopy category** $\mathsf{h}X$ of X is the quotient of the free category on its underlying reflexive directed graph by the congruence[3] generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

This implies in particular that homotopic 1-simplices represent the same arrow in the homotopy category.

1.1.11. Proposition. *The nerve embedding admits a left adjoint, namely the functor which sends a simplicial set to its homotopy category:*

$$\begin{tikzcd} \mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathsf{h}"'] \end{tikzcd}$$

Proof. Using the description of $\mathsf{h}X$ as a quotient of the free category on the underlying reflexive directed graph of X, we argue that the data of a functor $\mathsf{h}X \to C$ can be extended uniquely to a simplicial map $X \to C$. Presented as a quotient in this way, the functor $\mathsf{h}X \to C$ defines a map from the 1-skeleton of X into C, and since every 2-simplex in X witnesses a composite in $\mathsf{h}X$, this map extends to the 2-skeleton. Now C is 2-coskeletal, so via the adjunction $\mathsf{sk}_2 \dashv \mathsf{cosk}_2$ of Definition C.5.2, this map from the 2-truncation of X into C extends uniquely to a simplicial map $X \to C$. ☐

The homotopy category of a quasi-category admits a simplified description.

1.1.12. Lemma (the homotopy category of a quasi-category). *If A is a quasi-category then its* ***homotopy category*** *$\mathsf{h}A$ has*

- *the set of 0-simplices A_0 as its objects*
- *the set of homotopy classes of 1-simplices A_1 as its arrows*
- *the identity arrow at $a \in A_0$ represented by the degenerate 1-simplex $a \cdot \sigma_0 \in A_1$*
- *a composition relation $h = g \circ f$ in $\mathsf{h}A$ if and only if, for any choices of 1-simplices representing these arrows, there exists a 2-simplex with boundary*

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

Proof. Exercise 1.1.ii. ☐

1.1.13. Definition (isomorphisms in a quasi-category). A 1-simplex in a quasi-category is an **isomorphism** just when it represents an isomorphism in the homotopy category. By Lemma 1.1.12 this means that $f : a \to b$ is an isomorphism if and only if there exists a 1-simplex $f^{-1} : b \to a$ together with a pair of 2-simplices

$$\begin{tikzcd} & b \arrow[dr, "f^{-1}"] & \\ a \arrow[ur, "f"] \arrow[rr, equal] & & a \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dr, "f"] & \\ b \arrow[ur, "f^{-1}"] \arrow[rr, equal] & & b \end{tikzcd}$$

[3] A relation on parallel pairs of arrows of a 1-category is a **congruence** if it is an equivalence relation that is closed under pre- and post-composition: if $f \sim g$ then $hfk \sim hgk$.

The properties of the isomorphisms in a quasi-category are most easily proved by arguing in a slightly different category where simplicial sets have the additional structure of a "marking" on a specified subset of the 1-simplices subject to the condition that all degenerate 1-simplices are marked; maps of these so-called **marked simplicial sets** must then preserve the markings. Because these objects will seldom appear outside of the proofs of certain combinatorial lemmas about the isomorphisms in quasi-categories, we save the details for Appendix D.

Let us now motivate the first of several results proven using marked techniques. Quasi-categories are defined to have extensions along all *inner horns*. But if in an outer horn $\Lambda^0[2] \to A$ or $\Lambda^2[2] \to A$, the initial or final edges, respectively, are isomorphisms, then intuitively a filler should exist

$$\begin{tikzcd} & a_1 \arrow[dr, dashed, "hf^{-1}"] & \\ a_0 \arrow[ur, "f", "\sim"'] \arrow[rr, "h"'] & & a_2 \end{tikzcd} \qquad \begin{tikzcd} & a_1 \arrow[dr, "g", "\sim"'] & \\ a_0 \arrow[ur, dashed, "g^{-1}h"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

and similarly for the higher-dimensional outer horns.

1.1.14. Proposition (special outer horn lifting).

(i) Let A be a quasi-category. Then for $n \geq 2$ any outer horns

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "g"] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r, "h"] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.

(ii) Let A and B be quasi-categories and $f \colon A \to B$ a map that lifts against the inner horn inclusions. Then for $n \geq 2$ any outer horns

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "g"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta[n] \arrow[ur, dashed] \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r, "h"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta[n] \arrow[ur, dashed] \arrow[r] & B \end{tikzcd}$$

in which the edges $g|_{\{0,1\}}$ and $h|_{\{n-1,n\}}$ are isomorphisms admit fillers.

The proof of Proposition 1.1.14 requires clever combinatorics, due to Joyal, and is deferred to Proposition D.4.5 and Theorem D.4.18 in Appendix D.[4] Here, we enjoy its myriad consequences. Immediately:

1.1.15. Corollary. *A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.*

Proof. If the homotopy category of a quasi-category is a groupoid, then all of its 1-simplices are isomorphisms, and Proposition 1.1.14 then implies that all inner and outer horns have fillers. Thus, the quasi-category is a Kan complex. Conversely, in a Kan complex, all outer horns can be filled and in particular fillers for the horns $\Lambda^0[2]$ and $\Lambda^2[2]$ can be used to construct left and right inverses for any 1-simplex of the form displayed in Definition 1.1.13.[5] □

[4] The second statement subsumes the first, but the first is typically used to prove the second.

[5] In a quasi-category, any left and right inverses to a common 1-simplex are homotopic, but as Corollary 1.1.16 proves, any isomorphism in fact has a single two-sided inverse.

A quasi-category contains a canonical **maximal sub Kan complex**, the simplicial subset spanned by those 1-simplices that are isomorphisms. Just as the arrows in a quasi-category A are represented by simplicial maps $\mathbb{2} \to A$ whose domain is the nerve of the free-living arrow, the isomorphisms in a quasi-category are represented by diagrams $\mathbb{I} \to A$ whose domain is the free-living isomorphism:

1.1.16. Corollary. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a **homotopy coherent isomorphism***

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I} \arrow[ur, dashed] \end{tikzcd}$$

Proof. If f is an isomorphism, the map $f: \mathbb{2} \to A$ lands in the maximal sub Kan complex contained in A. The postulated extension also lands in this maximal sub Kan complex because the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ can be expressed as a sequential composite of outer horn inclusions; see Exercise 1.1.iv. ☐

The category of simplicial sets, like any category of presheaves, is cartesian closed. By the Yoneda lemma and the defining adjunction, an n-simplex in the exponential Y^X corresponds to a simplicial map $X \times \Delta[n] \to Y$, and its faces and degeneracies are computed by precomposing in the simplex variable. Our aim is now to show that the quasi-categories define an exponential ideal in the simplicially enriched category of simplicial sets: if X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. We will deduce this as a corollary of the "relative" version of this result involving a class of maps called isofibrations that we now introduce.

1.1.17. Definition (isofibrations between quasi-categories). A simplicial map $f: A \to B$ is a **isofibration** if it lifts against the inner horn inclusions, as displayed below left, and also against the inclusion of either vertex into the free-standing isomorphism $\mathbb{I}$.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

To notationally distinguish the isofibrations, we depict them as arrows "$\twoheadrightarrow$" with two heads.

By Theorem D.4.18, the isofibrations between quasi-categories can be understood as those maps that admit fillers for all inner horns as well as special outer horns in dimension $n \geq 1$, as opposed to only those horns with $n \geq 2$ appearing in the statement of Proposition 1.1.14.

1.1.18. Observation.

(i) For any simplicial set X, the unique map $X \to *$ whose codomain is the terminal simplicial set is an isofibration if and only if X is a quasi-category.
(ii) Any class of maps characterized by a right lifting property is automatically closed under composition, product, pullback, retract, and limits of towers; see Lemma C.2.3.
(iii) Combining (i) and (ii), if $A \twoheadrightarrow B$ is an isofibration, and B is a quasi-category, then so is A.
(iv) The isofibrations generalize the eponymous categorical notion. The nerve of any functor $f: A \to B$ between categories defines a map of simplicial sets that lifts against the inner horn inclusions. This map then defines an isofibration if and only if given any isomorphism in

B and specified object in A lifting either its domain or codomain, there exists an isomorphism in A with that domain or codomain lifting the isomorphism in B.

We typically only deploy the term "isofibration" for a map between quasi-categories because our usage of this class of maps intentionally parallels the classical categorical case.

Much harder to establish is the stability of the class of isofibrations under forming "Leibniz exponentials" as displayed in (1.1.20). The proof of this result is given in Proposition D.5.1 in Appendix D.

1.1.19. PROPOSITION. *If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map*

$$\begin{tikzcd} A^Y \arrow[dr, dashed, two heads, "i\widehat{\pitchfork}f"] \arrow[drr, bend left, two heads, "A^i"] \arrow[ddr, bend right, two heads, "f^Y"'] \\ & \bullet \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^X \arrow[d, two heads, "f^X"] \\ & B^Y \arrow[r, two heads, "B^i"'] & B^X \end{tikzcd} \tag{1.1.20}$$

is again an isofibration.[6]

1.1.21. COROLLARY. *If X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. Moreover, a 1-simplex in A^X is an isomorphism if and only if its components at each vertex of X are isomorphisms in A.*

PROOF. The first statement is a special case of Proposition 1.1.19; see Exercise 1.1.vi. The second statement is proven similarly by arguing with marked simplicial sets. See Corollary D.4.17. □

1.1.22. DEFINITION (equivalences of quasi-categories). A map $f\colon A \to B$ between quasi-categories is an **equivalence** if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g\colon B \to A$ and

$$\begin{tikzcd} & A & & & B \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & & B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & A & & & B \end{tikzcd}$$

We write "$\xrightarrow{\sim}$" to decorate equivalences and $A \simeq B$ to indicate the presence of an equivalence $A \xrightarrow{\sim} B$.

1.1.23. REMARK. If $f\colon A \to B$ is an equivalence of quasi-categories, then the functor $\mathsf{h}f\colon \mathsf{h}A \to \mathsf{h}B$ is an equivalence of categories, with equivalence inverse $\mathsf{h}g\colon \mathsf{h}B \to \mathsf{h}A$ and natural isomorphisms encoded by the composite functors

$$\mathsf{h}A \xrightarrow{\mathsf{h}\alpha} \mathsf{h}(A^{\mathbb{I}}) \longrightarrow (\mathsf{h}A)^{\mathbb{I}} \qquad \mathsf{h}B \xrightarrow{\mathsf{h}\beta} \mathsf{h}(B^{\mathbb{I}}) \longrightarrow (\mathsf{h}B)^{\mathbb{I}}$$

[6] Degenerate cases of this result, taking $X = \emptyset$ or $B = 1$, imply that the other six maps in this diagram are also isofibrations; see Exercise 1.1.vi.

1.1.24. Definition. A map $f\colon X \to Y$ between simplicial sets is a **trivial fibration** if it admits lifts against the boundary inclusions for all simplices

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad n \geq 0 \tag{1.1.25}$$

We write "$\overset{\sim}{\twoheadrightarrow}$" to decorate trivial fibrations.

1.1.26. Remark. The simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ "cellularly generate" the monomorphisms of simplicial sets — see Definition C.2.4 and Lemma C.5.9. Hence the dual of Lemma C.2.3 implies that trivial fibrations lift against any monomorphism between simplicial sets. In particular, applying this to the map $\emptyset \to Y$, it follows that any trivial fibration $X \overset{\sim}{\twoheadrightarrow} Y$ is a split epimorphism.

The notation "$\overset{\sim}{\twoheadrightarrow}$" is suggestive: the trivial fibrations between quasi-categories are exactly those maps that are both isofibrations and equivalences. This can be proven by a relatively standard although rather technical argument in simplicial homotopy theory, given as Proposition D.5.2 in Appendix D.

1.1.27. Proposition. *For a map $f\colon A \to B$ between quasi-categories the following are equivalent:*

(i) f is a trivial fibration
(ii) f is both an isofibration and an equivalence
*(iii) f is a **split fiber homotopy equivalence**: an isofibration admitting a section s that is also an equivalence inverse via a homotopy from sf to 1_A that composes with f to the constant homotopy from f to f.*

As a class characterized by a right lifting property, the trivial fibrations are also closed under composition, product, pullback, limits of towers, and contain the isomorphisms. The stability of these maps under Leibniz exponentiation will be verified along with Proposition 1.1.19 in Proposition D.5.1.

1.1.28. Proposition. *If $i\colon X \to Y$ is a monomorphism and $f\colon A \to B$ is an isofibration, then if either f is a trivial fibration or if i is in the class cellularly generated*[7] *by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ then the induced Leibniz exponential map*

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

a trivial fibration.

1.1.29. Digression (the Joyal model structure). The category of simplicial sets bears a model structure (see Appendix D) whose fibrant objects are exactly the quasi-categories; all objects are cofibrant. The fibrations, weak equivalences, and trivial fibrations between fibrant objects are precisely the classes of isofibrations, equivalences, and trivial fibrations defined above. Proposition 1.1.27 proves that the trivial fibrations are the intersection of the classes of fibrations and weak equivalences. Propositions 1.1.19 and 1.1.28 reflect the fact that the Joyal model structure is a *closed monoidal model category* with respect to the cartesian closed structure on the category of simplicial sets.

We have declined to elaborate on the Joyal model structure for quasi-categories alluded to in Digression 1.1.29 because the only aspects of it that we will need are those described above. The results proven here suffice to show that the category of quasi-categories defines an ∞-*cosmos*, a concept to which we now turn.

[7]See Definition C.2.4.

Exercises.

1.1.i. EXERCISE. Consider the set of 1-simplices in a quasi-category with initial vertex a_0 and final vertex a_1.

(i) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary
$$\begin{tikzcd} & a_1 \arrow[dr, equal] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & a_1 \end{tikzcd}$$
is an equivalence relation.

(ii) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary
$$\begin{tikzcd} & a_0 \arrow[dr, "f"] & \\ a_0 \arrow[ur, equal] \arrow[rr, "g"'] & & a_1 \end{tikzcd}$$
is an equivalence relation.

(iii) Prove that the equivalence relations defined by (i) and (ii) are the same.

This proves Lemma 1.1.9.

1.1.ii. EXERCISE. Consider the free category on the reflexive directed graph
$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & A_0, \arrow[l, "\sigma_0" description] \end{tikzcd}$$
underlying a quasi-category A.

(i) Consider the relation that identifies a pair of sequences of composable 1-simplices with common source and common target whenever there exists a simplex of A in which the sequences of 1-simplices define two paths from its initial vertex to its final vertex. Prove that this relation is stable under pre- and post-composition with 1-simplices and conclude that its transitive closure is a **congruence**: an equivalence relation that is closed under pre- and post-composition.[8]

(ii) Consider the congruence relation generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices
$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$
and prove that this coincides with the relation considered in (i).

(iii) In the congruence relations of (i) and (ii), prove that every sequence of composable 1-simplices in A is equivalent to a single 1-simplex. Conclude that every morphism in the quotient of the free category by this congruence relation is represented by a 1-simplex in A.

(iv) Prove that for any triple of 1-simplices f, g, h in A, $h = g \circ f$ in the quotient category if and only if there exists a 2-simplex with boundary
$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

This proves Lemma 1.1.12.

[8]Given a congruence relation on the hom-sets of a 1-category, the quotient category can be formed by quotienting each hom-set; see [**64**, §II.8].

1.1.iii. Exercise. Show that any quasi-category in which inner horns admit unique fillers is isomorphic to the nerve of its homotopy category.

1.1.iv. Exercise.

(i) Prove that $\mathbb{I}$ contains exactly two non-degenerate simplices in each dimension.

(ii) Inductively build $\mathbb{I}$ from $\mathbb{2}$ by expressing the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ as a sequential composite of pushouts of outer horn inclusions[9] $\Lambda^0[n] \hookrightarrow \Delta[n]$, one in each dimension starting with $n = 2$.[10]

1.1.v. Exercise. Prove the relative version of Corollary 1.1.16: for any isofibration $p\colon A \twoheadrightarrow B$ between quasi-categories and any $f\colon \mathbb{2} \to A$ that defines an isomorphism in A any homotopy coherent isomorphism in B extending pf lifts to a homotopy coherent isomorphism in A extending f.

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

1.1.vi. Exercise. Specialize Proposition 1.1.19 to prove the following:

(i) If A is a quasi-category and X is a simplicial set then A^X is a quasi-category.

(ii) If A is a quasi-category and $X \hookrightarrow Y$ is a monomorphism then $A^Y \twoheadrightarrow A^X$ is an isofibration.

(iii) If $A \twoheadrightarrow B$ is an isofibration and X is a simplicial set then $A^X \twoheadrightarrow B^X$ is an isofibration.

1.1.vii. Exercise. Anticipating Lemma 1.2.17:

(i) Prove that the equivalences defined in Definition 1.1.22 are closed under retracts.

(ii) Prove that the equivalences defined in Definition 1.1.22 satisfy the 2-of-3 property.

1.1.viii. Exercise. Prove that if $f\colon X \stackrel{\sim}{\twoheadrightarrow} Y$ is a trivial fibration between quasi-categories then the functor $\mathsf{h}f\colon \mathsf{h}X \stackrel{\sim}{\twoheadrightarrow} \mathsf{h}Y$ is a surjective equivalence of categories.

1.2. ∞-Cosmoi

In §1.1, we presented "analytic" proofs of a few of the basic facts about quasi-categories. The category theory of quasi-categories can be developed in a similar style, but we aim instead to develop the "synthetic" theory of infinite-dimensional categories, so that our results will apply to many models at once. To achieve this, our strategy is not to axiomatize what these infinite-dimensional categories *are*, but rather axiomatize the "universe" in which they *live*.

The following definition abstracts the properties of the quasi-categories and the classes of isofibrations, equivalences, and trivial fibrations introduced in §1.1. Firstly, the category of quasi-categories and simplicial maps is *enriched* over the category of simplicial sets — the set of morphisms from A to B coincides with the set of vertices of the simplicial set B^A — and moreover these hom-spaces are all quasi-categories. Secondly, a number of limit constructions that can be defined in the underlying 1-category of quasi-categories and simplicial maps satisfy universal properties relative to this simplicial enrichment, with the usual isomorphism of sets extending to an isomorphism of simplicial sets.

[9]By duality — the **opposite** of a simplicial set X is the simplicial set obtained by reindexing along the involution $(-)^{\mathrm{op}}\colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering in each ordinal — the outer horn inclusions $\Lambda^n[n] \hookrightarrow \Delta[n]$ can be used instead.

[10]This decomposition of the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ reveals which data can always be extended to a homotopy coherent isomorphism: for instance, the 1- and 2-simplices of Definition 1.1.13 together with a single 3-simplex that has these as its outer faces with its inner faces degenerate.

And finally, the classes of isofibrations, equivalences, and trivial fibrations satisfy properties that are familiar from abstract homotopy theory. In particular, the use of isofibrations in diagrams guarantees that their strict limits are equivalence invariant, so we can take advantage of up-to-isomorphism universal properties and strict functoriality of these constructions while still working "homotopically."

As will be explained in Digression 1.2.10, there are a variety of models of infinite-dimensional categories for which the category of "∞-categories," as we will call them, and "∞-functors" between them is enriched over quasi-categories and admits classes of isofibrations, equivalences, and trivial fibrations satisfying analogous properties. This motivates the following axiomatization:

1.2.1. Definition (∞-cosmoi). An **∞-cosmos** $\mathcal{K}$ is a category — whose objects A, B we call **∞-categories** and whose morphisms $f\colon A \to B$ we call **∞-functors** — that is enriched over quasi-categories,[11] meaning in particular that

- its morphisms $f\colon A \to B$ define the vertices of **functor-spaces** $\mathsf{Fun}(A, B)$, which are quasi-categories,

that is also equipped with a specified class of maps that we call **isofibrations** and denote by "$\twoheadrightarrow$" and satisfies the following two axioms:

(i) (completeness) The quasi-categorically enriched category $\mathcal{K}$ possesses a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.[12]

(ii) (isofibrations) The class of isofibrations contains all isomorphisms and any map whose codomain is the terminal object; is closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and has the property that if $f\colon A \twoheadrightarrow B$ is an isofibration and X is any object then $\mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

1.2.2. Definition. In an ∞-cosmos $\mathcal{K}$, we define a morphism $f\colon A \to B$ to be

- an **equivalence** if and only if the induced map $f_*\colon \mathsf{Fun}(X, A) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathcal{K}$, and
- a **trivial fibration** just when f is both an isofibration and an equivalence.

These classes are denoted by "$\xrightarrow{\sim}$" and "$\overset{\sim}{\twoheadrightarrow}$" respectively.

Put more concisely, one might say that an ∞-cosmos is a "quasi-categorically enriched category of fibrant objects." See Definition C.1.1 and Lemma C.1.3.

1.2.3. Digression (simplicial categories). A **simplicial category** $\mathcal{A}$ is given by categories $\mathcal{A}_n$, with a common set of objects and whose arrows are called **n-arrows**, that assemble into a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ of identity-on-objects functors

$$\cdots \qquad \mathcal{A}_3 \begin{array}{c} \xrightarrow{\delta_3} \\ \xleftarrow{\sigma_2} \\ \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{array} \mathcal{A}_2 \begin{array}{c} \xrightarrow{\delta_2} \\ \xleftarrow{\sigma_1} \\ \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{array} \mathcal{A}_1 \begin{array}{c} \xrightarrow{\delta_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\delta_0} \end{array} \mathcal{A}_0, \qquad =: \mathcal{A} \tag{1.2.4}$$

[11] This is to say $\mathcal{K}$ is a simplicially enriched category whose hom-spaces are all quasi-categories; this will be unpacked in 1.2.3.

[12] This will be elaborated upon in 1.2.5.

The data of a simplicial category can equivalently be encoded by a **simplicially enriched category** with a set of objects and a simplicial set $\mathcal{A}(x,y)$ of morphisms between each ordered pair of objects: an n-arrow in $\mathcal{A}_n$ from x to y corresponds to an n-simplex in $\mathcal{A}(x,y)$ (see Exercise 1.2.i). Each endo-hom-space contains a distinguished identity 0-arrow (the degenerate images of which define the corresponding identity n-arrows) and composition is required to define a simplicial map

$$\mathcal{A}(y,z)\times\mathcal{A}(x,y) \xrightarrow{\circ} \mathcal{A}(x,z)$$

the single map encoding the compositions in each of the categories $\mathcal{A}_n$ and also the functoriality of the diagram (1.2.4). The composition is required to be associative and unital, in a sense expressed by the commutative diagrams

$$\begin{tikzcd} \mathcal{A}(y,z)\times\mathcal{A}(x,y)\times\mathcal{A}(w,x) \arrow[r, "\circ\times 1"] \arrow[d, "1\times\circ"'] & \mathcal{A}(x,z)\times\mathcal{A}(w,x) \arrow[d, "\circ"] \\ \mathcal{A}(y,z)\times\mathcal{A}(w,y) \arrow[r, "\circ"'] & \mathcal{A}(w,z) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(x,y) \arrow[r, "\mathrm{id}_y\times 1"] \arrow[d, "1\times\mathrm{id}_x"'] \arrow[dr, equal] & \mathcal{A}(y,y)\times\mathcal{A}(x,y) \arrow[d, "\circ"] \\ \mathcal{A}(x,y)\times\mathcal{A}(x,x) \arrow[r, "\circ"'] & \mathcal{A}(x,y) \end{tikzcd}$$

the latter making use of natural isomorphisms $\mathcal{A}(x,y)\times 1\cong\mathcal{A}(x,y)\cong 1\times\mathcal{A}(x,y)$ in the domain vertex.

On account of the equivalence between these two presentations, the terms "simplicial category" and "simplicially-enriched category" are generally taken to be synonyms.[13] The category $\mathcal{A}_0$ of 0-arrows is the **underlying category** of the simplicial category $\mathcal{A}$, which forgets the higher dimensional simplicial structure.

In particular, the underlying category of an ∞-cosmos $\mathcal{K}$ is the category whose objects are the ∞-categories in $\mathcal{K}$ and whose morphisms are the 0-arrows in the functor spaces. In all of the examples to appear below, this recovers the expected category of ∞-categories in a particular model and functors between them.

1.2.5. Digression (simplicially enriched limits). Let $\mathcal{A}$ be a simplicial category. The **cotensor** of an object $A\in\mathcal{A}$ by a simplicial set U is characterized by an isomorphism of simplicial sets

$$\mathcal{A}(X,A^U)\cong\mathcal{A}(X,A)^U \tag{1.2.6}$$

natural in $X\in\mathcal{A}$. Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\begin{aligned} s\mathcal{S}et^{\mathrm{op}}\times\mathcal{A} &\longrightarrow \mathcal{A} \\ (U,A) &\longmapsto A^U \end{aligned}$$

in a unique way making the isomorphism (1.2.6) natural in U and A as well.

The other simplicial limit notions postulated by axiom 1.2.1(i) are **conical**, which is the term used for ordinary 1-categorical limit shapes that satisfy an enriched analog of the usual universal property; see Definition 7.1.14. When these limits exist they correspond to the usual limits in the underlying category, but the usual universal property is strengthened. Applying the covariant representable functor $\mathcal{A}(X,-)\colon\mathcal{A}_0\to s\mathcal{S}et$ to a limit cone $(\lim_{j\in J}A_j\to A_j)_{j\in J}$ in $\mathcal{A}_0$, there is natural comparison map

$$\mathcal{A}(X,\lim_{j\in J}A_j)\to\lim_{j\in J}\mathcal{A}(X,A_j) \tag{1.2.7}$$

[13] The phrase "simplicial object in $\mathcal{C}at$" is reserved for the more general yet less common notion of a diagram $\mathbf{\Delta}^{\mathrm{op}}\to\mathcal{C}at$ that is not necessarily comprised of identity-on-objects functors.

and we say that $\lim_{j\in J} A_j$ defines a **simplicially enriched limit** if and only if (1.2.7) is an isomorphism of simplicial sets for all $X \in \mathcal{A}$.

Considerably more details on the general theory of enriched categories can be found in [**56**] and in Appendix A. Enriched limits are the subjects of §A.4 and §A.5.

1.2.8. Remark (flexible weighted limits in ∞-cosmoi). The axiom 1.2.1(i) implies that any ∞-cosmos $\mathcal{K}$ admits all *flexible limits* (see Corollary 7.3.3), a much larger class of simplicially enriched "weighted" limits that will be introduced in §7.2.

Using the results of Joyal discussed in §1.1, we can easily verify:

1.2.9. Proposition. *The full subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ of quasi-categories defines an ∞-cosmos with the isofibrations, equivalences, and trivial fibrations of Definitions 1.1.17, 1.1.22, and 1.1.24.*

Proof. The subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ inherits its simplicial enrichment from the cartesian closed category of simplicial sets: note that for quasi-categories A and B, $\mathsf{Fun}(A, B) := B^A$ is again a quasi-category.

The limits postulated in 1.2.1(i) exist in the ambient category of simplicial sets.[14] The defining universal property of the simplicial cotensor is satisfied by the exponentials of simplicial sets. We now argue that the full subcategory of quasi-categories inherits all these limit notions.

Since the quasi-categories are characterized by a right lifting property, it is clear that they are closed under small products. Similarly, since the class of isofibrations is characterized by a right lifting property, Lemma C.2.3 implies that the isofibrations are closed under all of the limit constructions of 1.2.1(ii) except for the last two: Leibniz closure and closure under exponentiation $(-)^X$. These last closure properties are established in Proposition 1.1.19. This completes the proof of 1.2.1(i) and 1.2.1(ii).

It remains to verify that the classes of trivial fibrations and of equivalences coincide with those defined by 1.1.24 and 1.1.22. By Proposition 1.1.27 the former coincidence follows from the latter, so it remains only to show that the equivalences of 1.1.22 coincide with the **representably-defined equivalences**: those maps of quasi-categories $f: A \to B$ for which $A^X \to B^X$ is an equivalence of quasi-categories in the sense of 1.1.22. Taking $X = \Delta[0]$, we see immediately that representably-defined equivalences are equivalences, and the converse holds since the exponential $(-)^X$ preserves the data defining a simplicial homotopy. □

We mention a common source of ∞-cosmoi found in nature at the outside to help ground the intuition for readers familiar with Quillen's model categories, a popular framework for "abstract homotopy theory," but reassure others that model categories are not needed outside of Appendix E.

1.2.10. Digression (a source of ∞-cosmoi in nature). As explained in Appendix E, certain easily described properties of a model category imply that the full subcategory of fibrant objects defines an ∞-cosmos whose isofibrations, equivalences, and trivial fibrations are the fibrations, weak equivalences, and trivial fibrations between fibrant objects. Namely, any model category that is enriched as such over the Joyal model structure on simplicial sets and with the property that all fibrant objects are cofibrant has this property. This compatible enrichment in the Joyal model structure can be defined when the model category is cartesian closed and equipped with a right Quillen adjoint to the Joyal model structure on simplicial sets whose left adjoint preserves finite products. In this case, the right

[14] Any category of presheaves is cartesian closed, complete, and cocomplete — a "cosmos" in the sense of Bénabou. Our ∞-cosmoi are more similar to the fibrational cosmoi due to Street [**93**].

adjoint becomes the underlying quasi-category functor (see Proposition 1.3.3(ii)) and the ∞-cosmoi so-produced will then be cartesian closed (see Definition 1.2.20). The ∞-cosmoi listed in Example 1.2.21 all arise in this way.

The following results are consequences of the axioms of Definition 1.2.1. The first of these results tells us that the trivial fibrations enjoy all of the same stability properties satisfied by the isofibrations.

1.2.11. Lemma (stability of trivial fibrations). *The trivial fibrations in an ∞-cosmos define a subcategory containing the isomorphisms; are stable under product, pullback, forming inverse limits of towers; the Leibniz cotensors of any trivial fibration with a monomorphism of simplicial sets is a trivial fibration as is the Leibniz cotensor of an isofibration with a map in the class cellularly generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$; and if $E \xrightarrow{\sim} B$ is a trivial fibration then so is $\mathsf{Fun}(X, E) \xrightarrow{\sim} \mathsf{Fun}(X, B)$.*

Proof. We prove these statements in the reverse order. By axiom 1.2.1(ii) and the definition of the trivial fibrations in an ∞-cosmos, we know that if $E \xrightarrow{\sim} B$ is a trivial fibration then is $\mathsf{Fun}(X, E) \xrightarrow{\sim} \mathsf{Fun}(X, B)$ is both an isofibration and an equivalence, and hence by Proposition 1.1.27 a trivial fibration. For stability under the remaining constructions, we know in each case that the maps in question are isofibrations in the ∞-cosmos; it remains to show only that the maps are also equivalences. The equivalences in an ∞-cosmos are defined to be the maps that $\mathsf{Fun}(X, -)$ carries to equivalences of quasi-categories, so it suffices to verify that trivial fibrations of quasi-categories satisfy the corresponding stability properties. This is established in Proposition 1.1.28 and the fact that that class is characterized by a right lifting property. □

Additionally, every trivial fibration is "split" by a section.

1.2.12. Lemma (trivial fibrations split). *Every trivial fibration admits a section*

$$\begin{tikzcd} & E \arrow[d, two heads, "\wr"] \\ B \arrow[ur, dashed] \arrow[r, equal] & B \end{tikzcd}$$

Proof. If $p \colon E \xrightarrow{\sim} B$ is a trivial fibration, then by the final stability property of Lemma 1.2.11, so is $p_* \colon \mathsf{Fun}(B, E) \twoheadrightarrow Fun(B, B)$. By Definition 1.1.24, we may solve the lifting problem

$$\begin{tikzcd} \emptyset = \partial\Delta[0] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(B, E) \arrow[d, two heads, "\wr"', "p_*"] \\ \Delta[0] \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dashed, "s"] & \mathsf{Fun}(B, B) \end{tikzcd}$$

to find a map $s \colon B \to E$ so that $ps = \mathrm{id}_B$. □

A classical construction in abstract homotopy theory proves the following:

1.2.13. Lemma (Brown factorization lemma). *Any functor $f \colon A \to B$ in an ∞-cosmos may be factored as an equivalence followed by an isofibration, where this equivalence is constructed as a section of a trivial fibration.*

$$\begin{tikzcd} & Pf \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "s", "\sim"'] \arrow[rr, "f"'] & & B \end{tikzcd} \tag{1.2.14}$$

PROOF. The displayed factorization is constructed by the pullback of an isofibration formed by the simplicial cotensor of the inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{I}$ into the ∞-category B.

$$\begin{tikzcd} & A^{\mathbb{I}} \arrow[drr, "f^{\mathbb{I}}"] \\ A \arrow[ur, "\Delta"] \arrow[r, dashed, "\sim", "s"'] \arrow[dr, "(A,f)"'] & Pf \arrow[rr] \arrow[d, two heads, "(q,p)"'] \arrow[drr, phantom, "\lrcorner" very near start] & & B^{\mathbb{I}} \arrow[d, two heads, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ & A \times B \arrow[rr, "f \times B"'] & & B \times B \end{tikzcd}$$

Note the map q is a pullback of the trivial fibration $\mathrm{ev}_0 \colon B^{\mathbb{I}} \xrightarrow{\sim} B$ and is hence a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence. □

By a Yoneda-style argument, the "homotopy equivalence" characterization of the equivalences in the ∞-cosmos of quasi-categories extends to an analogous characterization of the equivalences in any ∞-cosmos:

1.2.15. LEMMA (equivalences are homotopy equivalences). *A map $f \colon A \to B$ between ∞-categories in an ∞-cosmos $\mathcal{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g \colon B \to A$ and*

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & B \end{tikzcd} \tag{1.2.16}$$

in the ∞-cosmos.

PROOF. By hypothesis, if $f \colon A \to B$ defines an equivalence in the ∞-cosmos $\mathcal{K}$ then the induced map on post-composition $f_* \colon \mathrm{Fun}(B, A) \xrightarrow{\sim} \mathrm{Fun}(B, B)$ is an equivalence of quasi-categories. Evaluating the equivalence inverse $\tilde{g} \colon \mathrm{Fun}(B, B) \xrightarrow{\sim} \mathrm{Fun}(B, A)$ and homotopy $\tilde{\beta} \colon \mathrm{Fun}(B, B) \to \mathrm{Fun}(B, B)^{\mathbb{I}}$ at the 0-arrow $1_B \in \mathrm{Fun}(B, B)$, we obtain a 0-arrow $g \colon B \to A$ together with an isomorphism $\mathbb{I} \to \mathrm{Fun}(B, B)$ from the composite fg to 1_B. By the defining universal property of the cotensor, this isomorphism internalizes to define the map $\beta \colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the right of (1.2.16).

Now the hypothesis that f is an equivalence also provides an equivalence of quasi-categories $f_* \colon \mathrm{Fun}(A, A) \xrightarrow{\sim} \mathrm{Fun}(A, B)$ and the map $\beta f \colon A \to B^{\mathbb{I}}$ represents an isomorphism in $\mathrm{Fun}(A, B)$ from fgf to f. Since f_* is an equivalence, we can conclude that 1_A and gf are isomorphic in the quasi-category $\mathrm{Fun}(A, A)$: such an isomorphism may be defined by applying the inverse equivalence $\tilde{h} \colon \mathrm{Fun}(A, B) \to \mathrm{Fun}(A, A)$ and composing with the components at $1_A, gf \in \mathrm{Fun}(A, A)$ of the isomorphism $\tilde{\alpha} \colon \mathrm{Fun}(A, A) \to \mathrm{Fun}(A, A)^{\mathbb{I}}$ from $1_{\mathrm{Fun}(A,A)}$ to $\tilde{h} f_*$. Now by Corollary 1.1.16 this isomorphism is represented by a map $\mathbb{I} \to \mathrm{Fun}(A, A)$ from 1_A to gf, which internalizes to a map $\alpha \colon A \to A^{\mathbb{I}}$ in $\mathcal{K}$ displayed on the left of (1.2.16).

The converse is easy: the simplicial cotensor construction commutes with $\mathsf{Fun}(X,-)$ so homotopy equivalences are preserved and by Definition 1.1.22 homotopy equivalences of quasi-categories define equivalences of quasi-categories. ☐

1.2.17. Lemma. *The class of equivalences in an ∞-cosmos are closed under retracts and satisfy the 2-of-3 property.*

For the reader who solved Exercise 1.1.vii, demonstrating the equivalences between quasi-categories are closed under retracts and have the 2-of-3 property, Lemma 1.2.17 follows easily from the representable definition of equivalences and functoriality. But for sake of completeness, we give an alternate proof of this result that makes use of Lemma 1.2.15 and subsumes Exercise 1.1.vii.

Proof. Let $f\colon A \xrightarrow{\sim} B$ be an equivalence equipped with an inverse "homotopy equivalence" as in (1.2.16) and consider a retract diagram

$$\begin{tikzcd} C \arrow[r, "u"'] \arrow[d, "h"'] \arrow[rr, bend left, equals] & A \arrow[r, "v"'] \arrow[d, "f"', "\wr"] & C \arrow[d, "h"] \\ D \arrow[r, "s"] \arrow[rr, bend right, equals] & B \arrow[r, "t"] & D \end{tikzcd}$$

By Lemma 1.2.15, to prove that $h\colon C \to D$ is an equivalence, it suffices to construct the data of an inverse homotopy equivalence. To that end define $k\colon D \to C$ to be the composite vgs and then observe from the commutative diagrams

$$\begin{tikzcd} & & A \arrow[r, "v"'] & C \\ C \arrow[r, "u"] \arrow[d, "h"'] \arrow[rrr, bend left, equals] & A \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[ur, equals] & A^{\mathbb{I}} \arrow[r, "v^{\mathbb{I}}"] \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & C^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ D \arrow[r, "s"] \arrow[rrr, bend right, "k"'] & B \arrow[r, "g"] & A \arrow[r, "v"] & C \end{tikzcd} \qquad \begin{tikzcd} & & C \arrow[dr, "h"] & \\ & A \arrow[ur, "v"] \arrow[r, "f"'] & B \arrow[r, "t"] & D \\ D \arrow[r, "s"] \arrow[uurr, bend left, "k"] \arrow[drrr, bend right, equals] & B \arrow[u, "g"] \arrow[r, "\beta"] \arrow[dr, equals] & B^{\mathbb{I}} \arrow[r, "t^{\mathbb{I}}"] \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] & D^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & & B \arrow[r, "t"] & D \end{tikzcd}$$

that $v^{\mathbb{I}}\alpha u\colon C \to C^{\mathbb{I}}$ and $t^{\mathbb{I}}\beta s\colon D \to D^{\mathbb{I}}$ define the required "homotopies."

Via Lemma 1.2.15, the 2-of-3 property for equivalence follows from the fact that the class of isomorphisms in a quasi-category is closed under composition. To prove that equivalences are closed under composition, consider a composable pair of equivalence with their equivalence inverses

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, phantom, "\simeq"] & B \arrow[l, shift left, "k"] \arrow[r, shift left, "g"] \arrow[r, phantom, "\simeq"] & C \arrow[l, shift left, "h"] \end{tikzcd}$$

The homotopies of Lemma 1.2.15 define isomorphisms $\alpha\colon \mathrm{id}_A \cong kf \in \mathsf{Fun}(A,A)$ and $\gamma\colon \mathrm{id}_B \cong hg \in \mathsf{Fun}(B,B)$, the latter of which composes to define $k\gamma f\colon kf \cong khgf \in \mathsf{Fun}(B,B)$. Composing these, we obtain an isomorphism $\mathrm{id}_A \cong khgf \in \mathsf{Fun}(A,A)$ defining one of the homotopies that witnesses that kh defines an equivalence inverse of gf. The construction of the other homotopy is dual.

To prove that the equivalences are closed under cancelation, now consider a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f", "\sim"'] & B \arrow[l, shift left, "k"] \arrow[r, "g"] & C \arrow[ll, bend right, "\ell"', "\sim"] \end{tikzcd}$$

with k an inverse equivalence to f and ℓ and inverse equivalence to gf. We will demonstrate that $f\ell$ defines an equivalence inverse to g. One of the required homotopies $\mathrm{id}_C \cong gf\ell$ is given already. The other is obtained by composing three isomorphisms in $\mathsf{Fun}(B, B)$

$$\mathrm{id}_B \xrightarrow[\cong]{\beta} fk \xrightarrow[\cong]{f\delta k} f\ell gfk \xrightarrow[\cong]{f\ell g\beta} f\ell g.$$

The proof of stability of equivalence under left cancelation is dual. □

1.2.18. Remark (equivalences satisfy the 2-of-6 property). In fact the class of equivalences in any ∞-cosmos satisfy the stronger **2-of-6 property**: for any composable triple of morphisms

$$\begin{tikzcd} & B \arrow[dr, "hg", "\sim"'] \arrow[dd, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "hgf"] \arrow[dr, "gf"', "\sim"] & & D \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

if gf and hg are equivalences then f, g, h, and hgf are too. The proof uses Lemma 1.2.17 together with the observation that in the case where $f\colon A \to B$ is an equivalence, the map p of (1.2.14) is also a trivial fibration, and in particular has a section by Lemma 1.2.12. Combining these facts, a result of Blumberg and Mandell [**15**, 6.4] reproduced in Proposition C.1.8 applies to prove that the equivalences have the 2-of-6 property. See Corollary C.1.9.

One of the key advantages of the ∞-cosmological approaches to abstract category theory is that there are a myriad varieties of "fibered" ∞-cosmoi that can be built from a given ∞-cosmos, which means that any theorem proven in this axiomatic framework specializes and generalizes to those contexts. The most basic of these derived ∞-cosmos is the ∞-cosmos of isofibrations over a fixed base, which we introduce now. Other examples of ∞-cosmoi will be introduced in §**??**, once we have developed a greater facility with the simplicial limits of axiom 1.2.1(i).

1.2.19. Proposition (sliced ∞-cosmoi). *For any ∞-cosmos $\mathcal{K}$ and any object $B \in \mathcal{K}$ there is an **∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over** B whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ with codomain B
(ii) functor-spaces, say from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$, are defined by pullback

$$\begin{tikzcd} \mathsf{Fun}_B(p\colon E \twoheadrightarrow B, q\colon F \twoheadrightarrow B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E, F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"] & \mathsf{Fun}(E, B) \end{tikzcd}$$

and abbreviated to $\mathsf{Fun}_B(E, F)$ when the specified isofibrations are clear from context

(iii) isofibrations are commutative triangles of isofibrations over B

$$\begin{tikzcd} E \arrow[rr, two heads, "r"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

(iv) terminal object is $\mathbb{1}\colon B \twoheadrightarrow B$ and products are defined by the pullback along the diagonal

$$\begin{tikzcd} \times_i^B E_i \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \prod_i E_i \arrow[d, two heads] \\ B \arrow[r, "\Delta"] & \prod_i B \end{tikzcd}$$

(v) pullbacks and limits of towers of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$

(vi) simplicial cotensors $p\colon E \twoheadrightarrow B$ by $U \in s\mathcal{S}et$ are denoted $U \pitchfork_B p$ and constructed by the pullback

$$\begin{tikzcd} U \pitchfork_B p \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E^U \arrow[d, two heads, "p^U"] \\ B \arrow[r, "\Delta"] & B^U \end{tikzcd}$$

(vii) and in which a map

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

over B is an equivalence in the ∞-cosmos $\mathcal{K}_{/B}$ if and only if f is an equivalence in $\mathcal{K}$.

Proof. Note first that the functor spaces are quasi-categories since axiom 1.2.1(ii) asserts that for any isofibration $q\colon F \twoheadrightarrow B$ in $\mathcal{K}$ the map $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration of quasi-categories. Other parts of this axiom imply that that each of the limit constructions define isofibrations over B. The closure properties of the isofibrations in $\mathcal{K}_{/B}$ follow from the corresponding ones in $\mathcal{K}$. The most complicated of these is the Leibniz cotensor stability of the isofibrations in $\mathcal{K}_{/B}$, which follows from the corresponding property in $\mathcal{K}$, since for a monomorphism of simplicial sets $i\colon X \hookrightarrow Y$ and an isofibration r over B as above, the map $i \mathbin{\widehat{\pitchfork}_B} r$ is constructed by pulling back $i \mathbin{\widehat{\pitchfork}} r$ along $\Delta\colon B \to B^Y$.

The fact that the above constructions define simplicially enriched limits in a simplicially enriched slice category are standard from enriched category theory. It remains only to verify that the equivalences in the ∞-cosmos of isofibrations are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$. First note that if $f\colon E \to F$ defines an equivalence in $\mathcal{K}$, then for any isofibration $s\colon A \twoheadrightarrow B$ the induced equivalence on functor-spaces in $\mathcal{K}$ pulls back to define an equivalence on corresponding functor spaces in $\mathcal{K}_{/B}$.

$$\begin{tikzcd} \mathsf{Fun}_B(A,E) \arrow[rr] \arrow[dr, "f_*", "\sim"'] \arrow[dd, two heads] & & \mathsf{Fun}(A,E) \arrow[dr, "f_*", "\sim"'] \arrow[dd, two heads, "p_*"' near start] & \\ & \mathsf{Fun}_B(A,F) \arrow[rr] \arrow[dl, two heads] & & \mathsf{Fun}(A,F) \arrow[dl, two heads, "q_*"] \\ \mathbb{1} \arrow[rr, "s"'] & & \mathsf{Fun}(A,B) & \end{tikzcd}$$

This can be verified either by appealing to Lemmas 1.2.11 and 1.2.13 and using standard techniques from simplicial homotopy theory[15] or by appealing to Lemma 1.2.15 and using the fact that pullback along r defines a simplicial functor.

For the converse implication, we appeal to Lemma 1.2.15. If $f: E \to F$ is an equivalence in $\mathcal{K}_{/B}$ then it admits a homotopy inverse in $\mathcal{K}_{/B}$. The inverse equivalence $g: F \to E$ also defines an inverse equivalence in $\mathcal{K}$ and the required simplicial homotopies in $\mathcal{K}$ are defined by composing

$$E \xrightarrow{\alpha} \mathbb{I} \pitchfork_B p \longrightarrow E^{\mathbb{I}} \qquad\qquad F \xrightarrow{\beta} \mathbb{I} \pitchfork_B q \to F^{\mathbb{I}}$$

with the top horizontal leg of the pullback defining the cotensor in $\mathcal{K}_{/B}$. ☐

1.2.20. Definition (cartesian closed ∞-cosmoi). An ∞-cosmos $\mathcal{K}$ is **cartesian closed** if the product bifunctor $- \times -: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ extends to a simplicially enriched two-variable adjunction

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A)$$

in which the right adjoint $(-)^A$ preserve the class of isofibrations.

For instance, the ∞-cosmos of quasi-categories is cartesian closed, with the exponentials defined as (special cases of) simplicial cotensors. This is one of the reasons that we use the same notation for cotensor and for exponential.[16]

1.2.21. Example (∞-cosmoi of $(\infty, 1)$-categories). The following models of $(\infty, 1)$-categories define cartesian closed ∞-cosmoi:

(i) Rezk's **complete Segal spaces** define the objects of an ∞-cosmos $\mathcal{CSS}$, in which the isofibrations, equivalences, and trivial fibrations are the corresponding classes of the model structure of **[75]**.[17]
(ii) The **Segal categories** defined by Dwyer, Kan, and Smith **[34]** and developed by Hirschowitz and Simpson **[45]** define the objects of an ∞-cosmos $\mathcal{S}egal$, in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure of **[69]** and **[9]**.[18]
(iii) The **1-complicial sets** of **[102]**, equivalently the "naturally marked quasi-categories" of **[61]**, define the objects of an ∞-cosmos $1\text{-}\mathcal{C}omp$ in which the isofibrations, equivalences and trivial fibrations are the corresponding classes of the model structure from either of these sources.

Proofs of these facts can be found in Appendix E.

Appendix E also proves that certain models of (∞, n)-categories or even (∞, ∞)-categories define ∞-cosmoi.

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[15] In more detail: any functor between the 1-categories underlying ∞-cosmoi that preserves trivial fibrations also preserves equivalences; see Lemma C.1.10 and Lemma C.1.11.

[16] Another reason for this convenient notational conflation will be explained in §2.3.

[17] Warning: the model category of complete Segal spaces is enriched over simplicial sets in two distinct "directions" — one enrichment makes the simplicial set of maps between two complete Segal spaces into a Kan complex that probes the "spacial" structure while another enrichment makes the simplicial set of maps into a quasi-category that probes the "categorical" structure **[50]**. It is this latter enrichment that we want.

[18] Here we reserve the term "Segal category" for those simplicial objects with a discrete set of objects that are Reedy fibrant and satisfy the Segal condition. The traditional definition does not include the Reedy fibrancy condition because it is not satisfied by the simplicial object defined as the nerve of a Kan complex enriched category. Since Kan complex enriched categories are not among our preferred models of $(\infty, 1)$-categories this does not bother us.

1.2.22. EXAMPLE ($\mathcal{C}at$ as an ∞-cosmos). The category $\mathcal{C}at$ of 1-categories defines a cartesian closed ∞-cosmos, inheriting its structure as a full subcategory $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ of the ∞-cosmos of quasi-categories via the nerve embedding, which preserves all limits and also exponentials: the nerve of the functor category B^A is the exponential of the nerves.

In the ∞-cosmos of categories, the isofibrations are the **isofibrations**: functors satisfying the displayed right lifting property:

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The equivalences are the equivalences of categories and the trivial fibrations are **surjective equivalences**: equivalences of categories that are also surjective on objects.

1.2.23. DEFINITION (co-dual ∞-cosmoi). There is an identity-on-objects functor $(-)^\circ \colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering of the elements in each ordinal $[n] \in \mathbf{\Delta}$. The functor $(-)^\circ$ sends a face map $\delta^i \colon [n-1] \rightarrowtail [n]$ to the face map $\delta^{n-i} \colon [n-1] \rightarrowtail [n]$ and sends the degeneracy map $\sigma^i \colon [n+1] \twoheadrightarrow [n]$ to the degeneracy map $\sigma^{n-i} \colon [n+1] \twoheadrightarrow [n]$. Precomposition with this involutive automorphism induces an involution $(-)^{\mathrm{op}} \colon s\mathcal{S}et \to s\mathcal{S}et$ that sends a simplicial set X to its **opposite simplicial set** X^{op}, with the orientation of the vertices in each simplex reversed. This construction preserves all conical limits and colimits and induces an isomorphism $(Y^X)^{\mathrm{op}} \cong (Y^{\mathrm{op}})^{X^{\mathrm{op}}}$ on exponentials.

For any ∞-cosmos $\mathcal{K}$, there is a **dual ∞-cosmos** $\mathcal{K}^{\mathrm{co}}$ with the same objects but with functor spaces defined by:

$$\mathsf{Fun}_{\mathcal{K}^{\mathrm{co}}}(A,B) \coloneqq \mathsf{Fun}_{\mathcal{K}}(A,B)^{\mathrm{op}}.$$

The isofibrations, equivalences, and trivial fibrations in $\mathcal{K}^{\mathrm{co}}$ coincide with those of $\mathcal{K}$.

Conical limits in $\mathcal{K}^{\mathrm{co}}$ coincide with those in $\mathcal{K}$, while the cotensor of $A \in \mathcal{K}$ with $U \in s\mathcal{S}et$ is defined to be $A^{U^{\mathrm{op}}}$.

A justification for this notation is given in Exercise 1.4.iii.

1.2.24. DEFINITION ('∞-categories). An object E in an ∞-cosmos $\mathcal{K}$ is **discrete** just when for all $X \in \mathcal{K}$ the functor-space $\mathsf{Fun}(X,E)$ is a Kan complex.

In the ∞-cosmos of quasi-categories, the discrete objects are exactly the Kan complexes: by Corollary D.3.11 the Kan complexes also define an exponential ideal in the category of simplicial sets. Similarly, in the ∞-cosmoi of Example 1.2.21 whose ∞-categories are $(\infty,1)$-categories in some model, the discrete objects are the ∞-groupoids.

1.2.25. PROPOSITION (∞-cosmos of discrete objects). *The full subcategory $\mathcal{D}isc(\mathcal{K}) \hookrightarrow \mathcal{K}$ spanned by the discrete objects in any ∞-cosmos form an ∞-cosmos.*

PROOF. We first establish this result for the ∞-cosmos of quasi-categories. By Proposition 1.1.14 an isofibration between Kan complexes is a **Kan fibration**: a map with the right lifting property with respect to all horn inclusions. Conversely, all Kan fibrations define isofibrations. Since Kan complexes are closed under simplicial cotensor (which coincides with exponentiation), It follows that the full subcategory $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ is closed under all of the limit constructions of axiom 1.2.1(i). The remaining axiom 1.2.1(ii) is inherited from the analogous properties established for quasi-categories in Proposition 1.2.9.

In a generic ∞-cosmos $\mathcal{K}$ we need only show that the discrete objects are closed in $\mathcal{K}$ under the limit constructions of 1.2.1(i). The definition natural isomorphism (1.2.7) characterizing these simplicial limits expresses the functor-space $\mathsf{Fun}(X, \lim_{j\in J} A_j)$ as an analogous limit of functor space $\mathsf{Fun}(X, A_j)$. If each A_j is discrete then these objects are Kan complexes and the previous paragraph then establishes that the limit is a Kan complex as well. This holds for all objects $X \in \mathcal{K}$ so it follows that $\lim_{j\in J} A_j$ is discrete as required. $\square$

Exercises.

1.2.i. Exercise. Prove that the following are equivalent:

(i) a simplicial category, as in 1.2.3,
(ii) a category enriched over simplicial sets.

1.2.ii. Exercise. Elaborate on the proof of Proposition 1.2.9 by proving that the simplicially enriched category $\mathcal{QC}at$ admits conical products satisfying the universal property of Digression 1.2.5. That is:

(i) For quasi-categories A, B, X, form the cartesian product $A \times B$ and prove that the projection maps $\pi_A \colon A \times B \to A$ and $\pi_B \colon A \times B \to B$ induce an isomorphism of quasi-categories
$$(A \times B)^X \xrightarrow{\cong} A^X \times B^X.$$
(ii) Explain how this relates to the universal property of Digression 1.2.5.
(iii) Express the usual 1-categorical universal property of the product $A \times B$ as the "0-dimensional aspect" of the universal property of (i).

1.2.iii. Exercise. Prove that any object in an ∞-cosmos has a **path object**
$$\begin{tikzcd} & B^{\mathbb{I}} \arrow[dl, bend right, two heads, "\sim"'] \arrow[dr, two heads, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] & \\ B \arrow[ur] \arrow[rr, "\Delta"'] & & B \times B \end{tikzcd}$$
constructed by cotensoring with the free-living isomorphism.

1.2.iv. Exercise.

(i) Use Exercise 1.1.iv and results from Appendix D to prove that a quasi-category Q is a Kan complex if and only if the map $Q^{\mathbb{I}} \twoheadrightarrow Q^{\mathbb{2}}$ induced by the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is a trivial fibration.
(ii) Conclude that an ∞-category A is discrete if and only if $A^{\mathbb{I}} \xrightarrow{\sim}\mathrel{\mkern-14mu}\rightarrow A^{\mathbb{2}}$ is a trivial fibration.

1.3. Cosmological functors

Certain "right adjoint type" constructions define maps between ∞-cosmoi that preserve all of the structures axiomatized in Definition 1.2.1. The simple observation that such constructions define *cosmological functors* between ∞-cosmoi will streamline many proofs.

1.3.1. Definition (cosmological functor). A **cosmological functor** is a simplicial functor $F \colon \mathcal{K} \to \mathcal{L}$ that preserves the specified classes of isofibrations and all of the simplicial limits enumerated in 1.2.1(i).

1.3.2. Lemma. *Any cosmological functor also preserves the equivalences and the trivial fibrations.*

Proof. By Lemma 1.2.15 the equivalences in an ∞-cosmos coincide with the homotopy equivalences defined relative to cotensoring with the free-living isomorphism. Since a cosmological functor preserves simplicial cotensors, it preserves the data displayed in (1.2.16) and hence carries equivalences to equivalences. The statement about trivial fibrations follows. $\square$

In general, cosmological functors preserve any ∞-categorical notion that can be characterized internally to the ∞-cosmos — for instance, as a property of a certain map — as opposed to externally — for instance, in a statement that involves a universal quantifier. From Definition 1.2.24 it is not clear whether cosmological functors preserve discrete objects, but using the internal characterization of Exercise 1.2.iv — an ∞-category A is discrete if and only if $A^{\mathbb{I}} \xrightarrow{\sim} A^{\mathbb{2}}$ is a trivial fibration — this follows easily: cosmological functors preserve simplicial cotensors and trivial fibrations.

1.3.3. Proposition.

(i) *For any object A in an ∞-cosmos $\mathcal{K}$, $\mathsf{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$ defines a cosmological functor.*

(ii) *Specializing, each ∞-cosmos has an* ***underlying quasi-category*** *functor*
$$(-)_0 \coloneqq \mathsf{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at.$$

(iii) *For any ∞-cosmos $\mathcal{K}$ and any simplicial set U, the simplicial cotensor defines a cosmological functor $(-)^U\colon \mathcal{K} \to \mathcal{K}$.*

(iv) *For any object A in a cartesian closed ∞-cosmos $\mathcal{K}$, exponentiation defines a cosmological functor $(-)^A\colon \mathcal{K} \to \mathcal{K}$.*

(v) *For any map $f\colon A \to B$ in an ∞-cosmos $\mathcal{K}$, pullback defines a cosmological functor $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$.*

(vi) *For any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map on slices $F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a cosmological functor.*

Proof. The first four of these statements are nearly immediate, the preservation of isofibrations being asserted explicitly as a hypothesis in each case and the preservation of limits following from standard categorical arguments.

For (v), pullback in an ∞-cosmos $\mathcal{K}$ is a simplicially enriched limit construction; one consequence of this is that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ defines a simplicial functor. The action of the functor f^* on a 0-arrow g in $\mathcal{K}_{/B}$ is also defined by a pullback square: since the front and back squares in the displayed diagram are pullbacks the top square is as well

$$\begin{tikzcd} f^*E \arrow[rr] \arrow[dr, dashed] \arrow[dd, two heads, "f^*(g)"'] & & E \arrow[dr, "g"] \arrow[dd, two heads, "p"'] & \\ & f^*F \arrow[rr] \arrow[dl] & & F \arrow[dl, two heads, "q"] \\ A \arrow[rr, "f"'] & & B & \end{tikzcd}$$

Since isofibrations are stable under pullback, it follows that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves isofibrations. It remains to prove that this functor preserves the simplicial limits constructed in Proposition 1.2.19. In the case of connected limits, which are created by the forgetful functors to $\mathcal{K}$, this is clear. For

products and simplicial cotensors, this follows from the commutative cubes

$$\begin{tikzcd}[row sep=small, column sep=small]
& \times_i^A f^*E_i \arrow[rr] \arrow[dl, dashed] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & \prod_i f^*E_i \arrow[dl] \arrow[dd, two heads] \\
\times_i^B E_i \arrow[rr, crossing over] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & \prod_i E_i & \\
& A \arrow[rr, "\Delta"] \arrow[dl, "f"] & & \prod_i A \arrow[dl, "\prod_i f"] \\
B \arrow[rr, "\Delta"'] & & \prod_i B \arrow[from=uu, two heads, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
& U \pitchfork_A f^*(p) \arrow[rr] \arrow[dl, dashed] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & (f^*E)^U \arrow[dl] \arrow[dd, two heads, "(f^*p)^U"] \\
U \pitchfork_B p \arrow[rr, crossing over] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & E^U & \\
& A \arrow[rr, "\Delta"] \arrow[dl, "f"] & & A^U \arrow[dl, "f^U"] \\
B \arrow[rr, "\Delta"] & & B^U \arrow[from=uu, two heads, crossing over, "p^U"] &
\end{tikzcd}$$

Since the front, back, and right faces are pullbacks, the left is as well, which is what we wanted to show.

The final statement (vi) is left as Exercise 1.3.i. ☐

1.3.4. Non-Example. The forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ is simplicial and preserves the class of isofibrations but does *not* define a cosmological functor, failing to preserve cotensors and products. However, by Proposition 1.3.3(v), $- \times B\colon \mathcal{K} \to \mathcal{K}_{/B}$ does define a cosmological functor.

1.3.5. Non-Example. By Proposition 1.2.25, the inclusion $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ is a cosmological functor. It has a right adjoint $\mathsf{core}\colon \mathcal{QC}at \to \mathcal{K}an$ that carries each quasi-category to its maximal sub Kan complex, the simplicial subset containing all n-simplices whose edges are all isomorphisms. Thus functor preserves isofibrations and 1-categorical limits but is *not* cosmological since it is not simplicially enriched: any functor $K \to Q$ whose domain is a Kan complex and whose codomain is a quasi-category factors through the inclusion $\mathsf{core}(Q) \hookrightarrow Q$ via a unique map $K \to \mathsf{core}(Q)$ but in general $\mathsf{Fun}(K, Q) \not\cong \mathsf{Fun}(K, \mathsf{core}(Q))$, since a natural transformation $K \times \Delta[1] \to Q$ will only factor through $\mathsf{core}(Q) \hookrightarrow Q$ in the case where its components are invertible. See Lemma 16.1.14 however.

1.3.6. Definition (biequivalences). A cosmological functor defines a *biequivalence* $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ if additionally it

(i) is **essentially surjective on objects up to equivalence**: for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$ and

(ii) it defines a **local equivalence**: for all $A, B \in \mathcal{K}$, the action of F on functor quasi-categories defines an equivalence

$$\mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(FA, FB).$$

1.3.7. Remark. Cosmological biequivalences will be studied more systematically in Chapter 14, where we think of them as "change-of-model" functors. A basic fact is that any biequivalence of ∞-cosmoi not only preserves equivalences but also *creates* them: a pair of objects in an ∞-cosmos are equivalent if and only if their images in any biequivalent ∞-cosmos are equivalent (Exercise 1.3.ii). It follows that the cosmological biequivalences satisfy the 2-of-3 property.

1.3.8. Example (biequivalences between ∞-cosmoi of $(\infty, 1)$-categories).

(i) The underlying quasi-category functors defined on the ∞-cosmoi of complete Segal spaces, Segal categories, and 1-complicial sets

$$\mathcal{CSS} \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \qquad \mathcal{S}egal \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \qquad 1\text{-}\mathcal{C}omp \xrightarrow[\sim]{(-)_0} \mathcal{QC}at$$

are all biequivalences. In the first two cases these are defined by "evaluating at the 0th row" and in the last case this is defined by "forgetting the markings."

(ii) There is also a cosmological biequivalence $\mathcal{QC}at \rightsquigarrow \mathcal{CSS}$ defined by Joyal and Tierney [**50**].

(iii) The functor $\mathcal{CSS} \rightsquigarrow \mathcal{S}egal$ defined by Bergner [**11**] that "discretizes" a complete Segal spaces also defines a cosmological biequivalence.

(iv) There is a further cosmological biequivalence $(-)^\natural \colon \mathcal{QC}at \rightsquigarrow 1\text{-}\mathcal{C}omp$ that gives each quasi-category its "natural marking," with all invertible 1-simplices and all simplices in dimension greater than 1 marked.

Proofs of these facts can be found in Appendix E.

1.3.9. Remark. The underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{QC}at$ carries the internal homs of a cartesian closed ∞-cosmos $\mathcal{K}$ to the corresponding functor spaces: for any ∞-categories A and B in $\mathcal{K}$, we have

$$(B^A)_0 \coloneqq \mathsf{Fun}(1, B^A) \cong \mathsf{Fun}(A, B).$$

In the case where the ∞-cosmos $\mathcal{K}$ is biequivalent to $\mathcal{QC}at$, we will see in Chapters 14 and 15 entails no essential loss of categorical information.

Exercises.

1.3.i. Exercise. Prove that for any cosmological functor $F \colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map $F \colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a cosmological functor.

1.3.ii. Exercise. Let $F \colon \mathcal{K} \rightsquigarrow \mathcal{L}$ be a cosmological biequivalence and let $A, B \in \mathcal{K}$. Sketch a proof that if $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$ (and see Exercise 1.4.i).

1.4. The homotopy 2-category

Small 1-categories define the objects of a strict 2-category[19] $\mathcal{C}at$ of categories, functors, and natural transformations. Many basic categorical notions — those defined in terms of categories, functors, and natural transformations and their various composition operations — can be defined internally to the 2-category $\mathcal{C}at$. This suggests a natural avenue for generalization: reinterpreting these same definitions in a generic 2-category using its objects in place of small categories, its 1-cells in place of functors, and its 2-cells in place of natural transformations.

In Chapter 2, we will develop a non-trivial portion of the theory of ∞-categories in any fixed ∞-cosmos following exactly this outline, working internally to a strict 2-category that we refer to as the *homotopy 2-category* that we associate to any ∞-cosmos. The homotopy 2-category of an ∞-cosmos is a quotient of the full ∞-cosmos, replacing each quasi-categorical functor-space by its homotopy category. Surprisingly, this rather destructive quotienting operation preserves quite a lot of information. Indeed, essentially all of the work in Part I will take place in the homotopy 2-category of an ∞-cosmos. This said, we caution the reader against becoming overly seduced by homotopy 2-categories, for that structure is more of a technical convenience for reducing the complexity of our arguments than a fundamental notion of ∞-category theory.

[19]A comprehensive introduction to strict 2-categories appears as Appendix B. Succinctly, in parallel with Digression 1.2.3, 2-categories can be understood equally as

- "two-dimensional" categories, with objects, 0-arrows (typically called **1-cells**), and 1-arrows (typically called **2-cells**)
- or as categories enriched over $\mathcal{C}at$.

The homotopy 2-category for the ∞-cosmos of quasi-categories was first introduced by Joyal in his work on the foundations of quasi-category theory.

1.4.1. Definition (homotopy 2-category). Let $\mathcal{K}$ be an ∞-cosmos. Its **homotopy 2-category** is the strict 2-category $\mathfrak{h}\mathcal{K}$ whose

- objects are the ∞-categories, i.e., the objects A, B of $\mathcal{K}$;
- 1-cells $f\colon A \to B$ are the 0-arrows in the functor space $\mathsf{Fun}(A, B)$ of $\mathcal{K}$, i.e., the ∞-functors; and
- 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ are homotopy classes of 1-simplices in $\mathsf{Fun}(A, B)$, which we call **∞-natural transformations**.

Put another way $\mathfrak{h}\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$ and with hom-categories defined by

$$\mathsf{hFun}(A, B) := \mathsf{h}(\mathsf{Fun}(A, B)),$$

that is, as the homotopy category of the quasi-category $\mathsf{Fun}(A, B)$.

The **underlying category** of a 2-category is defined by simply forgetting its 2-cells. Note that an ∞-cosmos $\mathcal{K}$ and its homotopy 2-category $\mathfrak{h}\mathcal{K}$ share the same underlying category of ∞-categories and ∞-functors in $\mathcal{K}$.

1.4.2. Digression. The homotopy category functor $\mathsf{h}\colon s\mathcal{S}et \to \mathcal{C}at$ preserves finite products, as of course does its right adjoint. It follows that the adjunction of Proposition 1.1.11 induces a change-of-base adjunction

$$\begin{tikzcd} 2\text{-}\mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend right, "\mathsf{h}_*"'] \end{tikzcd}$$

whose left and right adjoints change the enrichment by applying the homotopy category functor or the nerve functor to the hom objects of the enriched category. Here $2\text{-}\mathcal{C}at$ and $s\mathcal{S}et\text{-}\mathcal{C}at$ can each be understood as 2-categories — of enriched categories, enriched functors, and enriched natural transformations — and both change of base constructions define 2-functors [**18**, 6.4.3].

1.4.3. Observation (functors representing (invertible) 2-cells). By definition, every 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in the homotopy category $\mathfrak{h}\mathcal{K}$ is represented by a map $\mathbb{2} \to \mathsf{Fun}(A, B)$ defining a 1-simplex in the functor space $\mathsf{Fun}(A, B)$ and two such maps represent the same 2-cell if and only if their images are homotopic as 1-simplices in $\mathsf{Fun}(A, B)$ in the sense defined by Lemma 1.1.9.

Now a 2-cell in a 2-category is **invertible** if and only if it defines an isomorphism in the appropriate hom-category $\mathsf{hFun}(A, B)$. By Definition 1.1.13 and Corollary 1.1.16 it follows that each invertible 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a map $\mathbb{I} \to \mathsf{Fun}(A, B)$.

1.4.4. Lemma. *Any simplicial functor $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi induces a 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ between their homotopy 2-categories.*

Proof. This follows immediately from the remarks on change of base in Digression 1.4.2 but we can also argue directly. The action of the induced 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ on objects and 1-cells is given by the corresponding action of $F\colon \mathcal{K} \to \mathcal{L}$; recall an ∞-cosmos and its homotopy 2-category

have the same underlying 1-category. Each 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a 1-simplex in $\mathsf{Fun}(A,B)$ which is mapped via

$$\begin{tikzcd} \mathsf{Fun}(A,B) \arrow[r, "F"] & \mathsf{Fun}(FA,FB) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, mapsto] & FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \end{tikzcd}$$

to a 1-simplex representing a 2-cell in $\mathfrak{h}\mathcal{L}$. Since the action $F\colon \mathsf{Fun}(A,B) \to \mathsf{Fun}(FA,FB)$ on functor spaces defines a morphism of simplicial sets, it preserves faces and degeneracies. In particular, homotopic 1-simplices in $\mathsf{Fun}(A,B)$ are carried to homotopic 1-simplices in $\mathsf{Fun}(FA,FB)$ so the action on 2-cells just described is well-defined. The 2-functoriality of these mappings follows from the simplicial functoriality of the original mapping. □

We now begin to relate the simplicially enriched structures of an ∞-cosmos to the 2-categorical structures in its homotopy 2-category. The first result proves that homotopy 2-categories inherit products from their ∞-cosmoi, which satisfy a 2-categorical universal property. To illustrate, recall that the terminal ∞-category $1 \in \mathcal{K}$ has the universal property $\mathsf{Fun}(X,1) \cong \mathbb{1}$ for all $X \in \mathcal{K}$. Applying the homotopy category functor we see that $1 \in \mathfrak{h}\mathcal{K}$ has the universal property $\mathsf{hFun}(X,1) \cong \mathbb{1}$ for all $X \in \mathfrak{h}\mathcal{K}$. This 2-categorical universal property has both a 1-dimensional and a 2-dimensional aspect. Since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ is a category with a single object, there exists a unique morphism $X \to 1$ in $\mathcal{K}$. And since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ has only a single identity morphism, we see that the only 2-cells in $\mathfrak{h}\mathcal{K}$ with codomain 1 are identities.

1.4.5. Proposition (cartesian (closure)).

(i) The homotopy 2-category of any ∞-cosmos has 2-categorical products.
(ii) The homotopy 2-category of a cartesian closed ∞-cosmos is cartesian closed as a 2-category.

Proof. While the functor $\mathsf{h}\colon s\mathcal{S}et \to \mathcal{C}at$ only preserves finite products, the restricted functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ preserves *all* products on account of the simplified description of the homotopy category of a quasi-category given in Lemma 1.1.12. Thus for any set I and family of ∞-categories $(A_i)_{i\in I}$ in $\mathcal{K}$, the homotopy category functor carries the isomorphism of quasi-categories displayed below left to an isomorphisms of hom-categories displayed below right

$$\mathsf{Fun}(X,\textstyle\prod_{i\in I} A_i) \xrightarrow{\cong} \prod_{i\in I}\mathsf{Fun}(X,A_i) \qquad \overset{\mathsf{h}}{\mapsto} \qquad \mathsf{hFun}(X,\prod_{i\in I} A_i) \xrightarrow{\cong} \prod_{i\in I}\mathsf{hFun}(X,A_i).$$

This proves that the homotopy 2-category $\mathfrak{h}\mathcal{K}$ has products whose universal properties have both a 1- and 2-dimensional component, as described for terminal objects above.

If $\mathcal{K}$ is a cartesian closed ∞-cosmos, then for any triple of ∞-categories $A,B,C \in \mathcal{K}$ there exist exponential objects $C^A, C^B \in \mathcal{K}$ characterized by natural isomorphisms

$$\mathsf{Fun}(A\times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A).$$

Passing to homotopy categories we have natural isomorphisms

$$\mathsf{hFun}(A\times B, C) \cong \mathsf{hFun}(A, C^B) \cong \mathsf{hFun}(B, C^A),$$

which demonstrates that $\mathfrak{h}\mathcal{K}$ is cartesian closed as a 1-category: functors $A\times B \to C$ transpose to define functors $A \to C^B$ and $B \to C^A$, and 2-cells transpose similarly. □

There is a standard definition of *isomorphism* between two objects in any 1-category. Similarly, there is a standard definition of *equivalence* between two objects in any 2-category:

1.4.6. Definition (equivalence). An **equivalence** in a 2-category is given by

- a pair of objects A and B
- a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$
- a pair of invertible 2-cells

$$\begin{tikzcd} A \arrow[r, equal, bend left] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg"] \arrow[r, equal, bend right] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

When A and B are **equivalent**, we write $A \simeq B$ and refer to the 1-cells f and g as **equivalences**, denoted by "$\xrightarrow{\sim}$."

In the case of the homotopy 2-category of an ∞-cosmos we have a competing definition of equivalence from 1.2.1: namely a 1-cell $f\colon A \xrightarrow{\sim} B$ that induces an equivalence $f_*\colon \mathsf{Fun}(X,A) \xrightarrow{\sim} \mathsf{Fun}(X,B)$ on functor-spaces — or equivalently, by Lemma 1.2.15, a homotopy equivalence defined relative to the interval $\mathbb{I}$. Crucially, all three notions of equivalence coincide:

1.4.7. Theorem (equivalences are equivalences). *In any ∞-cosmos $\mathcal{K}$, the following are equivalent and characterize what it means for a functor $f\colon A \to B$ between ∞-categories to define an **equivalence**.*

(i) For all $X \in \mathcal{K}$, the post-composition map $f_*\colon \mathsf{Fun}(X,A) \xrightarrow{\sim} \mathsf{Fun}(X,B)$ defines an equivalence of quasi-categories.
(ii) There exists a functor $g\colon B \to A$ and natural isomorphisms $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$ in the homotopy 2-category.
(iii) There exist maps $g\colon B \to A$ and

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, two heads, "\mathrm{ev}_0", "\wr"'] \arrow[d, two heads, "\mathrm{ev}_1", "\wr"'] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, two heads, "\mathrm{ev}_0", "\wr"'] \arrow[d, two heads, "\mathrm{ev}_1", "\wr"'] \\ & B \end{tikzcd}$$

in the ∞-cosmos in $\mathcal{K}$.

To continue our theme of comparing 2-categorical and quasi-categorical techniques, rather than appealing to, Lemma 1.2.15 we re-prove it.

Proof. For (i)⇒(ii), if the induced map on post-composition $f_*\colon \mathsf{Fun}(X,A) \xrightarrow{\sim} \mathsf{Fun}(X,B)$ defines an equivalence of quasi-categories, then by Remark 1.1.23, $f_*\colon \mathsf{hFun}(X,A) \xrightarrow{\sim} \mathsf{hFun}(X,B)$ defines an equivalence of categories. In particular, $f_*\colon \mathsf{hFun}(B,A) \xrightarrow{\sim} \mathsf{hFun}(B,B)$ is essentially surjective so there exists $g \in \mathsf{hFun}(B,A)$ and an isomorphism $\beta\colon fg \cong \mathrm{id}_B \in \mathsf{hFun}(B,B)$. Now since $f_*\colon \mathsf{hFun}(A,A) \xrightarrow{\sim} \mathsf{hFun}(A,B)$ is fully faithful, the isomorphism $\beta f\colon fgf \cong f \in \mathsf{hFun}(A,B)$ can be lifted to define an isomorphism $\alpha^{-1}\colon gf \cong \mathrm{id}_A \in \mathsf{hFun}(A,A)$. This defines the data of a 2-categorical equivalence in Definition 1.4.6.

To see that (ii)⇒(iii) recall from Observation 1.4.3 that the natural isomorphisms $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$ in $\mathfrak{h}\mathcal{K}$ are represented by maps $\alpha\colon A \to A^{\mathbb{I}}$ and $\beta\colon B \to B^{\mathbb{I}}$ in $\mathcal{K}$ as in (1.2.16).

Finally, (iii)⇒(i) since $\mathsf{Fun}(X,-)$ carries the data of (iii) to the data of an equivalence of categories as in Definition 1.1.22. ☐

1.4.8. Digression (on the importance of Theorem 1.4.7). It is hard to overstate the importance of Theorem 1.4.7 to the work that follows. The categorical constructions that we will introduce for ∞-categories, ∞-functors, and ∞-natural transformations are invariant under 2-categorical equivalence in the homotopy 2-category and the universal properties we develop similarly characterize a 2-categorical equivalence class of ∞-categories. Theorem 1.4.7 then asserts that such constructions are "homotopically correct": both invariant under equivalence in the ∞-cosmos and precisely identifying equivalence classes of objects.

The equivalence invariance of the functor space in the codomain variable is axiomatic, but equivalence invariance in the domain variable is not.[20] But using 2-categorical techniques, there is now a short proof:

1.4.9. Corollary. *Equivalences of ∞-categories $A' \xrightarrow{\sim} A$ and $B \xrightarrow{\sim} B'$ induce an equivalence of functor spaces* $\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(A',B')$.

Proof. The simplicial functors $\mathsf{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$ and $\mathsf{Fun}(-,B)\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{QC}at$ induce 2-functors $\mathsf{Fun}(A,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ and $\mathsf{Fun}(-,B)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$, which preserve the 2-categorical equivalences of Definition 1.4.6. By Theorem 1.4.7 this is what we wanted to show. ☐

Similarly, there is a standard 2-categorical notion of an isofibration, defined in the statement of Proposition 1.4.10 and elaborated upon in Definition B.4.6, and any isofibration in an ∞-cosmos defines an isofibration in its homotopy 2-category.[21]

1.4.10. Proposition (isofibrations define isofibrations). *Any isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$, also defines an isofibration in the homotopy 2-category $\mathfrak{h}\mathcal{K}$: given any invertible 2-cell as displayed below left abutting to B with a specified lift of one of its boundary 1-cells through p, there exists an invertible 2-cell abutting to E with this boundary 1-cell as displayed below right that whiskers with p to the original 2-cell.*

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\cong\Downarrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\bar{e}"'] \arrow[r, phantom, "\cong\Downarrow\gamma"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

Proof. Put another way, the universal property of the statement says that the functor

$$p_*\colon \mathsf{hFun}(X,E) \twoheadrightarrow \mathsf{hFun}(X,B)$$

is an isofibration of categories in the sense defined in Example 1.2.22. By axiom 1.2.1(ii), since $p\colon E \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, the induced map $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ is an isofibration of quasi-categories. So it suffices to show that the functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries isofibrations of quasi-categories to isofibrations of categories.[22]

[20] Lemma 1.3.2 does not apply since $\mathsf{Fun}(-,B)$ is not cosmological.

[21] In this case, the converse does not hold, nor is it the case that a representably-defined isofibration of quasi-categories is necessarily an isofibration in the ∞-cosmos; consider the case of sliced ∞-cosmoi for instance.

[22] Alternately, argue directly using Observation 1.4.3.

So let us now consider an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories. By Corollary 1.1.16, every isomorphism β in the homotopy category $\mathsf{h}B$ of the quasi-category B is represented by a simplicial map $\beta\colon \mathbb{I} \to B$. By Definition 1.1.17, the lifting problem

$$\begin{tikzcd} \mathbb{1} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & B \end{tikzcd}$$

can be solved, and the map $\gamma\colon \mathbb{I} \to E$ so-produced represents a lift of the isomorphism from $\mathsf{h}B$ to an isomorphism in $\mathsf{h}E$ with domain e. □

1.4.11. Convention (on "isofibrations" in homotopy 2-categories). Since the converse to Proposition 1.4.10 does not hold, there is a potential ambiguity when using the term "isofibration" to refer to a map in the homotopy 2-category of an ∞-cosmos. We adopt the convention that when we declare that a map in $\mathfrak{h}\mathcal{K}$ is an isofibration we always mean this is the stronger sense of defining an isofibration in $\mathcal{K}$. This stronger condition gives us access to the 2-categorical lifting property of Proposition 1.4.10 but also to the many homotopical properties axiomatized in Definition 1.2.1, which guarantee that the strictly defined limits of 1.2.1(i) are automatically equivalence invariant constructions.

The 1- and 2-cells in the homotopy 2-category from the terminal ∞-category $1 \in \mathcal{K}$ to a generic ∞-category $A \in \mathcal{K}$ define the objects and morphisms in the *homotopy category* of A.

1.4.12. Definition (homotopy category of an ∞-category). The **homotopy category** of an ∞-category A in an ∞-cosmos $\mathcal{K}$ is defined to be the homotopy category of its underlying quasi-category, that is:

$$\mathsf{h}A := \mathsf{hFun}(1, A) := \mathsf{h}(\mathsf{Fun}(1, A)).$$

As we shall discover, homotopy categories generally bear "derived" analogues of structures present at the level of ∞-categories. See the remark after the statement Proposition 2.1.7 for an early example of this.

Exercises.

1.4.i. Exercise. Let $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ be a cosmological biequivalence and let $A, B \in \mathcal{K}$. Prove that if $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$ and ruminate on why this exercise is considerably easier than Exercise 1.3.ii).

1.4.ii. Exercise.

(i) What is the homotopy 2-category of the ∞-cosmos $\mathcal{C}at$ of 1-categories?
(ii) Prove that the nerve defines a 2-functor $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$ that is *locally fully faithful*.

1.4.iii. Exercise. Demonstrate that the homotopy 2-category of the dual cosmos $\mathcal{K}^{\mathrm{co}}$ of an ∞-cosmos $\mathcal{K}$ is the co-dual of the homotopy 2-category $\mathfrak{h}\mathcal{K}$, with the domains and codomains of 2-cells but not 1-cells reversed: in symbols $\mathfrak{h}(\mathcal{K}^{\mathrm{co}}) \cong (\mathfrak{h}\mathcal{K})^{\mathrm{co}}$.

1.4.iv. Exercise. Let B be an ∞-category in the ∞-cosmos $\mathcal{K}$ and let $\mathfrak{h}\mathcal{K}_{/B}$ denote the 2-category whose

- objects are isofibrations $E \twoheadrightarrow B$ in $\mathcal{K}$ with codomain B

- 1-cells are 1-cells in $\mathfrak{h}\mathcal{K}$ over B

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

- 2-cells are 2-cells in $\mathfrak{h}\mathcal{K}$ over B

$$\begin{tikzcd} E \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

in the sense that $q\alpha = \mathrm{id}_p$.

Argue that the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos has the same underlying 1-category but different 2-cells. How do these compare with the 2-cells of $\mathfrak{h}\mathcal{K}_{/B}$?[23]

[23]A more systematic comparison will be given in Proposition 3.6.3.

CHAPTER 2

Adjunctions, limits, and colimits I

Heuristically, ∞-categories generalize ordinary 1-categories by adding in higher dimensional morphisms and weakening the composition law. The dream is that proofs establishing the theory of 1-categories similarly generalize to give proofs for ∞-categories, just by adding a prefix "∞-" everywhere. In this chapter, we make this dream a reality — at least for a library of basic propositions concerning equivalences, adjunctions, limits, and colimits and the relationships between these notions.

Recall that categories, functors, and natural transformations assemble into a 2-category $\mathcal{C}at$. Similarly, the ∞-categories, ∞-functors, and ∞-natural transformations in any ∞-cosmos assemble into a 2-category, namely the *homotopy 2-category* of the ∞-cosmos, introduced in §1.4. By Exercise 1.4.ii, $\mathcal{C}at$ can be regarded as a special case of a homotopy 2-category. In this chapter, we will use strict 2-categorical techniques to define *adjunctions* between ∞-categories and *limits* and *colimits* of diagrams valued in an ∞-category and prove that these notions interact in the expected ways. In the homotopy 2-category of categories, these recover the classical results from 1-category theory. As these proofs are equally valid in any homotopy 2-category, our arguments also establish the desired generalizations by simply appending the prefix "∞-."

2.1. Adjunctions and equivalences

In §1.4, we encountered the definition of an *equivalence* between a pair of objects in a 2-category. In the case where the ambient 2-category is the homotopy 2-category of an ∞-cosmos, we observed in Theorem 1.4.7 that the 2-categorical notion of equivalence precisely recaptures the notion of equivalence introduced in Definition 1.2.1 between ∞-categories in the full ∞-cosmos. In each of the examples of ∞-cosmoi we have considered, the representably-defined equivalences in the ∞-cosmos coincide with the standard notion of equivalences between ∞-categories as presented in that particular model.[1] Thus, the 2-categorical notion of equivalence is the "correct" notion of equivalence between ∞-categories.

Similarly, there is a standard definition of an *adjunction* between a pair of objects in a 2-category, which, when interpreted in the homotopy 2-category of ∞-categories, functors, and natural transformations in an ∞-cosmos, will define the correct notion of adjunction between ∞-categories.

2.1.1. Definition (adjunction). An **adjunction** between ∞-categories is comprised of:

- a pair of ∞-categories A and B,
- a pair of functors $u\colon A \to B$ and $f\colon B \to A$,
- and a pair of natural transformations $\eta\colon 1_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow 1_A$, called the **unit** and **counit** respectively,

[1] For instance, as outlined in Digression 1.2.10, the equivalences in the ∞-cosmoi of Example 1.2.21 recapture the weak equivalences between fibrant-cofibrant objects in the usual model structure.

so that the triangle equalities hold:[2]

$$
\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & B \arrow[dl, phantom, "\Downarrow\eta"] \\
A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] &
\end{tikzcd}
=
\begin{tikzcd}
B \\
A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
& A \arrow[ur, "u"] \arrow[rr, equal] & {} & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
A
\end{tikzcd}
$$

The functor f is called the **left adjoint** and u is called the **right adjoint**, a relationship that is denoted symbolically in text by writing $f \dashv u$ or in a displayed diagram such as[3]

$$
\begin{tikzcd}
A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"']
\end{tikzcd}
$$

2.1.2. Digression (why this is the right definition). For readers who find Definition 2.1.1 implausible — perhaps too simple to be trusted — we offer a few words of justification. Firstly, the correct notion of *adjunction* between quasi-categories is well established, though the definition appearing in [**61**, §5.2] takes a quite different form. In Appendix F, we prove that in the ∞-cosmos of quasi-categories, our definition of adjunction precisely recovers Lurie's. As explained in Part IV, each of the models of $(\infty, 1)$-categories described in Example 1.2.21 "has the same category theory," so Definition 2.1.1 agrees with the community consensus notion of adjunction between $(\infty, 1)$-categories.

But what about those ∞-cosmoi whose objects model (∞, n)- or (∞, ∞)-categories? For instance in the ∞-cosmos of complicial sets, the adjunctions defined in the homotopy 2-category are the "pseudo-style" adjunctions. While these are not the most general adjunctions that might be considered — for instance, one could have (op)lax units and counits — they are an important class of adjunctions. One reason for the relevance of Definition 2.1.1 in all ∞-cosmoi is its formal properties vis-a-vis the related notion of equivalence, which Theorem 1.4.7 has established is morally "correct," and with the notions of limits and colimits to be introduced.

Finally, a reasonable objection is that Definition 2.1.1 appears too "low dimensional," comprised of data found entirely in the homotopy 2-category and ignoring the higher dimensional morphisms in an ∞-cosmos. This deficiency will be addressed in Chapter 9, when we prove that any adjunction between ∞-categories extends to a *homotopy coherent adjunction*, and moreover such extensions are homotopically unique.

The definition of an adjunction given in Definition 2.1.1 is "equational" in character: stated in terms of the objects, 1-cells, and 2-cells of a 2-category and their composites. Immediately:

2.1.3. Lemma. *Adjunctions in a 2-category are preserved by any 2-functor.* □

Lemma 2.1.3 provides an easy source of examples of adjunctions between quasi-categories. The 2-functors underlying the cosmological functors of Example 1.3.8 then transfer adjunctions defined in one model of $(\infty, 1)$-categories to adjunctions defined in each of the other models.

2.1.4. Example (adjunctions between 1-categories). Via the nerve embedding $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$, any adjunction between 1-categories induces an adjunction between their nerves regarded as quasi-categories.

[2]The left-hand equality of pasting diagrams asserts the composition relation $u\epsilon \cdot \eta u = \mathrm{id}_u$ in the hom-category $\mathsf{hFun}(A, B)$, while the right-hand equality asserts that $\epsilon f \cdot f\eta = \mathrm{id}_f$ in $\mathsf{hFun}(B, A)$.

[3]Some authors contort adjunction diagrams so that the left adjoint is always on the left; we instead use the turnstile symbol "$\perp$" to indicate which adjoint is the left adjoint.

2.1.5. Example (adjunctions between topological categories). The *homotopy coherent nerve* of Definition 6.3.1 defines a 2-functor $\mathfrak{N} \colon \mathcal{K}an\text{-}\mathcal{C}at \to \mathfrak{h}\mathcal{QC}at$ from the 2-category of Kan complex enriched categories, simplicially enriched functors, and simplicial natural transformations, to the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$. In this way, topologically enriched adjunctions define adjunctions between quasi-categories.

2.1.6. Remark. Topologically enriched adjunctions are relatively rare. More prevalent are "up-to-homotopy" topologically enriched adjunctions, such as those given by Quillen adjunctions between simplicial model categories. These also define adjunctions between quasi-categories, though the proof will have to wait until Part II.

The preservation of adjunctions by 2-functors proves:

2.1.7. Proposition. *Given any adjunction* $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ *between ∞-categories then:*

(i) for any ∞-category X,

$$\begin{tikzcd} \mathsf{Fun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

defines an adjunction between quasi-categories;

(ii) for any ∞-category X,

$$\begin{tikzcd} \mathsf{hFun}(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

defines an adjunction between categories;

(iii) for any simplicial set U,

$$\begin{tikzcd} A^U \arrow[r, bend right, "u^U"'] \arrow[r, phantom, "\perp"] & B^U \arrow[l, bend right, "f^U"'] \end{tikzcd}$$

defines an adjunction between ∞-categories;

(iv) and if the ambient ∞-cosmos is cartesian closed, then for any ∞-category C,

$$\begin{tikzcd} A^C \arrow[r, bend right, "u^C"'] \arrow[r, phantom, "\perp"] & B^C \arrow[l, bend right, "f^C"'] \end{tikzcd}$$

defines an adjunction between ∞-categories.

For instance, taking $X = 1$ in (ii) yields a "derived" adjunction between the homotopy categories of the ∞-categories A and B.

Proof. Any adjunction $f \dashv u$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ is preserved by the 2-functors $\mathsf{Fun}(X,-) \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, $\mathsf{hFun}(X,-) \colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, $(-)^U \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$, and $(-)^C \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$. $\square$

2.1.8. Remark. There are contravariant versions of each of the adjunction-preservation results of Proposition 2.1.7, the first of which we explain in detail. Fixing the codomain variable of the functor-space at any ∞-category $C \in \mathcal{K}$ defines a 2-functor

$$\mathsf{Fun}(-, C) \colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$$

that is contravariant on 1-cells and covariant on 2-cells.[4] Similarly, the cotensor or exponential $C^{(-)}$ is contravariant on 1-cells and covariant on 2-cells.[5] Such 2-functors preserve adjunctions, but exchange left and right adjoints: for instance, given $f \dashv u$ in $\mathcal{K}$, we obtain an adjunction

$$\begin{tikzcd} \mathsf{Fun}(A, C) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "f^*"'] & \mathsf{Fun}(B, C) \arrow[l, bend right, "u^*"'] \end{tikzcd}$$

between the functor- spaces.

2.1.9. Proposition. *Adjunctions compose: given adjoint functors*

$$\begin{tikzcd} C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"] \end{tikzcd}$$

the composite functors are adjoint.

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_{\mathrm{c}} \Rightarrow u'f'$, and $\epsilon'\colon f'u' \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams

$$\begin{tikzcd} C \arrow[rrrr, equal] \arrow[dr, "f'"'] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \\ & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"'] & \\ & & A \arrow[ur, "u"'] & & \end{tikzcd} \qquad \begin{tikzcd} & & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \\ & B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[dr, "f"] & \\ A \arrow[ur, "u"] \arrow[rrrr, equal] & & & & A \end{tikzcd}$$

define the unit and counit of $ff' \dashv u'u$ so that the triangle equalities

$$\begin{tikzcd}[column sep=small] C \arrow[rrrr, equal] \arrow[dr, "f'"'] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \\ & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[dr, "f"] & \\ & & A \arrow[ur, "u"] \arrow[rrrr, equal] & & & & A \end{tikzcd} = \begin{tikzcd} C \arrow[d, bend right=50, "ff'"'] \arrow[d, bend left=50, "ff'"] \arrow[d, phantom, "="] \\ A \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & C \arrow[rrrr, equal] \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & & \arrow[d, phantom, "\Downarrow\eta'"] & & C \\ & B \arrow[ur, "u'"] \arrow[rr, equal] & \arrow[d, phantom, "\Downarrow\epsilon"] & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[ur, "u'"'] & \\ A \arrow[ur, "u"] \arrow[rrrr, equal] & & & & A \arrow[ur, "u"'] & & \end{tikzcd} = \begin{tikzcd} C \\ A \arrow[u, bend left=50, "u'u"] \arrow[u, bend right=50, "u'u"'] \arrow[u, phantom, "="] \end{tikzcd}$$

hold. $\square$

An adjoint to a given functor is unique up to natural isomorphism:

[4]On a strict 2-category, the superscript "op" is used to signal that the 1-cells should be reversed but not the 2-cells, the superscript "co" is used to signal that the 2-cells should be reversed but not the 1-cells, and the superscript "coop" is used to signal that both the 1- and 2-cells should be reversed; see Definition B.1.7.

[5]In the case of the simplicial cotensor, the domain can safely be restricted to the homotopy 2-category of quasi-categories or can be regarded as an analogously-defined homotopy 2-category of simplicial sets.

2.1.10. Proposition (uniqueness of adjoints).

(i) *If $f \dashv u$ and $f' \dashv u$, then $f \cong f'$.*

(ii) *Conversely, if $f \dashv u$ and $f \cong f'$ then $f' \dashv u$.*

Proof. Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_c \Rightarrow uf'$, and $\epsilon'\colon f'u \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams

$$\begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f'"'] \arrow[r, phantom, "\Downarrow\eta'"] & & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] \arrow[r, phantom, "\Downarrow\eta"] & & B \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd}$$

define 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f$. The composites $f \Rightarrow f' \Rightarrow f$ and $f' \Rightarrow f \Rightarrow f$ are computed by pasting these diagrams together horizontally on one side or the other. Applying the triangle equalities for the adjunctions $f \dashv u$ and $f' \dashv u$ both composites are easily seen to be identities. Hence $f \cong f'$ as functors from B to A.

Part (ii) is left as Exercise 2.1.i. ☐

We will make repeated use of the following standard 2-categorical result, which says that any equivalence in a 2-category can be promoted to an equivalence that also defines an adjunction:

2.1.11. Proposition (adjoint equivalences). *Any equivalence can be promoted to an* ***adjoint equivalence*** *by modifying one of the 2-cells. That is, the invertible 2-cells in an equivalence can be chosen so as to satisfy the triangle equalities. Hence, if f and g are inverse equivalences then $f \dashv g$ and $g \dashv f$.*

Proof. Consider an equivalence comprised of functors $f\colon A \to B$ and $g\colon B \to A$ and invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal, ""{name=U}] \arrow[r, bend right, "gf"', ""{name=D}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "fg", ""{name=U}] \arrow[r, bend right, equal, ""{name=D}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

We will construct an adjunction $f \dashv g$ with unit $\eta \coloneqq \alpha$ by modifying β. The "triangle identity composite"

$$\phi \coloneqq f \xRightarrow{f\alpha} fgf \xRightarrow{\beta f} f$$

is an isomorphism, though likely not an identity. Define

$$\epsilon \coloneqq fg \xRightarrow{\phi^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B \coloneqq fg \xRightarrow{\beta^{-1}fg} fgfg \xRightarrow{f\alpha^{-1}g} fg \xRightarrow{\beta} \mathrm{id}_B$$

This "corrects" the counit so that now the composite $\epsilon f \cdot f\eta$, displayed on the top of the diagram

$$\begin{tikzcd} & fgf \arrow[dr, Rightarrow, "\phi^{-1}gf"'] \arrow[drr, Rightarrow, "\epsilon f"] & & \\ f \arrow[ur, Rightarrow, "f\alpha"] \arrow[dr, Rightarrow, "\phi^{-1}"'] & & fgf \arrow[r, Rightarrow, "\beta f"] & f \\ & f \arrow[ur, Rightarrow, "f\alpha"] \arrow[urr, Rightarrow, "\phi"'] & & \end{tikzcd}$$

which agrees with the bottom composite by "naturality of whiskering," is the identity id_f.

Now by another diagram chase, the other triangle composite $g\epsilon \cdot \eta g$ is an idempotent:

$$\begin{tikzcd}
g \arrow[r, Rightarrow, "\eta g"] \arrow[d, Rightarrow, "\eta g"'] & gfg \arrow[r, Rightarrow, "g\epsilon"] \arrow[d, Rightarrow, "\eta gfg"] & g \arrow[d, Rightarrow, "\eta g"] \\
gfg \arrow[r, Rightarrow, "gf\eta g"] \arrow[dr, equal] & gfgfg \arrow[r, Rightarrow, "gfg\epsilon"] \arrow[d, Rightarrow, "g\epsilon fg"] & gfg \arrow[d, Rightarrow, "g\epsilon"] \\
& gfg \arrow[r, Rightarrow, "g\epsilon"'] & g
\end{tikzcd}$$

By cancelation, any idempotent isomorphism is the identity, proving that $g\epsilon \cdot \eta g = \mathrm{id}_g$. □

One use of Proposition 2.1.11 is to show that adjunctions are equivalence invariant:

2.1.12. Proposition (equivalence-invariance of adjunctions). *A functor $u\colon A \to B$ between ∞-categories admits a left adjoint if and only if, for any pair of equivalent ∞-categories $A' \simeq A$ and $B' \simeq B$, the equivalent functor $u'\colon A' \to B'$ admits a left adjoint.*

Proof. Exercise 2.1.ii. □

As we will discover, all of ∞-category theory is equivalence invariant in this way.

2.1.13. Lemma. *For any ∞-category A, the "composition" functor*

$$\begin{tikzcd}
A^2 \underset{A}{\times} A^2 \arrow[r, "\circ"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & A^2 \arrow[l, bend right, "{(-,\mathrm{id}_{\mathrm{dom}(-)})}"'] \arrow[l, bend left, "{(\mathrm{id}_{\mathrm{cod}(-)},-)}"]
\end{tikzcd} \tag{2.1.14}$$

admits left and right adjoints, which, respectively, "extend an arrow into a composable pair" by pairing it with the identities at its domain or its codomain.

Proof. There is a dual adjunction in $\mathcal{C}at$ whose functors we describe using notation for simplicial operators introduced in 1.1.1; the full subcategory of $\mathcal{C}at$ spanned by the finite non-empty ordinals is isomorphic to $\mathbf{\Delta}$.

$$\begin{tikzcd}
\mathbb{3} \arrow[r, bend left, "s^0"] \arrow[r, bend right, "s^1"'] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"] & \mathbb{2} \arrow[l, "d^1"']
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
A^3 \arrow[r, "A^{d^1}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & A^2 \arrow[l, bend right, "A^{s^0}"'] \arrow[l, bend left, "A^{s^1}"]
\end{tikzcd}$$

Any ∞-category A in an ∞-cosmos $\mathcal{K}$ defines a 2-functor $A^{(-)}\colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$ carrying the adjoint triple displayed above-left to the one displayed above-right.

Now we claim there is a trivial fibration $A^3 \overset{\sim}{\twoheadrightarrow} A^2 \times_A A^2$ constructed as follows. The pushout diagram of simplicial sets displayed below-left is carried by the simplicial cotensor $A^{(-)}\colon s\mathcal{S}et^{\mathrm{op}} \to \mathcal{K}$ to a pullback diagram displayed below-right; since the legs of the pushout square are monomorphisms, the legs of the pullback square are isofibrations

$$\begin{tikzcd}
\Lambda^1[2] \arrow[dr, phantom, "\lrcorner"] & \mathbb{2} \arrow[l] \\
\mathbb{2} \arrow[u, hook] & \mathbb{1} \arrow[l, hook, "d^0"] \arrow[u, hook, "d^1"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A^{\Lambda^1[2]} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, two heads, "\mathrm{ev}_0"] \\
A^2 \arrow[r, two heads, "\mathrm{ev}_1"'] & A
\end{tikzcd}$$

Lemma 1.2.11 tells us that the cotensor of the inner horn inclusion $\Lambda^1[2] \hookrightarrow \mathbb{3}$ with the ∞-category A defines a trivial fibration $A^{\mathbb{3}} \xrightarrow{\sim} A^{\Lambda^1[2]}$ and the pullback square above-left recognizes its codomain as the desired ∞-category of "composable pairs." Any section s to $q\colon A^{\mathbb{3}} \xrightarrow{\sim} A^{\mathbb{2}} \times_A A^{\mathbb{2}}$ can be made into an equivalence inverse. By Proposition 2.1.11, these functors are both left and right adjoints. Composing the adjunction $q \dashv s \dashv q$ with the adjunction constructed above defines the desired adjunction. $\square$

Note that the adjoint functors of (2.1.14) commute with the "endpoint evaluation" functors to $A \times A$. In fact, the units and counits can similarly be fibered over $A \times A$; see Example 3.6.13.

Exercises.

2.1.i. Exercise. Prove Proposition 2.1.10(ii).

2.1.ii. Exercise. Prove Proposition 2.1.12: given an adjunction $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ and equivalences $A \simeq A'$ and $B \simeq B'$ construct an adjunction between A' and B'.

2.2. Initial and terminal elements

Employing the tactic used to define the homotopy category of A in Definition 1.4.12, we use the terminal ∞-category 1 to probe inside the ∞-category A. The objects $a \in \mathsf{h}A$ of the homotopy category of A were defined to be maps of ∞-categories $a\colon 1 \to A$, but to avoid the proliferation of the term "objects" we refer to maps $a\colon 1 \to A$ as **elements** of the ∞-category A instead.

Before introducing limits and colimits of general diagram shapes, we warm up by defining initial and terminal elements in an ∞-category A.

2.2.1. Definition (initial/terminal element). An **initial element** in an ∞-category A is a left adjoint to the unique functor $!\colon A \to 1$, as displayed below-left, while a **terminal element** in an ∞-category A is a right adjoint, as displayed below-right.

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \end{tikzcd} \qquad\qquad \begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

Let us unpack the definition of an initial element; dual remarks apply to terminal elements.

2.2.2. Lemma (the minimal data required to present an initial element). *To define an initial element in A, it suffices to specify*

- *an element $i\colon 1 \to A$ and*
- *a natural transformation*

$$\begin{tikzcd} & 1 \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ A \arrow[ur, "!"] \arrow[rr, equal] & {} & A \end{tikzcd}$$

so that the component $\epsilon i\colon i \Rightarrow i$ is the identity in $\mathsf{h}A$.

Proof. Proposition 1.4.5 demonstrates that the ∞-category $1 \in \mathcal{K}$ is terminal in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. The 1-dimensional aspect of this universal property implies that i defines a section of the unique map $A \to 1$ and from the 2-dimensional aspects, we see that there exist no non-identity

2-cells with codomain 1. In particular, the unit of the adjunction $i \dashv !$ is necessarily an identity and one of the triangle equalities comes for free. The data enumerated above is what remains of Definition 2.1.1 in this setting. $\square$

Put more concisely, an initial element i defines a **left adjoint right inverse** to the functor $! \colon A \to 1$. Such adjunctions are studied more systematically in §B.4. In fact, it suffices to assume that the counit component ϵi is an isomorphism, not necessarily the identity; see Lemma B.4.2.

In a cartesian closed ∞-cosmos, an initial element may also be characterized as a *limit* for the identity functor $\mathrm{id}_A \to \mathrm{id}_A$, with the universal property of the counit $\epsilon \colon i! \Rightarrow \mathrm{id}_A$ transposing across the 2-adjunction $A \times - \dashv (-)^A$ to define the limit cone. Corollary 13.5.4 deduces this result as a special case of more general formal category theory developed there, but it can also be proven directly as a 2-categorical pasting diagram calculation; see Exercise 2.2.iii.

2.2.3. Remark. Applying the 2-functor $\mathsf{Fun}(X,-) \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ to an initial or terminal element of an ∞-category $A \in \mathcal{K}$ yields adjunctions

$$\begin{tikzcd} \mathbb{1} \cong \mathsf{Fun}(X,1) \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,A) \arrow[l, bend left, "!"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \cong \mathsf{Fun}(X,1) \arrow[r, bend right, "t_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,A) \arrow[l, bend right, "!"'] \end{tikzcd}$$

Via the isomorphisms $\mathsf{Fun}(X,1) \cong \mathbb{1}$ that express the universal property of the terminal ∞-category 1, we see that initial or terminal elements of A define initial or terminal elements of the functor-space $\mathsf{Fun}(X,A)$, namely the composite functors

$$X \xrightarrow{!} 1 \xrightarrow{i} A \qquad \text{or} \qquad X \xrightarrow{!} 1 \xrightarrow{t} A\ .$$

In particular, initial or terminal elements are representably initial or terminal at the level of the ∞-cosmos.

This representable universal property is also captured at the level of the homotopy 2-category. The next lemma shows that the initial element $i \colon 1 \to A$ is initial among all *generalized elements* $f \colon X \to A$ in the following precise sense.

2.2.4. Lemma. *An element $i \colon 1 \to A$ is initial if and only if for all $f \colon X \to A$ there exists a unique 2-cell with boundary*

$$\begin{tikzcd} & 1 \arrow[dr, "i"] & \\ X \arrow[ur, "!"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\Downarrow\exists!"] & & A \end{tikzcd}$$

Proof. If $i \colon 1 \to A$ is initial, then the adjunction of Definition 2.2.1 is preserved by the 2-functor $\mathsf{hFun}(X,-) \colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, defining an adjunction

$$\begin{tikzcd} \mathbb{1} \cong \mathsf{hFun}(X,1) \arrow[r, bend left, "i_*"] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,A) \arrow[l, bend left, "!"] \end{tikzcd}$$

Via the isomorphism $\mathsf{hFun}(X,1) \cong \mathbb{1}$, this adjunction proves that the element $i! \colon X \to A$ is initial in $\mathsf{hFun}(X,A)$ and thus has the universal property of the statement.

Conversely, if $i: 1 \to A$ satisfies the universal property of the statement, applying this to the generic element of A (the identity map $\mathrm{id}_A : A \to A$) easily produces the data of Lemma 2.2.2. $\square$

2.2.5. Remark. Lemma 2.2.4 says that initial elements are *representably initial* in the homotopy 2-category. Specializing the generalized elements to ordinary elements, we see that initial and terminal elements in A respectively define initial and terminal elements in the homotopy category $\mathsf{h}A$.

2.2.6. Lemma. *If A has an initial element and $A \simeq A'$ then A' has an initial element and these elements are preserved up to isomorphism by the equivalences.*

Proof. By Proposition 2.1.11, the equivalence $A \simeq A'$ can be promoted to an adjoint equivalence, which can immediately be composed with the adjunction characterizing an initial element i of A:

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \arrow[r, bend left, "\sim"] \arrow[r, phantom, "\perp"] & A' \arrow[l, bend left, "\sim"] \end{tikzcd}$$

The composite adjunction provided by Proposition 2.1.9 proves that the image of i defines an initial element of A', which by construction is preserved by the equivalence $A \xrightarrow{\sim} A'$.

To see that the equivalence $A' \xrightarrow{\sim} A$ also preserves initial elements, we can use the invertible 2-cells of the equivalence to see that i is isomorphic to the image of the image of i in A'. In case the initial objects in mind are not the ones being considered here, we can appeal to the uniqueness of initial elements proven in Exercise 2.2.ii. $\square$

Exercises.

2.2.i. Exercise. Prove that initial elements are preserved by left adjoints and terminal elements are preserved by right adjoints.

2.2.ii. Exercise. Prove that any two initial elements in an ∞-category A are isomorphic in $\mathsf{h}A$.

2.2.iii. Exercise. Prove that in a cartesian closed ∞-cosmos, initial elements in A may be characterized as limits of the identity functor $\mathrm{id}_A : A \to A$ by transposing the universal property of the counit of Definition 2.2.1.

2.3. Limits and colimits

Our aim is now to introduce limits and colimits of diagram valued *inside* an ∞-category A in some ∞-cosmos. We will consider two varieties of diagrams:

- In a generic ∞-cosmos $\mathcal{K}$, we shall consider diagrams indexed by a simplicial set J and valued in an ∞-category A.
- In a cartesian closed ∞-cosmos $\mathcal{K}$, we shall also consider diagram indexed by an ∞-category J and valued in an ∞-category A.[6]

[6]In Proposition 14.3.4 proven in Part IV, we shall discover that in the case of the ∞-cosmoi of $(\infty, 1)$-categories, there is no essential difference between these notions: in $\mathcal{QC}at$ they are tautologically the same, and in all biequivalent ∞-cosmsoi the ∞-category of diagrams indexed by an ∞-category A is equivalent to the ∞-category of diagrams indexed by its underlying quasi-category, regarded as a simplicial set.

2.3.1. Definition (diagram ∞-categories). For a simplicial set J — or possibly, in the case of a cartesian closed ∞-cosmos, an ∞-category J — and an ∞-category A, we refer to A^J as the **∞-category of J-shaped diagrams in A**. Both constructions define bifunctors

$$\begin{array}{ccc} s\mathcal{S}et^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} & \qquad & \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} \\ (J, A) \longmapsto A^J & & (J, A) \longmapsto A^J \end{array}$$

In either indexing context, there is a terminal object 1 with the property that $A^1 \cong A$ for any ∞-category A. Restriction along the unique map $!\colon J \to 1$, induces the **constant diagram functor** $\Delta\colon A \to A^J$.

We are deliberately conflating the notation for ∞-categories of diagrams indexed by a simplicial set or by another ∞-category because all of the results we will prove in Part I about the former case will also apply to the latter. For economy of language, we refer only to simplicial set indexed diagrams for the remainder of this section.

2.3.2. Definition. An ∞-category A **admits all colimits** of shape J if the constant diagram functor $\Delta\colon A \to A^J$ admits a left adjoint, while A **admits all limits** of shape J if the constant diagram functor admits a right adjoint:

$$\begin{tikzcd} A^J \arrow[r, bend left, "\mathrm{colim}", "\perp"'] \arrow[r, bend right, "\mathrm{lim}"', "\perp"] & A \arrow[l, "\Delta"'] \end{tikzcd}$$

2.3.3. Warning. Limits or colimits of set-indexed diagrams — the case where the indexing shape is a coproduct of the terminal object 1 indexed by a set J — are called **products** or **coproducts**, respectively. In this case the ∞-category of diagrams itself decomposes as a product $A^J \cong \prod_J A$. As the functor

$$\begin{array}{c} \mathfrak{h}\mathcal{K} \xrightarrow{\mathsf{hFun}(1,-)} \mathcal{C}at \\ A \longmapsto \mathsf{h}A \end{array}$$

that carries an ∞-category to its homotopy category preserves products, when J is a set there is a chain of isomorphisms

$$\mathsf{h}(A^J) \cong \mathsf{h}(\textstyle\prod_J A) \cong \textstyle\prod_J \mathsf{h}A \cong (\mathsf{h}A)^J$$

Thus, in this special case the adjunctions of Definition 2.3.2 that define products or coproducts in an ∞-category descend to the adjunctions that define products or coproducts in its homotopy category.

However, this argument does *not* extend to more general limit or colimit notions, and such ∞-categorical limits or colimits are generally not limits or colimits in the homotopy category.[7] In §3.2, we shall see that the homotopy category construction fails to preserve more complicated cotensors, even in the relatively simple case of $J = \mathbb{2}$.

The problem with Definition 2.3.2 is that it is insufficiently general: many ∞-categories will have certain, but not all, limits of diagrams of a particular indexing shape. So it would be desirable to re-express Definition 2.3.2 in a form that allows us to define the limit of a single diagram $d\colon 1 \to A^J$

[7]This sort of behavior is expected in abstract homotopy theory: homotopy limits and colimits are not generally limits or colimits in the homotopy category.

or of a family of diagrams. To achieve this, we make use of the following 2-categorical notion that op-dualizes the more familiar absolute extension diagrams.

2.3.4. Definition (absolute lifting diagrams). Given a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in a 2-category, an **absolute left lifting** of g through f is given by a 1-cell and 2-cell as displayed below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, shift left=3, "\exists!\Uparrow\zeta"] & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

so that any 2-cell as displayed above-center factors uniquely through (ℓ, λ) as displayed above-right.

Dually, an **absolute right lifting** of g through f is given by a 1-cell and 2-cell as displayed below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, shift left=3, "\exists!\Downarrow\zeta"] & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd}$$

so that any 2-cell as displayed above-center factors uniquely through (r, ρ) as displayed above-right.

The adjectives "left" and "right" refer to the handedness of the adjointness of these constructions: left and right liftings respectively define left and right adjoints to the composition functor $f_* \colon \mathsf{hFun}(C, B) \to \mathsf{hFun}(C, A)$, with the 2-cells defining the components of the unit and counit of these adjunctions, respectively, at the object g. The adjective "absolute" refers to the following stability property.

2.3.5. Lemma. *Absolute left or right lifting diagrams are stable under restriction of their domain object: if (ℓ, λ) defines an absolute left lifting of g through f, then for any $c \colon X \to C$, the restricted diagram $(\ell c, \lambda c)$ defines an absolute left lifting of gc through f.*

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"'] & C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

Proof. Exercise 2.3.i. □

Units and counits of adjunctions provide important examples of absolute left and right lifting diagrams respectively:

2.3.6. Lemma. *A 2-cell $\eta \colon \mathrm{id}_B \Rightarrow uf$ defines the unit of an adjunction $f \dashv u$ if and only if (f, η) defines an absolute left lifting diagram, displayed below-left.*

$$\begin{tikzcd} & A \arrow[d, "u"] \\ B \arrow[ur, "f"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

Dually a 2-cell $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ defines the counit of an adjunction if and only if (u, ϵ) defines an absolute right lifting diagram, displayed above-right.

Proof. We prove the universal property of the counit. Given a 2-cell $\alpha\colon fb \Rightarrow a$ as displayed below-left

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[d, "f"] \\ A \arrow[r, equal] \arrow[ur, "u"] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

there exists a unique transpose $\beta\colon b \Rightarrow ua$ as displayed above-right across the induced adjunction

$$\begin{tikzcd} \mathsf{hFun}(X,B) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathsf{hFun}(X,A) \arrow[l, bend left, "u_*"] \end{tikzcd}$$

between the hom-categories of the homotopy 2-category; see Proposition 2.1.7(ii). From right to left, transposes are composed by pasting with the counit; hence the left-hand side above equals the right-hand side. The converse is left as Exercise 2.3.ii. $\square$

In particular, the unit of the adjunction $\mathrm{colim} \dashv \Delta$ of Definition 2.3.2 defines an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, "\mathrm{colim}"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Uparrow\eta"] & A^J \end{tikzcd}$$

By Lemma 2.3.5, this universal property is retained upon restricting to any subobject of the ∞-category of diagrams. This motivates the following definitions:.

2.3.7. Definition. A **colimit** of a family of diagrams $d\colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute left lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\mathrm{colim}"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Uparrow\eta"] & A^J \end{tikzcd}$$

comprised of a **colimit functor** $\mathrm{colim}\colon D \to A$ and a **colimit cone** $\eta\colon d \Rightarrow \Delta\,\mathrm{colim}$.

Dually, a **limit** of a family of diagrams $d\colon D \to A^J$ indexed by J in an ∞-category A is given by an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

comprised of a **limit functor** $\lim\colon D \to A$ and a **limit cone** $\epsilon\colon \Delta\lim \Rightarrow d$.

2.3.8. Remark. If A has all limits of shape J, then Lemma 2.3.5 implies that any family of diagrams $d\colon D \to A^J$ has a limit, defined by evaluating the limit functor $\lim\colon A^J \to A$ at d, i.e., by restricting $\lim$ along d. In certain ∞-cosmoi, such as $\mathcal{QC}at$, if every diagram $d\colon 1 \to A^J$ has a limit, then A has all J-indexed limits, because the quasi-category 1 generates the ∞-cosmos of quasi-categories in a suitable sense, but this result is not true for all ∞-cosmoi. See Corollary 16.2.10.

For example, a 2-categorical lemma enables general proof of a classical result from homotopy theory that computes geometric realizations of "split" simplicial objects. Before proving this, we introduce the indexing shapes involved.

2.3.9. Definition (split augmented (co)simplicial objects). Recall $\mathbf{\Delta}$ is the simplex category of finite non-empty ordinals and order-preserving maps introduced in 1.1.1. It defines a full subcategory of a category $\mathbf{\Delta}_+$ which freely appends the empty ordinal "$[-1]$" as an initial object. This in turn defines a wide subcategory of a category $\mathbf{\Delta}_\perp$, which adds an "extra" degeneracy $\sigma^{-1}\colon [n+1] \twoheadrightarrow [n]$ between each pair of consecutive ordinals, including $\sigma^{-1}\colon [0] \twoheadrightarrow [-1]$. The category $\mathbf{\Delta}_+$ also defines a wide subcategory of a category $\mathbf{\Delta}_\top$, which adds an "extra" degeneracy $\sigma^{n+1}\colon [n+1] \twoheadrightarrow [n]$ on the other side between each pair of consecutive ordinals, including $\sigma^0\colon [0] \twoheadrightarrow [-1]$.

Diagrams indexed by $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp, \mathbf{\Delta}_\top$ are respectively called **cosimplicial objects**, **coaugmented cosimplicial objects**, and **split coaugmented cosimplicial objects** (in the case of either $\mathbf{\Delta}_\perp$ or $\mathbf{\Delta}_\top$), if they are covariant, and **simplicial objects**, **augmented simplicial objects**, and **split augmented simplicial objects**, if they are contravariant. When it is useful to disambiguate between $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ we refer to the former category as a "bottom splitting" and the latter category as a "top splitting," but this terminology is not standard.

A simplicial object $d\colon 1 \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ in an ∞-category A **admits an augmentation** or **admits a splitting**, if it lifts along the restriction functors

$$\begin{tikzcd} & A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \arrow[d, two heads] \\ & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[d, two heads] \\ 1 \arrow[r, "d"'] \arrow[ur, dashed] \arrow[uur, dotted] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

where in the case of a top splitting, $\mathbf{\Delta}_\perp$ is replaced by $\mathbf{\Delta}_\top$. The family of simplicial objects admitting an augmentation and splitting is then represented by the generic element $A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \twoheadrightarrow A^{\mathbf{\Delta}^{\mathrm{op}}}$. The following proposition proves that for any simplicial object admitting a splitting, the augmentation defines the colimit cone; dual results apply to colimits of split cosimplicial objects. The limit and colimit cones are defined by cotensoring with the unique natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, hook] \arrow[dr, "!"'] & \Uparrow\nu & \mathbf{\Delta}_+ \\ & \mathbb{1} \arrow[ur, "{[-1]}"'] & \end{tikzcd} \tag{2.3.10}$$

that exists because $[-1]\colon \mathbb{1} \to \mathbf{\Delta}_+$ is initial; see Lemma 2.2.4.

2.3.11. Proposition (geometric realizations). *Let A be any ∞-category. For every cosimplicial object in A that admits a coaugmentation and a splitting, the coaugmentation defines a limit cone. Dually, for every simplicial object in A that admits an augmentation and a splitting, the augmentation defines a colimit cone.*

That is, there exist absolute right and left lifting diagrams

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}", near end, shift left=3ex] & A^{\mathbf{\Delta}} \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp^{\mathrm{op}}} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow A^{\nu^{\mathrm{op}}}", near end, shift left=3ex] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\top} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}", near end, shift left=3ex] & A^{\mathbf{\Delta}} \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\top^{\mathrm{op}}} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow A^{\nu^{\mathrm{op}}}", near end, shift left=3ex] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

in which the 2-cells are obtained as restrictions of the cotensor of the 2-cell (2.3.10) *with A. Moreover, such limits and colimits are* ***absolute****, preserved by any functor $f\colon A \to B$ of ∞-categories.*

Proof. By Example B.5.2, the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\perp$ admits a right adjoint, which can automatically be regarded as an adjunction "over" $\mathbb{1}$ since $\mathbb{1}$ is 2-terminal. The initial element $[-1] \in \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp$ defines a left adjoint to the constant functor:

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook] \arrow[d, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] & \mathbf{\Delta}_\perp \arrow[ll, dashed, bend right, "\top"'] \arrow[dll, "!"] \\ \mathbb{1} \arrow[ur, dashed, "{[-1]}"] \arrow[ur, phantom, "\perp", shift left=3ex] & & \end{tikzcd}$$

with the counit of this adjunction (2.3.10) defining the colimit cone under the constant functor at the initial element. These adjunctions are preserved by the 2-functor $A^{(-)}\colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$, yielding a diagram

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \arrow[dll, dotted, bend right, "!"'] \arrow[dll, phantom, "\perp", near end] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}", near end, shift left=3ex] & A^{\mathbf{\Delta}} \arrow[ll, dotted, bend left, "\top"] \end{tikzcd}$$

By Lemma B.5.1 these adjunctions witness the fact that evaluation at $[-1]$ and the 2-cell from (2.3.10) define an absolute right lifting of the canonical restriction functor $A^{\mathbf{\Delta}_\perp} \to A^{\mathbf{\Delta}}$ through the constant diagram functor, as claimed. The colimit case is proven similarly by applying the composite 2-functor

$$\mathcal{C}at^{\mathrm{coop}} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}at^{\mathrm{op}} \xrightarrow{A^{(-)}} \mathfrak{h}\mathcal{K}$$

A similar argument, starting from Example B.5.3, constructs the absolute lifting diagrams from the top splitting.

Finally, by the 2-functoriality of the simplicial cotensor, any $f\colon A \to B$ commutes with the 2-cells defined by cotensoring with ν or its opposite.

$$\begin{tikzcd} & & A \arrow[r, "f"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & A^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}", near end, shift left=3ex] & A^{\mathbf{\Delta}} \arrow[r, "f^{\mathbf{\Delta}}"'] & B^{\mathbf{\Delta}} \end{tikzcd} = \begin{tikzcd} & & & B \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[r, "f^{\mathbf{\Delta}_\perp}"'] & B^{\mathbf{\Delta}_\perp} \arrow[r, "\mathrm{res}"'] & B^{\mathbf{\Delta}_+} \arrow[ur, "\mathrm{ev}_{[-1]}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^{\nu}", near end, shift left=3ex] & B^{\mathbf{\Delta}} \end{tikzcd}$$

Since the right-hand composite is an absolute right lifting diagram, so is the left-hand composite, which says that $f\colon A \to B$ preserves the totalization of any split coaugmented cosimplicial object in A. $\square$

Exercises.

2.3.i. Exercise. Prove Lemma 2.3.5.

2.3.ii. Exercise. Re-prove the forwards implication of Lemma 2.3.6 by following your nose through a pasting diagram calculation and prove the converse similarly.

2.4. Preservation of limits and colimits

Famously, right adjoint functors preserve limits and left adjoints preserve colimits. Our aim in this section is to prove this in the ∞-categorical context and exhibit the first examples of initial and final functors, in the sense introduced in Definition 2.4.6 below.

The commutativity of right adjoints and limits is very easily established in the case where the ∞-categories in question admit *all* limits of a given shape: under these hypotheses, the limit functor is right adjoint to the constant diagram functor, which commutes with all functors between the base ∞-categories. Since the left adjoints commute, the uniqueness of adjoints (Proposition 2.1.10) implies that the right adjoints do as well. This outline gives a hint for Exercise 2.4.i.

A slightly more delicate argument is needed in the general case, involving, say, the preservation of a single limit diagram without a priori assuming that any other limits exist. This follows easily from a general lemma about composition and cancelation of absolute lifting diagrams:

2.4.1. Lemma (composition and cancelation of absolute lifting diagrams). *Suppose (r, ρ) defines an absolute right lifting of h through f:*

$$\begin{tikzcd}
& C \arrow[d, "g"] \\
& B \arrow[d, "f"] \\
D \arrow[uur, bend left, "s"] \arrow[uur, phantom, "\Downarrow\sigma"] \arrow[ur, dashed, "r"] \arrow[r, "h"'] \arrow[ur, phantom, "\Downarrow\rho"', near end] & A
\end{tikzcd}$$

Then (s, σ) defines an absolute right lifting of r through g if and only if $(s, \rho \cdot f\sigma)$ defines an absolute right lifting of h through fg.

Proof. Exercise 2.4.ii. $\square$

2.4.2. Theorem (RAPL/LAPC). *Right adjoints preserve limits and left adjoints preserve colimits.*

The usual argument that right adjoints preserve limits proceeds like this: a cone over a J-shaped diagram in the image of u transposes across the adjunction $f^J \dashv u^J$ to a cone over the original diagram, which factors through the designated limit cone. This factorization transposes across the adjunction $f \dashv u$ to define the sought-for unique factorization through the image of the limit cone. The use of absolute lifting diagrams to express the universal properties of limits and colimits (Definition 2.3.7) and adjoint transposition (Lemma 2.3.6) allows us to economize on the usual proof by suppressing consideration of a generic test cone that must be shown to uniquely factor through the limit cone.

Proof. We prove that right adjoints preserve limits. By taking "co" duals the same argument demonstrates that left adjoints preserve colimits.

Suppose $u\colon A \to B$ admits a left adjoint $f\colon B \to A$ with unit $\eta\colon \mathrm{id}_B \Rightarrow uf$ and counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$. Our aim is to show that any absolute right lifting diagram as displayed below-left is carried to an absolute right lifting diagram as displayed below-right:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \end{tikzcd} \qquad\qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd} \tag{2.4.3}$$

The cotensor $(-)^J\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ carries the adjunction $f \dashv u$ to an adjunction $f^J \dashv u^J$ with unit η^J and counit ϵ^J. In particular, by Lemma 2.3.6, (u^J, ϵ^J) defines an absolute right lifting of the identity through f^J, which is then preserved by restriction along the functor d. Thus, by Lemma 2.4.1, the diagram on the right of (2.4.3) is an absolute right lifting diagram if and only if the pasted composite displayed below-left

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "u"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"] \arrow[dr, equal] & B^J \arrow[d, "f^J"] \arrow[dl, phantom, shift right=2, "\Downarrow\epsilon^J"] \\ & & A^J \end{tikzcd} = \begin{tikzcd} & & B \arrow[d, "f"] \\ & A \arrow[ur, "u"] \arrow[r, equal] \arrow[d, "\Delta"] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, equal] & A^J \end{tikzcd} = \begin{tikzcd} & B \arrow[d, "f"] \\ & A \arrow[d, "\Delta"] \\ D \arrow[uur, bend left, "u\lim"] \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] \arrow[uur, phantom, shift right=4, "\Downarrow\epsilon\lim"] & A^J \end{tikzcd}$$

defines an absolute right lifting diagram. Pasting the 2-cell on the right of (2.4.3) with the counit in this way amounts to transposing the cone under $u\lim$ across the adjunction $f^J \dashv u^J$.

We'll now observe that this transposed cone factors through the limit cone $(\lim, \rho)$ in a canonical way. From the 2-functoriality of the simplicial cotensor in its exponent variable, $f^J\Delta = \Delta f$ and $\epsilon^J\Delta = \Delta\epsilon$. Hence, the pasting diagram displayed above-left equals the one displayed above-center and hence also, by naturality of whiskering, the diagram above-right.[8] This latter diagram is a pasted composite of two absolute right lifting diagrams, and is hence an absolute right lifting diagram in its own right; this universal property says that any cone over d whose summit factors through f factors uniquely through the limit cone $(\lim, \rho)$ through a map that then transposes along the adjunction $f \dashv u$. Hence all of the diagrams in the statement are absolute right lifting diagrams, including in particular the one on the right-hand side of (2.4.3). □

By combining Theorem 2.4.2 with Proposition 2.1.11, we have immediately that:

2.4.4. Corollary. *Equivalences preserve limits and colimits.* □

We can also prove a more refined result:

2.4.5. Proposition. *If $A \simeq B$, then any family of diagrams in A admitting a limit or colimit in B also admits a limit or colimit in A that is preserved by the equivalence.*

Proof. By Proposition 2.1.11 the equivalence $B \xrightarrow{\sim} A$ is both left and right adjoint to its equivalence inverse, preserving both limits and colimits of the composite family of diagrams $D \to A^J \xrightarrow{\sim} B^J$.

[8] By naturality of whiskering, $\epsilon^J d \cdot f^J u^J \rho = \rho \cdot \epsilon^J \Delta \lim$, and since $\epsilon^J\Delta = \Delta\epsilon$, this composite equals $\rho \cdot \Delta\epsilon\lim$.

Via the invertible 2-cells of the equivalence $A^J \simeq B^J$ constructed by applying $(-)^J : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ to the equivalence $A \simeq B$, the preserved diagram $D \to A^J \rightsquigarrow B^J \rightsquigarrow A^J$ is isomorphic to the original family of diagrams $D \to A^J$. Thus, we conclude that a family of diagrams in A has a limit or colimit if and only if its image in an equivalent ∞-category B does, and such limits and colimits are preserved by the equivalence. $\square$

The following definition makes sense between small quasi-categories or equally between arbitrary ∞-categories in a cartesian closed ∞-cosmos.

2.4.6. Definition (initial and final functor). A functor $k : I \to J$ is **final** if J-indexed colimits exist if and only if, and in such cases coincide with, the restricted I-indexed colimits. That is, $k : I \to J$ is final if and only if for any ∞-category A, the square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\ A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

preserves and reflects all absolute left lifting diagrams.

Dually a functor $k : I \to J$ is **initial** if this square preserves and reflects all absolute right lifting diagrams: or informally, if a generalized element defines a limit of a J-indexed diagram if and only if it defines a limit of the restricted I-indexed diagrams.

Historically, final functors were called "cofinal" with no obvious name for the dual notion. Our preferred terminology hinges on the following mnemonic: the inclusion of an initial element defines an initial functor, while the inclusion of a terminal (aka final) element defines a final functor. These results are special cases of a more general result we now establish, using exactly the same tactics as taken to prove Theorem 2.4.2.

2.4.7. Proposition. *Left adjoints define initial functors and right adjoints define final functors.*

Proof. If $k \dashv r$ with unit $\eta : \mathrm{id}_I \Rightarrow rk$ and counit $\epsilon : kr \Rightarrow \mathrm{id}_J$, then cotensoring into A yields an adjunction

$$\begin{tikzcd} A^J \arrow[r, bend right, "A^k"'] \arrow[r, phantom, "\perp"] & A^I \arrow[l, bend right, "A^r"'] \end{tikzcd}$$

with unit $A^\eta : \mathrm{id}_{A^I} \Rightarrow A^k A^r$ and counit $A^\epsilon : A^r A^k \Rightarrow \mathrm{id}_{A^J}$.

To prove that k is initial we must show that for any $(d, \lim, \rho)$ as displayed below-left,

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, equal] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

the left-hand diagram is an absolute right lifting diagram if and only if the right-hand diagram is an absolute right lifting diagram.

By Lemmas 2.3.6 and 2.4.1, the right-hand diagram is an absolute right lifting diagram if and only if the pasted composite displayed below-left

$$
\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J \arrow[r, "A^k"] \arrow[dr, equal] \arrow[dr, phantom, "\Downarrow A^\epsilon"] & A^I \arrow[d, "A^r"] \\
 & & A^J
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A \arrow[d, "\Delta"] \\
D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & A^J
\end{tikzcd}
$$

is also an absolute right lifting diagram. On noting that $A^r\Delta = \Delta$ and $A^\epsilon\Delta = \mathrm{id}_\Delta$, the left-hand side reduces to the right-hand side, which proves the claim. $\square$

Exercises.

2.4.i. Exercise. Show that any left adjoint $f\colon B \to A$ between ∞-categories admitting all J-shaped colimits preserves them in the sense that the square of functors

$$
\begin{tikzcd}
B^J \arrow[r, "f^J"] \arrow[d, "\mathrm{colim}"'] \arrow[dr, phantom, "\cong"] & A^J \arrow[d, "\mathrm{colim}"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}
$$

commutes up to isomorphism.

2.4.ii. Exercise. Prove Lemma 2.4.1.

2.4.iii. Exercise. Give a proof of Theorem 2.4.2 that does not appeal to Lemma 2.4.1 by directly verifying that the diagram on the right of (2.4.3) is an absolute right lifting diagram.

2.4.iv. Exercise. Use Lemma 2.4.1 to give a new proof of Proposition 2.1.9.

CHAPTER 3

Weak 2-limits in the homotopy 2-category

In Chapter 2, we introduced adjunctions between ∞-categories and limits and colimits of diagrams valued within an ∞-category through definitions that are particularly expedient for establishing the expected interrelationships. But neither 2-categorical definition clearly articulates the universal properties of these notions. Definition 2.3.7 does not obviously express the expected universal property of the limit cone: namely, that the limit cone over a diagram d defines the terminal element of the ∞-category of cones over d, yet-to-be-defined. Nor have we understood how an adjunction $f \dashv u$ induces an equivalence on as-yet-to-be-defined hom-spaces $\mathsf{Hom}_A(fb, a) \simeq \mathsf{Hom}_B(b, ua)$ for a pair of generalized elements.[1] In this section, we make use of the completeness axiom in the definition of an ∞-cosmos to exhibit a general construction that will specialize to give a definition of this ∞-category of cones and also specialize to define these hom-spaces. This construction will also permit us to represent a functor between ∞-categories *as* an ∞-category, in dual "left" or "right" fashions. Using this, we can redefine an adjunction to consist of a pair of functors $f \colon B \to A$ and $u \colon A \to B$ so that the left representation of f is equivalent to the right representation of u over $A \times B$.

Our vehicle for all of these new definitions is the *comma ∞-category* associated to a cospan

$$C \xrightarrow{g} A \xleftarrow{f} B \qquad \rightsquigarrow \qquad \begin{array}{c} \mathsf{Hom}_A(f,g) \\ \big\downarrow\!\!\!\downarrow{\scriptstyle (p_1,p_0)} \\ C \times B \end{array}$$

Our aim in this chapter is to develop the general theory of comma constructions from the point of view of the homotopy 2-category of an ∞-cosmos. Our first payoff for this work will occur in Chapter 4 where we study the universal properties of adjunctions, limits, and colimits in the sense of the ideas just outlined. The comma construction will also provide the essential vehicle for establishing the model-independence of the categorical notions we will introduce throughout this text.

There is a standard definition of a "comma object" that can be stated in any strict 2-category, defined as a particular weighted limit (see Example 7.1.17). Comma ∞-categories do *not* satisfy this universal property in the homotopy 2-category, however. Instead, they satisfy a somewhat peculiar "weak" variant of the usual 2-categorical universal property that to our knowledge has not been discovered elsewhere in the categorical or homotopical literature, expressed in terms of something we call a *smothering functor*. To introduce these universal properties in a concrete rather than abstract framework, we start in §3.1 by considering smothering functors involving homotopy categories of quasi-categories. The intrepid and impatient reader may skip the entirety of §3.1 if they wish to instead first encounter these notions in their full generality.

[1] A 2-categorical version of this result — exhibiting a bijection between *sets* of 2-cells — appears as Lemma 2.3.6, but in an ∞-category we'd hope for a similar equivalence of hom-spaces.

3.1. Smothering functors

Let Q be a quasi-category. Recall from Lemma 1.1.12 that its homotopy category $\mathsf{h}Q$ has

- the elements $\mathbb{1} \to Q$ of Q as its objects;
- the set of homotopy classes of 1-simplices of Q as its arrows, where parallel 1-simplices are homotopic just when they bound a 2-simplex with the remaining outer edge degenerate; and
- a composition relation if and only if any chosen 1-simplices representing the three arrows bound a 2-simplex.

For a 1-category J, it is well-known in classical homotopy theory that the homotopy category of diagrams $\mathsf{h}(Q^J)$ is not equivalent to the category $(\mathsf{h}Q)^J$ of diagrams in the homotopy category — except in very special cases, such as when J is a set (see Warning 2.3.3). The objects of $\mathsf{h}(Q^J)$ are *homotopy coherent diagrams* of shape J in Q, while the objects of $(\mathsf{h}Q)^J$ are mere *homotopy commutative diagrams*. There is, however, a canonical comparison functor

$$\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$$

defined by applying $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ to the evaluation functor $Q^J \times J \to Q$ and then transposing; a homotopy coherent diagram is in particular homotopy commutative.

Our first aim in this section is to better understand the relationship between the arrows in the homotopy category $\mathsf{h}Q$ and what we'll refer to as the **arrows of** Q, namely, the 1-simplices in the quasi-category. To study this we'll be interested in the quasi-category in which the arrows of Q live as elements, namely $Q^{\mathbb{2}}$, where $\mathbb{2} = \Delta[1]$ is the nerve of the "walking" arrow. Our notation deliberately imitates the notation commonly used for the **category of arrows**: if C is a 1-category, then $\mathsf{C}^{\mathbb{2}}$ is the category whose objects are arrows in C and whose morphisms are commutative squares, regarded as a morphism from the arrow displayed vertically on the left-hand side to the arrow displayed vertically on the right-hand side. This notational conflation suggests our first motivating question: how does the homotopy category of $Q^{\mathbb{2}}$ relate to the category of arrows in the homotopy category of Q?

3.1.1. Lemma. *The canonical functor* $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ *is*

- *surjective on objects,*
- *full, and*
- ***conservative**, i.e., reflects invertibility of morphisms,*

but not injective on objects nor faithful.

Proof. Surjectivity on objects asserts that every arrow in the homotopy category $\mathsf{h}Q$ is represented by a 1-simplex in Q. This is the conclusion of Exercise 1.1.ii(iii) which outlines the proof of Lemma 1.1.12.

To prove fullness, consider a commutative square in $\mathsf{h}Q$ and choose arbitrary 1-simplices representing each morphism and their common composite:

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "\ell"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "k"'] & \bullet \end{tikzcd}$$

By Lemma 1.1.12, every composition relation in $\mathsf{h}Q$ is witnessed by a 2-simplex in Q; choosing a pair of such 2-simplices defines a diagram $\mathbb{2} \to Q^{\mathbb{2}}$, which represents a morphism from f to g in $\mathsf{h}(Q^{\mathbb{2}})$, proving fullness.

Surjectivity on objects and fullness of the functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ are special properties having to do with the diagram shape $\mathbb{2}$. Conservativity is much more general as a consequence of the second statement of Corollary 1.1.21. $\square$

The properties of the canonical functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ will reappear frequently so are worth giving a name:

3.1.2. Definition. A functor $f \colon A \to B$ between 1-categories is **smothering** if it is surjective on objects, full, and conservative. That is, a functor is smothering if and only if it has the right lifting property with respect to the set of functors:

$$\left\{ \begin{array}{c} \emptyset \\ \downarrow \\ \mathbb{1} \end{array} , \begin{array}{c} \mathbb{1}+\mathbb{1} \\ \downarrow \\ \mathbb{2} \end{array} , \begin{array}{c} \mathbb{2} \\ \downarrow \\ \mathbb{I} \end{array} \right\}$$

Some elementary properties of smothering functors are established in Exercise 3.1.i. The most important of these is:

3.1.3. Lemma. *Each fibre of a smothering functor is a non-empty connected groupoid.*

Proof. Suppose $f \colon A \to B$ is smothering and consider the fiber

$$\begin{tikzcd} A_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd}$$

over an object b of B. By surjectivity on objects, the fiber is non-empty. Its morphisms are defined to be arrows between objects in the fiber of b that map to the identity on b. By fullness, any two objects in the fiber are connected by a morphism, indeed, by morphisms pointing in both directions. By conservativity, all the morphisms in the fiber are necessarily invertible. $\square$

The argument used to prove Lemma 3.1.1 generalizes to:

3.1.4. Lemma. *If J is a 1-category that is free on a reflexive directed graph and Q is a quasi-category, then the canonical functor $\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$ is smothering.*

Proof. Exercise 3.1.ii. $\square$

Cotensors are one of the simplicial limit constructions enumerated in axiom 1.2.1(i). Other limit constructions listed there also give rise to smothering functors.

3.1.5. Lemma. *For any pullback diagram of quasi-categories in which p is an isofibration*

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

the canonical functor $\mathsf{h}(A \times_B E) \to \mathsf{h}A \times_{\mathsf{h}B} \mathsf{h}E$ is smothering.

Proof. As $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ does not preserve pullbacks, the canonical comparison functor of the statement is not an isomorphism. It is however bijective on objects since the composite functor

$$\mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at \xrightarrow{\mathsf{obj}} \mathcal{S}et$$

is given by evaluation on the set of vertices of each quasi-category, and this functor *does* preserve pullbacks.

For fullness, note that a morphism in $\mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is represented by a pair of 1-simplices $\alpha\colon a \to a'$ and $\epsilon\colon e \to e'$ in A and E whose images in B are homotopic, a condition that implies in particular that $f(a) = p(e)$ and $f(a') = p(e')$. By Lemma 1.1.9, we can arrange this homotopy however we like, and thus we choose a 2-simplex witness β so as to define a lifting problem

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] & \ni & e \arrow[r, equals] & e \arrow[r, "\epsilon"] & e' \\ \Delta[2] \arrow[r, "\beta"'] \arrow[ur, dashed] & B & \ni & f(a) \arrow[r, "f(\alpha)"'] \arrow[r, equals] & p(e) \arrow[r, "p(\epsilon)"] & f(a') = p(e') \end{tikzcd}$$

Since p is an isofibration, a solution exists, defining an arrow $\tilde{\epsilon}\colon e \to e'$ in E in the same homotopy class as ϵ so that $p(\tilde{\epsilon}) = f(\alpha)$. The pair $(\alpha, \tilde{\epsilon})$ now defines the lifted arrow in $\mathsf{h}(E \times_B A)$.

Finally, consider an arrow $\mathbb{2} \to A \underset{B}{\times} E$ whose image in $\mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is an isomorphism, which is the case just when the projections to E and A define isomorphisms. By Corollary 1.1.16, we may choose a homotopy coherent isomorphism $\mathbb{I} \to A$ extending the given isomorphism $\mathbb{2} \to A$. This data presents us with a lifting problem

$$\begin{tikzcd} \mathbb{2} \arrow[r] \arrow[d, hook] & A \underset{B}{\times} E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, two heads, "p"] \\ \mathbb{I} \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & A \arrow[r, "f"'] & B \end{tikzcd}$$

which Exercise 1.1.v tells us we can solve. This proves that $\mathsf{h}(A \underset{B}{\times} E) \to \mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is conservative and hence also smothering. ☐

A similar argument proves:

3.1.6. Lemma. *For any tower of isofibrations between quasi-categories*

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathsf{h}(\lim_n E_n) \to \lim_n \mathsf{h}E_n$ *is smothering.*

Proof. Exercise 3.1.iii. ☐

3.1.7. LEMMA. *For any cospan between quasi-categories* $C \xrightarrow{g} A \xleftarrow{f} B$ *consider the quasi-category defined by the pullback*

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[r] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^2 \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd}$$

The canonical functor $\mathsf{hHom}_A(f,g) \to \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f,\mathsf{h}g)$ *is smothering.*

PROOF. Here, the codomain is the category defined by an analogous pullback

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{h}A}(\mathsf{h}f,\mathsf{h}g) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (\mathsf{h}A)^2 \arrow[d, "(\mathrm{cod},\mathrm{dom})"] \\ \mathsf{h}C \times \mathsf{h}B \arrow[r, "\mathsf{h}g\times \mathsf{h}f"'] & \mathsf{h}A \times \mathsf{h}A \end{tikzcd}$$

in $\mathcal{C}at$ and the canonical functor factors as

$$\mathsf{hHom}_A(f,g) \to \mathsf{h}(A^2) \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B) \to (\mathsf{h}A)^2 \times_{\mathsf{h}A\times\mathsf{h}A} (\mathsf{h}C \times \mathsf{h}B)$$

By Lemma 3.1.5 the first of these functors is smothering. By Lemma 3.1.1 the second is a pullback of a smothering functor. By Exercise 3.1.i(i) it follows that the composite functor is smothering. ☐

In the sections that follow, we will discover that the smothering functors just constructed express particular "weak" universal properties of arrow, pullback, and comma constructions in the homotopy 2-category of any ∞-cosmos. It is to the first of these that we now turn.

Exercises.

3.1.i. EXERCISE. Prove that:

(i) The class of smothering functors is closed under composition, retract, product, and pullback.
(ii) The class of smothering functors contains all surjective equivalences of categories.
(iii) All smothering functors are isofibrations, that is, maps that have the right lifting property with respect to $\mathbb{1} \hookrightarrow \mathbb{I}$.
(iv) Prove that if f and gf are smothering functors, then g is a smothering functor.[2]

3.1.ii. EXERCISE. Prove Lemma 3.1.4.

3.1.iii. EXERCISE. Prove Lemma 3.1.6.

3.2. ∞-categories of arrows

In this section, we replicate the discussion from the start of the previous section using an arbitrary ∞-category A in place of the quasi-category Q. The analysis of the previous section could have been developed natively in this general setting but at the cost of an extra layer of abstraction and more confusing notation — with a functor space $\mathsf{Fun}(X, A)$ replacing the quasi-category Q.

Recall an **element** of an ∞-category is defined to be a functor $1 \to A$. Tautologically, the elements of A are the vertices of the **underlying quasi-category** $\mathsf{Fun}(1, A)$ of A. In this section, we will define

[2]It suffices, in fact, to merely assume that f is surjective on objects and arrows.

and study an ∞-category $A^{\mathbb{2}}$ whose elements are the 1-simplices in the underlying quasi-category of A. We refer to $A^{\mathbb{2}}$ as the *∞-category of arrows* in A and call its elements simply *arrows* of A.

In fact, we've tacitly introduced this construction already. Recall $\mathbb{2}$ is our preferred notation for the quasi-category $\Delta[1]$, as this coincides with the nerve of the 1-category $\mathbb{2}$ with a single non-identity morphism $0 \to 1$.

3.2.1. Definition (arrow ∞-category). Let A be an ∞-category. The **∞-category of arrows** in A is the simplicial cotensor $A^{\mathbb{2}}$ together with the canonical endpoint-evaluation isofibration

$$A^{\mathbb{2}} := A^{\Delta[1]} \xrightarrow{(p_1,p_0)} A^{\partial\Delta[1]} \cong A \times A$$

induced by the inclusion $\partial\Delta[1] \hookrightarrow \Delta[1]$. For conciseness, we write $p_0 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the domain-evaluation induced by the inclusion $0 \colon \mathbb{1} \hookrightarrow \mathbb{2}$ and write $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ for the codomain-evaluation induced by $1 \colon \mathbb{1} \hookrightarrow \mathbb{2}$.

As an object of the homotopy 2-category $\mathfrak{h}\mathcal{K}$, the ∞-category of arrows comes equipped with a canonical 2-cell that we now construct.

3.2.2. Lemma. *For any ∞-category A, the ∞-category of arrows comes equipped with a canonical 2-cell*

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A \end{tikzcd} \tag{3.2.3}$$

that we refer to as the **generic arrow** *with codomain A.*

Proof. The simplicial cotensor has a strict universal property described in Digression 1.2.5: namely $A^{\mathbb{2}}$ is characterized by the natural isomorphism

$$\mathsf{Fun}(X, A^{\mathbb{2}}) \cong \mathsf{Fun}(X, A)^{\mathbb{2}}. \tag{3.2.4}$$

By the Yoneda lemma, the data of the natural isomorphism (3.2.4) is encoded by its "universal element", which is defined to be the image of the identity at the representing object. Here the identity functor $\mathrm{id} \colon A^{\mathbb{2}} \to A^{\mathbb{2}}$ is mapped to an element of $\mathsf{Fun}(A^{\mathbb{2}}, A)^{\mathbb{2}}$, a 1-simplex in $\mathsf{Fun}(A^{\mathbb{2}}, A)$ which represents a 2-cell in the homotopy 2-category defining (3.2.3).

To see that its source and target must be the domain-evaluation and codomain-evaluation maps, note that the action of the simplicial cotensor $A^{(-)}$ on morphisms of simplicial sets is defined so that the isomorphism (3.2.4) is natural in the cotensor variable as well. Thus, by restricting along the endpoint inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{2}$, we may regard the isomorphism (3.2.4) as lying over $\mathsf{Fun}(X, A \times A) \cong \mathsf{Fun}(X, A) \times \mathsf{Fun}(X, A)$. $\square$

There is a 2-categorical limit notion that is analogous to Definition 3.2.1, which constructs, for any object A, the universal 2-cell with codomain A: namely the cotensor with the 1-category $\mathbb{2}$. Its universal property is analogous to (3.2.4) but with the hom-categories of the 2-category in place of the functor spaces. In $\mathcal{C}at$ this constructs the arrow category associated to a 1-category.

In the homotopy 2-category $\mathfrak{h}\mathcal{K}$, by the Yoneda lemma again, the data (3.2.3) encodes a natural transformation

$$\mathsf{hFun}(X, A^{\mathbb{2}}) \to \mathsf{hFun}(X, A)^{\mathbb{2}}$$

of categories but this is *not* a natural isomorphism, nor even a natural equivalence of categories but does express the arrow ∞-category as a "weak" arrow object with a universal property of the following form:

3.2.5. Proposition (the weak universal property of the arrow ∞-category). *The generic arrow* (3.2.3) *with codomain A has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction****: Given a 2-cell over A as below-left*

$$\begin{tikzcd} X \arrow[d, bend right=50, "t"'] \arrow[d, bend left=50, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\ A \end{tikzcd} = \begin{tikzcd} X \arrow[d, dashed, "a"] \arrow[dd, bend right=70, "t"'] \arrow[dd, bend left=70, "s"] \\ A^2 \arrow[d, bend right, "p_1"'] \arrow[d, bend left, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\ A \end{tikzcd}$$

there exists a 1-cell $a\colon X \to A^2$ so that $s = p_0a$, $t = p_1a$, and $\alpha = \kappa a$.

(ii) ***2-cell induction****: Given a pair of functors $a, a'\colon X \rightrightarrows A^2$ and a pair of 2-cells τ_0 and τ_1 so that*

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\ A^2 \arrow[dr, "p_1"'] & {} & A^2 \arrow[dl, bend right, "p_1"'] \arrow[dl, bend left, "p_0"] \arrow[dl, phantom, "\overset{\kappa}{\Leftarrow}"] \\ & A & \end{tikzcd} = \begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ A^2 \arrow[dr, bend left, "p_0"] \arrow[dr, bend right, "p_1"'] \arrow[dr, phantom, "\overset{\kappa}{\Leftarrow}"] & {} & A^2 \arrow[dl, "p_0"] \\ & A & \end{tikzcd}$$

there exists a 2-cell $\tau\colon a \Rightarrow a'$ so that

$$\begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\ A^2 \arrow[dr, "p_1"'] & {} & A^2 \arrow[dl, "p_1"] \\ & A & \end{tikzcd} = \begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^2 \arrow[d, two heads, "p_1"] \\ A \end{tikzcd} \text{ and } \begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^2 \arrow[d, two heads, "p_0"] \\ A \end{tikzcd} = \begin{tikzcd} & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[d, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\ A^2 \arrow[dr, "p_0"'] & {} & A^2 \arrow[dl, "p_0"] \\ & A & \end{tikzcd}$$

(iii) ***2-cell conservativity****: Any 2-cell*

$$\begin{tikzcd} X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\ A^2 \end{tikzcd}$$

with the property that both $p_1\tau$ and $p_0\tau$ are isomorphisms is an isomorphism.

Proof. Let $Q = \mathsf{Fun}(X, A)$ and apply Lemma 3.1.1 to observe that the natural map of hom-categories

$$\begin{tikzcd} \mathsf{hFun}(X, A^2) \arrow[rr] \arrow[dr, "((p_1)_*,(p_0)_*)"'] & & \mathsf{hFun}(X, A)^2 \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_0)"] \\ & \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A) & \end{tikzcd}$$

over $\mathsf{hFun}(X, A \times A) \cong \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A)$ is a smothering functor. Surjectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. $\square$

Note that the functors $X \to A^2$ that represent a fixed 2-cell with domain X and codomain A are not unique. However, they are unique up to "fibered" isomorphisms that whisker with $(p_1, p_0)\colon A^2 \twoheadrightarrow A \times A$ to an identity 2-cell:

3.2.6. Proposition. *Whiskering with* (3.2.3) *induces a bijection between 2-cells with domain X and codomain A as displayed below-left*

$$\left\{ \begin{tikzcd} X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[d, "a"] & \\ A & A^2 \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & A \end{tikzcd} \right\}_{/\cong}$$

and fibered isomorphism classes of functors $X \to A^2$ as displayed above-right, where the fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\gamma}{\cong}"] & \\ A & A^2 \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & A \end{tikzcd}$$

so that $p_0\gamma = \mathrm{id}_s$ and $p_1\gamma = \mathrm{id}_t$.

Proof. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.2.5 are connected groupoids. The objects of these fibers are functors $X \to A^2$ and the morphisms are invertible 2-cells that whisker with $(p_1, p_0)\colon A^2 \twoheadrightarrow A \times A$ to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

Our final task is to observe that the universal property of Proposition 3.2.5 is also enjoyed by any object $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ that is equivalent to the arrow ∞-category $(p_1, p_0)\colon A^2 \twoheadrightarrow A \times A$ in the slice ∞-cosmos $\mathcal{K}_{/A\times A}$. We have special terminology to allow us to concisely express the type of equivalence we have in mind.

3.2.7. Definition (fibered equivalence). A **fibered equivalence** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd} \tag{3.2.8}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$. We write $E \simeq_B F$ to indicate that that specified isofibrations with these domains are equivalent **over** B.

By Proposition 1.2.19(vii), a fibered equivalence is just a map between a pair of isofibrations over a common base that defines an equivalence in the underlying ∞-cosmos: the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences. Note, however, that it does not create them: it is possible for two ∞-categories E and F to be equivalent without there existing any equivalence compatible with a pair of specified isofibration $E \twoheadrightarrow B$ and $F \twoheadrightarrow B$.

3.2.9. Remark. At this point, there is some ambiguity about the 2-categorical data that presents a fibered equivalence related to the question posed in Exercise 1.4.iv. But since Proposition 1.2.19(vii) tells us that a mere equivalence in $\mathfrak{h}\mathcal{K}$ involving a functor of the form (3.2.8) is sufficient to guarantee that this as-yet-unspecified 2-categorical data exists, we defer a careful analysis of this issue to §3.6.

3.2.10. Proposition (uniqueness of arrow ∞-categories). *For any isofibration $(e_1, e_0): E \twoheadrightarrow A \times A$ equipped with a fibered equivalence $e: E \xrightarrow{\sim} A^2$, the corresponding 2-cell*

$$\begin{tikzcd} E \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$$

satisfies the weak universal property of Proposition 3.2.5. Conversely, if $(d_1, d_0): D \twoheadrightarrow A \times A$ and $(e_1, e_0): E \twoheadrightarrow A \times A$ are equipped with 2-cells

$$\begin{tikzcd} D \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\delta"] & A \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} E \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$$

satisfying the weak universal property of Proposition 3.2.5, then D and E are fibered equivalent over $A \times A$.

Proof. We prove the first statement. By the definition equation of 1-cell induction $\epsilon = \kappa e$, where κ is the canonical 2-cell of (3.2.3). Hence, pasting with ϵ induces a functor

$$\begin{tikzcd} \mathsf{hFun}(X, E) \arrow[r, "e_*"] \arrow[dr, "((p_1)_*, (p_0)_*)"'] & \mathsf{hFun}(X, A^2) \arrow[r] & \mathsf{hFun}(X, A)^2 \arrow[dl, "(\mathrm{ev}_1, \mathrm{ev}_0)"] \\ & \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A) & \end{tikzcd}$$

and our task is to prove that this composite functor is smothering. We see that the first functor, defined by post-composing with the equivalence $e: E \to A^2$, is an equivalence of categories, and the second functor is smothering. Thus, the composite is clearly full and conservative. To see that it is also surjective on objects, note first that by 1-cell induction any 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

is represented by a functor $a: X \to A^2$ over $A \times A$. Composing with any fibered inverse equivalence e' to e yields a functor

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "(t,s)"'] & A^2 \arrow[r, "e'", "\sim"'] \arrow[d, two heads, "(p_1,p_0)"'] & E \arrow[dl, two heads, "(e_1,e_0)"] \\ & A \times A & \end{tikzcd}$$

whose image after post-composing with e is isomorphic to a over $A \times A$. Because this isomorphism is fibered (see Proposition 3.2.6), the image of ae' under the functor $\mathsf{hFun}(X, E) \to \mathsf{hFun}(X, A)^2$ returns the 2-cell α. This proves that this mapping is surjective on objects and hence defines a smothering functor as claimed.

The converse is left to Exercise 3.2.ii and proven in a more general context in Proposition 3.4.11. □

3.2.11. Convention. On account of Proposition 3.2.10, we extend the appellation "∞-category of arrows" from the strict model constructed in Definition 3.2.1 to any ∞-category that is fibered equivalent to it.

Via Lemma 3.1.4, the discussion of this section extends to establish corresponding weak universal properties for the cotensors A^J of an ∞-category A with a free category J. We leave the exploration of this to the reader.

Exercises.

3.2.i. Exercise.

(i) Prove that a parallel pair of 1-simplices in a quasi-category Q are homotopic if and only if they are isomorphic as elements of $Q^{\mathbb{2}}$ via an isomorphism that projects to an identity along $(p_1, p_0)\colon Q^{\mathbb{2}} \twoheadrightarrow Q \times Q$.

(ii) Conclude that a parallel pair of 1-arrows in the functor space $\mathsf{Fun}(X, A)$ between two ∞-categories X and A in any ∞-cosmos represent the same natural transformation if and only if they are isomorphic as elements of $\mathsf{Fun}(X, A)^{\mathbb{2}} \cong \mathsf{Fun}(X, A^{\mathbb{2}})$ via an isomorphism whose domain and codomain components are an identity.

(iii) Conclude that a parallel pair of 1-arrows in the functor space $\mathsf{Fun}(X, A)$, which may be encoded as functors $X \rightrightarrows A^{\mathbb{2}}$, represent the same natural transformation if and only if they are connected by a fibered isomorphism:

$$\begin{tikzcd} X \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\cong"] \arrow[dr] & & A^{\mathbb{2}} \arrow[dl, "{(p_1,p_0)}"] \\ & A \times A & \end{tikzcd}$$

3.2.ii. Exercise. Prove the second statement of Proposition 3.2.10.

3.3. Pullbacks and limits of towers

Pullbacks in an ∞-cosmos also have a weak 2-dimensional universal property in the homotopy 2-category. For the most part, we won't make heavy use of this, preferring to exploit the strict universal property of the simplicially enriched limit instead. However, the weak 2-dimensional universal property can be used to prove that equivalences pull back along isofibrations to equivalence and generalize our previous results about the equivalence-invariance of pullbacks in an ∞-cosmos.

3.3.1. Proposition (the weak universal property of the pullback). *The pullback of an isofibration along a functor in an ∞-cosmos*

$$\begin{tikzcd} A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

has a weak universal property in the homotopy 2-category given by three operations:

*(i) **1-cell induction**: Commutative squares $pe = fa$ over the cospan underlying a pullback diagram factor uniquely through the pullback square*

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists ! x"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "a"'] & & \\ & A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

*(ii) **2-cell induction**: Given a pair of functors $x, x' \colon X \rightrightarrows A \times_B E$ and a pair of 2-cells $\alpha \colon qx \Rightarrow qx'$ and $\epsilon \colon gx \Rightarrow gx'$ as below-left so that $p\epsilon = f\alpha$ there exists a 2-cell $\tau \colon x \Rightarrow x'$ as below-right so that $q\tau = \alpha$ and $g\tau = \epsilon$.*

$$\begin{tikzcd} X \arrow[drr, bend left, "gx"] \arrow[drr, bend right=10, "gx'"'] \arrow[ddr, bend right=10, "qx"] \arrow[ddr, bend right, "qx'"'] \arrow[drr, phantom, "\epsilon \Downarrow"] \arrow[ddr, phantom, "\alpha \Downarrow"] & & \\ & & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] \arrow[ur, phantom, "="] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, bend left, "x"] \arrow[dr, bend right, "x'"'] \arrow[dr, phantom, "\exists \tau \Downarrow"] & & \\ & A \times_B E \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

*(iii) **2-cell conservativity**: Any 2-cell*

$$\begin{tikzcd} X \arrow[r, bend left, "x"] \arrow[r, bend right, "x'"'] \arrow[r, phantom, "\tau \Downarrow"] & A \times_B E \end{tikzcd}$$

with the property that both $q\tau$ and $g\tau$ are isomorphisms is an isomorphism.

PROOF. Apply Lemma 3.1.5 to the pullback diagram of quasi-categories

$$\begin{tikzcd} \mathsf{Fun}(X, A \times_B E) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p_*"] \\ A \arrow[r, "f_*"'] & B \end{tikzcd}$$

to observe that the natural map of hom-categories

$$\mathsf{hFun}(X, A \times_B E) \longrightarrow \mathsf{hFun}(X, A) \times_{\mathsf{hFun}(X,B)} \mathsf{hFun}(X, E)$$

is a bijective-on-objects smothering functor. Bijectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. □

Using the weak 2-categorical universal property of the pullback, we can show that ∞-cosmoi are **right proper**, meaning that the pullback of any equivalence along an isofibration defines an equivalence.

3.3.2. Lemma. *In any ∞-cosmos, the pullback of an equivalence along an isofibration is an equivalence.*

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

Proof. By Proposition 2.1.11, we may choose an inverse adjoint equivalence to f and pick invertible 2-cells $\alpha\colon \mathrm{id}_A \cong f^{-1}f$ and $\beta\colon ff^{-1} \cong \mathrm{id}_B$ satisfying the triangle equalities. It is for this reason that we work with the 2-categorical universal property of the pullback rather than the simplicially enriched universal property. Now since the map p is an isofibration, we may use Proposition 1.4.10 to lift the isomorphism $\beta p\colon ff^{-1}p \cong p$ along p to define an isomorphism $\epsilon\colon e \cong \mathrm{id}_E$ with codomain $\mathrm{id}_E\colon E \to E$. By construction $pe = ff^{-1}p$, so by 1-cell induction the pair $(f^{-1}p, e)$ induces a map $g^{-1}\colon E \to F$ so that $qg^{-1} = f^{-1}p$ and $gg^{-1} = e$. In this way we obtain an isomorphism $\epsilon\colon gg^{-1} \cong \mathrm{id}_E$ with $p\epsilon = \beta p$.

Now by 2-cell induction and conservativity of Proposition 3.3.1, to define an isomorphism $\mathrm{id}_F \cong g^{-1}g$, it suffices to exhibit a pair of isomorphisms $\alpha q\colon q \cong f^{-1}fq = f^{-1}pg = qg^{-1}g$ and $\epsilon^{-1}g\colon g \cong gg^{-1}g$ so that $f\alpha q = p\epsilon g$. This latter equation holds because $p\epsilon^{-1}g = \beta^{-1}pg = \beta^{-1}fq = f\alpha q$ by the triangle equality $\beta f \cdot f\alpha = \mathrm{id}_f$ for the adjoint equivalence $f \dashv f^{-1}$. Thus, we may lift the data of an inverse equivalence to f to define an inverse equivalence to its pullback g. $\square$

As a consequence of right properness, we can show that pullback is an equivalence invariant construction in any ∞-cosmos.

3.3.3. Proposition. *Given a diagram of isofibrations and equivalences in any ∞-cosmos*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"', "\wr"] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "\wr"', "q"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"] \end{tikzcd}$$

the induced map $C \times_A B \to \bar{C} \times_{\bar{A}} \bar{B}$ between the pullbacks of the horizontal rows is again an equivalence.

Proof. By factoring via Lemma 1.2.13, we can replace the map $\bar{g}$ by an isofibration. By the 2-of-3 property and the right properness of Lemma 3.3.2, the pullback of this isofibration along the equivalence p is equivalent to the map g:

$$\begin{tikzcd} C \arrow[r, dashed, "\sim"'] \arrow[rr, bend left, "g"] \arrow[d, "r"', "\wr"] & P \arrow[r, two heads] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "\wr"', "p"] \\ \bar{C} \arrow[r, "\sim"] \arrow[rr, bend right, "\bar{g}"'] & \bar{P} \arrow[r, two heads] & \bar{A} \end{tikzcd}$$

By right properness again, the pullback of $P \twoheadrightarrow A$ along f is equivalent to the pullback of $C \to A$ along f and similarly for the lower-horizontal maps. So without loss of generality, we may assume that the maps g and $\bar{g}$ of the statement are fibrations and the left-hand square is a pullback.

Under these new hypothesis, the top, bottom, and front faces of the cube are pullback squares:

$$\begin{tikzcd}
& C \times_A B \arrow[rr, two heads] \arrow[dl, two heads] \arrow[dd, dashed] \arrow[dr, phantom, "\lrcorner"] & & B \arrow[dl, two heads, "f"'] \arrow[dd, "q", "\wr"'] \\
C \arrow[rr, two heads, "g" near start] \arrow[dd, "r"', "\wr"] \arrow[dr, phantom, "\lrcorner"] & & A \arrow[dd, "p", "\wr"' near start] & \\
& \bar{C} \times_{\bar{A}} \bar{B} \arrow[rr, two heads] \arrow[dl, two heads] \arrow[dr, phantom, "\lrcorner"] & & \bar{B} \arrow[dl, two heads, "\bar{f}"] \\
\bar{C} \arrow[rr, two heads, "\bar{g}"'] & & \bar{A} &
\end{tikzcd}$$

so by pullback composition and cancelation, the back face is a pullback square as well. Now the induced map $C \times_A B \to \bar{C} \times_{\bar{A}} \bar{B}$ is the pullback of the equivalence q along an isofibration and hence is an equivalence by Lemma 3.3.2. □

Exercises.

3.3.i. Exercise. State and prove an analogous result to Proposition 3.3.1 that describes the weak 2-categorical universal property of limits of towers of isofibrations.

3.4. The comma construction

The *comma ∞-category* is defined by restricting the domain and codomain of the ∞-category of arrows A^2 along specified functors with codomain A.

3.4.1. Definition (comma ∞-category). Let $C \xrightarrow{g} A \xleftarrow{f} B$ be a diagram of ∞-categories. The **comma ∞-category** is constructed as a pullback of the simplicial cotensor A^2 along $g \times f$

$$\begin{tikzcd}
\mathsf{Hom}_A(f,g) \arrow[r, "\phi"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^2 \arrow[d, two heads, "{(p_1,p_0)}"] \\
C \times B \arrow[r, "g \times f"'] & A \times A
\end{tikzcd} \tag{3.4.2}$$

This construction equips the comma ∞-category with a specified isofibration $(p_1, p_0) \colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ and a canonical 2-cell

$$\begin{tikzcd}
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\Leftarrow\phi"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd} \tag{3.4.3}$$

in the homotopy 2-category called the **comma cone**.

3.4.4. Example (arrow ∞-categories as comma ∞-categories). The arrow ∞-category arises as a special case of the comma construction applied to the identity span. This provides us with alternate

notation for the generic arrow of (3.2.3), which may be regarded as a particular instance of a comma cone.

$$\begin{tikzcd} & \mathsf{Hom}_A \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[dr, equal] & & A \arrow[dl, equal] \\ & A & \end{tikzcd}$$

The following proposition encodes the homotopical properties of the comma construction. The first statement is a special case of Proposition 3.3.3. The proof of the remaining statements is by a standard argument in abstract homotopy theory, which appears as Proposition C.1.12. A hint for this proof is given in Exercise 3.4.i.

3.4.5. Proposition (maps between commas). *A commutative diagram*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] & B \arrow[l, "f"'] \arrow[d, "q"] \\ \bar{C} \arrow[r, "\bar{g}"] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"'] \end{tikzcd}$$

induces a map between the comma ∞*-categories*

$$\begin{tikzcd} \mathsf{Hom}_A(f,g) \arrow[r, dashed, "\mathsf{Hom}_p(q,r)"] \arrow[d, two heads, "(p_1,p_0)"'] & \mathsf{Hom}_{\bar{A}}(\bar{f},\bar{g}) \arrow[d, two heads, "(p_1,p_0)"] \\ C \times B \arrow[r, "r\times q"] & \bar{C} \times \bar{B} \end{tikzcd}$$

Moreover, if p, q, and r are all

(i) equivalences,
(ii) isofibrations, or
(iii) trivial fibrations

then the induced map is again an equivalence, isofibration, or trivial fibration, respectively.

There is a 2-categorical limit notion that is analogous to Definition 3.4.1, which constructs the universal 2-cell inhabiting a square over a specified cospan. In $\mathcal{C}at$ the category so-constructed is referred to as a *comma category*, from when we borrow the name. As with the case of ∞-categories of arrow, comma ∞-categories do *not* satisfy this 2-universal property strictly. Instead:

3.4.6. Proposition (the weak universal property of the comma ∞-category). *The comma cone* (3.4.3) *has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction***: *Given a 2-cell over* $C \xrightarrow{g} A \xleftarrow{f} B$ *as below-left*

$$
\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[d, dashed, "a"] \arrow[ddl, bend right, "c"'] \arrow[ddr, bend left, "b"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

there exists a 1-cell $a\colon X \to \mathsf{Hom}_A(f,g)$ *so that* $b = p_0 a$, $c = p_1 a$, *and* $\alpha = \phi a$.

(ii) ***2-cell induction***: *Given a pair of functors* $a, a' \colon X \rightrightarrows \mathsf{Hom}_A(f,g)$ *and a pair of 2-cells* τ_0 *and* τ_1 *so that*

$$
\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[dd, phantom, "\overset{\tau_1}{\Leftarrow}"] & & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& & A &
\end{tikzcd}
=
\begin{tikzcd}
& & X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[dd, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] & \\
& A & &
\end{tikzcd}
$$

there exists a 2-cell $\tau\colon a \Rightarrow a'$ *so that*

$$
\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[dd, phantom, "\overset{\tau_1}{\Leftarrow}"] & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_1"'] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"] \\
& C &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g) \arrow[d, "p_1"] \\
C
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g) \arrow[d, "p_0"] \\
B
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dl, "a'"'] \arrow[dr, "a"] \arrow[dd, phantom, "\overset{\tau_0}{\Leftarrow}"] & \\
\mathsf{Hom}_A(f,g) \arrow[dr, "p_0"'] & & \mathsf{Hom}_A(f,g) \arrow[dl, "p_0"] \\
& B &
\end{tikzcd}
$$

(iii) ***2-cell conservativity***: *Any 2-cell*

$$
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\tau}{\Leftarrow}"] \\
\mathsf{Hom}_A(f,g)
\end{tikzcd}
$$

with the property that both $p_1\tau$ *and* $p_0\tau$ *are isomorphisms is an isomorphism.*

PROOF. The cosmological functor $\mathsf{Fun}(X,-)\colon \mathcal{K} \to \mathcal{QC}at$ carries the pullback (3.4.2) to a pullback

$$\begin{tikzcd}
\mathsf{Fun}(X,\mathsf{Hom}_A(f,g)) \cong \mathsf{Hom}_{\mathsf{Fun}(X,A)}(\mathsf{Fun}(X,f),\mathsf{Fun}(X,g)) \arrow[r, "\phi"] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X,A)^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\
\mathsf{Fun}(X,C)\times\mathsf{Fun}(X,B) \arrow[r, "\mathsf{Fun}(X,g)\times\mathsf{Fun}(X,f)"'] & \mathsf{Fun}(X,A)\times\mathsf{Fun}(X,A)
\end{tikzcd}$$

of quasi-categories. Now Lemma 3.1.7 demonstrates that the canonical 2-cell (3.4.3) induces a natural map of hom-categories

$$\begin{tikzcd}
\mathsf{hFun}(X,\mathsf{Hom}_A(f,g)) \arrow[rr] \arrow[dr, "((p_1)_*,(p_0)_*)"'] & & \mathsf{Hom}_{\mathsf{hFun}(X,A)}(\mathsf{hFun}(X,f),\mathsf{hFun}(X,g)) \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_0)"] \\
& \mathsf{hFun}(X,C)\times\mathsf{hFun}(X,B) &
\end{tikzcd}$$

over $\mathsf{hFun}(X,C\times B) \cong \mathsf{hFun}(X,C)\times\mathsf{hFun}(X,B)$ that is a smothering functor. The properties of 1-cell induction, 2-cell induction, and 2-cell conservativity follow from surjectivity on objects, fullness, and conservativity of this smothering functor respectively. □

The 1-cells $X \to \mathsf{Hom}_A(f,g)$ that are induced by a fixed 2-cell $\alpha\colon fb \Rightarrow gc$ are unique up to fibered isomorphism over $C\times B$.

3.4.7. PROPOSITION. *Whiskering with the comma cone* (3.4.3) *induces a bijection between 2-cells as displayed below-left*

$$\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}\right\}
\quad\leftrightsquigarrow\quad
\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, "a"] & \\
C & \mathsf{Hom}_A(f,g) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B
\end{tikzcd}\right\}_{/\cong}$$

and fibered isomorphism classes of maps of spans from C to B as displayed above-right, where the fibered isomorphisms are given by invertible 2-cells

$$\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\gamma}{\cong}"] & \\
C & \mathsf{Hom}_A(f,g) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B
\end{tikzcd}$$

so that $p_0\gamma = \mathrm{id}_b$ and $p_1\gamma = \mathrm{id}_c$.

PROOF. Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.4.6 are connected groupoids. The objects of these fibers are functors $X \to \mathsf{Hom}_A(f,g)$ and the morphisms are invertible 2-cells that whisker with

$$(p_1,p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C\times B$$

to an identity 2-cell. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

The construction of the comma ∞-category is also pseudo-functorial in lax maps defined in the homotopy 2-category:

3.4.8. Observation. By 1-cell induction a diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"'] \arrow[dr, phantom, "\swArrow\gamma"] & A \arrow[d, "p"] & B \arrow[l, "f"'] \arrow[d, "q"] \arrow[dl, phantom, "\swArrow\beta"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"] \end{tikzcd}$$

induces a map between comma ∞-categories as displayed below-right:

$$\begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\swArrow\phi"] & \\ C \arrow[d, "r"'] \arrow[dr, "g"] & & B \arrow[d, "q"] \arrow[dl, "f"'] \\ \bar{C} \arrow[r, phantom, "\swArrow\gamma"] \arrow[dr, "\bar{g}"'] & A \arrow[d, "p"] \arrow[r, phantom, "\Leftarrow\beta"] & \bar{B} \arrow[dl, "\bar{f}"] \\ & \bar{A} & \end{tikzcd} = \begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[d, dashed, "\beta\downarrow\gamma"] \arrow[ddl, bend right, two heads, "rp_1"'] \arrow[ddr, bend left, two heads, "qp_0"] & \\ & \mathsf{Hom}_{\bar{A}}(\bar{f},\bar{g}) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ \bar{C} \arrow[dr, "\bar{g}"'] & & \bar{B} \arrow[dl, "\bar{f}"] \\ & \bar{A} & \end{tikzcd}$$

that is well-defined and functorial up to fibered isomorphism.

One of many uses of comma ∞-categories is to define the internal mapping spaces between two elements of an ∞-category A. This is one motivation for our notation "Hom_A."

3.4.9. Definition. For any two elements $x, y \colon 1 \rightrightarrows A$ of an ∞-category A, their **mapping space** is the comma ∞-category $\mathsf{Hom}_A(x,y)$ defined by the pullback diagram

$$\begin{tikzcd} \mathsf{Hom}_A(x,y) \arrow[r, "\phi"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ 1 \arrow[r, "{(y,x)}"'] & A \times A \end{tikzcd}$$

The mapping spaces in any ∞-category are discrete in the sense of Definition 1.2.24.

3.4.10. Proposition (internal mapping spaces are discrete). *For any pair of elements $x, y \colon 1 \rightrightarrows A$ of an ∞-category A, the mapping space $\mathsf{Hom}_A(x,y)$ is discrete.*

Proof. Our task is to prove that for any ∞-category X, the functor space $\mathsf{Fun}(X, \mathsf{Hom}_A(x,y))$ is a Kan complex. This is so just when $\mathsf{hFun}(X, \mathsf{Hom}_A(x,y))$ is a groupoid, i.e., when any 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible. By 2-cell conservativity, a 2-cell with codomain $\mathsf{Hom}_A(x,y)$ is invertible just when its whiskered composite with the isofibration $(p_1, p_0) \colon \mathsf{Hom}_A(x,y) \twoheadrightarrow 1 \times 1$ is an invertible 2-cell, but in fact this whiskered composite is an identity since 1 is terminal. □

As in our convention for ∞-categories of arrows, it will be convenient to weaken the meaning of "comma ∞-category" to extend this appellation to any object of $\mathcal{K}_{/C\times B}$ that is fibered equivalent (see Definition 3.2.7) to the strict model $(p_1, p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ defined by 3.4.1. This is justified because such objects satisfy the weak universal property of Proposition 3.4.6 and conversely any two objects satisfying this weak universal property are equivalent over $C \times B$.

3.4.11. Proposition (uniqueness of comma ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ that is fibered equivalent to $\mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ the 2-cell*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \arrow[dd, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

encoded by the equivalence $E \xrightarrow{\sim} \mathsf{Hom}_A(f,g)$ satisfies the weak universal property of Proposition 3.4.6. Conversely, if $(d_1, d_0)\colon D \twoheadrightarrow C \times B$ and $(e_1, e_0)\colon E \twoheadrightarrow C \times B$ are equipped with 2-cells

$$\begin{tikzcd} & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] \arrow[dd, phantom, "\overset{\delta}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \arrow[dd, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{3.4.12}$$

satisfying the weak universal property of Proposition 3.4.6, then D and E are fibered equivalent over $C \times B$.

Proof. The proof of the first statement proceeds exactly as in the special case of Proposition 3.2.10. We prove the converse, solving Exercise 3.2.ii.

Consider a pair of 2-cells (3.4.12) satisfying the weak universal properties enumerated in Proposition 3.4.6. 1-cell induction supplies maps of spans

$$\begin{tikzcd} & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] \arrow[dd, phantom, "\overset{\delta}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & D \arrow[d, "d"] \arrow[ddl, bend right, "d_1"'] \arrow[ddr, bend left, "d_0"] & \\ & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \arrow[dd, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \arrow[dd, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} = \begin{tikzcd} & E \arrow[d, "e"] \arrow[ddl, bend right, "e_1"'] \arrow[ddr, bend left, "e_0"] & \\ & D \arrow[dl, "d_1"'] \arrow[dr, "d_0"] \arrow[dd, phantom, "\overset{\delta}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

with the property that $\epsilon de = \epsilon$ and $\delta ed = \delta$. By Proposition 3.4.7 it follows that $de \cong \mathrm{id}_E$ over $C \times B$ and $ed \cong \mathrm{id}_D$ over $C \times B$. This defines the data of a fibered equivalence $D \simeq E$.[3] $\square$

3.4.13. Convention. On account of Proposition 3.4.11, we extend the appellation "comma ∞-category" from the strict model constructed in Definition 3.4.1 to any ∞-category that is fibered equivalent to it and refer to its accompanying 2-cell as the "comma cone."

[3]For the reader uncomfortable with Remark 3.2.9, Proposition 3.6.3 and Lemma 3.6.4 provides a small boost to finish the proof.

For example, in §4.3 we define the ∞-category of cones over a fixed diagram as a comma ∞-category. Proposition 3.4.11 gives us the flexibility to use multiple models for this ∞-category, which will be useful in characterizing the universal properties of limits and colimits.

Exercises.

3.4.i. EXERCISE. Prove Proposition 3.4.5 by observing that the map $\mathsf{Hom}_p(q,r)$ factors as a pullback of the Leibniz cotensor of $\partial\Delta[1] \hookrightarrow \Delta[1]$ with p followed by a pullback of $r \times q$.

3.4.ii. EXERCISE. Use Proposition 3.4.7 to justify the pseudofunctoriality of the comma construction in lax morphisms described in Observation 3.4.8.

3.5. Representable comma ∞-categories

Definition 3.4.1 constructs a comma ∞-category for any cospan. Of particular importance, are the special cases of this construction where one of the legs of the cospan is an identity:

3.5.1. DEFINITION (left and right representations). Any functor $f\colon A \to B$ admits a **left representation** and a **right representation** as a comma ∞-category, displayed below-left and below-right respectively:

$$
\begin{tikzcd}[column sep=small]
& \mathsf{Hom}_B(f,B) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
B \arrow[dr, equal] & \overset{\phi}{\Leftarrow} & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
A \arrow[dr, "f"'] & \overset{\phi}{\Leftarrow} & B \arrow[dl, equal] \\
& B &
\end{tikzcd}
$$

To save space, we typically depict the **left comma cone over** p displayed above-left and the **right comma cone over** p displayed above-right as inhabiting triangles rather than squares.

By Proposition 3.4.11, the weak universal property of the comma cone characterizes the comma span up to fibered equivalence over the product of the codomain objects. Thus:

3.5.2. DEFINITION. A comma ∞-category $\mathsf{Hom}_A(f,g) \twoheadrightarrow C \times B$ is

- **left representable** if there exists a functor $\ell\colon B \to C$ so that $\mathsf{Hom}_A(f,g) \simeq \mathsf{Hom}_C(\ell, C)$ over $C \times B$ and
- **right representable** if there exists a functor $r\colon C \to B$ so that $\mathsf{Hom}_A(f,g) \simeq \mathsf{Hom}_B(B,r)$ over $C \times B$.

In this section, we prove the first of many representability theorems: demonstrating that a functor $g\colon C \to A$ admits an absolute right lifting along $f\colon B \to A$ if and only if the comma ∞-category $\mathsf{Hom}_A(f,g)$ is right representable, the representing functor then defining the postulated lifting. We prove this over the course of three theorems, each strengthening the previous statement. The first theorem characterizes 2-cells

$$
\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
$$

that define absolute right lifting diagrams via an induced equivalence $\mathrm{Hom}_B(B, r) \simeq_{C\times B} \mathrm{Hom}_A(f, g)$ between comma ∞-categories. The second theorem proves that a functor r defines an absolute right lifting of g through f just when $\mathrm{Hom}_A(f, g)$ is right-represented by r; the difference is that no 2-cell $\rho\colon fr \Rightarrow g$ need be postulated a priori to exist. The final theorem gives a general right-representability criterion that can be applied to construct a right representation to $\mathrm{Hom}_A(f, g)$ without a priori specifying the representing functor r.

3.5.3. Theorem. *The triangle below-left defines an absolute right lifting diagram if and only if the induced 1-cell below-right*

$$
\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_0"'] \arrow[ddr, bend left, two heads, "p_1"] & \\
& \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\tag{3.5.4}
$$

defines a fibered equivalence $\mathrm{Hom}_B(B, r) \simeq \mathrm{Hom}_A(f, g)$ *over* $C \times B$.

In [**97**], Street and Walters interpret the equivalence $\mathrm{Hom}_B(B, r) \simeq \mathrm{Hom}_A(f, g)$ encoding an absolute right lifting diagram as asserting that "f is left adjoint to r relative to g." This notion of relative adjunction, first studied by Ulmer [**99**], should be compared with the definition of adjunction given in Proposition 4.1.1.

Proof. Suppose that (r, ρ) defines an absolute right lifting of g through f and consider the corresponding unique factorization of the comma cone under $\mathrm{Hom}_A(f, g)$ through ρ as displayed below-center

$$
\begin{tikzcd}
& \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\zeta}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathrm{Hom}_A(f,g) \arrow[d, dashed, "z"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
& \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\tag{3.5.5}
$$

By 1-cell induction, the 2-cell ζ factors through the right comma cone over r as displayed above-right. Substituting the right-hand side of (3.5.4) into the bottom portion of the above-right diagram, we see that $yz\colon \mathrm{Hom}_A(f, g) \to \mathrm{Hom}_A(f, g)$ is a 1-cell that factors the comma cone for $\mathrm{Hom}_A(f, g)$ through itself. Applying the universal property of Proposition 3.4.7, it follows that there is a fibered isomorphism $yz \cong \mathrm{id}_{\mathrm{Hom}_A(f,g)}$ over $C \times B$.

To prove that $zy \cong \mathrm{id}_{\mathrm{Hom}_B(B,r)}$ it suffices to argue similarly that the right comma cone over r restricts along zy to itself. Since ρ is absolute right lifting, it suffices to verify the equality $\phi zy = \phi$ after pasting below with ρ. But now reversing the order of the equalities in (3.5.5) and (3.5.4) we have

$$
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[dddl, bend right, two heads, "p_1"'] \arrow[dddr, bend left, two heads, "p_0"] & \\
 & \mathrm{Hom}_A(f,g) \arrow[d, dashed, "z"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\zeta}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

which is exactly what we wanted to show. Thus, we see that if (r,ρ) is an absolute right lifting of g through f, then the induced map (3.5.4) defines a fibered equivalence $\mathrm{Hom}_B(B,r) \simeq \mathrm{Hom}_A(f,g)$.

Now, conversely, suppose the 1-cell y defined by (3.5.4) is a fibered equivalence and let us argue that (r,ρ) is an absolute right lifting of g through f. By Proposition 3.4.11, via this fibered equivalence the 2-cell displayed on the left-hand side of (3.5.4) inherits the weak universal property of a comma cone from $\mathrm{Hom}_A(f,g)$. So Proposition 3.4.7 supplies a bijection displayed below-left-center

$$
\left\{
\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\right\}
\leftrightsquigarrow
\left\{
\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "a"] & \\
C & & B \\
 & \mathrm{Hom}_B(B,r) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] &
\end{tikzcd}
\right\}_{/\cong}
\leftrightsquigarrow
\left\{
\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[d, phantom, "\overset{\xi}{\Leftarrow}"] & \\
C \arrow[rr, "r"'] & {} & B
\end{tikzcd}
\right\}
$$

between 2-cells over the cospan and fibered isomorphism classes of maps of spans that is implemented, from center to left, by whiskering with the 2-cell $\rho p_1 \cdot f\phi \colon fp_0 \Rightarrow gp_1$ in the center of (3.5.4). Proposition 3.4.7 also applies to the right comma cone ϕ over $r \colon C \to B$ giving us a second bijection, displayed above center-right between the same fibered isomorphism classes of maps of spans and 2-cells over r. This second bijection is implemented, from center to right, by pasting with the right comma cone $\phi \colon p_0 \Rightarrow rp_1$. Combining these yields a bijection between the 2-cells displayed on the right and the 2-cells displayed on the left implemented by pasting with ρ, which is precisely the universal property that characterizes absolute right lifting diagrams. $\square$

As a special case of this result, we can now present several equivalent characterizations of fully faithful functors between ∞-categories.

3.5.6. Corollary. *The following are equivalent, and define what it means for a functor $f \colon A \to B$ between ∞-categories to be* ***fully faithful****:*

(i) The identity defines an absolute right lifting diagram:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] & B \end{tikzcd}$$

(ii) The identity defines an absolute left lifting diagram:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] & B \end{tikzcd}$$

(iii) For all $X \in \mathcal{K}$, the induced functor

$$f_*\colon \mathsf{hFun}(X, A) \to \mathsf{hFun}(X, B)$$

is a fully faithful functor of 1-categories.

(iv) The functor induced by the identity 2-cell id_f is an equivalence

$$\begin{tikzcd} & \mathcal{A}^2 \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "\wr"', "\ulcorner \mathrm{id}_f \urcorner"] & \\ A & & A \\ & \mathsf{Hom}_B(f, f) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}$$

Proof. The statement (iii) is an unpacking of the meaning of both (i) and (ii). Theorem 3.5.3 specializes to prove (i)⇔(iv) or dually (ii)⇔(iv). □

It is not surprising that post-composition with a fully faithful functor of ∞-categories should induce a fully-faithful functor of hom-categories in the homotopy 2-category. What is surprising is that this definition is strong enough. This result, together with the general case of Theorem 3.5.3 should be provide some retroactive justification for our use of absolute lifting diagrams in Chapter 2.

Having proven Theorem 3.5.3 our immediate aim is to strengthen it to show that a fibered equivalence $\mathsf{Hom}_B(B, r) \simeq \mathsf{Hom}_A(f, g)$ over $C \times B$ implies that $r\colon C \to B$ defines an absolute right lifting of g through f without a previously specified 2-cell $\rho\colon fr \Rightarrow g$.

3.5.7. Theorem. *Given a trio of functors $r\colon C \to B$, $f\colon B \to A$, and $g\colon C \to A$ there is a bijection between 2-cells as displayed below-left and fibered isomorphism classes of maps of spans as displayed below-right*

$$\left\{ \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, bend right, "\Downarrow\rho"] & A \end{tikzcd} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{tikzcd} & \mathsf{Hom}_B(B, r) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "y"] & \\ C & & B \\ & \mathsf{Hom}_A(f, g) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd} \right\}_{/\cong}$$

that is constructed by pasting the right comma cone over r and then applying 1-cell induction to factor through the comma cone for $\mathsf{Hom}_A(f,g)$.

$$
\begin{tikzcd}
 & \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r" description] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
 & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

Moreover, a 2-cell $\rho\colon fr \Rightarrow g$ displays r as an absolute right lifting of g through f if and only if the corresponding map of spans $y\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ is an equivalence.

The second clause is the statement of Theorem 3.5.3, so it remains only to prove the first. We show the claimed construction is a bijection by exhibiting its inverse, the construction of which involves a rather mysterious lemma the significance of which will gradually reveal itself. For instance, Lemma 3.5.8 figures prominently in the proof of the external Yoneda lemma in §5.5 and is also the main ingredient in a "cheap" version of the Yoneda lemma appearing as Corollary 3.5.10.

3.5.8. Lemma. *Let $f\colon A \to B$ be any functor and denote the right comma cone over f by*

$$
\begin{tikzcd}
 & \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
$$

Then the codomain-projection functor $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ admits a right adjoint right inverse[4] *induced from the identity 2-cell id_f, defining an adjunction*

$$
\begin{tikzcd}
A \arrow[rr, phantom, "\perp"] \arrow[dr, equal] \arrow[rr, dashed, bend right, "i"'] & & \mathsf{Hom}_B(B,f) \arrow[ll, bend right, "p_1"'] \arrow[dl, "p_1"] \\
 & A &
\end{tikzcd}
$$

over A whose counit is an identity and whose unit $\eta\colon \mathrm{id} \Rightarrow ip_1$ satisfies the conditions $\eta i = \mathrm{id}_i$, $p_1\eta = \mathrm{id}_{p_1}$ and $p_0\eta = \phi$.

[4] A functor admits a **right adjoint right inverse** just when it admits a right adjoint in an adjunction whose counit is the identity. When the original functor is an isofibration, as is the case here, it suffices to merely assume that the counit is invertible; see Lemma B.4.7 and Appendix B.

PROOF. This adjunction will be constructed using the weak universal properties of the right comma cone over f. The identity 2-cell id_f induces a 1-cell over the right comma cone over f:

$$\begin{tikzcd}
& A \arrow[d, dashed, "i"] \arrow[ddl, equals, bend right] \arrow[ddr, bend left, "f"] & \\
& \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& A \arrow[dr, "f"] \arrow[d, phantom, "="] & \\
A \arrow[ur, equals] \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Note that $p_1 i = \mathrm{id}_A$, so we may take the counit to be the identity 2-cell. Since $\phi i = \mathrm{id}_f$, we have a pasting equality:

$$\begin{tikzcd}
& \mathrm{Hom}_B(B,f) \arrow[dl, "ip_1"'] \arrow[dr, equals] & & \\
\mathrm{Hom}_B(B,f) \arrow[dr, "p_1"'] \arrow[rr, phantom, "="] & & \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dddr, bend left, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& A \arrow[d, "i"] \arrow[ddl, equals, bend right] \arrow[ddrr, dotted, "f"] & {} & \\
& \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & & \\
A \arrow[rr, "f"'] & {} & B &
\end{tikzcd}$$

while allows us to induce a 2-cell $\eta\colon \mathrm{id} \Rightarrow ip_1$ with defining equations $p_1\eta = \mathrm{id}_{p_1}$ and $p_0\eta = \phi$. The first of these conditions ensures one triangle identity; for the other, we must verify that $\eta i = \mathrm{id}_i$. By 2-cell conservativity, ηi is an isomorphism since $p_1\eta i = \mathrm{id}_A$ and $p_0\eta i = \phi i = \mathrm{id}_f$ are both invertible. By naturality of whiskering, we have

$$\begin{tikzcd}
i \arrow[r, Rightarrow, "\eta i"] \arrow[d, Rightarrow, "\eta i"'] & i \arrow[d, Rightarrow, "\eta i"] \\
i \arrow[r, Rightarrow, "ip_1\eta i"'] & i
\end{tikzcd}$$

and since $p_1\eta = \mathrm{id}_{p_1}$ the bottom edge is an identity. So $\eta i \cdot \eta i = \eta i$ and since ηi is an isomorphism cancelation implies that $\eta i = \mathrm{id}_i$ as required. ☐

One interpretation of Lemma 3.5.8 is best revealed though a special case:

3.5.9. COROLLARY. *For any element $b\colon 1 \to B$, the identity at b defines a terminal element in $\mathrm{Hom}_B(B,b)$.*

PROOF. By Lemma 3.5.8, the codomain-projection from the right representation of any functor admits a right adjoint right inverse induced from its identity 2-cell. In this case, the codomain-projection is the unique functor $!\colon \mathrm{Hom}_B(B,b) \to 1$, so by Definition 2.2.1, this right adjoint identifies a terminal element of $\mathrm{Hom}_B(B,b)$ corresponding to the identity morphism id_b in the homotopy category $\mathrm{h}B$. ☐

The general version of Lemma 3.5.8 has a similar interpretation: in the sliced ∞-cosmos $\mathcal{K}_{/A}$, the identity functor at A defines the terminal object, and Lemma 3.5.8 asserts that id_f induces a terminal element of $\mathrm{Hom}_B(B,f)$ "over A."

Proof of Theorem 3.5.7. The inverse to the function that takes a 2-cell $fr \Rightarrow g$ and produces an isomorphism class of maps $\mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$ is constructed by applying Lemma 3.5.8 to the functor $r\colon C \to B$: given a map of spans, restrict along the right adjoint $i\colon C \to \mathsf{Hom}_B(B,r)$ and paste with the comma cone for $\mathsf{Hom}_A(f,g)$ to define a 2-cell $fr \Rightarrow g$.

Starting from a 2-cell $\rho\colon fr \Rightarrow g$, the composite of these two functions constructs the 2-cell displayed below-left

$$
\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[dddl, bend right=40, equals] \arrow[dddr, bend left=40, "r"] & \\
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "y"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[ddl, bend right=40, equals] \arrow[ddr, bend left=40, "r"] & \\
& \mathsf{Hom}_B(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& C \arrow[dl, bend right=40, equals] \arrow[dr, bend left=40, "r"] \arrow[d, phantom, "="] & \\
C \arrow[rr, "r"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

which equals the above-center pasted composite by the definition of y from ρ, and equals the above-right composite since $\phi i = \mathrm{id}_r$. Thus, when a 2-cell $\rho\colon fr \Rightarrow g$ is encoded as a map $y\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$, and then re-converted into a 2-cell, the original 2-cell ρ is recovered.

For the converse, starting with a map $z\colon \mathsf{Hom}_B(B,r) \to \mathsf{Hom}_A(f,g)$ over $C \times B$, the composite of these two functions constructs an isomorphism class of maps of spans w displayed below-left by applying 1-cell induction for the comma cone $\mathsf{Hom}_A(f,g)$ to the composite 2-cell pasted below-center-left:

$$
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, dashed, "w"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, "p_1"'] \arrow[ddddr, bend left=40, "p_0"] & \\
& C \arrow[d, "i"] \arrow[dddl, bend right=40, equals] \arrow[dddr, bend left=30, "r"] \arrow[ur, phantom, "\overset{\phi}{\Leftarrow}"] & \\
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, "p_1"'] \arrow[dd, bend left=60, equals] & \\
& C \arrow[d, "i"] \arrow[dddl, bend right=40, equals] \arrow[r, phantom, "\overset{\eta}{\Leftarrow}"] & {} \\
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_B(B,r) \arrow[d, "z"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

Applying Lemma 3.5.8, there exists a 2-cell $\eta\colon \mathrm{id} \Rightarrow ip_1$ so that $p_0\eta = \phi$ — this gives the pasting equality above center — and $p_1\eta = \mathrm{id}$ — which gives the pasting equality above right. Proposition 3.4.7 now implies that $w \cong z$ over $C \times B$. $\square$

A dual version of Theorem 3.5.7 represents 2-cells $g \Rightarrow f\ell$ as fibered isomorphism classes of maps $\mathrm{Hom}_B(\ell, B) \to \mathrm{Hom}_A(g, f)$ over $B \times C$. Specializing these results to the case where one of f or g is the identity, we immediately recover a "cheap" form of the Yoneda lemma:

3.5.10. Corollary. *Given a parallel pair of functors, $f, g\colon A \rightrightarrows B$, there are bijections between 2-cells as displayed below-center and fibered isomorphism classes of maps between their left and right representations as comma ∞-categories, as displayed below-left and below-right, respectively:*

$$
\left\{\begin{tikzcd} & \mathrm{Hom}_B(g,B) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "a"] & \\ B & & A \\ & \mathrm{Hom}_B(f,B) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}\right\}_{/\cong}
\quad \leftrightsquigarrow \quad
\left\{\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}\right\}
\quad \leftrightsquigarrow \quad
\left\{\begin{tikzcd} & \mathrm{Hom}_B(B,f) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[dd, "a"] & \\ A & & B \\ & \mathrm{Hom}_B(B,g) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}\right\}_{/\cong}
$$

that are constructed by pasting with the left comma cone over g and right comma cone over f, respectively:

$$
\begin{tikzcd} & \mathrm{Hom}_B(g,B) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ B & {} & A \arrow[ll, "g" description] \arrow[ll, bend left=40, "f"] \arrow[ll, phantom, bend left=20, "\Uparrow\alpha"] \end{tikzcd}
=
\begin{tikzcd} & \mathrm{Hom}_B(g,B) \arrow[d, dashed, "a"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\ & \mathrm{Hom}_B(f,B) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ B & {} & A \arrow[ll, "f"] \end{tikzcd}
\qquad
\begin{tikzcd} & \mathrm{Hom}_B(B,f) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "f" description] \arrow[rr, bend right=40, "g"'] \arrow[rr, phantom, bend right=20, "\Downarrow\alpha"] & {} & B \end{tikzcd}
=
\begin{tikzcd} & \mathrm{Hom}_B(B,f) \arrow[d, dashed, "a"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\ & \mathrm{Hom}_B(B,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "g"'] & {} & B \end{tikzcd}
$$

and then applying 1-cell induction to factor through the left comma cone over f in the former case or the right comma cone over g in the latter. $\square$

Combining the results of this section, we prove one final representability theorem that allows us to recognize when a comma ∞-category is right representable in the absence of a predetermined representing functor. This result specializes to give existence theorems for adjoint functors and limits and colimits in the next chapter.

3.5.11. Theorem. *The comma ∞-category $\mathrm{Hom}_A(f, g)$ associated to a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ is right representable if and only if its codomain-projection functor admits a right adjoint right inverse*

$$
\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ C \arrow[ur, dashed, bend right, "i"'] \arrow[ur, phantom, "\perp"] & & B \end{tikzcd}
$$

in which case the composite $p_0 i\colon C \to B$ defines the representing functor and the 2-cell represented by the functor $i\colon C \to \mathrm{Hom}_A(f, g)$ defines an absolute right lifting of g through f.

Proof. If $\mathrm{Hom}_A(f, g)$ is represented on the right by a functor $r\colon C \to B$, then $\mathrm{Hom}_A(f, g) \simeq \mathrm{Hom}_B(B, r)$ over $C \times B$ and the codomain-projection functor is equivalent to $p_1\colon \mathrm{Hom}_B(B, r) \twoheadrightarrow C$,

which admits a right adjoint right inverse i by Lemma 3.5.8. The proof of Theorem 3.5.3 then shows that i represents an absolute right lifting diagram. Thus, it remains only to prove the converse.

To that end, suppose we are given a right adjoint right inverse adjunction $p_1 \dashv i$. Unpacking the definition, this provides an adjunction

$$\begin{tikzcd}
C \arrow[rr, phantom, "\perp"] \arrow[dr, equals] \arrow[rr, dashed, bend right, "i"'] & & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, "p_1"'] \arrow[dl, two heads, "p_1"] \\
& C &
\end{tikzcd}$$

over C whose counit is an identity and whose unit $\eta\colon \mathrm{id} \Rightarrow ip_1$ satisfies the conditions $\eta i = \mathrm{id}_i$ and $p_1\eta = \mathrm{id}_{p_1}$. By Theorem 3.5.7, to construct the fibered equivalence $\mathsf{Hom}_B(B,r) \simeq_{C\times B} \mathsf{Hom}_A(f,g)$ with $r := p_0 i$, it suffices to demonstrate that the 2-cell defined by restricting the comma cone for $\mathsf{Hom}_A(f,g)$ along i

$$\begin{tikzcd}
& C \arrow[d, dashed, "i"] \arrow[ddl, equals, bend right] \arrow[ddr, two heads, bend left, "r"] & \\
& \mathsf{Hom}_A(f,g) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

defines an absolute right lifting diagram.

By 1-cell induction any 2-cell as displayed below-left induces a 1-cell m as displayed below-center:

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "m"'] \arrow[rr, bend left, "b"] \arrow[dr, bend right, "c"'] & \mathsf{Hom}_A(f,g) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "m"'] \arrow[rrr, bend left, "b"] \arrow[dr, bend right, "c"'] & \mathsf{Hom}_A(f,g) \arrow[r, equals] \arrow[d, two heads, "p_1"] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{Hom}_A(f,g) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\Downarrow\phi"] & B \arrow[d, "f"] \\
& C \arrow[r, equals] \arrow[ur, "i"'] & C \arrow[r, "g"'] & A
\end{tikzcd}$$

Inserting the triangle equality $p_1\eta = \mathrm{id}_{p_1}$ as displayed above-right constructs the desired factorization $p_0\eta m\colon b \Rightarrow rc$ of χ through ϕi.

In fact, by 2-cell induction for the comma cone ϕ, any 2-cell $\tau_0\colon b \Rightarrow rc$ defining a factorization of $\chi\colon fb \Rightarrow gc$ through ϕi must have the form $\tau_0 = p_0\tau$ for some 2-cell $\tau\colon m \Rightarrow ic$ so that $\pi_1\tau = \mathrm{id}_c$. The pair (τ_0, id_c) satisfies the compatibility condition of Proposition 3.4.6(ii) to induce a 2-cell $\tau\colon m \Rightarrow ic$. We'll argue that the 2-cell τ is unique, proving that the factorization $p_0\tau\colon b \Rightarrow rc$ is also unique.

To see this, note that the adjunction $p_1 \dashv i$ over C exhibits the right adjoint as a terminal element of the object $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ in the slice 2-category $(\mathfrak{h}\mathcal{K})_{/C}$. It follows, as in Lemma 2.2.4, that for any object $c\colon X \to C$ and any morphism $m\colon X \to \mathsf{Hom}_A(f,g)$ over C, there exists a unique 2-cell $m \Rightarrow ic$ over C. Thus, there is a unique 2-cell $\tau\colon m \Rightarrow ic$ with the property that $p_1\tau = \mathrm{id}_c$, and so the factorization $p_0\tau\colon b \Rightarrow rc$ of χ through ϕi must also be unique. $\square$

More concisely, Theorem 3.5.11 shows that a comma ∞-category $\mathsf{Hom}_A(f,g)$ is right representable just when its codomain-projection functor $p_1 \colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a terminal element as an object of the sliced ∞-cosmos $\mathcal{K}_{/C}$; dually, $\mathsf{Hom}_A(f,g)$ is left representable just when its domain-projection functor admits an initial element as an object of the sliced ∞-cosmos $\mathcal{K}_{/B}$; see Corollary 3.6.11. There is a small gap between this statement and the version proven in Theorem 3.5.11 having to do with the discrepancy between the homotopy 2-category of $\mathcal{K}_{/C}$ and the slice of the homotopy 2-category $\mathfrak{h}\mathcal{K}$ over C. This is the subject to which we now turn.

Exercises.

3.5.i. Exercise. How might one encode the existence of an adjunction $f \dashv u$ between a given opposing pair of functors using comma ∞-categories?

3.6. Sliced homotopy 2-categories and fibered equivalences

The ∞-category $A^{\mathbb{2}}$ of arrows in A together with its domain- and codomain-evaluation functors $(p_0, p_1) \colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ satisfies a weak universal property in the homotopy 2-category that characterizes the ∞-category up to equivalence *over* $A \times A$; see Proposition 3.2.10. Similarly the comma ∞-category is characterized up to fibered equivalence, as defined in Definition 3.2.7.

As commented upon in Remark 3.2.9 there is some ambiguity regarding the 2-categorical data required to specify a fibered equivalence, that we shall now address head-on. The issue is that, for an ∞-category B in an ∞-cosmos $\mathcal{K}$, the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos of Proposition 1.2.19 is not isomorphic to the 2-category $(\mathfrak{h}\mathcal{K})_{/B}$ of isofibrations, functors, and 2-cells over B in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of $\mathcal{K}$; see Exercise 1.4.iv.

However, there is a canonical comparison functor relating this pair of 2-categories that satisfies a property we now introduce:

3.6.1. Definition (smothering 2-functor). A 2-functor $F \colon \mathcal{A} \to \mathcal{B}$ is **smothering** if it is

- surjective on 0-cells;
- full on 1-cells: for any pair of objects A, A' in $\mathcal{A}$ and 1-cell $k \colon FA \to FA'$ in $\mathcal{B}$, there exists $f \colon A \to A'$ in $\mathcal{A}$ with $Ff = k$;
- full on 2-cells: for any parallel pair $f, g \colon A \rightrightarrows A'$ in $\mathcal{A}$ and 2-cell $\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow\beta"] & FA' \end{tikzcd}$ in $\mathcal{B}$, there exists a 2-cell $\alpha \colon f \Rightarrow g$ in $\mathcal{A}$ with $F\alpha = \beta$; and
- conservative on 2-cells: for any 2-cell α in $\mathcal{A}$ if $F\alpha$ is invertible in $\mathcal{B}$ then α is invertible in $\mathcal{A}$.

3.6.2. Remark. Note that smothering 2-functors are surjective on objects 2-functors that are "locally smothering": meaning that the action on hom-categories is by a smothering functor, as defined in 3.1.2.

The prototypical example of a smothering 2-functor solves Exercise 1.4.iv.

3.6.3. Proposition. *Let B be an ∞-category in an ∞-cosmos $\mathcal{K}$. There is a canonical 2-functor*

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

from the homotopy 2-category of the sliced ∞-cosmos $\mathcal{K}_{/B}$ to the 2-category of isofibrations, functors, and 2-cells over B in $\mathfrak{h}\mathcal{K}$ and this 2-functor is smothering.

This follows more-or-less immediately from Lemma 3.1.5 but we spell out the details nonetheless.

Proof. The 2-categories $\mathfrak{h}(\mathcal{K}_{/B})$ and $(\mathfrak{h}\mathcal{K})_{/B}$ have the same objects — isofibrations with codomain B — and 1-cells — functors between the "total spaces" that commute with these isofibrations to B — so the canonical mapping may be defined to act as the identity on underlying 1-categories.

By the definition of the sliced ∞-cosmos given in Proposition 1.2.19, a 2-cell between functors $f, g\colon E \rightrightarrows F$ from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$ is a homotopy class of 1-simplices in the quasi-category defined by the pullback of simplicial sets below-left

$$\begin{tikzcd} \mathsf{Fun}_B(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"'] & \mathsf{Fun}(E,B) \end{tikzcd} \qquad \begin{tikzcd} (\mathsf{hFun})_{/B}(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{hFun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{1} \arrow[r, "p"'] & \mathsf{hFun}(E,B) \end{tikzcd}$$

Unpacking, a 2-cell $\alpha\colon f \Rightarrow g$ is represented by a 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whiskers with q to the degenerate 1-simplex on the vertex $p \in \mathsf{Fun}(E,B)$, and two such 1-simplices represent the same 2-cell if and only if they bound a 2-simplex of the form displayed in (1.1.8) that also whiskers with q to the degenerate 2-simplex on p.

By contrast, a 2-cell in $(\mathfrak{h}\mathcal{K})_{/B}$ is a morphism in the category defined by the pullback of categories above-right. Such 2-cells are represented by 1-simplices $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$ that whisker with q to 1-simplices in $\mathsf{Fun}(E,B)$ that are homotopic to the degenerate 1-simplex on p, and two such 1-simplices represent the same 2-cell if and only if they are homotopic in $\mathsf{Fun}(E,F)$.

Applying the homotopy category functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ to the above-left pullback produces a cone over the above-right pullback, inducing a canonical map

$$\mathfrak{h}(\mathsf{Fun}_B(E,F)) \to (\mathsf{hFun})_{/B}(E,F),$$

which is the action on homs of the canonical 2-functor $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$.

The 2-functor just constructed is bijective on 0- and 1-cells. To see that it is full on 2-cells we must show that any 1-simplex $\alpha\colon f \to g$ in $\mathsf{Fun}(E,F)$, for which $q\alpha\colon p \to p$ is homotopic to $p \cdot \sigma^0\colon p \to p$ in $\mathsf{Fun}(E,B)$, is homotopic in $\mathsf{Fun}(E,F)$ to a 1-simplex from f to g over $p \cdot \sigma^0$. By Lemma 1.1.9, any such α defines a lifting problem

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E,B) \end{tikzcd} \quad \begin{matrix} \ni \\ \\ \ni \end{matrix} \quad \begin{tikzcd}[row sep=small] & f \arrow[dr, "\alpha"] & \\ f \arrow[ur, equal] & & g \\ & \mapsto & \\ & p \arrow[dr, "q\alpha"] & \\ p \arrow[ur, equal] \arrow[rr, equal] & & p \end{tikzcd}$$

A solution exists since $q_*\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,B)$ is an isofibration, proving that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ is full on 2-cells.

Now suppose $\alpha\colon f \to g$ represents a 2-cell in $\mathsf{Fun}_B(E,F)$ whose image in $(\mathsf{hFun})_{/B}(E,F)$ is an isomorphism. A map in a 1-category defined by a pullback is invertible if and only if its projections along the legs of the pullback cone are isomorphisms. Thus the image of α is invertible if and only if $\alpha\colon f \to g$ defines an isomorphism in $\mathsf{hFun}(E,F)$, which by Definition 1.1.13 is the case if and only if

$\alpha\colon f \to g$ represents an isomorphism in $\mathsf{Fun}(E,F)$. Since α is fibered over the degenerate 1-simplex at p, this presents us with a lifting problem

$$\begin{tikzcd} \mathbb{2} \arrow[r, "\alpha"] \arrow[d, hook] & \mathsf{Fun}_B(E,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q_*"] \\ \mathbb{I} \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & \mathbb{1} \arrow[r, "p"'] & \mathsf{Fun}(E,B) \end{tikzcd}$$

which Exercise 1.1.v tells us we can solve. This proves that $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ reflects invertibility of 2-cells and hence defines a smothering 2-functor. □

Smothering 2-functors are not strictly speaking invertible, but nevertheless 2-categorical structures from the codomain can be lifted to the domain:

3.6.4. Lemma. *Smothering 2-functors reflect equivalences: for any smothering 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ and 1-cell $f\colon A \to B$ in $\mathcal{A}$, if $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence in $\mathcal{B}$ then f is an equivalence in $\mathcal{A}$.*

Proof. By fullness on 1-cells, an equivalence inverse $g'\colon FB \xrightarrow{\sim} FA$ to Ff lifts to a 1-cell $g\colon B \to A$ in $\mathcal{A}$. By fullness on 2-cells, the isomorphisms $\mathrm{id}_{FA} \cong g' \circ Ff$ and $Ff \circ g' \cong \mathrm{id}_{FB}$ also lift to $\mathcal{A}$ and by conservativity on 2-cells these lifted 2-cells are also invertible. □

Applying Lemma 3.6.4 to the smothering 2-functor

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

we resolve the ambiguity about the 2-categorical data of a fibered equivalence.

3.6.5. Proposition.

(i) Any equivalence in $(\mathfrak{h}\mathcal{K})_{/B}$ lifts to an equivalence in $\mathfrak{h}(\mathcal{K}_{/B})$. That is, fibered equivalences over B may be specified by defining an opposing pair of 1-cells $f\colon E \to F$ and $g\colon F \to E$ over B together with invertible 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$ that lie over B in $\mathfrak{h}\mathcal{K}$.

(ii) Moreover, if $f\colon E \to F$ is a map between isofibrations over B that admits an not-necessarily fibered equivalence inverse $g\colon F \to E$ with not-necessarily fibered 2-cells $\mathrm{id}_E \cong gf$ and $fg \cong \mathrm{id}_F$, then this data is isomorphic to a genuine fibered equivalence.

Proof. The first statement is proven by Lemma 3.6.4 and Proposition 3.6.3. The second statement asserts that the forgetful 2-functor $(\mathfrak{h}\mathcal{K})_{/B} \to \mathfrak{h}\mathcal{K}$ reflects equivalences. Exercise 3.6.i shows that for any map between isofibrations over B that admits an equivalence inverse in the underlying 2-category, the inverse equivalence and invertible 2-cells can be lifted to also lie over B. □

This gives a 2-categorical proof of Proposition 1.2.19(vii), that for any ∞-category B in an ∞-cosmos $\mathcal{K}$, the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences.

The smothering 2-functor $\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$ can also be used to lift adjunctions that are fibered 2-categorically over B to adjunctions in the sliced ∞-cosmos $\mathcal{K}_{/B}$.

3.6.6. Definition (fibered adjunction). A **fibered adjunction** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an adjunction

$$\begin{tikzcd} E \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads] & & F \arrow[ll, bend left, "u"] \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$. We write $f \dashv_B u$ to indicate that specified maps over B are **adjoint over** B.

3.6.7. Lemma (pullback and pushforward of fibered adjunctions).

(i) A fibered adjunction over B can be pulled back along any functor $k\colon A \to B$ to define a fibered adjunction over A.

(ii) A fibered adjunction over A can be pushed forward along any isofibration $p\colon A \twoheadrightarrow B$ to define a fibered adjunction over B.

Proof. By Proposition 1.3.3(v), pullback defines a cosmological functor $k^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$, which descends to a 2-functor $k^*\colon \mathfrak{h}(\mathcal{K}_{/B}) \to \mathfrak{h}(\mathcal{K}_{/A})$ that carries fibered adjunctions over B to fibered adjunctions over A. This proves (i).

Composition with an isofibration $p\colon A \twoheadrightarrow B$ also defines a 2-functor $p_*\colon \mathfrak{h}(\mathcal{K}_{/A}) \to \mathfrak{h}(\mathcal{K}_{/B})$; the reason we ask p to be an isofibration is due to our convention that the objects in the sliced ∞-cosmoi are isofibrations over a fixed base. Thus, composition with an isofibration carries a fibered adjunction over A to a fibered adjunction over B proving (ii). ☐

In analogy with Lemma 3.6.4, we have:

3.6.8. Lemma. *If $F\colon \mathcal{A} \to \mathcal{B}$ is a smothering 2-functor, then any adjunction in $\mathcal{B}$ may be lifted to an adjunction in $\mathcal{A}$.*

Proof. Exercise 3.6.ii. ☐

3.6.9. Remark. A direct proof of Lemma 3.6.8 proceeds as follows: since a smothering 2-functor is surjective on objects and full on both 1- and 2-cells, the data of an adjunction in $\mathcal{B}$ may be lifted to an adjunction in $\mathcal{A}$. Since smothering 2-functors are not in general faithful at the level of 2-cells, there is no reason why the triangle identity composites should be identities, but by 2-cell conservativity they are both invertible. Now either the unit or counit may be modified as in the proof of Proposition 2.1.11 by composing with the inverse of one of these triangle identity composite isomorphisms. Now that triangle equality holds and the other triangle identity composite is an idempotent isomorphism and hence also an identity.

John Bourke pointed out that this proof invokes a recharacterization of adjunctions that makes the conclusion of Lemma 3.6.8 obvious: a pair of 1-cells $f\colon B \to A$ and $u\colon A \to B$ in a 2-category form an adjoint pair $f \dashv u$ if and only if there exist 2-cells $\mathrm{id}_B \Rightarrow uf$ and $fu \Rightarrow \mathrm{id}_A$ so that the composites $f \Rightarrow fuf \Rightarrow f$ and $u \Rightarrow ufu \Rightarrow u$ are both invertible.

Many of the examples of fibered adjunctions we will encounter are right adjoint right inverses or left adjoint right inverses to a given isofibration. The next result shows that whenever an isofibration $p\colon E \twoheadrightarrow B$ admits a left adjoint with unit an isomorphism, then this left adjoint may be modified so as to define a left adjoint right inverse, making the adjunction fibered over B. The dual also holds:

3.6.10. Lemma. *Let $p\colon E \twoheadrightarrow B$ be any isofibration that admits a right adjoint $r'\colon B \to E$ with counit $\epsilon\colon pr' \cong \mathrm{id}_B$ an isomorphism. Then r' is isomorphic to a functor r that lies strictly over B and defines a right adjoint right inverse to p. Thus any such p defines a fibered adjunction*

$$\begin{tikzcd} E \arrow[rr, two heads, bend left, "p"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & B \arrow[ll, dashed, bend left, "r"] \arrow[dl, equal] \\ & B & \end{tikzcd}$$

in $\mathcal{K}_{/B}$ whose right adjoint r lies strictly over B, whose counit is the identity 2-cell, and in which the unit η lies over B in the sense that $p\eta = \mathrm{id}_p$.

PROOF. Exercise 3.6.iii. □

Since the identity on B defines the terminal object of the sliced ∞-cosmos $\mathcal{K}_{/B}$, Lemma B.4.7 can be summarized more compactly as follows:

3.6.11. COROLLARY. *An isofibration $p\colon E \twoheadrightarrow B$ admits a right adjoint right inverse if and only if it admits a terminal element as an object of $\mathcal{K}_{/B}$. Dually, $p\colon E \twoheadrightarrow B$ admits a left adjoint right inverse if and only if it admits an initial element as an object of $\mathcal{K}_{/B}$*

3.6.12. EXAMPLE. Lemma 3.5.8 constructs an adjunction in the sliced 2-category $\mathfrak{h}\mathcal{K}_{/A}$. Lemma 3.6.8 now allows us to lift it to a genuine adjunction

$$\begin{tikzcd} A \arrow[dr, equals] \arrow[rr, bend right, dashed, "i"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Hom}_B(B,f) \arrow[ll, bend right, "p_1"'] \arrow[dl, two heads, "p_1"] \\ & A & \end{tikzcd}$$

in the sliced ∞-cosmos $\mathcal{K}_{/A}$. By Corollary 3.6.11 this situation may be summarized by saying that $p_1\colon \mathrm{Hom}_B(B,f) \twoheadrightarrow A$ admits a terminal element over A.

By Lemma 3.6.7(i), we may pull back the fibered adjunction along any element $a\colon 1 \to A$ to obtain an adjunction

$$\begin{tikzcd} 1 \arrow[rr, bend right, dashed, "i_a"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Hom}_B(B,fa) \arrow[ll, bend right, "!"'] \end{tikzcd}$$

that identifies a terminal element in the fiber $\mathrm{Hom}_B(B,fa)$ of $p_1\colon \mathrm{Hom}_B(B,f) \twoheadrightarrow A$ over a. This generalizes the result of Corollary 3.5.9.

3.6.13. EXAMPLE (the fibered adjoints to composition). For any ∞-category A, the adjoints to the "composition" functor $\circ\colon A^2 \times_A A^2 \to A^2$ constructed in Lemma 2.1.13 may be constructed by composing a triple of adjoint functors that are fibered over the endpoint-evaluation functors

$$\begin{tikzcd} A^3 \arrow[rr, "A^{d^1}" description] \arrow[dr, two heads, "(\mathrm{ev}_2{,}\mathrm{ev}_0)"'] & & A^2 \arrow[ll, bend right, "A^{s^0}"'] \arrow[ll, bend left, "A^{s^1}"] \arrow[dl, two heads, "(\mathrm{ev}_1{,}\mathrm{ev}_0)"] \\ & A \times A & \end{tikzcd}$$

with an adjoint equivalence involving a functor $A^3 \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow A^2 \times_A A^2$, which also lies over $A \times A$. Lemma B.4.7 and its dual implies that these adjoint equivalences can be lifted to fibered adjoint equivalences

over $A \times A$, and now both adjoint triples and hence also the composite adjunctions

$$\begin{tikzcd}
A^2 \underset{A}{\times} A^2 \arrow[rr, "\circ" description] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[dr, "(\mathrm{ev}_2,\mathrm{ev}_0)"'] & & A^2 \arrow[ll, bend right, "(-,\mathrm{id}_{\mathrm{dom}(-)})"'] \arrow[ll, bend left, "(\mathrm{id}_{\mathrm{cod}(-)},-)"] \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_0)"] \\
& A \times A &
\end{tikzcd}$$

lie in $\mathcal{K}_{/A\times A}$.

This fibered adjunction figures in the proof of a result that will allow us to convert limit and colimit diagrams into right and left Kan extension diagrams in the next chapter.

3.6.14. Proposition. *A cospan as displayed below-left admits an absolute right lifting if and only if the cospan displayed below-right admits an absolute right lifting*

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, dashed, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathsf{Hom}_A(f,A) \arrow[d, "p_1"] \\
C \arrow[ur, dashed, "i"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A
\end{tikzcd}$$

in which case the 2-cell ϵ is necessarily an isomorphism and can be chosen to be an identity.

Proof. By Theorem 3.5.11, a cospan admits an absolute right lifting if and only if the codomain-projection functor from the associated comma ∞-category admits a right adjoint right inverse. Our task is thus to show that this right adjoint right inverse exists for $\mathsf{Hom}_A(f,g)$ if and only if this right adjoint right inverse exists for $\mathsf{Hom}_A(p_1,g)$.

From the defining pullback (3.4.2) that constructs the comma ∞-category $\mathsf{Hom}_A(p_1,g)$ reproduced below-left, we have the below-right pullback square

$$\begin{tikzcd}
\mathsf{Hom}_A(p_1,g) \arrow[r] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Hom}_A(A,g) \arrow[r] \arrow[d, two heads, "(p_1,p_0)"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^2 \arrow[d, two heads] \\
C \times \mathsf{Hom}_A(f,A) \arrow[r, "C\times p_1"'] \arrow[d, two heads, "\pi"'] & C \times A \arrow[r, "g\times A"'] \arrow[d, two heads, "\pi"] & A \times A \\
\mathsf{Hom}_A(f,A) \arrow[r, "p_1"'] & A &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathsf{Hom}_A(p_1,g) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Hom}_A(A,g) \arrow[d, two heads, "p_0"] \\
\mathsf{Hom}_A(f,A) \arrow[r, two heads, "p_1"'] & A
\end{tikzcd}
\tag{3.6.15}$$

By Lemma 3.6.7, the composition-identity fibered adjunction of Example 3.6.13 pulls back along $g \times f \colon C \times B \to A \times A$ to define a fibered adjunction

$$\begin{tikzcd}
\mathsf{Hom}_A(p_1,g) \cong \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[rr, "\circ" description] \arrow[dr, two heads, "(p_1,p_0)"'] & & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, "\perp"'] \arrow[ll, bend left, "\perp"] \arrow[dl, two heads, "(p_1,p_0)"] \\
& C \times B &
\end{tikzcd}$$

which then pushes forward along the projection $\pi \colon C \times B \twoheadrightarrow C$ to a fibered adjunction over C

$$\begin{tikzcd} \mathsf{Hom}_A(p_1,g) \arrow[rr, "\circ" description] \arrow[dr, two heads, "p_1"'] & & \mathsf{Hom}_A(f,g) \arrow[ll, bend right, "\perp"'] \arrow[ll, bend left, "\perp"] \arrow[dl, two heads, "p_1"] \\ & C & \end{tikzcd}$$

between the codomain-projection for $\mathsf{Hom}_A(p_1,g)$ and the codomain projection for $\mathsf{Hom}_A(f,g)$. Now by Corollary 3.6.11, $p_1 \colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a right adjoint right inverse just when the object on the right admits a terminal element, while $p_1 \colon \mathsf{Hom}_A(p_1,g) \twoheadrightarrow C$ similarly admits a right adjoint right inverse just when the object on the left admits a terminal element. By Theorem 2.4.2, a terminal element on either side is carried by the appropriate right adjoint to a terminal element on the other side. This proves the equivalence of these conditions.

It remains only to prove that the 2-cell for the absolute right lifting of g through p_1 is invertible. By Theorem 3.5.11, this 2-cell is constructed as by restricting the comma cone along the terminal element, so it is given by the composite

$$\begin{tikzcd} C \arrow[r] & \mathsf{Hom}_A(p_1,g) \arrow[r] & A^{\mathbb{2}} \arrow[r, bend left, two heads, "p_0"] \arrow[r, bend right, two heads, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A \end{tikzcd}$$

where the left-hand map is the terminal element just constructed and the middle one comes comes from the defining pullback diagram displayed on the left of (3.6.15). As just argued, that terminal element may be chosen to be in the image of the right adjoint $\mathsf{Hom}_A(f,g) \to \mathsf{Hom}_A(A,g) \times_A \mathsf{Hom}_A(f,A) \cong \mathsf{Hom}_A(p_1,g)$, whose component on the left factor is the identity. Simultaneously, the pullback defining the map $\mathsf{Hom}_A(p_1,g) \to A^{\mathbb{2}}$ factors through the projection onto the left factor, so we see that the 2-cell in the absolute right lifting diagram is represented by the composite

$$\begin{tikzcd} C \arrow[r, "i"] & \mathsf{Hom}_A(f,g) \arrow[r, "(\mathrm{id}_{\mathrm{cod}(-)}, -)"] & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[r, "\pi"] & \mathsf{Hom}_A(A,g) \arrow[r] & A^{\mathbb{2}} \arrow[r, bend left, two heads, "p_0"] \arrow[r, bend right, two heads, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A \end{tikzcd}$$

and hence that this cell is invertible. ☐

Exercises.

3.6.i. Exercise. Let B be an object in a 2-category $\mathcal{C}$ and consider a map

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads] & & F \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

between isofibrations over B. Prove that if f is an equivalence in $\mathcal{C}$ then f is also an equivalence in the slice 2-category $\mathcal{C}_{/B}$ of isofibrations over B, 1-cells that form commutative triangles over B, and 2-cells that lie over B in the sense that they whisker with the codomain isofibration to the identity 2-cell on the domain isofibration.

3.6.ii. Exercise. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a smothering 2-functor. Show that any adjunction in $\mathcal{B}$ can be lifted to an adjunction in $\mathcal{A}$. Demonstrate furthermore that if we have previously specified a lift of

the objects, 1-cells, and either the unit or counit of the adjunction in $\mathcal{B}$, then there is a lift of the remaining 2-cell that combines with the previously specified data to define an adjunction in $\mathcal{A}$. This proves a more precise version of Lemma 3.6.8.

3.6.iii. Exercise. Prove Lemma B.4.7.

CHAPTER 4

Adjunctions, limits, and colimits II

Comma ∞-categories provide a vehicle for encoding the universal properties of categorical constructions that restrict to define equivalences between the internal mapping spaces introduced in Definition 3.4.9 between suitable pairs of elements. Using the theory developed in Chapter 3, we quickly prove a variety of results of this type first for adjunctions in §4.1 and then for limits and colimits in §4.3. In an interlude in §4.2, we introduce the ∞-categories of cones over or under a diagram as a comma ∞-category and then give a second model for these ∞-categories of cones in the case of diagrams indexed by simplicial sets built from Joyal's join construction. Then we conclude in §4.4 with an application, constructing the loops $\vdash$ suspension adjunction for *pointed* ∞-categories, containing an element that is both initial and terminal.

4.1. The universal property of adjunctions

Our first result shows that an adjunction between an opposing pair of functors can equally be encoded by a "transposing equivalence" between their left and right representations as comma ∞-categories.

4.1.1. Proposition. *An opposing pair of functors* $u\colon A \to B$ *and* $f\colon B \to A$ *define an adjunction* $f \dashv u$ *if and only if* $\mathsf{Hom}_A(f, A) \simeq \mathsf{Hom}_B(B, u)$ *over* $A \times B$.

Proof. This is a special case of Theorem 3.5.7. If $f \dashv u$, then Lemma 2.3.6 tells us that its counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ defines an absolute right lifting diagram. Theorem 3.5.7 then tells us that the 1-cell induced by the left-hand pasted composite

$$
\begin{tikzcd}
 & \mathsf{Hom}_B(B,u) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "u" description] \arrow[dr, equal] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathsf{Hom}_B(B,u) \arrow[d, dashed, "\ulcorner \epsilon \circ f(-) \urcorner"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "p_0"] & \\
 & \mathsf{Hom}_A(f,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
A \arrow[dr, equal] \arrow[rr, phantom, "\underset{\Leftarrow}{\phi}"] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

defines a fibered equivalence $\mathsf{Hom}_B(B, u) \xrightarrow{\sim} \mathsf{Hom}_A(f, A)$ over $A \times B$. We interpret this result as saying that in the presence of an adjunction $f \dashv u$, the right comma cone over u transposes to define the left comma cone over f.[1]

[1]If desired, an inverse equivalence can be constructed by applying the dual of Theorem 3.5.7 to the absolute left lifting diagram presented by the unit.

Conversely, Theorem 3.5.7 tells us that from a fibered equivalence $\mathsf{Hom}_B(B,u) \stackrel{\sim}{\Rightarrow} \mathsf{Hom}_A(f,A)$ over $A \times B$ one can extract a 2-cell that defines an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equals] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

Lemma 2.3.6 then tells us that this 2-cell defines the counit of an adjunction $f \dashv u$. ☐

4.1.2. Observation (the transposing equivalence). To justify referring to the induced functor

$$\ulcorner \epsilon \circ f(-) \urcorner \colon \mathsf{Hom}_B(B,u) \stackrel{\sim}{\Rightarrow} \mathsf{Hom}_A(f,A)$$

as a **transposing equivalence**, recall that the transpose of a 2-cell $\chi\colon b \Rightarrow ua$ across the adjunction $f \dashv u$ is computed by the left-hand pasting diagram below:

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\overset{\chi}{\Leftarrow}"] & \\ A \arrow[rr, "u" description] \arrow[dr, equals] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}
=
\begin{tikzcd}[column sep=small] & X \arrow[d, dashed, "x"] \arrow[ddl, bend right, "a"'] \arrow[ddr, bend left, "b"] & \\ & \mathsf{Hom}_B(B,u) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A \arrow[rr, "u" description] \arrow[dr, equals] & {} \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & B \arrow[dl, "f"] \\ & A & \end{tikzcd}
=
\begin{tikzcd}[column sep=small] & X \arrow[d, dashed, "x"] \arrow[dddl, bend right, "a"'] \arrow[dddr, bend left, "b"] & \\ & \mathsf{Hom}_B(B,u) \arrow[d, dashed, "\wr"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_0"] & \\ & \mathsf{Hom}_A(f,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ A & {} & B \arrow[ll, "f"] \end{tikzcd}$$

By the weak universal property of the right comma cone over u, the 2-cell χ is represented by the induced functor $X \to \mathsf{Hom}_B(B,u)$, which then composes with the transposing equivalence to define a functor $X \to \mathsf{Hom}_A(f,A)$ that represents the transpose of χ, by the pasting diagram equalities from right to left. This observation also justifies our notation, in which we name the fibered equivalence $\ulcorner \epsilon \circ f(-) \urcorner$ after the formula for adjoint transposition.

4.1.3. Corollary. *An adjunction* $\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd}$ *induces an equivalence* $\mathsf{Hom}_A(fb,a) \simeq \mathsf{Hom}_B(b,ua)$ *over* $X \times Y$ *for any pair of generalized elements* $a\colon X \to A$ *and* $b\colon Y \to B$.

Proof. By the pullback construction of comma ∞-categories given in (3.4.2), the equivalence $\mathsf{Hom}_A(f,A) \simeq \mathsf{Hom}_B(B,u)$ in $\mathcal{K}_{/A\times B}$ pulls back along $a \times b\colon X \times Y \to A \times B$ to define an equivalence $\mathsf{Hom}_A(fb,a) \simeq \mathsf{Hom}_B(b,ua)$ in $\mathcal{K}_{/X\times Y}$. ☐

In particular, the equivalence of Proposition 4.1.1 pulls back to define an equivalence of internal mapping spaces, introduced in 3.4.9.

4.1.4. Proposition (the universal property of units and counits). *Consider an adjunction*

$$\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] \end{tikzcd} \qquad \textit{with unit } \eta\colon \mathrm{id}_B \Rightarrow uf \textit{ and counit } \epsilon\colon fu \Rightarrow \mathrm{id}_A\,.$$

Then for each element $a\colon 1 \to A$, the component ϵa defines a terminal element of $\mathsf{Hom}_A(f,a)$, and for each element $b\colon 1 \to B$, the component ηb defines an initial element of $\mathsf{Hom}_B(b,u)$.

PROOF. By Corollary 4.1.3, the fibered equivalence $\mathsf{Hom}_A(f,A) \simeq_{A\times B} \mathsf{Hom}_B(B,u)$ of Proposition 4.1.1 pulls back to define equivalences

$$\mathsf{Hom}_A(f,a) \simeq_B \mathsf{Hom}_B(B,ua) \qquad \text{and} \qquad \mathsf{Hom}_A(fb,A) \simeq_A \mathsf{Hom}_B(b,u).$$

By Corollary 3.5.9, id_{ua} induces a terminal element of $\mathsf{Hom}_B(B,ua)$ and by Observation 4.1.2 its image across the equivalence $\mathsf{Hom}_B(B,ua) \xrightarrow{\sim} \mathsf{Hom}_A(f,a)$ is again a terminal element, which represents the transposed 2-cell: the component of the counit ϵ at the element a. The proof that the unit component defines a terminal element of $\mathsf{Hom}_B(b,u)$ is dual. □

A more sophisticated formulation of the universal property of unit and counit components will appear in Proposition 9.3.2 where it will form a key step in the proof that any adjunction extends to a *homotopy coherent adjunction*.

The universal property of unit and counit components captured in Proposition 4.1.4 gives the main idea behind the adjoint functor theorems: a functor $f\colon B \to A$ admits a right adjoint just when for each element $a\colon 1 \to A$, the ∞-category $\mathsf{Hom}_A(f,a)$ admits a terminal element. The image of this terminal element under the domain-projection functor $p_0\colon \mathsf{Hom}_A(f,a) \twoheadrightarrow B$ then defines the element $ua\colon 1 \to B$ and the comma cone defines the component of the counit at a. The universal property of these unit components is then used to extend the mapping on elements to a functor $u\colon A \to B$.

The result just stated is true in the ∞-cosmos of quasi-categories and in other ∞-cosmoi where universal properties are generated by the terminal ∞-category 1; see Corollary 16.2.7.[2] What is true in all ∞-cosmoi is the version of the result just stated where the quantifier "for each element $a\colon 1 \to A$" is replaced with "for each generalized element $a\colon X \to A$," in which case the meaning of "terminal element" should be enhanced to "terminal element over X"; see the remark after Corollary 3.5.9. Since every generalized element factors through the universal generalized element, namely the identity functor at A, it suffices to prove:

4.1.5. PROPOSITION. *A functor $f\colon B \to A$ admits a right adjoint if and only if $\mathsf{Hom}_A(f,A)$ admits a terminal element over A. Dually, $f\colon B \to A$ admits a left adjoint if and only if $\mathsf{Hom}_A(A,f)$ admits an initial element over A.*

PROOF. By Proposition 4.1.1, $f\colon B \to A$ admits a right adjoint if and only if the comma ∞-category $\mathsf{Hom}_A(f,A)$ is right representable. Theorem 3.5.11 specializes to tell us that this is the case if and only if the codomain-projection functor $p_1\colon \mathsf{Hom}_A(f,A) \twoheadrightarrow A$ admits a right adjoint right inverse, which by Corollary 3.6.11 is equivalent to postulating a terminal element over A. □

The same suite of results from §3.5 specialize to theorems that encode the universal properties of limits and colimits. Before proving these, we first construct the ∞-category of cones over a fixed diagram and also construct alternate models for the ∞-categories of cones over varying J-indexed diagrams, in the case where J is a simplicial set.

---

[2] We delay the discussion of "analytically-proven" theorems about quasi-categories until we demonstrate in Part IV that such results apply also in biequivalent ∞-cosmoi. Various "pointwise-determined" universal properties that hold in ∞-cosmoi whose objects are $(\infty,1)$-categories are established in §16.2.

Exercises.

4.1.i. Exercise. Prove that the transposing equivalence of Proposition 4.1.1, as elaborated upon in Observation 4.1.2, is natural with respect to pre-composing with a 2-cell $\beta\colon b' \Rightarrow b$ or post-composing with a 2-cell $\alpha\colon a \Rightarrow a'$.

4.2. ∞-categories of cones

4.2.1. Definition (the ∞-category of cones). Let $d\colon 1 \to A^J$ be a J-shaped diagram in an ∞-category A. The **∞-category of cones over** d is the comma ∞-category $\mathsf{Hom}_{A^J}(\Delta, d)$ with comma cone displayed below-left, while the **∞-category of cones under** d is the comma ∞-category $\mathsf{Hom}_{A^J}(d, \Delta)$ with comma cone displayed below-right:

$$\begin{tikzcd} & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ 1 \arrow[dr, "d"'] & \overset{\phi}{\Leftarrow} & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Hom}_{A^J}(d, \Delta) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ A \arrow[dr, "\Delta"'] & \overset{\phi}{\Leftarrow} & 1 \arrow[dl, "d"] \\ & A^J & \end{tikzcd}$$

By replacing the "d" leg of the cospans, Definition 4.2.1 can be modified to allow $d\colon D \to A^J$ to be a family of diagrams or to define ∞-categories of cones over any diagram of shape J: an element of $\mathsf{Hom}_{A^J}(\Delta, A^J)$ is a cone with any summit over any J-indexed diagram.

In the case where the indexing shape J is a simplicial set (and not an ∞-category in a cartesian closed ∞-cosmos), there is another model of the ∞-categories of cones over or under a diagram that may be constructed using Joyal's join construction. The reason for the equivalence is that joins of simplicial sets are known to be equivalent to so-called "fat joins" of simplicial sets, and a particular instance of the fat join construction gives the shape of the cones appearing in Definition 4.2.1. We now introduce these notions.

4.2.2. Definition (fat join). The **fat join** of simplicial sets I and J is the simplicial set constructed by the following pushout:

$$\begin{tikzcd} (I \times J) \sqcup (I \times J) \arrow[r, "\pi_I \sqcup \pi_J"] \arrow[d, hook] & I \sqcup J \arrow[d, hook] \\ I \times \mathbb{2} \times J \arrow[r] & I \diamond J \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

from which it follows that

$$(I \diamond J)_n := I_n \sqcup \Big(\coprod_{[n] \twoheadrightarrow [1]} I_n \times J_n \Big) \sqcup J_n.$$

Note there is a natural map $I \diamond J \twoheadrightarrow \mathbb{2}$ induced by the projection $\pi\colon I \times \mathbb{2} \times J \twoheadrightarrow \mathbb{2}$ so that I is the fiber over 0 and J is the fiber over 1:

$$\begin{tikzcd} I \sqcup J \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & I \diamond J \arrow[d, two heads] \\ \mathbb{1} + \mathbb{1} \arrow[r, hook, "(0{,}1)"] & \mathbb{2} \end{tikzcd}$$

4.2.3. LEMMA. *For any simplicial set J and ∞-category A we have natural isomorphisms*

$$\mathsf{Hom}_{A^J}(\Delta, A^J) \cong A^{\mathbb{1}\star J} \qquad \textit{and} \qquad \mathsf{Hom}_{A^J}(A^J, \Delta) \cong A^{J\star\mathbb{1}}.$$

PROOF. The simplicial cotensor $A^{(-)}\colon s\mathcal{S}et^{\mathrm{op}} \to \mathcal{K}$ carries the pushout of Definition 4.2.2 to the pullback squares that define the left and right representations of $\Delta\colon A \to A^J$ as a comma ∞-category:

$$\begin{tikzcd} A^{\mathbb{1}\star J} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (A^J)^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\ A^J \times A \arrow[r, "\mathrm{id}\times\Delta"'] & A^J \times A^J \end{tikzcd} \qquad \begin{tikzcd} A^{J\star\mathbb{1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & (A^J)^{\mathbb{2}} \arrow[d, two heads, "(p_1,p_0)"] \\ A \times A^J \arrow[r, "\Delta\times\mathrm{id}"'] & A^J \times A^J \end{tikzcd}$$

□

4.2.4. DEFINITION (join). The **join** of simplicial sets I and J is the simplicial set $I \star J$

$$\begin{tikzcd} I \sqcup J \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & I \star J \arrow[d] \\ \mathbb{1}+\mathbb{1} \arrow[r, hook, "{(0,1)}"] & \mathbb{2} \end{tikzcd}$$

with

$$(I \star J)_n := I_n \sqcup \Big(\coprod_{0\le k<n} I_{n-k-1} \times J_k\Big) \sqcup J_n$$

and with the vertices of these n-simplices oriented so that there is a canonical map $I \star J \to \mathbb{2}$ so that I is the fiber over 0 and J is the fiber over 1. See Definitions D.2.2 and D.2.3 or the original [**48**, §3] for more details.

The join functor $-\star J\colon s\mathcal{S}et \to s\mathcal{S}et$ preserves connected colimits but not the initial object or other coproducts, but this issue can be rectified by replacing the codomain by the slice category under J; see Lemma D.2.7 for a precise statement in proof. Contextualized in this way, the join admits a right adjoint, defined by Joyal's **slice** construction:

4.2.5. PROPOSITION. *The join functors admit right adjoints*

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "I\star-"] \arrow[r, phantom, "\top"] & {}^{I/}s\mathcal{S}et \arrow[l, bend left, "-/\_"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star J"] \arrow[r, phantom, "\top"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "\_/-"] \end{tikzcd}$$

defined by the natural bijections

$$\left\{ \begin{tikzcd}[column sep=small] & I \arrow[dl] \arrow[dr, "h"] & \\ I\star\Delta[n] \arrow[rr] & & X \end{tikzcd} \right\} \cong \left\{ \Delta[n] \to {}^{h/}X \right\} \quad \textit{and} \quad \left\{ \begin{tikzcd}[column sep=small] & J \arrow[dl] \arrow[dr, "k"] & \\ \Delta[n]\star J \arrow[rr] & & X \end{tikzcd} \right\} \cong \left\{ \Delta[n] \to X_{/k} \right\}.$$

PROOF. As in the statement, the simplicial set $X_{/k}$ is defined to have n-simplices corresponding to maps $\Delta[n] \star J \to X$ under J, with the right action by the simplicial operators $[m] \to [n]$ given by pre-composition with $\Delta[m] \to \Delta[n]$. Since the join functor $-\star J\colon s\mathcal{S}et \to {}^{J/}s\mathcal{S}et$ preserves colimits, this extends to a bijection between maps $I \to X_{/k}$ and maps $I \star J \to X$ under J that is natural in I and in $k\colon J \to X$. □

4.2.6. Notation. For any simplicial set J, we write

$$J^{\triangleleft} := \mathbb{1} \star J \qquad \text{and} \qquad J^{\triangleright} := J \star \mathbb{1}$$

and write $\perp$ for the **cone vertex** of $J^{\triangleleft}$ and $\top$ for the **cone vertex** of $J^{\triangleright}$. These simplicial sets are equipped with canonical inclusions

$$J^{\triangleleft} \hookleftarrow J \hookrightarrow J^{\triangleright}$$

4.2.7. Proposition (join vs fat join). *For any simplicial sets I and J and any ∞-category A, there is a natural equivalence*

$$\begin{tikzcd} A^{I\star J} \arrow[rr, "\sim"] \arrow[dr, two heads, "\text{res}"'] & & A^{I\diamond J} \arrow[dl, two heads, "\text{res}"] \\ & A^{I\sqcup J} & \end{tikzcd}$$

Proof. There is a canonical map of simplicial sets

$$\begin{tikzcd} (I\times J)\sqcup(I\times J) \arrow[r, two heads, "\pi_I\sqcup\pi_J"] \arrow[d, hook] & I\sqcup J \arrow[d, hook] \arrow[dr] & \\ I\times\mathbb{2}\times J \arrow[r, two heads] \arrow[dr, two heads] \arrow[ur, phantom, "\ulcorner"] & I\diamond J \arrow[r, dashed] \arrow[d, two heads] & I\star J \arrow[dl, two heads] \\ & \mathbb{2} & \end{tikzcd}$$

that commutes with the inclusions of the fibers $I \sqcup J$ over the endpoints of $\mathbb{2}$. This dashed map displayed above is defined on those n-simplices over $I \diamond J$ that map surjectively onto $\mathbb{2}$ to send a triple $(\alpha\colon [n] \twoheadrightarrow [1], \sigma \in I_n, \tau \in J_n)$ representing an n-simplex of $I \diamond J$ to the pair $(\sigma|_{\{0,\ldots,k\}} \in I_k, \tau|_{\{k+1,\ldots,n\}} \in J_{n-k-1})$ representing an n-simplex of $I \star J$, where $k \in [n]$ is the maximal vertex in $\alpha^{-1}(0)$. Proposition D.6.4 of Appendix D proves that this map induces a natural equivalence $Q^{I\star J} \xrightarrow{\sim\!\!\!\!\to} Q^{I\diamond J}$ of quasi-categories over $Q^J \times Q^I$. Taking Q to be the functor space $\mathsf{Fun}(X, A)$ proves the claimed equivalence for general ∞-categories. $\square$

By Lemma 4.2.3 $\mathsf{Hom}_{A^J}(\Delta, A^J) \cong A^{\mathbb{1}\diamond J}$ and $\mathsf{Hom}_{A^J}(A^J, \Delta) \cong A^{J\diamond\mathbb{1}}$. Thus Proposition 4.2.7 specializes to give an alternate model for the ∞-categories of cones over or under a diagram.

4.2.8. Corollary. *In particular, there are comma squares*

$$\begin{tikzcd}[column sep=small] & A^{J^{\triangleleft}} \arrow[dl, two heads, "\text{res}"'] \arrow[dr, two heads, "\text{ev}_\perp"] & \\ A^J \arrow[dr, equal] & \overset{\phi}{\Leftarrow} & A \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & A^{J^{\triangleright}} \arrow[dl, two heads, "\text{ev}_\top"'] \arrow[dr, two heads, "\text{res}"] & \\ A \arrow[dr, "\Delta"'] & \overset{\phi}{\Leftarrow} & A^J \arrow[dl, equal] \\ & A^J & \end{tikzcd} \tag{4.2.9}$$

which pullback over a diagram $d\colon 1 \to A^J$ to define alternate models for the ∞-categories of cones over or under d.

Proof. By Lemma 4.2.3, $A^{\mathbb{1} \diamond J}$ and $A^{J \diamond \mathbb{1}}$ are comma ∞-categories. Thus, Proposition 3.4.11 implies that the fibered equivalence of Proposition 4.2.7 equips $A^{J^{\triangleleft}}$ and $A^{J^{\triangleright}}$ with comma cone squares. The 2-cells in (4.2.9) are represented by the maps

$$
\begin{tikzcd}
J \sqcup J \arrow[r, two heads] \arrow[d, hook] & \mathbb{1} \sqcup J \arrow[d, hook] \arrow[dr, hook] \\
J \times \mathbb{2} \arrow[r, two heads] \arrow[dr] & \mathbb{1} \diamond J \arrow[r, dashed] \arrow[d, two heads] \arrow[ul, phantom, "\ulcorner" very near start] & J^{\triangleleft} \arrow[dl] \\
& \mathbb{2} &
\end{tikzcd}
\qquad
\begin{tikzcd}
J \sqcup J \arrow[r, two heads] \arrow[d, hook] & J \sqcup \mathbb{1} \arrow[d, hook] \arrow[dr, hook] \\
J \times \mathbb{2} \arrow[r, two heads] \arrow[dr] & J \diamond \mathbb{1} \arrow[r, dashed] \arrow[d, two heads] \arrow[ul, phantom, "\ulcorner" very near start] & J^{\triangleright} \arrow[dl] \\
& \mathbb{2} &
\end{tikzcd}
$$

which yield 2-cells

$$
\begin{tikzcd}
A^{J^{\triangleleft}} \arrow[r] \arrow[rr, bend left=50, "\Delta\,\mathrm{ev}_{\perp}"] \arrow[rr, bend right=50, "\mathrm{res}"'] & (A^J)^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A^J
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{J^{\triangleright}} \arrow[r] \arrow[rr, bend left=50, "\mathrm{res}"] \arrow[rr, bend right=50, "\Delta\,\mathrm{ev}_{\top}"'] & (A^J)^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & A^J
\end{tikzcd}
$$

upon cotensoring into A. □

4.2.10. Warning. By Lemma 4.2.3, the fibers of $A^{\mathbb{1} \diamond J} \twoheadrightarrow A^J$ and $A^{J \diamond \mathbb{1}} \twoheadrightarrow A^J$ over a diagram $d \colon 1 \to A^J$ are isomorphic to the ∞-categories of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ and $\mathsf{Hom}_{A^J}(d, \Delta)$. In the ∞-cosmos of quasi-categories, it is tempting to believe that the fibers of the equivalent isofibrations $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ and $A^{J^{\triangleright}} \twoheadrightarrow A^J$ over d recover Joyal's slice quasi-categories $A_{/d}$ and ${}^{d/}A$, but from a direct comparison of the defining universal properties left as Exercise 4.2.ii, this can easily be seen not to be the case. However, the fiber of $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ over d is equivalent to $A_{/d}$ and the fiber of $A^{J^{\triangleright}} \twoheadrightarrow A^J$ is equivalent to ${}^{d/}A$. Proposition D.6.5 will prove that in the ∞-cosmos of quasi-categories that $A_{/d} \simeq \mathsf{Hom}_{A^J}(\Delta, d)$ and ${}^{d/}A \simeq \mathsf{Hom}_{A^J}(d, \Delta)$ over A.

Exercises.

4.2.i. Exercise. Compute $\Delta[n] \star \Delta[m]$ and $\Delta[n] \diamond \Delta[m]$ and define a section

$$\Delta[n] \star \Delta[m] \to \Delta[n] \diamond \Delta[m]$$

to the map constructed in the proof of Proposition 4.2.7.

4.2.ii. Exercise. Compute the fiber of $A^{J^{\triangleleft}} \twoheadrightarrow A^J$ over $d \colon 1 \to A^J$ in the ∞-cosmos of quasi-categories and prove that this quasi-category is not isomorphic to $A_{/d}$.

4.3. The universal property of limits and colimits

We now return to the general context of Definition 2.3.1, simultaneously considering diagrams valued in an ∞-category A that are indexed either by a simplicial set or by another ∞-category in the case where the ambient ∞-cosmos is cartesian closed. As was the case for Proposition 4.1.1, Theorem 3.5.7 specializes immediately to prove:

4.3.1. Proposition (co/limits represent cones). *A family of diagrams $d\colon D \to A^J$ admits a limit if and only if the ∞-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ over d is right representable*

$$\mathsf{Hom}_{A^J}(\Delta, d) \simeq_{D\times A} \mathsf{Hom}_A(A, \ell),$$

in which case the representing functor $\ell\colon D \to A$ defines the limit functor. Dually, $d\colon D \to A^J$ admits a colimit if and only if the ∞-category of cones $\mathsf{Hom}_{A^J}(d, \Delta)$ under d is left representable

$$\mathsf{Hom}_{A^J}(d, \Delta) \simeq_{A\times D} \mathsf{Hom}_A(c, A),$$

in which case the representing functor $c\colon D \to A$ defines the colimit functor. $\square$

Theorem 3.5.11 now specializes to tell us that such representations can be encoded by terminal or initial elements, a result which is easiest to interpret in the case of a single diagram rather than a family of diagrams.

4.3.2. Proposition (limits as terminal elements). *Consider a diagram $d\colon 1 \to A^J$ of shape J in an ∞-category A.*

(i) If d admits a limit, then the 1-cell $1 \to \mathsf{Hom}_{A^J}(\Delta, d)$ induced by the limit cone $\epsilon\colon \Delta\ell \Rightarrow d$ defines a terminal element of the ∞-category of cones.

(ii) Conversely, if the ∞-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ admits a terminal element, then the cone represented by this element defines a limit cone.

Dually d admits a colimit if and only if the ∞-category $\mathsf{Hom}_{A^J}(d, \Delta)$ of cones under d admits an initial element, in which case the initial element defines the colimit cone.

Proof. By Definition 2.3.7, a limit cone defines an absolute right lifting diagram, which by Theorem 3.5.3, induces an equivalence $\mathsf{Hom}_A(A, \ell) \xrightarrow{\sim} \mathsf{Hom}_{A^J}(\Delta, d)$ over A. By Corollary 3.5.9, the identity at ℓ induces a terminal element of $\mathsf{Hom}_A(A, \ell)$ which the equivalence carries to a terminal element of the ∞-category of cones, this being the element that represents the limit cone $\epsilon\colon \Delta\ell \Rightarrow d$.

Conversely, if $\mathsf{Hom}_{A^J}(\Delta, d)$ admits a terminal element, this defines a right adjoint right inverse to the codomain-projection functor $\mathsf{Hom}_{A^J}(\Delta, d)$. Theorem 3.5.11 then tells us that the cone represented by this element $1 \to \mathsf{Hom}_{A^J}(\Delta, d)$ defines an absolute right lifting of d through Δ. $\square$

4.3.3. Remark. The proof of Proposition 4.3.2 extends without change to the case of a family of diagrams $d\colon D \to A^J$ in place of a single diagram since Theorem 3.5.11 applies at this level of generality. For a family of diagrams d parametrized by D, the ∞-category of cones defines an object $p_1\colon \mathsf{Hom}_{A^J}(\Delta, d) \twoheadrightarrow D$ of the sliced ∞-cosmos $\mathcal{K}_{/D}$ and the terminal elements referred to in both (i) and (ii) should be interpreted as terminal elements in $\mathcal{K}_{/D}$.

4.3.4. Proposition. *An ∞-category A admits a limit of a family of diagrams $d\colon D \to A^J$ indexed by a simplicial set[3] J if and only if there exists an absolute right lifting of d through the restriction functor*

$$\begin{tikzcd} & A^{J^\triangleleft} \arrow[d, two heads, "\mathrm{res}"] \\ D \arrow[ur, "\mathrm{ran}"] \arrow[r, "d"'] \arrow[r, phantom, bend left=20, "\Downarrow\epsilon"] & A^J \end{tikzcd}$$

[3]We have stated this result for diagrams indexed by simplicial sets because its means is easiest to interpret, but we actually prove it with the codomain-projection functor $p_1\colon \mathsf{Hom}_{A^J}(\Delta, A^J) \twoheadrightarrow A^J$ in place of the equivalent isofibration $A^{J^\triangleleft} \twoheadrightarrow A^J$, and this proof applies equally in the case of diagrams indexed by ∞-categories J in cartesian closed ∞-cosmoi that may or may not have a join operation available.

When these equivalent conditions hold, ϵ is necessarily an isomorphism and may be chosen to be the identity.

PROOF. By Definition 2.3.7, the family of diagrams admits a limit if and only if d admits an absolute right lifting through $\Delta\colon A \to A^J$. By Proposition 3.6.14, such an absolute lifting diagram exists if and only if d admits an absolute right lifting through codomain-projection functor $p_1\colon \mathrm{Hom}_{A^J}(\Delta, A^J) \twoheadrightarrow A^J$, in which case the 2-cell of this latter absolute right lifting diagram is invertible. By Corollary 4.2.8, the restriction functor $\mathrm{res}\colon A^{J^{\triangleleft}} \twoheadrightarrow A^J$ is equivalent to this codomain-projection functor, so absolute right liftings of d through p_1 are equivalent to absolute right liftings of d through res. If this absolute lifting diagram is inhabited by an invertible 2-cell, the isomorphism lifting property of the isofibration proven in Proposition 1.4.10 can be used to replace the functor $\mathrm{ran}\colon D \to A^{J^{\triangleleft}}$ with an isomorphic functor making the triangle commute strictly. $\square$

Recall from Lemma 2.3.6 that absolute lifting diagrams can be used to encode the existence of adjoint functors. Combining this with Definition 2.3.2, Proposition 4.3.4 specializes to prove:

4.3.5. COROLLARY. *An ∞-category A admits all limits indexed by a simplicial set J if and only if the restriction functor*

$$\begin{tikzcd} A^{J^{\triangleleft}} \arrow[r, two heads, bend left, "\mathrm{res}"] \arrow[r, phantom, "\top"] & A^J \arrow[l, dashed, bend left, "\mathrm{ran}"] \end{tikzcd}$$

admits a right adjoint. Dually, an ∞-category A admits all colimits indexed by a simplicial set J if and only if the restriction functor

$$\begin{tikzcd} A^{J^{\triangleright}} \arrow[r, two heads, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\top"] & A^J \arrow[l, dashed, bend right, "\mathrm{lan}"'] \end{tikzcd}$$

admits a left adjoint. $\square$

4.3.6. REMARK. Since the restriction functor is an isofibration, Lemma B.4.7 applies and the adjunctions of Corollary 4.3.5 can be defined so as to be fibered over the ∞-category of diagrams A^J.

The adjunctions of Corollary 4.3.5 are particular useful in the case of pullbacks and pushouts.

4.3.7. DEFINITION (pushouts and pullbacks). A **pushout** in an ∞-category A is a colimit indexed by the simplicial set

$$\ulcorner := \Lambda^0[2].$$

Dually, a **pullback** in an ∞-category A is a limit indexed by the simplicial set

$$\lrcorner := \Lambda^2[2].$$

Cones over diagrams of shape $\lrcorner$ or cones under diagrams of shape $\ulcorner$ define **commutative squares**, diagrams of shape

$$\boxdot := \Delta[1] \times \Delta[1] \cong \ulcorner^{\triangleright} \cong \lrcorner^{\triangleleft}\ .$$

A **pullback square** in an ∞-category A is an element of $A^{\boxdot}$ in the essential image of the functor ran of Proposition 4.3.4 for some diagram of shape $\lrcorner$. When A admits all pullbacks, these are exactly those elements of $A^{\boxdot}$ at which the component of the unit of the adjunction $\mathrm{res} \dashv \mathrm{ran}$ of Corollary 4.3.5 is an isomorphism. Dually, a **pushout square** in A is an element in the essential image of the dual functor lan for some diagram of shape $\ulcorner$, i.e., those elements for which the component of the counit of the adjunction $\mathrm{lan} \dashv \mathrm{res}$ is an isomorphism.

An X-indexed commutative square in A is a diagram $X \to A^{\boxdot}$, or equivalently, an element of $\mathsf{Fun}(X,A)^{\boxdot}$. We label the 0- and 1-simplex components as follows:

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, "w" description] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & a \end{tikzcd} \qquad \in \qquad \mathsf{Fun}(X,A)$$

The diagram also determines a pair of 2-simplices that witness commutativity $fu = w = gv$ in $\mathsf{hFun}(X,A)$, but the names of these witnesses won't matter for this discussion.

4.3.8. Lemma. *An X-indexed commutative square valued in an ∞-category A in $\mathcal{K}$ as below-left is a pullback square if and only if the induced 2-cell below-right is an absolute right lifting diagram in $\mathcal{K}_{/X}$:*

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, "w" description] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & a \end{tikzcd} \qquad \begin{tikzcd} & & \mathsf{Hom}_A(A,b) \arrow[d, "\ulcorner f_* \urcorner"] \arrow[dl, dotted, two heads, bend left, "p_1"] \\ X \arrow[urr, bend left, "\ulcorner u \urcorner"] \arrow[rr, "\ulcorner g \urcorner"'] \arrow[dr, dotted, equal] \arrow[urr, phantom, "\Downarrow(\mathrm{id}_a, v)"] & & \mathsf{Hom}_A(A,a) \arrow[dl, dotted, two heads, "p_1"] \\ & X & \end{tikzcd}$$

The statement requires some explanation. The 1-simplex $g\colon c \to a$ represents a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "c"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow g"] & A \end{tikzcd}$, inducing the map $\ulcorner g \urcorner \colon 1 \to \mathsf{Hom}_A(A,a)$. The map $\ulcorner u \urcorner \colon X \to \mathsf{Hom}_A(A,b)$ is defined similarly. The map $\ulcorner f_* \urcorner \colon \mathsf{Hom}_A(A,b) \to \mathsf{Hom}_A(A,a)$ is characterized by the pasting diagram

$$\begin{tikzcd} & \mathsf{Hom}_A(A,b) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ X \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] \arrow[rr, phantom, "\Downarrow f"] & {} & A \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathsf{Hom}_A(A,b) \arrow[d, dashed, "\ulcorner f_* \urcorner"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\ & \mathsf{Hom}_A(A,a) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ X \arrow[rr, "a"'] & {} & A \end{tikzcd}$$

By Proposition 3.4.7, the composite $\ulcorner f_* \urcorner \ulcorner u \urcorner$ is isomorphic to $\ulcorner fu \urcorner$. By 2-cell induction, the 2-cell may be constructed by specifying its domain and codomain components, the former of which we take to be $\begin{tikzcd} X \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow v"] & A \end{tikzcd}$ and the latter of which we take to be id_a. Note that the 2-cell just constructed lies in $\mathfrak{h}\mathcal{K}_{/X}$ and so can be lifted to $\mathfrak{h}(\mathcal{K}_{/X})$ by Proposition 3.6.3.

Proof. We prove the result in the case $X = 1$ and then deduce the result for families of pullback diagrams from this case. By Corollary 4.2.8, the pullback

$$\begin{tikzcd} A^{\boxdot}_{g \vee f} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[r, "g \vee f"'] & A^{\lrcorner} \end{tikzcd}$$

is equivalent to the ∞-category of cones over the cospan diagram $g \vee f$. By Proposition 4.3.2, to show that the commutative square defines a pullback diagram is to show that $(u, v, f, g) \colon 1 \to A^{\boxdot}_{g \vee f}$ defines a terminal element in the pullback. We will show that this pullback $A^{\boxdot}_{g\vee f}$ is also equivalent to the comma ∞-category $\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \ulcorner g \urcorner)$. By Theorem 3.5.11, the pair $(\ulcorner u \urcorner, (\mathrm{id}_a, v))$ defines an absolute right lifting if and only if it represents a terminal element in this comma ∞-category, which will prove the claimed equivalence.

To see this first consider the diagram, which induces a map between the two pullbacks

$$\begin{tikzcd}
 & A_{/f} \arrow[rr] \arrow[dl, dashed, two heads, "p_f"'] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner"] & & A^{3} \arrow[r, two heads, "p_{01}"] \arrow[dl, two heads, "p_{02}"'] \arrow[dd, two heads, "p_{12}"] & A^{2} \arrow[dd, two heads, "p_1"] \\
\mathsf{Hom}_A(A,a) \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner"] & & A^{2} \arrow[dd, two heads, "p_1" near start] & & \\
 & X \arrow[r, "f"] \arrow[dl, equal] & A^{2} \arrow[r, "p_1"] \arrow[d, phantom] & A^{2} \arrow[r, two heads, "p_0"] \arrow[dl, two heads, "p_1"] & A \\
X \arrow[rr, "a"'] & & A & &
\end{tikzcd}$$

Since $A^{3} \simeq A^{\Lambda^1[2]}$, the right-hand back square is equivalent to a pullback. Composing the pullback squares in the back face of the diagram, we obtain an equivalence $A_{/f} \xrightarrow{\sim\!\!\!\!\rightarrow} \mathsf{Hom}_A(A, b)$ and by inspection see that the map $p_f \colon A_{/f} \twoheadrightarrow \mathsf{Hom}_A(A, a)$ is equivalent to the map $\ulcorner f_* \urcorner \colon \mathsf{Hom}_A(A, b) \to \mathsf{Hom}_A(A, a)$ over $\mathsf{Hom}_A(A, a)$.

By applying $(-)^{2}$ to the pullback diagram that defines $\mathsf{Hom}_A(A, a)$ we obtain a pullback square that factors as:

$$\begin{tikzcd}
\mathsf{Hom}_A(A,a)^{2} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{2 \diamond \mathbb{1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{2 \times 2} \arrow[d, two heads, "p_1^{2}"] \\
1 \arrow[r, "a"] & A \arrow[r, "\Delta"] & A^{2}
\end{tikzcd}$$

By the equivalence $A^{2 \diamond \mathbb{1}} \simeq A^{2 \star \mathbb{1}}$ of Corollary 4.2.8, the left-hand pullback square shows that $\mathsf{Hom}_A(A, a)^{2}$ is equivalent to the pullback of $p_2 \colon A^{3} \twoheadrightarrow A$ along $a \colon 1 \to A$. Modulo this equivalence, the map $p_0 \colon \mathsf{Hom}_A(A, a)^{2} \twoheadrightarrow \mathsf{Hom}_A(A, a)$ is the pullback of the fibered map

$$\begin{tikzcd}
A^{3} \arrow[rr, two heads, "p_{02}"] \arrow[dr, two heads, "p_2"'] & & A^{2} \arrow[dl, two heads, "p_1"] \\
 & A &
\end{tikzcd}$$

along $a \colon 1 \to A$ and the codomain projection is similarly the pullback of the fibered map $p_{12} \colon A^{3} \twoheadrightarrow A^{2}$.

Putting this together, it follows that the pullback

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \mathsf{Hom}_A(A,a)) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_A(A,a)^{2} \arrow[d, two heads, "p_0"] \\
\mathsf{Hom}_A(A,b) \arrow[r, "\ulcorner f_* \urcorner"'] & \mathsf{Hom}_A(A,a)
\end{tikzcd}$$

is equivalently computed by forming the limit

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads] & A^{\boxdot} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{3} \arrow[r, two heads, "p_{12}"] \arrow[d, two heads, "p_{02}"] & A^{2} \\
& A^{3} \arrow[r, two heads, "p_{02}"'] \arrow[d, two heads, "p_{12}"] & A^{2} \\
1 \arrow[r, "f"'] & A^{2}
\end{tikzcd}$$

The codomain projection $p_1 \colon \mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \mathsf{Hom}_A(A,a)) \twoheadrightarrow \mathsf{Hom}_A(A,a)$ is the pullback of the top-horizontal composite in the above diagram along $\mathsf{Hom}_A(A,a) \to A^{2}$. So we see that the comma ∞-category $\mathsf{Hom}_{\mathsf{Hom}_A(A,a)}(\ulcorner f_* \urcorner, \ulcorner g \urcorner)$ is equivalently computed by the limit below-left, or equivalently by the limit below right, exactly as we claimed:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[rr, two heads] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads] & A^{\boxdot} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{3} \arrow[r, two heads, "p_{12}"] \arrow[d, two heads, "p_{02}"] & A^{2} \\
& A^{3} \arrow[r, two heads, "p_{02}"'] \arrow[d, two heads, "p_{12}"] & A^{2} \\
1 \arrow[r, "f"'] & A^{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\boxdot}_{g \vee f} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\
1 \arrow[r, "g \vee f"'] & A^{\lrcorner}
\end{tikzcd}$$

The same computation proves the general case for $X \neq 1$ when the comma ∞-category is constructed in $\mathcal{K}_{/X}$; see Proposition 1.2.19 for a description of the simplicial limits in sliced ∞-cosmoi. Alternatively, a diagram $s \colon X \to A^{\boxdot}$ in $\mathcal{K}$ also defines a X-indexed diagram in the ∞-cosmos $\mathcal{K}_{/X}$ valued in the ∞-category $\pi \colon A \times X \to X$. This takes the form of a functor $(d, \mathrm{id}_X) \colon X \to A^{\boxdot} \times X$ over X. It's easy to verify that a diagram valued in $\pi \colon A \times X \twoheadrightarrow X$ whose component at X is the identity has a limit in $\mathcal{K}_{/X}$ if and only if the A-component of the diagram has a limit in $\mathcal{K}$. Since id_X is the terminal object of $\mathcal{K}_{/X}$, this object is the ∞-category $1 \in \mathcal{K}_{/X}$, so the proof just give applies to prove the general case of X-indexed families of commutative squares. □

There is an automorphism of the simplicial set $\mathbb{2} \times \mathbb{2}$ that swaps the "intermediate" vertices $(0,1)$ and $(1,0)$, which induces a "transposition" automorphism of $A^{\boxdot}$. By symmetry, a commutative square in A is a pullback if and only if its transposed square is a pullback. This gives a dual form of Lemma 4.3.8 with the roles of f and g and of u and v interchanged. As a corollary, we can easily prove that pullback squares compose both "vertically" and "horizontally" and can be cancelled from the "right" and "bottom":

4.3.9. Proposition (composition and cancelation of pullback squares). *Given a composable pair of X-indexed commutative squares in A and their composite rectangle defined via the equivalence $A^{3 \times 2} \simeq$*

$A^{\boxdot} \underset{A^2}{\times} A^{\boxdot}$

$$\begin{tikzcd} p \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, "z"] & d \arrow[r, "u"] \arrow[d, "v"] \arrow[dr, "w"] & b \arrow[d, "f"] \\ e \arrow[r, "h"'] & c \arrow[r, "g"'] & a \end{tikzcd}$$

if the right-hand square is a pullback, then the left-hand square is a pullback if and only if the composite rectangle is a pullback.

Proof. By Lemma 4.3.8, we are given an absolute right lifting diagram in $\mathcal{K}_{/X}$

$$\begin{tikzcd} & \mathrm{Hom}_A(A,c) \arrow[d, "\ulcorner g_* \urcorner"] \\ X \arrow[ur, "\ulcorner v \urcorner"] \arrow[r, "\ulcorner f \urcorner"'] \arrow[r, phantom, shift left=4, "\Downarrow(\mathrm{id}_a, u)"] & \mathrm{Hom}_A(A,a) \end{tikzcd}$$

By Lemma 2.4.1, the composite diagram

$$\begin{tikzcd} & \mathrm{Hom}_A(A,e) \arrow[d, "\ulcorner h_* \urcorner"] \\ & \mathrm{Hom}_A(A,c) \arrow[d, "\ulcorner g_* \urcorner"] \\ X \arrow[uur, bend left, "\ulcorner y \urcorner"] \arrow[ur, "\ulcorner v \urcorner"] \arrow[r, "\ulcorner f \urcorner"'] \arrow[r, phantom, shift left=4, "\Downarrow(\mathrm{id}_a, u)"] & \mathrm{Hom}_A(A,a) \end{tikzcd}$$

(with $\Downarrow(\mathrm{id}_c, x)$ in the top triangle)

is an absolute right lifting diagram in $\mathcal{K}_{/X}$ if and only if the top triangle is an absolute right lifting diagram in $\mathcal{K}_{/X}$. By Lemma 4.3.8, this is exactly what we wanted to show. ☐

Terminal elements are special cases of limits where the diagram shape is empty. For any ∞-category A, the ∞-category of diagrams $A^{\varnothing} \cong 1$, which tells us that there is a unique $\varnothing$-indexed diagram in A. In this context, the ∞-categories of cones over or under the unique diagram constructed in Definition 4.2.1 are isomorphic to A. In the case of cones over an empty diagram, the domain-evaluation functor, carrying a cone to its summit, is the identity on A, while in the case of cones under the empty diagram, the codomain-evaluation functor, carrying a cone to its nadir, is the identity on A. The following characterization of terminal elements can be deduced as a special case of Proposition 4.3.1, though we find it easier to argue from Proposition 4.1.1.

4.3.10. Proposition. *For an element $t\colon 1 \to A$ of an ∞-category A,*

(i) t defines a terminal element of A if and only if the domain-projection functor $p_0\colon \mathrm{Hom}_A(A,t) \twoheadrightarrow A$ is a trivial fibration.

(ii) t defines an initial element of A if and only if the codomain-projection functor $p_1\colon \mathrm{Hom}_A(t,A) \twoheadrightarrow A$ is a trivial fibration.

Proof. Recall from Definition 2.2.1, that an element is terminal if and only if it is right adjoint to the unique functor

$$\begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\top"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

By Proposition 4.1.1, $! \dashv t$ if and only if there is an equivalence $\mathrm{Hom}_1(!, 1) \simeq_A \mathrm{Hom}_A(A, t)$. By the defining pullback (3.4.2) for the comma ∞-category, the left representation of $!\colon A \to 1$ is A itself, with domain-projection functor the identity. So the component of the equivalence $\mathrm{Hom}_A(A, t) \rightsquigarrow A$ over A must be the domain projection functor $p_0\colon \mathrm{Hom}_A(A, t) \twoheadrightarrow A$, and we conclude that t is a terminal element if and only if this isofibration is a trivial fibration. $\square$

4.3.11. Digression (terminal elements of a quasi-category). In the ∞-cosmos of quasi-categories, the isofibration $p_0\colon \mathrm{Hom}_A(A, t) \twoheadrightarrow A$ is equivalent over A to the *slice quasi-category* $A_{/t}$, defined as a right adjoint to the join construction of Definition 4.2.4. Proposition 4.3.10 proves that t is terminal if and only if the projection $A_{/t} \twoheadrightarrow A$ is a trivial fibration in the sense of Definition 1.1.24, which transposes to Joyal's original definition of a terminal element of a quasi-category. See Appendix F for a full proof.

We conclude with two results that could have been proven in Chapter 2, were it not for one small step of the argument, as we explain. A functor $f\colon A \to B$ *preserves* limits if the image of a limit cone in A also defines a limit cone in B. In the other direction, a functor $f\colon A \to B$ *reflects* limits if a cone in A that defines a limit cone in B is also a limit cone in A.

4.3.12. Proposition. *A fully faithful functor $f\colon A \to B$ reflects any limits or colimits that exist in B.*

Proof. The statement for limits asserts that given any family of diagrams $d\colon D \to A^J$ of shape J in A, any functor $\ell\colon D \to A$ and cone $\rho\colon \Delta\ell \Rightarrow d$ as below-left so that the whiskered composite with $f^J\colon A^J \to B^J$ displayed below is an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd}$$

then (ℓ, ρ) defines an absolute right lifting of $d\colon D \to A^J$ through $\Delta\colon A \to A^J$. Our proof strategy mirrors the results of §2.4. By Corollary 3.5.6(i), to say that f is fully faithful is to say that $\mathrm{id}_A\colon A \to A$ defines an absolute right lifting of f through itself. So by Lemma 2.4.1 and the hypothesis just stated, the composite diagram below-left is an absolute right lifting diagram, and by 2-functoriality of the simplicial cotensor with J, the diagram below-left coincides with the diagram below-right:

$$\begin{tikzcd} & & A \arrow[d, "f"] \\ & A \arrow[ur, equal] \arrow[r, "f"'] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd} = \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ & & A^J \arrow[d, "f^J"] \\ D \arrow[uurr, "\ell"] \arrow[r, "d"'] & A^J \arrow[ur, equal] \arrow[u, phantom, "\Downarrow\rho"] \arrow[r, "f^J"'] & B^J \end{tikzcd}$$

By Corollary 3.5.6(iv) to say that f is fully faithful is to say that $\ulcorner \mathrm{id}_f \urcorner\colon A^2 \rightsquigarrow \mathrm{Hom}_B(f, f)$ is a fibered equivalence over $A \times A$. Applying $(-)^J\colon \mathcal{K} \to \mathcal{K}$, this maps to a fibered equivalence $\ulcorner \mathrm{id}_{f^J} \urcorner\colon (A^J)^2 \rightsquigarrow \mathrm{Hom}_{B^J}(f^J, f^J)$ over $A^J \times A^J$, proving that if $f\colon A \to B$ is fully faithful, then $f^J\colon A^J \to B^J$ is also.[4] Hence by Corollary 3.5.6(i), $\mathrm{id}_{A^J}\colon A^J \to A^J$ defines an absolute right lifting of f^J through itself.

[4]This is the statement that we could not yet prove in Chapter 2.

Applying Lemma 2.4.1 again, we now conclude that (ℓ, ρ) is an absolute right lifting of d through Δ as required. $\square$

An alternate approach to proving this result is suggested as Exercise 4.3.iii.

Our final result, proves that, for I and J simplicial sets, whenever we are given a J-indexed diagram valued in the ∞-category A^I of I-indexed diagrams in A, its limit may be computed pointwise in the vertices of I as the limit of the corresponding J-indexed diagram in A. Our argument requires the following representable characterization of absolute lifting diagrams whose proof again makes use of the fact that they are preserved by cosmological functors.

4.3.13. Proposition. *A natural transformation defined in an ∞-cosmos $\mathcal{K}$ as below-left is an absolute right lifting diagram if and only if its "externalization" displayed below-right defines a right lifting diagram in $\mathcal{QC}at$*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & \mathsf{Fun}(C,A) \end{tikzcd}$$

that is preserved by precomposition with any functor $c\colon X \to C$ in $\mathcal{K}$ in the sense that the diagram below is also right lifting:

$$\begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \arrow[r, "c^*"] & \mathsf{Fun}(X,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & \mathsf{Fun}(C,A) \arrow[r, "c^*"'] & \mathsf{Fun}(X,A) \end{tikzcd} = \begin{tikzcd} & \mathsf{Fun}(X,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "rc"] \arrow[r, "gc"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho c"] & \mathsf{Fun}(X,C) \end{tikzcd}$$

Moreover the externalized right lifting diagrams in $\mathcal{QC}at$ are in fact absolute.

Proof. Since $\mathsf{Fun}(\mathbb{1}, -)\colon \mathcal{QC}at \to \mathcal{QC}at$ is naturally isomorphic to the identity functor, for any ∞-categories $X, A \in \mathcal{K}$ we have $\mathsf{Fun}(X, A) \cong \mathsf{Fun}(\mathbb{1}, \mathsf{Fun}(X, A))$ and hence

$$\mathsf{hFun}(X, A) \cong \mathsf{hFun}(\mathbb{1}, \mathsf{Fun}(X, A)). \tag{4.3.14}$$

This justifies our use of the same name ρ for the 2-cell in $\mathfrak{h}\mathcal{K}$ and the 2-cell in $\mathfrak{h}\mathcal{QC}at$ in the statement.

To say that (r, ρ) defines a right lifting of g through f in $\mathfrak{h}\mathcal{K}$ asserts a bijection between 2-cells in $\mathsf{hFun}(C, B)$ with codomain r and 2-cells in $\mathsf{hFun}(C, A)$ with codomain g and domain factoring through f implemented by pasting with ρ. Under the correspondence of (4.3.14), this asserts equally that r defines a right lifting of g through f_* in $\mathfrak{h}\mathcal{QC}at$. To say that (r, ρ) defines an *absolute* right lifting of g through f is to assert the analogous right lifting property for the pair $(rc, \rho c)$ defined by restricting along any $c\colon X \to C$. This is exactly the first claim of the statement.

It remains only to argue that if (r, ρ) is an absolute right lifting diagram in $\mathcal{K}$, then its externalized right lifting diagram of quasi-categories is also absolute. To see this, first note that the cosmological functor $\mathsf{Fun}(C, -)\colon \mathcal{K} \to \mathcal{QC}at$ preserves this absolute right lifting diagram, yielding an absolute right lifting diagram of quasi-categories as below left:

$$\begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathsf{Fun}(C,C) \arrow[ur, "r_*"] \arrow[r, "g_*"'] \arrow[r, phantom, shift left=3, "\Downarrow\mathsf{Fun}(C,\rho)"] & \mathsf{Fun}(C,A) \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & \mathsf{Fun}(C,A) \end{tikzcd} = \begin{tikzcd} & & \mathsf{Fun}(C,B) \arrow[d, "f_*"] \\ \mathbb{1} \arrow[r, "\mathrm{id}_C"] & \mathsf{Fun}(C,C) \arrow[ur, "r_*"] \arrow[r, "g_*"'] \arrow[r, phantom, shift left=3, "\Downarrow\mathsf{Fun}(C,\rho)"] & \mathsf{Fun}(C,A) \end{tikzcd}$$

We obtain the desired absolute right lifting diagram by evaluating at the identity. $\square$

4.3.15. Proposition. *Let I and J be simplicial sets and let A be an ∞-category. Then a diagram as below-left is an absolute right lifting diagram*

$$\begin{tikzcd} & A^I \arrow[d, "\Delta^I"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^{I\times J} \end{tikzcd} \qquad \begin{tikzcd} & A^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & A^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & A^J \end{tikzcd}$$

if for each vertex $i \in I$, the diagram above right is an absolute right lifting diagram.

Note this statement is not a biconditional. Even in the case of strict 1-categories, there may exist coincidental limits of diagram valued in functor categories that are not defined pointwise [**17**, 2.17.10].

Proof. By Proposition 4.3.13, we may externalize and instead show that the diagram of quasi-categories displayed below left

$$\begin{tikzcd} & \mathsf{Fun}(D,A)^I \arrow[d, "\Delta^I"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & \mathsf{Fun}(D,A)^{I\times J} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{Fun}(D,A)^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & \mathsf{Fun}(D,A) \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & \mathsf{Fun}(D,A)^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & \mathsf{Fun}(D,A)^J \end{tikzcd}$$

is absolute right lifting lifting if the diagrams above-right are for each vertex $i \in I$. By naturality of our methods, our proof will show that the absolute right lifting diagrams on the left are preserved by postcomposition with the restriction functors induced by $d\colon X \to D$ if the same is true on the right.

We simplify our notation and write Q for the quasi-category $\mathsf{Fun}(D,A)$ and assume that for each $i \in I$ the diagram

$$\begin{tikzcd} & Q^I \arrow[d, "\Delta^I"] \arrow[r, "\mathrm{ev}_i"] & Q \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^{I\times J} \arrow[r, "\mathrm{ev}_i"'] & Q^J \end{tikzcd}$$

is absolute right lifting. By 2-adjunction $(-\times I) \dashv (-)^I$, a diagram as below-left is absolute right lifting if and only if the transposed diagram below-right is a right lifting diagram and this remains the case upon restricting along functors of the form $\pi\colon X \times I \to I$.[5]

$$\begin{tikzcd} & Q^I \arrow[d, "\Delta^I"] \\ \mathbb{1} \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^{I\times J} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & Q \arrow[d, "\Delta"] \\ I \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift right=3] & Q^J \end{tikzcd} \tag{4.3.16}$$

We'll argue that this right-hand diagram is in fact absolute right lifting, which implies that the left-hand diagram is absolute right lifting as well, as desired.

[5]If the reader is concerned that I is not a quasi-category, there are two ways to proceed. One is to replace I by a quasi-category I' by inductively attaching fillers for inner horns; note that I and I' will have the same sets of vertices. By Proposition 1.1.28, the diagram quasi-categories $Q^{I'}$ and Q^I are equivalent. The other option is to observe that it doesn't matter if I is a quasi-category or not, because we may define $\mathsf{hFun}(I,Q) := \mathsf{h}(Q^I)$ and by Corollary 1.1.21 Q^I is a quasi-category regardless of whether I is.

To see this we appeal to Theorem 16.2.9, which tells us that universal properties in $\mathcal{QC}at$ are detected pointwise. Specifically, this tells us that the triangle above right is an absolute right lifting diagram if and only if the restricted diagram is absolute right lifting for each vertex $i \in I$

$$\begin{tikzcd} & & Q \arrow[d, "\Delta"] \\ \mathbb{1} \arrow[r, "i"] & I \arrow[ur, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho", shift left=3] & Q^J \end{tikzcd}$$

and this is exactly what we have assumed in (4.3.16). ☐

Exercises.

4.3.i. Exercise. Prove that if A has a terminal element t then for any element a the mapping space $\mathsf{Hom}_A(a, t)$ is **contractible**, i.e., is equivalent to the terminal ∞-category $\mathbb{1}$.

4.3.ii. Exercise. Prove that a square in A is a pullback if and only if its "transposed" square, defined by composing with the involution $A^{\square} \cong A^{\square}$ induced from the automorphism of $\mathbb{2} \times \mathbb{2}$ that swaps the "off-diagonal" elements, is a pulllback square.

4.3.iii. Exercise ([**98**, 3.7]). Use Theorem 3.5.3 and Corollary 3.5.6(iv) to prove that a fully faithful functor $f\colon A \to B$ reflects all limits or colimits that exist in A. Why does this argument not also show that $f\colon A \to B$ preserves them?

4.4. Loops and suspension in pointed ∞-categories

4.4.1. Definition (pointed ∞-categories). An ∞-category A is **pointed** if it admits a **zero element**: an element $*\colon \mathbb{1} \to A$ that is both initial and terminal.

The counit of the adjunction $* \dashv !$ that witnesses the initiality of the zero element defines a natural transformation $\rho\colon *! \Rightarrow \mathrm{id}_A$ that we refer to as the **family of points** of A. Dually, the unit of the adjunction $! \dashv *$ that witnesses the terminality of the zero element defines a natural transformation $\xi\colon \mathrm{id}_A \Rightarrow *!$ that we refer to as the **family of copoints**.

Cospans and spans in an ∞-category A may be defined by gluing together a pair of arrows along their codomains or domains respectively:

$$\begin{tikzcd} A^{\lrcorner} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ A^{\mathbb{2}} \arrow[r, two heads, "p_1"'] & A \end{tikzcd} \qquad \begin{tikzcd} A^{\ulcorner} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_0"] \\ A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd}$$

For instance, the family of points in a pointed ∞-category A is represented by a functor $\rho\colon A \to A^{\mathbb{2}}$ whose domain-component is constant at $*$ and whose codomain component is id_A. Gluing two copies of this map along their codomain defines a diagram $\check{\rho}\colon A \to A^{\lrcorner}$. Dually, there is a diagram $\hat{\xi}\colon A \to A^{\ulcorner}$ defined by gluing the functor $\xi\colon A \to A^{\mathbb{2}}$ that represents the family of copoints to itself along their domains.

4.4.2. DEFINITION (loops and suspension). A pointed ∞-category A **admits loops** if it admits a limit of the family of diagrams

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[ur, dashed, "\Omega"] \arrow[r, "\check{\rho}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow"] & A^{\lrcorner} \end{tikzcd}$$

in which case the limit functor $\Omega\colon A \to A$ is called the **loops functor**. Dually, a pointed ∞-category A **admits suspensions** if it admits a colimit of the family of diagrams

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[ur, dashed, "\Sigma"] \arrow[r, "\hat{\xi}"'] \arrow[r, phantom, shift left=3ex, "\Uparrow"] & A^{\ulcorner} \end{tikzcd}$$

in which case the colimit functor $\Sigma\colon A \to A$ is called the **suspension functor**.

Importantly, if A admits loops and suspensions, then the loops and suspension functors are adjoint:

4.4.3. PROPOSITION (the loops-suspension adjunction). *If A is a pointed ∞-category that admits loops and suspensions, then the loops functor is right adjoint to the suspension functor*

$$\begin{tikzcd} A \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\top"] & A \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$$

The main idea of the proof is easy to describe. If A admits all pullbacks and all pushouts, then Corollary 4.3.5 supplies adjunctions

$$\begin{tikzcd} A^{\lrcorner} \arrow[r, bend right, "\mathrm{ran}"'] \arrow[r, phantom, "\top"] & A^{\square} \arrow[l, bend right, "\mathrm{res}"'] \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\top"] & A^{\ulcorner} \arrow[l, bend right, "\mathrm{lan}"'] \end{tikzcd}$$

that are fibered over $A \times A$ upon evaluating at the intermediate vertices of the commutative square. By pulling back along $(*,*)\colon 1 \to A \times A$, we can pin these vertices at the zero element. Since the zero element is initial and terminal, the ∞-categories of pullback and pushout diagrams of this form are both equivalent to A and the pulled-back adjoints now coincide with the loops and suspension functors.

The only subtlety in the proof that follows is that we have assumed weaker hypotheses: that A admits only loops and suspensions, but perhaps not all pullbacks and pushouts.

PROOF. The diagram $\check{\rho}$ lands in a subobject $A^{\lrcorner}_*$ of $A^{\lrcorner}$ defined below-left that is comprised of those pullback diagrams whose source elements are pinned at the zero element $*$ of A.

$$\begin{tikzcd} A \arrow[dr, dashed, "\check{\rho}_*"'] \arrow[drr, bend left, "\rho\vee\rho"] \arrow[ddr, bend right] & & \\ & A^{\lrcorner}_* \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\lrcorner} \arrow[d, two heads] \\ & 1 \arrow[r, "(*,*)"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, dashed, "\check{\rho}_*"'] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\rho"'] & & \\ & A^{\lrcorner}_* \arrow[r, two heads, "\sim"] \arrow[d, two heads, "\wr"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Hom}_A(*, A) \arrow[d, two heads, "\wr", "p_1"'] \\ & \mathrm{Hom}_A(*, A) \arrow[r, two heads, "\sim", "p_1"'] & A \end{tikzcd}$$

From a second construction of $A^{\lrcorner}_*$ displayed above-right and the characterization of initiality given in Proposition 4.3.10, we may apply the 2-of-3 property of equivalences to conclude first that $\rho\colon A \xrightarrow{\sim} \mathrm{Hom}_A(*, A)$ and then that the induced diagram $\check{\rho}_*\colon A \xrightarrow{\sim} A^{\lrcorner}_*$ are equivalences. Dually, the diagram $\hat{\xi}\colon A \to A^{\ulcorner}$ defines an equivalence $\hat{\xi}_*\colon A \xrightarrow{\sim} A^{\ulcorner}_*$ when its codomain is restricted to the subobject of pushout diagrams whose target elements are pinned at the zero element $*$.

By Proposition 4.3.4, a pointed ∞-category A admits loops or admits suspensions if and only if there exist absolute lifting diagrams as below-left and below-right respectively:

$$\begin{tikzcd}
& A^{\square} \arrow[d, two heads, "\mathrm{res}"] \\
A \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, "\check{\rho}"'] \arrow[r, phantom, shift left=3, "\cong\Downarrow"] & A^{\lrcorner}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& A^{\square} \arrow[d, two heads, "\mathrm{res}"] \\
A \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, "\hat{\xi}"'] \arrow[r, phantom, shift left=3, "\cong\Uparrow"] & A^{\ulcorner}
\end{tikzcd}$$

By Theorem 3.5.3 the absolute right lifting diagram defines a right representation $\mathrm{Hom}_{A^{\square}}(A^{\square}, \mathrm{ran}) \simeq \mathrm{Hom}_{A^{\lrcorner}}(\mathrm{res}, \check{\rho})$, this being a fibered equivalence over $A \times A^{\square}$. The represented comma ∞-category may be pulled back along the inclusion of the subobject $A^{\square}_* \hookrightarrow A^{\square}$ of commutative squares in A whose intermediate vertices are pinned at the zero object:

$$\begin{tikzcd}
& & \mathrm{Hom}_{A^{\lrcorner}}(\mathrm{res}, \check{\rho}) \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner"] & & (A^{\lrcorner})^2 \arrow[dd, two heads] \\
\mathrm{Hom}_{A^{\lrcorner}_*}(\mathrm{res}, \check{\rho}_*) \arrow[urr, hook] \arrow[rrr] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner"] & & & (A^{\lrcorner}_*)^2 \arrow[ur, hook] \arrow[dd, two heads] & \\
& & A^{\square} \times A \arrow[rr, "\mathrm{res} \times \check{\rho}"] & & A^{\lrcorner} \times A^{\lrcorner} \\
A^{\square}_* \times A \arrow[urr, hook] \arrow[rrr, "\mathrm{res} \times \check{\rho}_*"'] \arrow[ur, phantom, "\urcorner"] & & & A^{\lrcorner}_* \times A^{\lrcorner}_* \arrow[ur, hook] &
\end{tikzcd}$$

The right-hand face of this commutative cube is not strictly a pullback but the universal property of the zero element implies that the induced map from $(A^{\lrcorner}_*)^2$ to the pullback is an equivalence. It follows that $\mathrm{Hom}_{A^{\lrcorner}_*}(\mathrm{res}, \check{\rho}_*)$ is equivalent over $A^{\square}_* \times A$ to the pullback of $\mathrm{Hom}_{A^{\lrcorner}}(\mathrm{res}, \check{\rho})$ along this inclusion and so the fibered equivalence pulls back to define a right representation for $\mathrm{Hom}_{A^{\lrcorner}_*}(\mathrm{res}, \check{\rho}_*)$. Dually, the left representation for $\mathrm{Hom}_{A^{\ulcorner}}(\hat{\xi}, \mathrm{res})$ pulls back to a left representation for $\mathrm{Hom}_{A^{\ulcorner}_*}(\hat{\xi}_*, \mathrm{res})$. By Theorem 3.5.7 these unpack to define absolute lifting diagrams:

$$\begin{tikzcd}
& A^{\square}_* \arrow[d, two heads, "\mathrm{res}"] \\
A \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, "\check{\rho}_*"'] \arrow[r, phantom, shift left=3, "\cong\Downarrow"] & A^{\lrcorner}_*
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& A^{\square}_* \arrow[d, two heads, "\mathrm{res}"] \\
A^{\lrcorner}_* \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r, equals] \arrow[r, phantom, shift left=3, "\cong\Downarrow"] & A^{\lrcorner}_*
\end{tikzcd}
\qquad
\begin{tikzcd}
& A^{\square}_* \arrow[d, two heads, "\mathrm{res}"] \\
A \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, "\hat{\xi}_*"'] \arrow[r, phantom, shift left=3, "\cong\Uparrow"] & A^{\ulcorner}_*
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
& A^{\square}_* \arrow[d, two heads, "\mathrm{res}"] \\
A^{\ulcorner}_* \arrow[ur, dashed, "\mathrm{lan}"] \arrow[r, equals] \arrow[r, phantom, shift left=3, "\cong\Uparrow"] & A^{\ulcorner}_*
\end{tikzcd}$$

Restricting along the inverse equivalences $A^{\lrcorner}_* \xrightarrow{\sim} A$ and $A^{\ulcorner}_* \xrightarrow{\sim} A$ and pasting with the invertible 2-cell we obtain absolute lifting diagrams whose bottom edge is the identity.

By Lemma 2.3.6, these lifting diagrams define adjunctions:

$$\begin{tikzcd}
A \simeq A^{\lrcorner}_* \arrow[r, bend right, "\mathrm{ran}"'] \arrow[r, phantom, "\top"] & A^{\square}_* \arrow[l, bend right, "\mathrm{res}"'] \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\top"] & A^{\ulcorner}_* \simeq A \arrow[l, bend right, "\mathrm{lan}"']
\end{tikzcd}$$

which compose to the desired adjunction $\Sigma \dashv \Omega$. $\square$

4.4.4. Definition. An arrow $f\colon 1 \to A^2$ from x to y in a pointed ∞-category A **admits a fiber** if A admits a pullback of the diagram defined by gluing f to the component ρy of the family of points. By Proposition 4.3.4 such pullbacks give rise to a pullback square

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[ur, "\mathrm{ran}"] \arrow[r, "\rho y \vee f"'] \arrow[r, phantom, shift left=2, "\cong\Downarrow"] & A^{\lrcorner} \end{tikzcd}$$

that is referred to as the **fiber sequence** for f. Dually, f **admits a cofiber** if A admits a pushout of the diagram defined by gluing f to the component ξx of the family of copoints, in which case the pushout square

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[ur, "\mathrm{lan}"] \arrow[r, "\xi x \vee f"'] \arrow[r, phantom, shift left=2, "\cong\Uparrow"] & A^{\ulcorner} \end{tikzcd}$$

defines the **cofiber sequence** for f.

Fiber and cofiber sequences define commutative squares in A whose lower-left vertex is the zero element $*$. The data of such squares is given by a commutative triangle in A — an element of A^3 — together with a **nullhomotopy** of the diagonal edge, a witness that this edge factors through the zero element in $\mathfrak{h}A$. Borrowing a classical term from homological algebra, a commutative square in A whose lower-left vertex is the zero element is referred to as a **triangle** in A.

4.4.5. Definition (stable ∞-category). A **stable ∞-category** is a pointed ∞-category A in which

(i) every morphism admits a fiber and a cofiber: that is, there exist absolute lifting diagrams

$$\begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A^2 \arrow[ur, "\mathrm{ran}"] \arrow[r, "\rho \vee \mathrm{id}"'] \arrow[r, phantom, shift left=2, "\cong\Downarrow"] & A^{\lrcorner} \end{tikzcd} \qquad \begin{tikzcd} & A^{\boxdot} \arrow[d, two heads, "\mathrm{res}"] \\ A^2 \arrow[ur, "\mathrm{lan}"] \arrow[r, "\xi \vee \mathrm{id}"'] \arrow[r, phantom, shift left=2, "\cong\Uparrow"] & A^{\ulcorner} \end{tikzcd}$$

(ii) and a triangle in A defines a fiber sequence if and only if it also defines a cofiber sequence. Such triangles are called **exact**.

A stable ∞-category admits loops and admits suspensions, formed by taking fibers of the arrows in the family of points and cofibers of arrows in the family of copoints respectively.

4.4.6. Proposition. *If A is a stable ∞-category, then $\Sigma \dashv \Omega$ are inverse equivalences.*

Proof. In the proof of Proposition 4.4.3, the adjunction $\Sigma \dashv \Omega$ is constructed as a composite of adjunctions

$$\begin{tikzcd} A \simeq A_*^{\lrcorner} \arrow[r, bend right, "\mathrm{ran}"'] \arrow[r, phantom, "\top"] & A_*^{\boxdot} \arrow[l, bend right, "\mathrm{res}"'] \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\top"] & A_*^{\ulcorner} \simeq A \arrow[l, bend right, "\mathrm{lan}"'] \end{tikzcd}$$

that construct fiber and cofiber sequences. By Proposition 2.1.9, the unit and counit of this composite adjunction are given by

$$
\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[dr, "\mathrm{lan}"'] \arrow[ddrr, bend right, "\Sigma"'] \arrow[rrrrd, phantom, "\Downarrow\cong"] & & & & A \\
& A^{\boxdot}_{*} \arrow[rr, equal] \arrow[dr, "\mathrm{res}"'] \arrow[rrd, phantom, "\Downarrow\eta"] & & A^{\boxdot}_{*} \arrow[ur, "\mathrm{res}"'] & \\
& & A \arrow[ur, "\mathrm{ran}"'] \arrow[uurr, bend right, "\Omega"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A \arrow[dr, "\mathrm{lan}"] \arrow[ddrr, bend left, "\Sigma"] & & \\
& A^{\boxdot}_{*} \arrow[rr, equal] \arrow[ur, "\mathrm{res}"] \arrow[ur, phantom, ""] & \Downarrow\epsilon & A^{\boxdot}_{*} \arrow[dr, "\mathrm{res}"] & \\
A \arrow[rrrr, equal] \arrow[ur, "\mathrm{ran}"] \arrow[uurr, bend left, "\Omega"] & & \Downarrow\cong & & A
\end{tikzcd}
$$

By Definition 4.3.7, the unit of $\mathrm{res} \dashv \mathrm{ran}$ restricts to an isomorphism on the subobject of pushout squares. In a stable ∞-category, the cofiber sequences in the image of $\mathrm{lan}\colon A \to A^{\boxdot}_{*}$ are pullback squares, so this tells us that $\eta\,\mathrm{lan}$ is an isomorphism. Dually, the fiber sequences in the image of $\mathrm{ran}\colon A \to A^{\boxdot}_{*}$ are pushout squares, which tells us that $\epsilon\,\mathrm{ran}$ is an isomorphism. Hence, the unit and counit of $\Sigma \dashv \Omega$ are invertible, so these functors define an adjoint equivalence. ☐

4.4.7. Proposition (finite limits and colimits in stable ∞-categories). *A stable ∞-category admits all pushouts and all pullbacks, and moreover, a square is pushout if and only if it is a pullback.*

Proof. Given a cospan $g \vee f\colon X \to A^{\lrcorner}$ in A, form the cofiber of f followed by the fiber of the composite map $c \to a \to \operatorname{coker} f$:

$$
\begin{tikzcd}
\ker(qg) \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, dashed] \\
c \arrow[r, "g"'] & a \arrow[r, dashed, "q"'] \arrow[ur, phantom, "\ulcorner", very near end] & \operatorname{coker}(f)
\end{tikzcd}
$$

By Definition 4.4.5(ii), the cofiber sequence $b \to a \to \ker f$ is also a fiber sequence. By the pullback cancelation result of Proposition 4.3.9, we conclude that $\ker(qg)$ computes the pullback of the cospan $g \vee f$.

To see that this pullback square is also a pushout, form the fiber of the map v:

$$
\begin{tikzcd}
\ker(v) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \ker(qg) \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "f"'] \\
* \arrow[r] & c \arrow[r, "g"'] & a
\end{tikzcd}
$$

By the pullback composition result of Proposition 4.3.9, $\ker(v)$ is also the fiber of the map f. By Definition 4.4.5(ii), the fiber sequences $\ker(v) \to \ker(qg) \to c$ and $\ker(v) \to b \to a$ are also cofiber sequences. Now by the pushout cancelation result of Proposition 4.3.9, we see that the right-hand pullback square is also a pushout square. A dual argument proves that pushouts coincide with pullbacks. ☐

Exercises.

4.4.i. Exercise. Arguing in the homotopy category, show that if an ∞-category A

- admits an initial element i,
- admits a terminal element t, and
- there exists an arrow $t \to i$

then A is a pointed ∞-category.

CHAPTER 5

Fibrations and Yoneda's lemma

The fibers E_b of an isofibration $p\colon E \twoheadrightarrow B$ over an element $b\colon 1 \to B$ are necessarily ∞-categories, so it is natural to ask where the isofibration also encodes the data of functors between the fibers in this family of ∞-categories. Roughly speaking, an isofibration defines a *cartesian fibration* just when the arrows of B act contravariantly functorially on the fibers and a *cocartesian fibration* when the arrows of B act covariantly functorially on the fibers. This action is not strict but rather *pseudofunctorial* in a sense explored in Exercise 5.1.iii. A complete proof of pseudofunctoriality requires the verification of certain conditions, which in this context extend to higher dimensions. In Chapter 17 when we study the comprehension construction, we will construct a homotopy coherent functor from the underlying quasi-category of B to the ∞-cosmos $\mathcal{K}$, vastly generalizing the pseudofunctoriality described here.

One of the properties that characterizes cocartesian fibrations is an axiom that says that for any 2-cell with codomain B and specified lift of its source 1-cell, there is a lifted 2-cell with codomain E with that one cell as its source. In particular, this lifting property can be applied in the case where the 2-cell in question is a whiskered composite of an arrow in the homotopy category of B as below-left and the lift of the source 1-cell is the canonical inclusion of its fiber:

$$
\begin{tikzcd}
E_a \arrow[r, bend left, "\ell_a"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & B
\end{tikzcd}
=
\begin{tikzcd}
E_a \arrow[rr, bend left, "\ell_a"] \arrow[rr, phantom, bend left=15, "\Downarrow\chi_\beta"] \arrow[rr, dotted, "\beta_*(\ell_a)"] \arrow[dr, dashed, "\beta_*"'] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\
& E_b \arrow[ur, dashed, "\ell_b"'] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
1 \arrow[rr, bend right, "b"'] & & B
\end{tikzcd}
\tag{5.0.1}
$$

In this case the codomain $\beta_*(\ell_a)$ of the lifted cell χ_β displayed above right lies strictly above the codomain of the original 2-cell, and thus factors through the pullback defining its fiber. This defines a functor $\beta_*\colon E_a \to E$, the "action" of the arrow β on the fibers of p. The pseudofunctoriality of these action maps arises from a universal property required of the specified lifted 2-cells, namely that they are *cartesian* in a sense we now define.

5.1. The 2-category theory of cartesian fibrations

There is a standard notion of cartesian fibration in a 2-category developed by Street [**90**] that recovers the Grothendieck fibrations when specialized to the 2-category $\mathcal{C}at$. This is *not* the correct notion of cartesian fibration between ∞-categories as the universal property the usual notion demands of lifted 2-cells is too strict. Instead of referring to the notions defined here as "weak," we would prefer to refer to the classical notion of cartesian fibration in a 2-category as "strict" were we to refer to it again, which we largely will not.

To remind the reader of the interpretation of the data in the homotopy 2-category of an ∞-cosmos, we refer to 1- and 2-cells as "functors" and "natural transformations." Before defining the notion of cartesian fibration we describe the weak universal property enjoyed by a certain class of "upstairs" natural transformations.

5.1.1. Definition (p-cartesian transformations). Let $p\colon E \twoheadrightarrow B$ be an isofibration. A natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ with codomain E is p-**cartesian** if

(i) **induction**: Given any natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "pe''"] \arrow[r, bend right, "pe'"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ so that $p\tau = p\chi \cdot \gamma$, there exists a lift $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of γ so that $\tau = \chi \cdot \bar{\gamma}$.

$$\begin{tikzcd} e'' \arrow[rr, Rightarrow, "\tau"] \arrow[dr, Rightarrow, dotted, "\bar{\gamma}"'] & & e & \in & \mathsf{hFun}(X,E) \arrow[dd, "p_*"] \\ & e' \arrow[ur, Rightarrow, "\chi"'] \arrow[d, mapsto] & & & \\ pe'' \arrow[rr, Rightarrow, "p\tau"] \arrow[dr, Rightarrow, "\gamma"'] & {} & pe & \in & \mathsf{hFun}(X,B) \\ & pe' \arrow[ur, Rightarrow, "p\chi"'] & & & \end{tikzcd}$$

(ii) **conservativity**: Any fibered endomorphism of χ is invertible: if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\chi \cdot \zeta = \chi$ and $p\zeta = \mathrm{id}_{pe'}$ then ζ is invertible.

5.1.2. Remark (why "cartesian"). The induction property for a p-cartesian natural transformation $\chi\colon e' \Rightarrow e$ says that for any $e''\colon X \to E$, there is a surjective function from the set $\mathsf{hFun}(X,E)(e'',e')$ of natural transformations from e'' to e' to the pullback induced in the commutative square

$$\begin{tikzcd} \mathsf{hFun}(X,E)(e'',e') \arrow[r, "\chi\circ-"] \arrow[d, "p"'] & \mathsf{hFun}(X,E)(e'',e) \arrow[d, "p"] \\ \mathsf{hFun}(X,B)(pe'',pe') \arrow[r, "p\chi\circ-"'] & \mathsf{hFun}(X,B)(pe'',pe) \end{tikzcd}$$

Were the induction property strict and not weak, this square would be a pullback, i.e., a "cartesian" square.

It follows that p-cartesian lifts of a given 2-cell with specified codomain are unique up to fibered isomorphism:

5.1.3. Lemma (uniqueness of cartesian lifts). *If* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi'"] & E \end{tikzcd}$ *are p-cartesian lifts of a common 2-cell $p\chi$, then there exists an invertible 2-cell* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\cong\Downarrow\zeta"] & E \end{tikzcd}$ *so that $\chi' = \chi \cdot \zeta$ and $p\zeta = \mathrm{id}$.* □

Proof. By induction, there exists 2-cells $\zeta\colon e'' \Rightarrow e'$ and $\zeta'\colon e' \Rightarrow e''$ so that $\chi' = \chi \cdot \zeta$ and $\chi = \chi' \cdot \zeta'$ with $p\zeta = p\zeta' = \mathrm{id}$. The composites $\zeta \cdot \zeta'$ and $\zeta' \cdot \zeta$ are then fibered automorphisms of χ and χ' and thus invertible by conservativity. Now the 2-of-6 property for isomorphisms implies that ζ and ζ' are also isomorphisms (though perhaps not inverses). □

We frequently make use of the isomorphism stability of the p-cartesian transformations given by the following suite of observations:

5.1.4. Lemma. *Let $p\colon E \twoheadrightarrow B$ be an isofibration.*

(i) Isomorphisms define p-cartesian transformations.
(ii) Any p-cartesian lift of an identity is a natural isomorphism.
(iii) The class of p-cartesian transformations is closed under pre- and post-composition with natural isomorphisms.

Proof. Exercise 5.1.i. □

Furthermore:

5.1.5. Lemma (more conservativity). *If* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$, $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi'"] & E \end{tikzcd}$, *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ *are 2-cells so that χ and χ' are p-cartesian, $\chi' = \chi \cdot \zeta$, and $p\zeta$ is invertible, then ζ is invertible.*

Proof. Given the data in the statement we can use the induction property for χ' applied to the pair $(\chi, p\zeta^{-1})$ to induce a candidate inverse $\hat{\zeta}$ for ζ and then apply the conservativity property to conclude that $\zeta \cdot \hat{\zeta}$ and $\hat{\zeta} \cdot \zeta$ are both isomorphisms. By the 2-of-6 property, ζ is an isomorphism, as desired. □

We now introduce the class of cartesian fibrations.

5.1.6. Definition (cartesian fibration). An isofibration $p\colon E \twoheadrightarrow B$ is a **cartesian fibration** if

(i) Any natural transformation $\beta\colon b \Rightarrow pe$ as below-left admits a p-cartesian lift[1] $\chi_\beta\colon \beta^*e \Rightarrow e$ as below-right:

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[dr, bend right, "b"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\Uparrow\beta"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta^*e"'] \arrow[r, phantom, "\Uparrow\chi_\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

[1]To ask that $\chi_\beta\colon \beta^*e \Rightarrow e$ is a **lift** of $\beta\colon b \Rightarrow pe$ asserts that $p\chi_\beta = \beta$ and hence $p\beta^*e = b$.

(ii) The class of p-cartesian transformations is closed under restriction: that is, if $X \overset{e'}{\underset{e}{\Downarrow\chi}} E$ is p-cartesian and $f\colon Y \to X$ is any functor then $Y \overset{e'f}{\underset{ef}{\Downarrow\chi f}} E$ is p-cartesian.

The lifting property (i) implies that a p-cartesian transformation $\chi\colon e' \Rightarrow e$ is the "universal natural transformation over $p\chi$ with codomain e" in the following weak sense: any transformation $\psi\colon e' \Rightarrow e$ factors through a p-cartesian lift $\chi_\psi\colon (p\psi)^*e \Rightarrow e$ of $p\psi$ via a 2-cell $\gamma\colon e' \Rightarrow (p\psi)^*e$ over an identity, and moreover ψ is p-cartesian if and only if this factorization γ is invertible.

The reason for condition (ii) will become clearer in §5.3. For now, note that since all p-cartesian lifts of a given 2-cell β are isomorphic and the class of p-cartesian cells is stable under isomorphism, to verify the condition (ii) it suffices to show that for any functor $f\colon Y \to X$ there is *some* p-cartesian lift of β that restricts along f to another p-cartesian transformation.

5.1.7. Lemma (composites of cartesian fibrations). *If $p\colon E \twoheadrightarrow B$ and $q\colon B \twoheadrightarrow A$ are cartesian fibrations, then so is $qp\colon E \twoheadrightarrow A$. Moreover, a natural transformation $X \overset{e'}{\underset{e}{\Downarrow\chi}} E$ is qp-cartesian if and only if χ is p-cartesian and $p\chi$ is q-cartesian.*

Proof. The first claim follows immediately from the second, for the lifts required by Definition 5.1.6(i) can be constructed by first taking a q-cartesian lift χ_β and then taking a p-cartesian lift χ_{χ_β} of this lifted cell.

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[ddr, bend right, "a"'] \arrow[dr, phantom, "\Uparrow\beta"] & E \arrow[d, two heads, "p"] \\
& B \arrow[d, two heads, "q"] \\
& A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Uparrow\chi_\beta"] & E \arrow[d, two heads, "p"] \\
& B \arrow[d, two heads, "q"] \\
& A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta^*e"'] \arrow[r, phantom, "\Uparrow\chi_{\chi_\beta}"] & E \arrow[d, two heads, "p"] \\
& B \arrow[d, two heads, "q"] \\
& A
\end{tikzcd}
$$

and the stability condition 5.1.6(ii) is then inherited from the stability of p- and q-cartesian transformations.

To prove the second claim, first consider a natural transformation $X \overset{e'}{\underset{e}{\Downarrow\chi}} E$ that is p-cartesian and so that $p\chi$ is q-cartesian. Given any natural transformations $X \overset{e''}{\underset{e}{\Downarrow\tau}} E$ and $X \overset{qpe''}{\underset{qpe'}{\Downarrow\gamma}} A$ so that $qp\tau = qp\chi \cdot \gamma$, q-cartesianness of $p\chi$ induces a lift $X \overset{e''}{\underset{e'}{\Downarrow\hat{\gamma}}} B$ of γ so that $p\tau = p\chi \cdot \hat{\gamma}$. Now p-cartesianness of χ induces a further lift $X \overset{e''}{\underset{e'}{\Downarrow\bar{\gamma}}} E$ of $\hat{\gamma}$ so that $\tau = \chi \cdot \bar{\gamma}$. Moreover, if

$$\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$$
is any natural transformation so that $\chi \cdot \zeta = \chi$ and $qp\zeta = \mathrm{id}$ then $p\chi \cdot p\zeta = p\chi$ and by conservativity for $p\chi$, $p\zeta$ is invertible. Applying Lemma 5.1.5, we conclude that ζ is invertible. This proves that χ is qp-cartesian.

Conversely, if χ is qp-cartesian, then Lemma 5.1.3 implies it is isomorphic to all other qp-cartesian lifts of $qp\chi$. The construction given above produces a qp-cartesian lift of any 2-cell that is p-cartesian and whose image under p is q-cartesian. By the isomomorphism stability of p- and q-cartesian transformations of Lemma 5.1.4, χ must also have these properties. ☐

The following lemma proves that cartesian fibrations come equipped with a "generic p-cartesian transformation."

5.1.8. Lemma. *An isofibration $p\colon E \twoheadrightarrow B$ is a cartesian fibration if and only if the right comma cone over p displayed below-left admits a lift χ as displayed below-right:*

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[rr, bend left, "p_1"] \arrow[drr, bend right, "p_0"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow\phi"] & & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \tag{5.1.9}$$

with the property that the restriction of χ along any $X \to \mathsf{Hom}_B(B,p)$ is a p-cartesian transformation.

Proof. If $p\colon E \twoheadrightarrow B$ is cartesian, then the right comma cone ϕ admits a p-cartesian lift $\chi\colon r \Rightarrow p_1$ by 5.1.6(i), which by 5.1.6(ii) has the property that the restriction of this p-cartesian transformation along any $X \to \mathsf{Hom}_B(B,p)$ is also p-cartesian.

For the converse, suppose we are given the generic p-cartesian transformation $\chi\colon r \Rightarrow p_1$ of the statement and consider a 2-cell $\beta\colon b \Rightarrow pe$ as below-left.

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Downarrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, dashed, "\ulcorner\beta\urcorner"] \arrow[rr, bend left, "e"] \arrow[drr, bend right, "b"'] & \mathsf{Hom}_B(B,p) \arrow[r, "p_1"] \arrow[dr, "p_0"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\phi"] & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, dashed, "\ulcorner\beta\urcorner"] \arrow[rr, bend left, "e"] \arrow[drr, bend right, "b"'] & \mathsf{Hom}_B(B,p) \arrow[r, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, shift right=2ex, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd}$$

By 1-cell induction $\beta = \phi\ulcorner\beta\urcorner$ for some functor $\ulcorner\beta\urcorner\colon X \to \mathsf{Hom}_B(B,p)$ as above- center. By substituting the equation (5.1.9) as above-right, we see that $\chi\ulcorner\beta\urcorner$ is a lift of β whose codomain is $p_1\ulcorner\beta\urcorner = e$, as required. The hypothesis that restrictions of χ are p-cartesian implies that this lift is a p-cartesian transformation.

Now Lemma 5.1.3 implies that any p-cartesian natural transformation is isomorphic to a restriction of χ. Thus restrictions of p-cartesian transformations are isomorphic to restrictions of χ, and it follows from Lemma 5.1.4 that the class of p-cartesian transformation is closed under restriction. ☐

The first major result of this section is an internal characterization of cartesian fibrations inspired by a similar result of Street [**90, 94, 95**]; see also [**104**]. Before stating this result, recall from Lemma 3.5.8 that from a functor $p\colon E \twoheadrightarrow B$, we can build a fibered adjunction $p_1 \dashv_E i$, where the right adjoint

is induced from the identity 2-cell id_p:

$$
\begin{tikzcd}
& E \arrow[d, dashed, "i"] \arrow[ddr, bend left, two heads, "p"] \arrow[ddl, bend right, equal] \\
& \mathsf{Hom}_B(B,p) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
E \arrow[rr, "p"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& E \arrow[dl, equal] \arrow[dr, two heads, "p"] \arrow[d, phantom, "="] & \\
E \arrow[rr, "p"'] & {} & B
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
E \arrow[rr, dashed, bend right, "i"'] \arrow[dr, equal] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B,p) \arrow[ll, two heads, bend right, "p_1"'] \arrow[dl, two heads, "p_1"] \\
& E &
\end{tikzcd}
$$

Similarly, 1-cell induction for the right comma cone over p applied to the generic arrow for E induces a functor

$$
\begin{tikzcd}
E^{\mathbb{2}} \arrow[d, bend right, two heads, "p_1"'] \arrow[d, bend left, two heads, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] \\
E \arrow[d, two heads, "p"] \\
B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& E^{\mathbb{2}} \arrow[d, dashed, "k"] \arrow[ddl, bend right, two heads, "p_1"'] \arrow[ddr, bend left, two heads, "pp_0"] & \\
& \mathsf{Hom}_B(B,p) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
E \arrow[rr, "p"'] & {} & B
\end{tikzcd}
\tag{5.1.10}
$$

5.1.11. Theorem (an internal characterization of cartesian fibrations). *For an isofibration $p\colon E \twoheadrightarrow B$ the following are equivalent:*

(i) $p\colon E \twoheadrightarrow B$ defines a cartesian fibration.

(ii) The functor $i\colon E \to \mathsf{Hom}_B(B,p)$ admits a right adjoint over B:

$$
\begin{tikzcd}
E \arrow[rr, bend left, "i"] \arrow[dr, two heads, "p"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "p_0"] \\
& B &
\end{tikzcd}
$$

(iii) The functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ admits a right adjoint with invertible counit:

$$
\begin{tikzcd}
E^{\mathbb{2}} \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] & \mathsf{Hom}_B(B,p) \arrow[l, dashed, bend left, "\bar{r}"]
\end{tikzcd}
$$

When these equivalent conditions hold, then for a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\psi"] & E \end{tikzcd}$ *the following are equivalent:*

(iv) ψ is p-cartesian.

(v) *ψ factors through a restriction of the 2-cell $p_1\epsilon$, where ϵ is the counit of the adjunction $i \dashv r$, via a natural isomorphism $\bar{\gamma}$ so that $p\bar{\gamma} = \mathrm{id}$:*

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\psi"] & E \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=small] X \arrow[r, dashed, "\ulcorner p\psi \urcorner"] \arrow[ddrr, dashed, bend right, "e'"'] \arrow[rrrr, bend left, "e"] & \mathsf{Hom}_B(B,p) \arrow[rr, equal] \arrow[dr, "r"] \arrow[d, phantom, "\cong\Uparrow\bar{\gamma}"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{Hom}_B(B,p) \arrow[r, two heads, "p_1"] & E \\ & {} & E \arrow[ur, "i"] \arrow[urr, equal, bend right] & & \end{tikzcd}$$

(vi) *The component*

$$\begin{tikzcd} X \arrow[r, "\ulcorner \psi \urcorner"] & E^{\mathbb{2}} \arrow[r, bend left, equal] \arrow[r, bend right, "\bar{r}k"'] \arrow[r, phantom, "\Downarrow\bar{\eta}"] & E^{\mathbb{2}} \end{tikzcd}$$

of the unit for $k \dashv \bar{r}$ is invertible; that is, ψ is in the essential image of the right adjoint.

The right adjoint of (ii) is the domain-component of the generic cartesian lift of (5.1.9); that cartesian transformation is then recovered as $p_1\epsilon$, where ϵ is the counit of the fibered adjunction $i \dashv r$. This explains the statement of (v). By 1-cell induction, the generic cartesian lift χ can be represented by a functor $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$ and this defines the right adjoint of (iii) and explains the statement of (vi).

Before proving Theorem 5.1.11 we make two further remarks on these postulated adjunctions.

5.1.12. Remark. By Lemma 3.5.8, the functor $i\colon E \to \mathsf{Hom}_B(B,p)$ is itself a right adjoint over B to the codomain-projection functor. Since the counit of the adjunction $p_1 \dashv i$ is an isomorphism, it follows formally that the unit of the adjunction $i \dashv r$ must also be an isomorphism, whenever the adjunction postulated in (ii) exists; see Lemma B.3.8.

5.1.13. Remark. In the case where the ∞-categories $E^{\mathbb{2}}$ and $\mathsf{Hom}_B(B,p)$ are defined by the strict simplicial limits of Definitions 3.2.1 and 3.4.1, the 1-cell k induced in (5.1.10) can be modeled by an isofibration:

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[dr, dashed, two heads, "k"] \arrow[drr, bend left, "p^{\mathbb{2}}"] \arrow[ddr, bend right, two heads, "p_1"'] & & \\ & \mathsf{Hom}_B(B,p) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & B^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ & E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

namely the Leibniz cotensor of the codomain inclusion $1\colon \mathbb{1} \hookrightarrow \mathbb{2}$ and the isofibration $p\colon E \twoheadrightarrow B$. Now Lemma B.4.7 can be used to rectify the adjunction $k \dashv r$ of (iii) to a right adjoint right inverse adjunction, that is then fibered over $\mathsf{Hom}_B(B,p)$. So when Theorem 5.1.11(iii) holds, we may model the postulated adjunction by a right adjoint right inverse to the isofibration k.

Proof. We'll prove (i)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) and demonstrate the equivalences (iv)$\Leftrightarrow$(vi) and (iv)$\Leftrightarrow$(v) in parallel.

(i)⇒(iii): If $p\colon E \twoheadrightarrow B$ is cartesian, then the right comma cone over p admits a cartesian lift along p,

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[rr, bend left, "p_1"] \arrow[drr, bend right, "p_0"'] & \arrow[d, phantom, "\Uparrow\phi"] & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, phantom, "\Uparrow\chi"] \arrow[rr, bend left, "p_1"] \arrow[rr, bend right, "r"'] & & E \arrow[d, two heads, "p"] \\ & & B \end{tikzcd}$$

defining a functor $r\colon \mathsf{Hom}_B(B,p) \to E$ over B together with a p-cartesian transformation $\chi\colon r \Rightarrow p_1$. By 1-cell induction, this generic cartesian transformation is represented by a functor

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[r, dashed, "\bar{r}"] & E^2 \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\kappa"] & E \end{tikzcd}$$

which we take as our definition of the putative right adjoint $\bar{r}$. By the definition (5.1.10) of k, $\phi k\bar{r} = p\kappa\bar{r} = p\chi = \phi$, so Proposition 3.4.7 supplies a fibered isomorphism $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ with $p_0\bar{\epsilon} = \mathrm{id}_{p_0}$ and $p_1\bar{\epsilon} = \mathrm{id}_{p_1}$.

To prove that $k \dashv \bar{r}$, it remains to define the unit 2-cell $\bar{\eta}$, which we do by 2-cell induction from a pair given by an identity 2-cell $p_1\bar{\eta} = \mathrm{id}_{p_1}$ and a 2-cell $p_0\bar{\eta}$ that remains to be specified. The required compatibility condition of Proposition 3.4.6(ii) asserts that this $p_0\bar{\eta}$ must define a factorization of the generic arrow

$$\begin{tikzcd} p_0 \arrow[r, Rightarrow, "\kappa"] \arrow[d, Rightarrow, "p_0\bar{\eta}"'] & p_1 \arrow[d, equals, "p_1\bar{\eta}=\mathrm{id}"] \\ p_0\bar{r}k = rk \arrow[r, Rightarrow, "\chi k"'] & p_1 k \end{tikzcd} \tag{5.1.14}$$

through $\kappa\bar{r}k = \chi k$. Note $p\kappa = \phi k$ has χk as its p-cartesian lift, so we define $p_0\bar{\eta}$ by the induction property for the cartesian transformation χk applied to the generic arrow $\kappa\colon p_0 \Rightarrow p_1$. By construction $pp_0\bar{\eta} = \mathrm{id}$.

By Lemma B.4.2, once we verify that $k\bar{\eta}$ and $\bar{\eta}\bar{r}$ are invertible, then this data this together with $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ defines an adjunction with invertible counit.[2] We prove $k\bar{\eta}$ is invertible by 2-cell conservativity: $p_1k\bar{\eta} = p_1\bar{\eta} = \mathrm{id}$ and $p_0k\bar{\eta} = pp_0\bar{\eta} = \mathrm{id}$.

Similarly, by 2-cell conservativity, to conclude that $\bar{\eta}\bar{r}$ is invertible, it suffices to prove that $p_0\bar{\eta}\bar{r}$ is an isomorphism. Restricting (5.1.14) along $\bar{r}$, we see that $p_0\bar{\eta}\bar{r}$ defines a fibered isomorphism of p-cartesian transformations

$$\begin{tikzcd} r \arrow[rr, Rightarrow, "\chi"] \arrow[dr, Rightarrow, "p_0\bar{\eta}\bar{r}"'] & & p_1 \\ & p_0\bar{r}k = rk \arrow[ur, Rightarrow, "\chi k\bar{r}"'] \end{tikzcd}$$

so this follows from Lemma 5.1.5.

Finally note that a transformation $\psi\colon e' \Rightarrow e$ is p-cartesian if and only if its factorization through the generic p-cartesian lift of $p\psi$ is invertible. This factorization may be constructed by restricting the 2-cells of (5.1.14) along $\ulcorner\psi\urcorner\colon X \to E^2$ since $\kappa\ulcorner\psi\urcorner = \psi$, so we see that ψ is p-cartesian if and only if $p_0\bar{\eta}\ulcorner\psi\urcorner$ is invertible. By 2-cell conservativity, $\bar{\eta}\ulcorner\psi\urcorner$ is invertible if and only if its domain component is invertible. This proves that (iv)⇔(vi).

[2] By Remark B.4.3, we can make $\bar{\epsilon}$ the unit of this adjunction at the cost of modifying the counit by an isomorphism.

(iii)⇒(ii): By Remark 5.1.13, we can model the left adjoint of (iii) by an isofibration $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B,p)$ and use Lemma B.4.7 to rectify the adjunction $k \dashv \bar{r}$ with invertible counit into a right adjoint right inverse adjunction, that is then fibered over $\mathsf{Hom}_B(B,p)$. Composing with the projection $p_0\colon \mathsf{Hom}_B(B,p) \twoheadrightarrow B$, Lemma 3.6.7(ii) then gives us a fibered adjunction over B

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[rr, bend left, "k"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "pp_0"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, bend left, "\bar{r}"] \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

By the dual of Lemma 3.5.8, the 1-cell $j\colon E \to E^{\mathbb{2}}$ induced by the identity defines a left adjoint right inverse to the domain projection

$$\begin{tikzcd} E \arrow[r, "j" description] & E^{\mathbb{2}} \arrow[l, two heads, bend right, "p_1"', "\perp"] \arrow[l, two heads, bend left, "p_0", "\perp"'] \end{tikzcd} \tag{5.1.15}$$

supplying a fibered adjunction $j \dashv p_0$ over E that we push forward along $p\colon E \twoheadrightarrow B$ to a fibered adjunction over B:

$$\begin{tikzcd} E \arrow[rr, bend left, "j"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & E^{\mathbb{2}} \arrow[ll, two heads, bend left, "p_0"] \arrow[dl, two heads, "pp_0"] \\ & B & \end{tikzcd}$$

This pair of fibered adjunctions composes to define a fibered adjunction over B with left adjoint kj and right adjoint $r \coloneqq p_0\bar{r}$. Proposition 3.4.7 supplies a fibered isomorphism $i \cong kj$ since both i and kj define functors $E \rightrightarrows \mathsf{Hom}_B(B,p)$ are induced by the identity id_p over the right comma cone over p. Composing with this fibered isomorphism, we can replace the left adjoint of the composite adjunction by i

$$\begin{tikzcd} E \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "i", "\cong"'] \arrow[dr, two heads, bend right, "p"'] & E^{\mathbb{2}} \arrow[l, two heads, bend left, "p_0"] \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "pp_0"] & \mathsf{Hom}_B(B,p) \arrow[l, bend left, "\bar{r}"] \arrow[dl, two heads, bend left, "p_0"] \\ & B & \end{tikzcd} = \begin{tikzcd} E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, bend left, "r"] \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

proving (ii).

(ii)⇒(i): Now suppose given a fibered adjunction

$$\begin{tikzcd} E \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, bend left, "r"] \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

We will show that the codomain component of the counit

$$\begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[rr, equal, bend left] \arrow[dr, "r"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathrm{Hom}_B(B,p) \arrow[r, two heads, "p_1"] & E \\ & E \arrow[ur, "i"] \arrow[urr, equal, bend right] & & \end{tikzcd}$$

satisfies the conditions of the generic p-cartesian transformation described in Lemma 5.1.8. This will then also demonstrate the equivalence (iv)⇔(v).

The first thing to check is that $p_1\epsilon$ defines a lift of the right comma cone along p. To see this, consider the horizontal composite:

$$\begin{tikzcd} & & & & E \arrow[dd, two heads, "p"] \\ \mathrm{Hom}_B(B,p) \arrow[rr, equal, bend left] \arrow[dr, "r"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathrm{Hom}_B(B,p) \arrow[urr, two heads, "p_1"] \arrow[drr, two heads, "p_0"'] & \Uparrow\phi & \\ & E \arrow[ur, "i"] & & & B \end{tikzcd}$$

Naturality of whiskering provides a commutative square

$$\begin{tikzcd} p_0 ir \arrow[r, equal, "p_0\epsilon"] \arrow[d, equal, "\phi ir"'] & p_0 \arrow[d, Rightarrow, "\phi"] \\ pp_1 ir \arrow[r, Rightarrow, "pp_1\epsilon"'] & pp_1 \end{tikzcd}$$

in which $p_0\epsilon = \mathrm{id}$, as a fibered counit, and $\phi ir = \mathrm{id}$, since $\phi i = \mathrm{id}_p$. Thus $pp_1\epsilon = \phi$ and we see that $p_1\epsilon$ is a lift of ϕ along p.

It remains only to verify that the restriction of $p_1\epsilon$ along any functor defines a p-cartesian transformation. To that end, consider $\ulcorner\beta\urcorner\colon X \to \mathrm{Hom}_B(B,p)$ representing a 2-cell $\beta\colon b \Rightarrow pe$; our task is to verify that $p_1\epsilon\ulcorner\beta\urcorner$ is p-cartesian. Note that $pp_1\epsilon\ulcorner\beta\urcorner = \phi\ulcorner\beta\urcorner = \beta$, so to prove the induction property, consider a 2-cell $\tau\colon e'' \Rightarrow e$ and a factorization $p\tau = \beta \cdot \gamma$ for some $\gamma\colon pe'' \Rightarrow b$. Our task is to define a 2-cell $\bar{\gamma}\colon e'' \Rightarrow r\ulcorner\beta\urcorner$ so that the pasted composite

$$\begin{tikzcd} X \arrow[rrrr, bend left, "e"] \arrow[r, "\ulcorner\beta\urcorner"'] \arrow[drr, bend right, "e''"'] & \mathrm{Hom}_B(B,p) \arrow[rr, equal, bend left] \arrow[dr, "r"'] \arrow[d, phantom, "\Uparrow\bar{\gamma}"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathrm{Hom}_B(B,p) \arrow[r, two heads, "p_1"] & E \\ & {} & E \arrow[ur, "i"] \arrow[urr, equal, bend right] & & \end{tikzcd}$$

is τ. Transposing across the adjunction $i \dashv r$, it suffices instead to define a 2-cell $\hat{\gamma}\colon ie'' \Rightarrow \ulcorner\beta\urcorner$ so that $p_1\hat{\gamma} = \tau$. We define $\hat{\gamma}$ by 2-cell induction from this condition and $p_0\hat{\gamma} = \gamma$, a pair which satisfies the 2-cell induction compatibility condition

$$\begin{tikzcd} pe'' \arrow[r, equal] \arrow[d, Rightarrow, "\gamma = p_0\hat{\gamma}"'] & pe'' \arrow[d, Rightarrow, "pp_1\hat{\gamma} = p\tau"] \\ b \arrow[r, Rightarrow, "\beta"'] & pe \end{tikzcd}$$

precisely on account of the postulated factorization $p\tau = \beta \cdot \gamma$. This verifies the induction condition of Definition 5.1.1(i).

Now consider an endomorphism $\zeta\colon r\hat{\beta} \Rightarrow r\hat{\beta}$ so that $p_1\epsilon\ulcorner\beta\urcorner \cdot \zeta = p_1\epsilon\ulcorner\beta\urcorner$ and $p\zeta = \mathrm{id}_b$. Write $\bar{r} := \ulcorner p_1\epsilon\urcorner\colon \mathsf{Hom}_B(B,p) \to E^2$ for the functor induced by 1-cell induction from the generic p-cartesian transformation. Now the conditions defining ζ allow us to induce a 2-cell $\bar{\zeta}\colon \bar{r}\ulcorner\beta\urcorner \Rightarrow \bar{r}\ulcorner\beta\urcorner$ satisfying $p_0\bar{\zeta} = \zeta$ and $p_1\bar{\zeta} = \mathrm{id}$. To prove that ζ is invertible, will make use of the naturality of whiskering square for the horizontal composite

$$\begin{tikzcd} & \mathsf{Hom}_B(B,p) \arrow[dr, "\bar{r}"] \arrow[dd, phantom, "\Downarrow\bar{\zeta}"] & & & \\ X \arrow[ur, "\ulcorner\beta\urcorner"] \arrow[dr, "\ulcorner\beta\urcorner"'] & & E^2 \arrow[rr, bend left, "p_0"] \arrow[dr, "k"'] & \Downarrow\bar{\gamma} & E \\ & \mathsf{Hom}_B(B,p) \arrow[ur, "\bar{r}"'] & & \mathsf{Hom}_B(B,p) \arrow[ur, "r"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} p_0r\ulcorner\beta\urcorner \arrow[r, dotted, "\bar{\gamma}\bar{r}\ulcorner\beta\urcorner", "\cong"'] \arrow[d, Rightarrow, "\zeta=p_0\bar{\zeta}"'] & rk\bar{r}\ulcorner\beta\urcorner \arrow[d, Rightarrow, "rk\bar{\zeta}", "\cong"'] \\ p_0\bar{r}\ulcorner\beta\urcorner \arrow[r, dotted, "\bar{\gamma}\bar{r}\ulcorner\beta\urcorner", "\cong"'] & rk\bar{r}\ulcorner\beta\urcorner \end{tikzcd}$$

where $\bar{\gamma}$ is a special case of the 2-cell just given this name to be described momentarily for which we will demonstrate that $\bar{\gamma}\bar{r}$ is an isomorphism. The composite $k\bar{\zeta}$ is a 2-cell induced from $p_0k\bar{\zeta} = pp_0\bar{\zeta} = p\zeta = \mathrm{id}$ and $p_1k\bar{\zeta} = p_1\bar{\zeta} = \mathrm{id}$, so by 2-cell conservativity, this is an isomorphism. Now ζ is a composite of three isomorphisms and hence is invertible.

To complete the proof, we must define $\bar{\gamma}$ and prove that $\bar{\gamma}\bar{r}$ is invertible. Specializing the construction just given, we define $\bar{\gamma}$ to be the induced 2-cell satisfying $p\bar{\gamma} = \mathrm{id}$ and $\kappa = p_1\epsilon k \cdot \bar{\gamma}$. Transposing across the fibered adjunction $i \dashv r$, it suffices to define the transposed 2-cell $\hat{\gamma}\colon ip_0 \Rightarrow k$ so that $p_1\hat{\gamma} = \kappa$ and $p_0\hat{\gamma} = \mathrm{id}$. And we may transpose once more along an adjunction $j \dashv p_0$ of (5.1.15) constructed by the dual of Lemma 3.5.8. The counit $\nu\colon jp_0 \Rightarrow \mathrm{id}$ of this adjunction satisfied the defining conditions that $p_0\nu = \mathrm{id}$ and $p_1\nu = \kappa$, so to construct $\hat{\gamma}$ satisfying the conditions just described, it suffices to define instead a 2-cell $\xi\colon i \Rightarrow kj$ satisfying the conditions $p_1\xi = \mathrm{id}$ and $p_0\xi = \mathrm{id}$. These conditions are satisfied by the fibered isomorphism $\xi\colon i \cong kj$ that arises by Proposition 3.4.7 since both 1-cells $E \rightrightarrows \mathsf{Hom}_B(B,p)$ are induced by the identity id_p. Unpacking these transpositions, $\bar{\gamma}$ is defined to be the composite

$$\bar{\gamma} :=\ p_0 \xRightarrow[\cong]{\eta} rip_0 \xRightarrow[\cong]{\xi} rkjp_0 \xRightarrow{rk\nu} rk$$

To verify that $\bar{\gamma}\bar{r}$ is invertible, it thus suffices to demonstrate that $rk\nu\bar{r}$ is invertible. To see this, we consider another pasting diagram

$$\begin{tikzcd} & E \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & & & E \arrow[dr, "j"] \arrow[d, phantom, "\Downarrow\nu"] & & & \\ \mathsf{Hom}_B(B,p) \arrow[ur, "r"] \arrow[rr, equal] & {} & \mathsf{Hom}_B(B,p) \arrow[urr, bend left, "r"] \arrow[r, "\bar{r}"] \arrow[rrrr, bend right, equal, "\cong\Downarrow\bar{\epsilon}"'] & E^2 \arrow[ur, "p_0"] \arrow[rr, equal] & {} & E^2 \arrow[r, "k"] & \mathsf{Hom}_B(B,p) \arrow[r, "r"] & E \end{tikzcd}$$

By the definition (5.1.10) of k, $\phi k\bar{r} = p\kappa\bar{r} = p\chi = \phi$, so Proposition 3.4.7 supplies a fibered isomorphism $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$ with $p_0\bar{\epsilon} = \mathrm{id}_{p_0}$ and $p_1\bar{\epsilon} = \mathrm{id}_{p_1}$.

Now naturality of whiskering supplies a commutative diagram of 2-cells

$$\begin{tikzcd} rkjp_0\bar{r}ir \arrow[r, Rightarrow, "rk\nu\bar{r}ir"] \arrow[d, Rightarrow, "rkjr\epsilon"', "\cong"] & rk\bar{r}ir \arrow[r, Rightarrow, "r\bar{\epsilon}ir", "\cong"'] \arrow[d, Rightarrow, "rk\bar{r}\epsilon"] & rir \arrow[d, Rightarrow, "r\epsilon", "\cong"'] \\ rkjp_0\bar{r} \arrow[r, Rightarrow, "rk\nu\bar{r}"'] & rk\bar{r} \arrow[r, Rightarrow, "\cong", "r\bar{\epsilon}"'] & r \end{tikzcd}$$

Since ϵ is the counit of an adjunction $i \dashv r$ with invertible unit, $r\epsilon$ is an isomorphism, so we see that $rk\nu\bar{r}$ is an isomorphism if and only if $rk\nu\bar{r}ir$ is. And this is the case since $k\nu\bar{r}i$ is an isomorphism by

2-cell conservativity: $p_0 k v \bar{r} i = \mathrm{id}_p$, while $p_1 k v \bar{r} i = p_1 \epsilon i$, which is an isomorphism again because ϵ is the counit of an adjunction $i \dashv r$ with invertible unit. □

One of the myriad applications of Theorem 5.1.11 is:

5.1.16. COROLLARY. *Cosmological functors preserve cartesian fibrations and cartesian natural transformations.*

PROOF. By Theorem 5.1.11(i)⇔(iii), an isofibration $p: E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ is cartesian if and only if the isofibration $k: E^2 \twoheadrightarrow \mathsf{Hom}_B(B, p)$ defined in Remark 5.1.13 admits a right adjoint right inverse. A cosmological functor $F: \mathcal{K} \to \mathcal{L}$ preserves the class of isofibrations and the simplicial limits that define the domain and codomain of this k. Moreover, cosmological functors preserve adjunctions and natural isomorphisms, so if this adjoint exists in $\mathcal{K}$ it also does in $\mathcal{L}$. Similarly, the internal characterization of p-cartesian natural transformations given by Theorem 5.1.11(iv)⇔(vi) is also preserved by cosmological functors. □

Another application of Theorem 5.1.11 is that it allows us to conclude that cartesianness is an equivalence-invariant property of isofibrations.

5.1.17. COROLLARY. *Consider a commutative square between isofibrations whose horizontals are equivalences*

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

Then p is a cartesian fibration if and only if q is a cartesian fibration in which case g preserves and reflects cartesian transformations: χ is q-cartesian if and only if $g\chi$ is p-cartesian.

PROOF. The commutative square of the statement induces a commutative square up to isomorphism whose horizontals are equivalences

$$\begin{tikzcd} F^2 \arrow[r, "g^2", "\sim"'] \arrow[d, "k"'] \arrow[dr, phantom, "\cong"] & E^2 \arrow[d, "k"] \\ \mathsf{Hom}_A(A, q) \arrow[r, "\sim", "\mathsf{Hom}_f(f,g)"'] & \mathsf{Hom}_B(B, p) \end{tikzcd}$$

By the equivalence-invariance of adjunctions, the left-hand vertical admits a right adjoint with invertible counit if and only if the right-hand vertical does, these adjunctions being defined in such a way that the mate of the given square is an isomorphism built by composing with the natural isomorphisms of the horizontal equivalences (see Proposition B.3.9). By Theorem 5.1.11(i)⇔(iii), it follows that p is cartesian if and only if q is cartesian.

Supposing the postulated adjoints exist, via their construction, the whiskered composites $g^2\eta$ and ηg^2 of the units of the respective adjunctions are isomorphic. Hence the component at an element $\ulcorner \chi \urcorner : X \to F^2$ of $g^2\eta$ is invertible if and only if the component of ηg^2 is invertible; since g^2 is an equivalence, the former is the case if and only if the component of η is invertible. By Theorem 5.1.11(iv)⇔(vi), this proves that χ is p-cartesian if and only if $g\chi$ is q-cartesian. □

In terminology we now introduce, the square defined by the equivalences and also the square defined by their inverses[3], defines a *cartesian functor* from q to p.

5.1.18. Definition (cartesian functor). Let $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow A$ be cartesian fibrations. A commutative square

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

defines a **cartesian functor** if g preserves cartesian transformations: if χ is q-cartesian then $g\chi$ is p-cartesian.

The internal characterization of cartesian functors makes use of a map between right representable comma ∞-categories induced by the commutative square $fq = pg$ defined by Proposition 3.4.5.

5.1.19. Theorem (an internal characterization of cartesian functors). *For a commutative square*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

between cartesian fibrations the following are equivalent:

(i) The square (g, f) defines a cartesian functor from q to p.

(ii) The mate of the canonical isomorphism

$$\begin{tikzcd} F \arrow[rr, "g"] \arrow[d, "i"'] & \arrow[d, phantom, "\cong\swarrow"] & E \arrow[d, "i"] \\ \mathsf{Hom}_A(A,q) \arrow[rr, "\mathsf{Hom}_f(f,g)"'] & {} & \mathsf{Hom}_B(B,p) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F \arrow[rr, "g"] & \arrow[d, phantom, "\cong\searrow"] & E \\ \mathsf{Hom}_A(A,q) \arrow[u, "r"] \arrow[rr, "\mathsf{Hom}_f(f,g)"'] & {} & \mathsf{Hom}_B(B,p) \arrow[u, "r"'] \end{tikzcd}$$

in the diagram of functors over $f\colon A \to B$ is an isomorphism.

(iii) The mate of the canonical isomorphism

$$\begin{tikzcd} F^2 \arrow[rr, "g^2"] \arrow[d, "k"'] & \arrow[d, phantom, "\cong\swarrow"] & E^2 \arrow[d, "k"] \\ \mathsf{Hom}_A(A,q) \arrow[rr, "\mathsf{Hom}_f(f,g)"'] & {} & \mathsf{Hom}_B(B,p) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F^2 \arrow[rr, "g^2"] & \arrow[d, phantom, "\cong\searrow"] & E^2 \\ \mathsf{Hom}_A(A,q) \arrow[u, "\bar{r}"] \arrow[rr, "\mathsf{Hom}_f(f,g)"'] & {} & \mathsf{Hom}_B(B,p) \arrow[u, "\bar{r}"'] \end{tikzcd}$$

in the diagram of functors is an isomorphism.

Proof. We will prove (i)⇔(ii) and (i)⇔(iii). The idea in each case is similar. Conditions (ii) and (iii) imply that g preserves the explicitly chosen cartesian lifts up to isomorphism, which by Lemma 5.1.3 implies that g preserves all cartesian lifts. Conversely, assuming (i), we need to show that a whiskered copy of the counit of $i \dashv r$ and of the unit of $k \dashv \bar{r}$ are isomorphisms. The counit of $i \dashv r$

[3]For any inverse equivalences g' and f' to g and f, there is a natural isomorphism $qg' \cong f'fqg' = f'pgg' \cong f'p$. Using the isofibration property of q of Proposition 1.4.10, g' may be replaced by an isomorphic functor g'', which also defines an inverse equivalence to g and for which the square $qg'' = f'p$ commutes strictly.

and unit of $k \dashv \bar{r}$ each encode the data of the factorizations of a natural transformation through the cartesian lift of its projection. It will follow from (i) that the cells in question are cartesian and the factorizations live over identities, so Lemma 5.1.5 will imply that these natural transformations are invertible.

(i)⇔(iii): For convenience, we take the functors k to be the isofibrations of Remark 5.1.13, so the square on the left hand side of (iii) commutes strictly and its mate is the 2-cell $\bar{\eta}g^2\bar{r}$. By Theorem 5.1.11(iv)⇔(vi), this component of $\bar{\eta}$ is invertible if and only if $g\chi$ is p-cartesian, where χ is the generic q-cartesian lift of Lemma 5.1.8 for the cartesian fibration $q\colon F \twoheadrightarrow A$; recall $\bar{r} = \ulcorner\chi\urcorner$. By that lemma again, $g\chi$ is p-cartesian if and only if g preserves cartesian transformations, since the other canonical q-cartesian lifts are constructed as restrictions of χ.

(i)⇔(ii): Let us write $\bar{g}$ for $\mathrm{Hom}_f(f,g)$ to save space. Since the unit of $i \dashv r$ is an isomorphism by Remark 5.1.12, the mate of the isomorphism on the left hand side of (ii) is isomorphic to $r\bar{g}\epsilon$, so our task is to show that this natural transformation is invertible if and only if g defines a cartesian functor. Recall from the proof of Theorem 5.1.11(ii)⇒(i) that $p_1\epsilon$ defines the generic q-cartesian lift of Lemma 5.1.8 for the cartesian fibration $q\colon F \twoheadrightarrow A$, whiskering with $\bar{g} \coloneqq \mathrm{Hom}_f(f,g)\colon \mathrm{Hom}_A(A,q) \to \mathrm{Hom}_B(B,p)$ carries this to a 2-cell whose projection with p_0 is an identity, since $p_0\epsilon = \mathrm{id}$, and whose projection along p_1 is $gp_1\epsilon$, by the commutativity of the left-hand portion of the diagram below.

$$\begin{tikzcd}
& F \arrow[r, "g"] \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & E \arrow[dr, "i"] & & E \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
\mathrm{Hom}_A(A,q) \arrow[ur, "r"] \arrow[rr, equals] & & \mathrm{Hom}_A(A,q) \arrow[r, "\bar{g}"'] \arrow[d, two heads, "p_1"'] & \mathrm{Hom}_B(B,p) \arrow[ur, "r"] \arrow[rr, equals] \arrow[d, two heads, "p_1"] & & \mathrm{Hom}_B(B,p) \arrow[d, two heads, "p_1"] \\
& & F \arrow[r, "g"'] & E \arrow[rr, equals] & & E
\end{tikzcd}$$

Now naturality of whiskering provides a commutative square of natural transformations:

$$\begin{tikzcd}
p_1irigr \arrow[r, Rightarrow, "r\bar{g}\epsilon"] \arrow[d, Rightarrow, "p_1\epsilon igr"', "\wr\|"] & r\bar{g} \arrow[d, Rightarrow, "p_1\epsilon\bar{g}"] \\
p_1\bar{g}ir \arrow[r, Rightarrow, "p_1\bar{g}\epsilon"'] & p_1\bar{g}
\end{tikzcd}$$

where we've simplified some of the names since $p_1i = \mathrm{id}$. Since ϵ is the counit of an adjunction $i \dashv r$ with invertible unit, ϵi is an isomorphism. Note $pr\bar{g}\epsilon = p_0\bar{g}\epsilon = \mathrm{id}$ and the right-hand vertical $p_1\epsilon\bar{g}$ is a p-cartesian lift of the restriction of $\phi\bar{g}$, this ϕ being the right comma cone over p, which equals the whiskered right comma cone $f\phi$, this ϕ being the right comma cone over q, by the definition of $\bar{g}$. The bottom horizontal $p_1\bar{g}\epsilon$ is similarly a lift of $f\phi = \phi\bar{g}$. So if g is a cartesian functor, the right-hand vertical and bottom horizontal are both p-cartesian lifts of a common 2-cell and the conservativity property implies that $r\bar{g}\epsilon$ is invertible. Conversely, if $r\bar{g}\epsilon$ is invertible, the $p_1\bar{g}\epsilon = gp_1\epsilon$ is isomorphic to a p-cartesian transformation and is consequently p-cartesian. Since Lemma 5.1.8 constructs the other canonical q-cartesian lifts as restrictions of $p_1\epsilon$, this is the case if and only if g is a cartesian functor. ◻

5.1.20. Proposition (pullback stability). *If*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square and p is a cartesian fibration then q is a cartesian fibration. Moreover, a natural transformation χ with codomain F is q-cartesian if and only if $g\chi$ is p-cartesian, and in particular the pullback square defines a cartesian functor.

Proof. We apply Theorem 5.1.11(i)⇒(iii) in the form described in Remark 5.1.13 to the cartesian fibration p, which yields a fibered adjunction

$$\begin{tikzcd} E^2 \arrow[rr, bend left, "k"] \arrow[rr, phantom, "\perp"] \arrow[dr, "k"'] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "\bar{r}"] \arrow[dl, equal] \\ & \mathsf{Hom}_B(B,p) & \end{tikzcd} \tag{5.1.21}$$

We will now argue that this functor k pulls back to the corresponding functor for q. To that end, first note that the top face of the following cube is a pullback since the front, back, and bottom faces are.

$$\begin{tikzcd} & & F^2 \arrow[dll, "g^2"'] \arrow[ddrr, bend left, "q^2"] \arrow[dd, dashed, "k"] & & \\ E^2 \arrow[ddrrr, bend left, "p^2"] \arrow[dd, dashed, "k"'] & & & & \\ & & \mathsf{Hom}_A(A,q) \arrow[dll, "\mathsf{Hom}_f(f,g)"'] \arrow[rr, two heads] \arrow[dd, two heads, "p_1"'] & & A^2 \arrow[dl, "f^2"] \arrow[dd, two heads, "p_1"] \\ \mathsf{Hom}_B(B,p) \arrow[rrr, two heads] \arrow[dd, two heads, "p_1"'] & & & B^2 \arrow[dd, two heads, "p_1"] & \\ & & F \arrow[dll, "g"'] \arrow[rr, two heads, "q"] & & A \arrow[dl, "f"] \\ E \arrow[rrr, two heads, "p"'] & & & B & \end{tikzcd}$$

The right adjoint $(-)^2$ preserves pullbacks, so F^2 is the pullback of p^2 along f^2, and since this pullback square factors through the top face of the cube along the square inducing the maps k, we conclude that this last square is a pullback, as claimed.

Now pullback defines a cosmological functor $\mathsf{Hom}_f(f,g)^* \colon \mathcal{K}_{/\mathsf{Hom}_B(B,p)} \to \mathcal{K}_{/\mathsf{Hom}_A(A,q)}$ that carries the fibered adjunction (5.1.21) to a fibered adjunction

$$\begin{tikzcd} F^2 \arrow[rr, bend left, "k"] \arrow[rr, phantom, "\perp"] \arrow[dr, "k"'] & & \mathsf{Hom}_A(A,q) \arrow[ll, dashed, bend left, "\bar{r}"] \arrow[dl, equal] \\ & \mathsf{Hom}_A(A,q) & \end{tikzcd}$$

which by Theorem 5.1.11(iii)⇒(i) proves that q is a cartesian fibration. Moreover, by construction of the adjunction $k \dashv \bar{r}$ as a pullback of the adjunction $k \dashv \bar{r}$, both of the mates in Theorem 5.1.19(iii) are identities, proving that (g, f) defines a cartesian functor.

To see that (g, f) creates cartesian natural transformations, note that a natural transformation χ with codomain F is represented by an element $\ulcorner\chi\urcorner \colon X \to F^{\mathbb{2}}$ and $g\chi$ is represented by the image of this element under the functor $g^{\mathbb{2}} \colon F^{\mathbb{2}} \to E^{\mathbb{2}}$. By Theorem 5.1.11(iv)⇔(vi), χ is q-cartesian just when the component $\bar{\eta}\ulcorner\chi\urcorner$ of the unit of $k \dashv \bar{r}$. This unit component $\eta\ulcorner\chi\urcorner$ is the pullback of the corresponding unit component $\eta g^{\mathbb{2}}\ulcorner\chi\urcorner$ indexed by $g\chi$, and by conservativity of the smothering functor

$$\mathsf{hFun}(X, F^{\mathbb{2}}) \to \mathsf{hFun}(X, \mathsf{Hom}_A(A, q)) \underset{\mathsf{hFun}(X, \mathsf{Hom}_B(B,p))}{\times} \mathsf{hFun}(X, E^{\mathbb{2}})$$

if $\eta g^{\mathbb{2}}\ulcorner\chi\urcorner$ is invertible, then so is $\eta\ulcorner\chi\urcorner$. □

Pullback squares provide a key instance of cartesian functors. Another is given by the following lemma, which can be proven using Theorem 5.1.19.

5.1.22. Lemma. *If*

$$\begin{tikzcd} F \arrow[rr, "g"] \arrow[dr, two heads, "q"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

is a functor between cartesian fibrations that admits a left adjoint over B, then g defines a cartesian functor.

Proof. If $\ell \dashv_B g$, then the cosmological functor $p_1^* \colon \mathcal{K}_{/B} \to \mathcal{K}_{/B^{\mathbb{2}}}$ carries this fibered adjunction to a fibered adjunction

$$\begin{tikzcd} \mathsf{Hom}_B(B, q) \arrow[rr, bend right, "\mathsf{Hom}_{\mathrm{id}_B}(\mathrm{id}_B, g)"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B, p) \arrow[ll, bend right, "\mathsf{Hom}_{\mathrm{id}_B}(\mathrm{id}_B, \ell)"'] \end{tikzcd}$$

Now both horizontal functors in the commutative square

$$\begin{tikzcd} F^{\mathbb{2}} \arrow[r, "g^{\mathbb{2}}"] \arrow[d, "k"'] \arrow[dr, phantom, "\cong\Swarrow"] & E^{\mathbb{2}} \arrow[d, "k"] \\ \mathsf{Hom}_A(A, q) \arrow[r, "\mathsf{Hom}_f(f,g)"'] & \mathsf{Hom}_B(B, p) \end{tikzcd}$$

admit left adjoints and a standard result from the calculus of mates tells us that the mate with respect to the vertical adjunctions $k \dashv \bar{r}$ is an isomorphism if and only if the mate with respect to the horizontal adjunctions is an isomorphism, the latter natural transformation between left adjoints being the transpose of the former natural transformation between their right adjoints. This is the case because the mate with respect to the left adjoints lies is the fiber of the smothering functor of Proposition 3.2.5 for $F^{\mathbb{2}}$. □

Examples of cartesian fibrations are overdue.

5.1.23. Proposition (domain projection). *For any ∞-category A, the domain-projection functor $p_0 \colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cartesian fibration. Moreover, a natural transformation χ with codomain $A^{\mathbb{2}}$ is p_0-cartesian if and only if $p_1\chi$ is invertible.*

Before giving the proof, we explain the idea. A 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "\ulcorner\beta\urcorner"] \arrow[dr, bend right, "a"'] \arrow[dr, phantom, "\Uparrow\alpha"] & A^{\mathbb{2}} \arrow[d, two heads, "p_0"] \\ & A \end{tikzcd}$$

defines a composable pair of 2-cells $\alpha\colon a \Rightarrow x$ and $\beta\colon x \Rightarrow y$ in $\mathsf{hFun}(X, A)$. Composing these we induce a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "\ulcorner\beta\circ\alpha\urcorner"] \arrow[r, bend right, "\ulcorner\beta\urcorner"'] \arrow[r, phantom, "\Downarrow\chi"] & A^{\mathbb{2}} \end{tikzcd}$ representing the commutative square

$$\begin{tikzcd} a \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\beta\circ\alpha"'] & x \arrow[d, Rightarrow, "\beta"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

so that $p_0\chi = \alpha$, as required, and $p_1\chi = \mathrm{id}$.[4]

Proof. We use Theorem 5.1.11(i)⇔(iii) and prove that p_0 is cartesian by constructing an appropriate adjoint to the functor

$$\begin{tikzcd} (A^{\mathbb{2}})^{\mathbb{2}} \arrow[dr, dashed, "k"] \arrow[drr, bend left, "p_0^{\mathbb{2}}"] \arrow[ddr, bend right, "p_1"'] & & \\ & \mathsf{Hom}_A(A, p_0) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ & A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{2}\times\mathbb{2} & & \\ & \mathbb{3} \arrow[ul, dashed] & \mathbb{2} \arrow[l] \arrow[ull, bend right, "0\times\mathbb{2}"'] \\ & \mathbb{2} \arrow[u] \arrow[uul, bend left, "\mathbb{2}\times 1"] & \mathbb{1} \arrow[l, hook', "0"] \arrow[u, hook, "1"'] \end{tikzcd}$$

defined by cotensoring with the 1-categories displayed above right.[5]

To construct a right adjoint with invertible counit to the map k, it suffices to construct a left adjoint left inverse to the inclusion of 1-categories $\mathbb{3} \hookrightarrow \mathbb{2}\times\mathbb{2}$ with image $(0,0) \to (0,1) \to (1,1)$. The left adjoint $\ell\colon \mathbb{2}\times\mathbb{2} \to \mathbb{3}$ is a left inverse on the image of $\mathbb{3}$ and sends $(1,0)$ to the terminal element of $\mathbb{3}$:

$$\mathbb{2}\times\mathbb{2} \ni \left\{\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}\right\} \qquad \mapsto \qquad \left\{\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r, equal] & 2 \end{tikzcd}\right\} \in \mathbb{3}$$

[4]While the weak universal property of the arrow ∞-category can be used to induce 2-cells with codomain $A^{\mathbb{2}}$, it cannot be used to prove equations between the induced 2-cells, as required to demonstrate that induction condition of Definition 5.1.1. Thus, some sort of ∞-cosmos-level argument is necessary to establish this result.

[5]The cotensor $A^{(-)}$ carries pushouts of simplicial sets to pullbacks of ∞-categories, and the pushout of $\mathbb{2} \cup_{\mathbb{1}} \mathbb{2}$ of simplicial sets is $\Lambda^1[2]$, not $\mathbb{3} = \Delta[2]$. However, on account of the equivalence of ∞-categories $A^{\mathbb{3}} \simeq A^{\Lambda^1[2]}$, no harm comes from making the indicated substitution.

Now

$$\begin{tikzcd} (A^2)^2 \arrow[r, phantom, "\perp"] & A^3 \simeq \mathrm{Hom}_A(A, p_0) \arrow[l, bend left, dashed, "A^\ell"] \end{tikzcd}$$

with $k\colon (A^2)^2 \to A^3$ the upper arrow, defines the desired right adjoint with invertible counit.

The characterization of p_0-cartesian transformations now follows from Theorem 5.1.11(iv)⇔(vi). A cell χ with codomain A^2 is p_0-cartesian if and only if it its representing element of $(A^2)^2$ is in the essential image of the right adjoint $A^\ell\colon A^3 \to A^{2\times 2}$. Clearly if χ is in the essential image then its codomain component must be invertible.

Conversely, suppose the codomain component

$$X \xrightarrow{\ulcorner\chi\urcorner} A^{2\times 2} \xrightarrow{\mathrm{ev}_{(1,-)}} A^2$$

represents a natural isomorphism. Applying Lemma 1.1.12 to $\mathrm{Fun}(X, A)$ one can build a diagram in $\mathrm{Fun}(X, A)^{2\times 2} \cong \mathrm{Fun}(X, A^{2\times 2})$ whose top edge is the given isomorphism, whose vertical edges are isomorphisms, and whose bottom edge is an identity. This glues onto χ to define a diagram $X \to A^{\Lambda^1[2]\times 2}$ and now the composite

$$X \longrightarrow A^{\Lambda^1[2]\times 2} \xrightarrow{\sim} A^{3\times 2} \xrightarrow{A^c} A^{2\times 2\times 2}$$

where $c\colon 2\times 2 \to 3$ is the surjective functor that sends $(0,0)$ and $(0,1)$ to 0, witnesses an isomorphism between $\ulcorner\chi\urcorner$ and a diagram in the image of A^ℓ, whose codomain component is the identity. □

The same argument proves that for any $g\colon C \to A$, the domain-projection $p_0\colon \mathrm{Hom}_A(A, g) \twoheadrightarrow A$ is a cartesian fibration. For $f\colon B \to A$, the domain-projections $p_0\colon \mathrm{Hom}_A(f, A) \twoheadrightarrow B$ and $p_0\colon \mathrm{Hom}_A(f, g) \twoheadrightarrow B$ are obtained by pullback via Proposition 5.1.20. These cartesian fibrations figures in a final important lemma about the class of p-cartesian transformations.

5.1.24. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a composable pair of natural transformations* $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\psi"] & E \end{tikzcd}$ *and* $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\psi'"] & E \end{tikzcd}$ *with codomain E.*

(i) If ψ and ψ' are p-cartesian, then so is $\psi \cdot \psi'$.
(ii) If ψ and $\psi \cdot \psi'$ are p-cartesian, then so is ψ'.

Proof. For (i), recall from the proof of Lemma 5.1.8 that a p-cartesian lift of $p\psi$ is given by the composite

$$\begin{tikzcd} X \arrow[r, "\ulcorner p\psi\urcorner"] & \mathrm{Hom}_B(B, p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$$

with the natural transformation $\chi\colon r \Rightarrow p_1$ of (5.1.9) whose domain r is the right adjoint of Theorem 5.1.11(ii). Since we are assuming that ψ is p-cartesian, ψ is isomorphic to this whiskered natural transformation so we may redefine ψ to equal it and redefine ψ' to absorb the isomorphism. By Lemma 5.1.4, this new ψ' is still p-cartesian since we've assumed ψ' is. This modification does not change the composition transformation $\psi \cdot \psi'$ that we desire to show is p-cartesian.

Now by 2-cell induction, the diagram as below-left defines a 2-cell as below-right:

$$\begin{tikzcd} pe'' \arrow[r, Rightarrow, "p\psi'"] \arrow[d, Rightarrow, "p(\psi\cdot\psi')"'] & pe' \arrow[d, Rightarrow, "p\psi"] \\ pe \arrow[r, equal] & pe \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "\ulcorner p(\psi\cdot\psi')\urcorner"] \arrow[r, bend right, "\ulcorner p\psi \urcorner"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B,p) \end{tikzcd}$$

By the generalization of Proposition 5.1.23, γ is a p_0-cartesian cell. By Lemma 5.1.22, the fibered right adjoint r

$$\begin{tikzcd} \mathsf{Hom}_B(B,p) \arrow[rr, bend right, "r"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "p_0"'] & & E \arrow[ll, bend right, "i"'] \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

carries γ to a p-cartesian transformation $r\gamma$.

Now the horizontal composite

$$\begin{tikzcd} X \arrow[r, bend left, "\ulcorner p(\psi\cdot\psi')\urcorner"] \arrow[r, bend right, "\ulcorner p\psi \urcorner"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B,p) \arrow[r, bend left, "r"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$$

provides a commutative diagram of natural transformations

$$\begin{tikzcd} \bar{e} \arrow[r, Rightarrow, "r\gamma"] \arrow[d, Rightarrow, "\chi_{p(\psi\cdot\psi')}"'] & e' \arrow[d, Rightarrow, "\psi"] \\ e \arrow[r, equal] & e \end{tikzcd}$$

In particular, the composite $\psi \cdot r\gamma$ is a p-cartesian natural transformation. Since $pr\gamma = p_0\gamma = p\psi'$, $r\gamma$ is a p-cartesian lift of the cartesian transformation ψ', so ψ' and $r\gamma$ are isomorphic. But now $\psi \cdot \psi'$ is isomorphic to the p-cartesian transformation $\chi_{p(\psi\cdot\psi')}$, and Lemma 5.1.4 proves that $\psi\cdot\psi'$ is p-cartesian.

Now for (ii) suppose ψ and $\psi \cdot \psi'$ are p-cartesian and let $\chi' \colon \bar{e} \Rightarrow e'$ denote a p-cartesian lift of $p\psi'$. Consider the factorization $\psi' = \chi' \cdot \theta$ of ψ' through its p-cartesian lift with $p\theta = \mathrm{id}$. Now part (i) implies that $\psi \cdot \chi'$ is p-cartesian and $\psi \cdot \psi' = \psi \cdot \chi' \cdot \theta$, so Lemma 5.1.5 implies that θ is an isomorphism. Now ψ' is isomorphic to a p-cartesian cell so Lemma 5.1.4 implies that ψ' must be p-cartesian. □

This result allows us to strengthen the conclusion of Lemma 5.1.4(ii).

5.1.25. Corollary. *Let $p \colon E \twoheadrightarrow B$ be a cartesian fibration. Then any p-cartesian lift of an isomorphism is an isomorphism.*

Proof. Let $\chi \colon e' \Rightarrow e$ be a p-cartesian lift of an isomorphism $\beta \colon b' \cong b$. Applying the induction property of χ, we obtain a lift of $\chi' \colon e \Rightarrow e'$ of β^{-1} so that $\chi \cdot \chi' = \mathrm{id}_e$. Since χ and id_e are both p-cartesian, Lemma 5.1.24(ii) tells us that χ' is as well. Now by Lemma 5.1.24(i), $\chi' \cdot \chi$ is a p-cartesian lift of $\mathrm{id}_{b'}$ and hence by Lemma 5.1.4(ii) this composite is also an isomorphism. Therefore, we conclude that χ is invertible. □

For ∞-categories admitting pullbacks, the codomain-projection functor also defines a cartesian fibration:

5.1.26. Proposition (codomain projection). *Let A be an ∞-category that admits pullbacks in the sense of Definition 4.3.7. Then the codomain-projection functor $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ is a cartesian fibration and the p_1-cartesian arrows are just those 2-cells that represent pullback squares.*

Proof. Via Theorem 5.1.11(i)$\Leftrightarrow$(iii), we desire a right adjoint right inverse to the functor k defined below-left applying A^{-} to the diagram of simplicial sets appearing below-right:

$$\begin{tikzcd} (A^{\mathbb{2}})^{\mathbb{2}} \arrow[dr, dashed, two heads, "k"] \arrow[drr, bend left, two heads, "p_1^{\mathbb{2}}"] \arrow[ddr, bend right, two heads, "p_1"'] & & \\ & \mathrm{Hom}_A(A, p_1) \arrow[r, two heads] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ & A^{\mathbb{2}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{2} \times \mathbb{2} & & \\ & \lrcorner \arrow[ul, dashed] & \mathbb{2} \arrow[l] \arrow[ull, bend right, hook', "1\times\mathbb{2}"'] \\ & \mathbb{2} \arrow[u, hook] \arrow[uul, bend left, hook, "\mathbb{2}\times 1"] & \mathbb{1} \arrow[l, hook', "1"] \arrow[u, hook', "1"'] \end{tikzcd}$$

This is done in Corollary 4.3.5:

$$\begin{tikzcd} (A^{\mathbb{2}})^{\mathbb{2}} \cong A^{\square} \arrow[r, bend left, two heads, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & A^{\lrcorner} \cong \mathrm{Hom}_A(A, p_1) \arrow[l, bend left, dashed, "\mathrm{ran}"] \end{tikzcd}$$
$\square$

Exercises.

5.1.i. Exercise. Prove Lemma 5.1.4.

5.1.ii. Exercise. Attempt to prove directly from Definition 5.1.6 that cosmological functors preserve cartesian fibrations and explain what goes wrong.

5.1.iii. Exercise. Show that a cartesian fibration $p \colon E \twoheadrightarrow B$ defines an "incoherent pseudofunctor" $E \colon \mathrm{h}B^{\mathrm{op}} \rightsquigarrow \mathfrak{h}\mathcal{K}$ given by the data:

- a mapping on objects $b \in \mathrm{h}B \mapsto E_b \in \mathfrak{h}\mathcal{K}$;
- a mapping on 1-cells $\beta \colon a \to b \in \mathrm{h}B \mapsto \beta^* \colon E_b \to E_a \in \mathfrak{h}\mathcal{K}$ defined by

$$\begin{tikzcd} E_b \arrow[r, bend left, "\ell_b"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ 1 \arrow[r, bend left, "b"] \arrow[r, phantom, "\Uparrow\beta"] \arrow[r, bend right, "a"'] & B \end{tikzcd} = \begin{tikzcd} E_b \arrow[rr, bend left, "\ell_b"] \arrow[rr, phantom, bend left=20, "\Uparrow\chi_\beta"] \arrow[dr, dashed, "\beta^*"'] \arrow[rr, dotted, "\beta^*(\ell_b)"] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_a \arrow[ur, dashed, "\ell_a"'] \arrow[dl] & \\ 1 \arrow[rr, bend right, "a"'] & & B \end{tikzcd} \tag{5.1.27}$$

- an invertible 2-cell $\begin{tikzcd} E_b \arrow[r, bend left, equals] \arrow[r, phantom, "\iota_b \Downarrow\cong"] \arrow[r, bend right, "\mathrm{id}_b^*"'] & E_b \end{tikzcd} \in \mathfrak{h}\mathcal{K}$ for each $b \in \mathrm{h}B$; and

- an invertible 2-cell

$$\begin{tikzcd} & E_b \arrow[dr, "\beta^*"] & \\ E_c \arrow[ur, "\gamma^*"] \arrow[rr, "(\gamma\circ\beta)^*"'] & \arrow[phantom, "\alpha_{\beta,\gamma}\Downarrow\cong"] & E_a \end{tikzcd}$$

 in $\mathfrak{h}\mathcal{K}$ for each composable pair of morphisms in $\mathsf{h}B$.

The coherence conditions present in the full definition of a **pseudo-functor**, which appears as Definition 14.4.1, are not evidently satisfied here, but do follow from the extension of this construction to a homotopy coherent diagram indexed by the underlying quasi-category of B. This will be proven in Chapter 17.

5.1.iv. Exercise. Categorify the intuition that cartesian fibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow B$ define "contravariant B-indexed functors valued in ∞-categories" by proving that a cartesian functor

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

defines a "natural transformation": show that there exists a natural isomorphism in the square of fibers

$$\begin{tikzcd} E_b \arrow[r, "g"] \arrow[d, "\beta^*"'] \arrow[dr, phantom, "\exists\cong"] & F_b \arrow[d, "\beta^*"] \\ E_a \arrow[r, "g"'] & F_a \end{tikzcd}$$

where the action of an arrow β in the homotopy category of B on the fibers is defined by factoring the domain of a p- or q-cartesian lift of β as displayed in (5.1.27).

5.1.v. Exercise. Use Proposition 5.1.20 to prove that cartesian functors pull back.

5.2. Cocartesian fibrations and bifibrations

By the dual of Proposition 5.1.23, the codomain-projection functor is also a cocartesian fibration, a notion we now introduce.

5.2.1. Definition (p-cocartesian transformations). Let $p\colon E \twoheadrightarrow B$ be an isofibration. A natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ with codomain E is p-**cocartesian** if

(i) **induction**: Given any natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e''"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "pe'"] \arrow[r, bend right, "pe''"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ so that $p\tau = \gamma \cdot p\chi$, there exists a lift $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e''"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of γ so that $\tau = \bar{\gamma} \cdot \chi$.

$$\begin{tikzcd} e \arrow[rr, Rightarrow, "\tau"] \arrow[dr, Rightarrow, "\chi"'] & & e'' & & \mathsf{hFun}(X,E) \arrow[dd, two heads, "p_*"] \\ & e' \arrow[ur, Rightarrow, dotted, "\bar{\gamma}"'] & & \in & \\ & \arrow[d, maps to] & & & \\ pe \arrow[rr, Rightarrow, "p\tau"] \arrow[dr, Rightarrow, "p\chi"'] & & pe'' & \in & \mathsf{hFun}(X,B) \\ & pe' \arrow[ur, Rightarrow, "\gamma"'] & & & \end{tikzcd}$$

(ii) **conservativity**: Any fibered endomorphism of χ is invertible: if $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\zeta \cdot \chi = \chi$ and $p\zeta = \mathbf{id}_{pe'}$ then ζ is invertible.

5.2.2. Definition (cocartesian fibration). An isofibration $p\colon E \twoheadrightarrow B$ is a **cocartesian fibration** if

(i) Any natural transformation $\beta\colon pe \Rightarrow b$ as below-left admits a lift $\chi_\beta\colon e \Rightarrow \beta_* e$ as below-right

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Downarrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "\beta_* e"'] \arrow[r, phantom, "\Downarrow\chi_\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

that is a p-cartesian transformation so that $p\chi = \beta$.

(ii) The class of p-cocartesian transformations is closed under restriction along any functor: that is, if $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ is p-cocartesian and $f\colon Y \to X$ is any functor then $\begin{tikzcd} Y \arrow[r, bend left, "ef"] \arrow[r, bend right, "e'f"'] \arrow[r, phantom, "\Downarrow\chi f"] & E \end{tikzcd}$ is p-cocartesian.

The dual to Theorem 5.1.11 asks for left adjoints to $i\colon E \to \mathsf{Hom}_B(p,B)$ and $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ in place of right adjoints. See Exercise 5.2.i. This result can be deduced immediately by considering the isofibration $p\colon E \twoheadrightarrow B$ as a map in the dual ∞-cosmos of Definition 1.2.23. Recall that for any ∞-cosmos $\mathcal{K}$, there is a **dual ∞-cosmos** $\mathcal{K}^{\mathrm{co}}$ with the same objects but with functor spaces defined by:

$$\mathsf{Fun}_{\mathcal{K}^{\mathrm{co}}}(A,B) := \mathsf{Fun}_{\mathcal{K}}(A,B)^{\mathrm{op}}.$$

The isofibrations, equivalences, and trivial fibrations in $\mathcal{K}^{\mathrm{co}}$ coincide with those of $\mathcal{K}$. Conical limits in $\mathcal{K}^{\mathrm{co}}$ coincide with those in $\mathcal{K}$, while the cotensor of $A \in \mathcal{K}$ with $U \in s\mathcal{S}et$ is defined to be $A^{U^{\mathrm{op}}}$. In particular, the cotensor of an ∞-category with $\mathbb{2}$ is defined to be $A^{\mathbb{2}^{\mathrm{op}}}$, which exchanges the domain and codomain projections from arrow and comma ∞-categories.

5.2.3. Definition. An isofibration $p\colon E \twoheadrightarrow B$ defines a **bifibration** if p is both a cartesian fibration and a cocartesian fibration.

Projections give trivial examples of bifibrations.

5.2.4. Example. For any ∞-categories A and B the projection functor $\pi\colon A\times B \twoheadrightarrow B$ is a bifibration, in which a 2-cell with codomain $A\times B$ is π-cocartesian or π-cartesian if and only if its composite with the projection $\pi\colon A\times B \twoheadrightarrow A$ is an isomorphism.

5.2.5. Proposition. *Let $p\colon E \twoheadrightarrow B$ be a bifibration. Then any arrow* $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$ *induces a fibered adjunction*

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\beta_*"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p_a"'] & & E_b \arrow[ll, bend left, "\beta^*"] \arrow[dl, two heads, "p_b"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} E_a \arrow[r, "\ell_a"] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] & E_b \arrow[l, "\ell_b"'] \arrow[d, two heads, "p_b"] \arrow[dl, phantom, "\llcorner", very near start] \\ X \arrow[r, "a"'] & B & X \arrow[l, "b"] \end{tikzcd}$$

between the fibers of p over a and b.

As will be remarked upon following the proof of this result, the left adjoint β_* is the covariantly pseudofunctorial action of the arrow β on the fibers of p defined in (5.0.1), while the right adjoint β^* is the dual contravariantly pseudofunctorial action.

Proof. Write $\ulcorner\beta\urcorner\colon X \to B^{\mathbb{2}}$ for the functor induced by β. Note that the pullbacks defining the fibers over its domain edge a and codomain edge b factor as:

$$\begin{tikzcd} E_a \arrow[r] \arrow[rr, bend left, two heads, "\ell_a"] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_B(p,B) \arrow[r, two heads, "p_0"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] & \mathsf{Hom}_B(B,p) \arrow[l, two heads, "p_1"] \arrow[d, two heads, "q"] \arrow[dl, phantom, "\llcorner", very near start] & E_b \arrow[l] \arrow[ll, bend right, two heads, "\ell_b"'] \arrow[d, two heads, "p_b"] \arrow[dl, phantom, "\llcorner", very near start] \\ X \arrow[r, "\ulcorner\beta\urcorner"] \arrow[rr, bend right, "a"'] & B^{\mathbb{2}} \arrow[r, two heads, "p_0"] & B & B^{\mathbb{2}} \arrow[l, two heads, "p_1"'] & X \arrow[l, "\ulcorner\beta\urcorner"'] \arrow[ll, bend left, "b"] \end{tikzcd}$$

Now via Remark 5.1.13, Theorem 5.1.11(i)$\Rightarrow$(iii) and its dual provide a right adjoint right inverse to $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ and a left adjoint right inverse to $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$. Composing the former fibered adjunction with $q\colon \mathsf{Hom}_B(B,p) \twoheadrightarrow B^{\mathbb{2}}$ and the latter fibered adjunction with $q\colon \mathsf{Hom}_B(p,B) \twoheadrightarrow B^{\mathbb{2}}$ we obtain a composable pair of adjunctions

$$\begin{tikzcd} \mathsf{Hom}_B(p,B) \arrow[r, bend left, "\bar{\ell}"] \arrow[r, phantom, "\perp"] \arrow[dr, two heads, "q"'] & E^{\mathbb{2}} \arrow[l, bend left, two heads, "k"] \arrow[r, bend left, two heads, "k"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "p^{\mathbb{2}}"] & \mathsf{Hom}_B(B,p) \arrow[l, bend left, "\bar{r}"] \arrow[dl, two heads, "q"] \\ & B^{\mathbb{2}} & \end{tikzcd}$$

fibered over $B^{\mathbb{2}}$; note in both cases that $qk = p^{\mathbb{2}}$. Pulling back the composite adjunction along $\ulcorner\beta\urcorner\colon X \to B^{\mathbb{2}}$ yields the fibered adjunction of the statement. $\square$

5.2.6. Remark (action of arrows on the fibers of a co/cartesian fibration). If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration but not a cartesian fibration, the construction in the proof of Proposition 5.2.5 still defines the functor $\beta_*\colon E_a \to E_b$. Examining the details of this construction, we see that it produces the

functor given this name in (5.0.1). The functor just constructed as the pullback over $\ulcorner\beta\urcorner$ of $k\bar{\ell}$ is induced by the composite commutative square

$$\begin{tikzcd} E_a \arrow[r] \arrow[d, two heads, "p_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_B(p,B) \arrow[r, "p_1\bar{\ell}"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "\ulcorner\beta\urcorner"'] & B^{\mathbb{2}} \arrow[r, two heads, "p_1"'] & B \end{tikzcd}$$

which defines a cone over the pullback defining E_b. The top-horizontal functor takes the codomain-component of the p-cocartesian lift of β with domain ℓ_a. This recovers the description given at the start of this chapter.

Exercises.

5.2.i. Exercise. Formulate the dual to Theorem 5.1.11, providing an internal characterization of cocartesian fibrations.

5.2.ii. Exercise. Prove that for any ∞-category A, the codomain-projection functor $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cocartesian fibration.

5.3. The quasi-category theory of cartesian fibrations

In this section, we reinterpret the notion of cartesian fibration and cartesian natural transformation from the point of view of the ∞-cosmos $\mathcal{K}$, rather than its quotient homotopy 2-category $\mathfrak{h}\mathcal{K}$. In doing so, we recall that the functors $e \colon X \to E$ between ∞-categories are precisely the vertices, or *0-arrows*, in the quasi-categorical functor space $\mathsf{Fun}(X,E)$. The 1-simplices, or *1-arrows*, $\chi \colon e' \to e$ of $\mathsf{Fun}(X,E)$ represent natural transformations $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$. Every natural transformation from e' to e is represented by a 1-arrow $e' \to e$ and a parallel pair of 1-arrows represent the same natural transformation if and only if they are *homotopic*, bounding a 2-arrow in $\mathsf{Fun}(X,E)$ whose 0th or 2nd edge is degenerate.

Before analyzing the 0- and 1-arrows in the functor spaces in particular, we consider the collection of functor spaces of an ∞-cosmos globally and prove another important corollary of the internal characterization of cartesian fibrations. The notion of cartesian fibrations are representably defined in the following sense:

5.3.1. Proposition. *Let $p \colon E \twoheadrightarrow B$ be an isofibration between ∞-categories in an ∞-cosmos $\mathcal{K}$. Then p is a cartesian fibration if and only if:*

(i) For all $X \in \mathcal{K}$, the isofibration $p_* \colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ is a cartesian fibration between quasi-categories.

(ii) For all $f \colon Y \to X \in \mathcal{K}$, the square

$$\begin{tikzcd} \mathsf{Fun}(X,E) \arrow[r, "f^*"] \arrow[d, two heads, "p_*"'] & \mathsf{Fun}(Y,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X,B) \arrow[r, "f^*"'] & \mathsf{Fun}(Y,B) \end{tikzcd}$$

is a cartesian functor.

Proof. If $p\colon E \twoheadrightarrow B$ is a cartesian fibration, then Theorem 5.1.11(i)⇒(iii) constructs a right adjoint right inverse to $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B,p)$. The simplicial bifunctor $\mathsf{Fun}(-,-)\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{QC}at$ defines a 2-functor $\mathsf{Fun}(-,-)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \times \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, which transposes to a Yoneda-type embedding $\mathsf{Fun}(-,-)\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at^{\mathfrak{h}\mathcal{K}^{\mathrm{op}}}$ from the homotopy 2-category of $\mathcal{K}$ to the 2-category of 2-functors, 2-natural transformations, and modifications. This 2-functor carries the adjunction $k \dashv \bar{r}$ to an adjunction in the 2-category $\mathfrak{h}\mathcal{QC}at^{\mathfrak{h}\mathcal{K}^{\mathrm{op}}}$. This latter adjunction defines, for each $X \in \mathcal{K}$, a right adjoint right inverse adjunction

$$\begin{tikzcd} \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}}) \arrow[r, bend left, two heads, "k_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \cong \mathsf{Hom}_{\mathsf{Fun}(X,B)}(\mathsf{Fun}(X,B),p_*) \arrow[l, bend left, "\bar{r}_*"] \end{tikzcd}$$

and for each $f\colon Y \to X$ in $\mathcal{K}$, a strict adjunction morphism[6] commuting strictly with the left adjoints and with the right adjoints:

$$\begin{tikzcd} \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}}) \arrow[r, "f^*"] \arrow[d, two heads, "k_*"] & \mathsf{Fun}(Y,E^{\mathbb{2}}) \cong \mathsf{Fun}(Y,E)^{\mathbb{2}} \arrow[d, two heads, "k_*"'] \\ \mathsf{Hom}(\mathsf{Fun}(X,B),p_*) \cong \mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \arrow[r, "f^*"'] \arrow[u, bend left, "\bar{r}_*"] \arrow[u, phantom, "\vdash"] & \mathsf{Fun}(Y,\mathsf{Hom}_B(B,p)) \cong \mathsf{Hom}(\mathsf{Fun}(Y,B),p_*) \arrow[u, bend right, "\bar{r}_*"'] \arrow[u, phantom, "\dashv"] \end{tikzcd} \tag{5.3.2}$$

By Theorems 5.1.11(iii)⇒(i) and 5.1.19(iii)⇒(i), this demonstrates the two conditions of the statement.

Conversely, supposing $p\colon E \twoheadrightarrow B$ satisfies conditions (i) and (ii) by Theorems 5.1.11(i)⇒(iii) and 5.1.19(i)⇒(iii) there is a commutative square $k_*f^* = f^*k_*$ where both verticals k_* admit right adjoint right inverses $k_* \dashv \bar{r}$ and the mate of the identity $k_*f^* = f^*k_*$ defines an isomorphism $f^*\bar{r} \cong \bar{r}f^*$. By Proposition B.6.2 in Appendix B, this data suffices to internalize the right adjoints to the representable functors $k_*\colon \mathsf{Fun}(X,E^{\mathbb{2}}) \to \mathsf{Fun}(X,\mathsf{Hom}_B(B,p))$ to a functor $\bar{r}_*\colon \mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \to \mathsf{Fun}(X,E^{\mathbb{2}})$ arising from post-composition with some $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$. This right adjoint $\bar{r}$ is extracted as the image of the identity element

$$\begin{array}{ccc} \mathsf{Fun}(\mathsf{Hom}_B(B,p),\mathsf{Hom}_B(B,p)) & \xrightarrow{\bar{r}} & \mathsf{Fun}(\mathsf{Hom}_B(B,p),E^{\mathbb{2}}) \\ \mathrm{id} & \longmapsto & \bar{r} \end{array}$$

and the unit and counit are internalized similarly; the condition on mates is used to verify the triangle equalities that demonstrate that $k \dashv \bar{r}$. Now Theorem 5.1.11(iii)⇒(i) proves that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. □

An easier argument along the same lines demonstrates:

[6] A **strict adjunction morphism** is given by a pair of functors, the horizontals of (5.3.2), that define strictly commutative squares with both the left and with the right adjoints and so that the units of each adjunction whisker along these functors to each other and the counits of each adjunction whisker along these functors to each other. See Proposition B.6.2 for more.

5.3.3. Corollary. *A commutative square between cartesian fibrations as displayed below-left*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{Fun}(X,F) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] & \mathsf{Fun}(X,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X,A) \arrow[r, "f_*"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

defines a cartesian functor in an ∞-cosmos $\mathcal{K}$ if and only if for all $X \in \mathcal{K}$, the square displayed above right defines a cartesian functor between cartesian fibrations of quasi-categories.

Proof. Exercise 5.3.i. □

Our aim is now to characterize those 1-arrows in $\mathsf{Fun}(X,E)$ that represent p-cartesian natural transformation for some cartesian fibration $p\colon E \twoheadrightarrow B$. Recall that the 1-arrows in $\mathsf{Fun}(X,E)$ are in bijection with the 0-arrows of $\mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}})$.

5.3.4. Definition (p-cartesian 1-arrow). Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a 1-arrow χ in $\mathsf{Fun}(X,E)$, defining an element $\chi \in \mathsf{Fun}(X,E)^{\mathbb{2}} \cong \mathsf{Fun}(X,E^{\mathbb{2}})$. Say χ is a p-**cartesian 1-arrow** if it is isomorphic to some object in the image of the right adjoint right inverse functor

$$\mathsf{Fun}(X,\mathsf{Hom}_B(B,p)) \xrightarrow{\bar{r}_*} \mathsf{Fun}(X,E^{\mathbb{2}})$$

of Theorem 5.1.11(iii).

The new notion of p-cartesian 1-arrow coincides exactly with the previous notion of p-cartesian natural transformation:

5.3.5. Lemma. *Consider a cartesian fibration $p\colon E \twoheadrightarrow B$ between ∞-categories. For a 1-arrow $\chi\colon e' \to e$ in $\mathsf{Fun}(X,E)$, the following are equivalent:*

(i) χ is a p-cartesian 1-arrow.

(ii) χ represents a p-cartesian natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$.

(iii) χ represents a natural transformation

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd}$$

that is cartesian for $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$.

Conversely, a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ *is p-cartesian if and only if any representing 1-arrow $\ulcorner\chi\urcorner\colon X \to E^{\mathbb{2}}$ satisfies all of these equivalent conditions.*

Proof. We start with the final clause. Note from Exercise 3.2.i that homotopic 1-arrows are isomorphic as objects of $\mathsf{Fun}(X,E^{\mathbb{2}})$ so if some 1-arrow representing a p-cartesian transformation is a p-cartesian 1-arrow then all representatives of that natural transformation are.

Now the equivalence (i)⇔(ii) follows from Theorem 5.1.11(iv)⇔(vi) once we establish that a 1-arrow of $\mathsf{Fun}(X,E)$, when encoded as a functor $\chi\colon X \to E^{\mathbb{2}}$ is in the essential image of $\bar{r}_*$ if and only if the component $\bar{\eta}\chi$ of the unit of $k \dashv \bar{r}$ is invertible.

If $\chi\colon e' \to e$ is a p-cartesian 1-arrow, then by definition there exists some $\beta\colon X \to \mathsf{Hom}_B(B,p)$ and an invertible 2-cell

$$\begin{tikzcd} X \arrow[rr, bend left, "\chi"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\Downarrow\cong"] & E^2 \\ & \mathsf{Hom}_B(B,p) \arrow[ur, "\bar{r}"'] & \end{tikzcd}$$

The unit $\bar{\eta}$ of the adjunction $k \dashv \bar{r}$ of Theorem 5.1.11(iii) has the property that $\bar{\eta}\bar{r}$ is invertible, so the component $\bar{\eta}\bar{r}\beta$ is invertible and so $\bar{\eta}\chi$ is also invertible. By Theorem 5.1.11(vi)⇒(iv), this implies that χ represents a p-cartesian natural transformation.

Conversely, if χ defines a p-cartesian natural transformation then Theorem 5.1.11(iv)⇒(vi) tells us that for any representing 1-arrow $\ulcorner\chi\urcorner\colon X \to E^2$, the component $\bar{\eta}\ulcorner\chi\urcorner$ is an isomorphism. In particular, $\ulcorner\chi\urcorner$ is isomorphic to $\bar{r}k\ulcorner\chi\urcorner$, which proves that $\ulcorner\chi\urcorner$ is in the essential image of $\bar{r}_*$ and thus defines a p-cartesian 1-arrow.

Finally, via the characterization of cartesian transformations given in Theorem (vi) and the fact that the adjunction $k \dashv \bar{r}$ in $\mathfrak{h}\mathcal{K}$ induces an adjunction $k_* \dashv \bar{r}_*$ in $\mathfrak{h}\mathcal{QC}at$, the conditions (ii) and (iii) are tautologically equivalent. The former refers to the adjunction between categories $\mathsf{hFun}(X,E)$ and $\mathsf{hFun}(X,B)$, while the latter refers to the adjunction between categories $\mathsf{hFun}(\mathbb{1},\mathsf{Fun}(X,E))$ and $\mathsf{hFun}(\mathbb{1},\mathsf{Fun}(X,B))$ and these are isomorphic. ☐

Combining Lemma 5.3.5 with Proposition 5.3.1 we arrive at a new equivalent definition of cartesian fibrations.

5.3.6. Corollary. *An isofibration $p\colon E \twoheadrightarrow B$ is a* ***cartesian fibration*** *if and only if*

(i) Any 1-arrow with codomain B admits a p-cartesian lift with specified codomain 0-arrow.
(ii) p-cartesian 1-arrows are stable under precomposition with 0-arrows. ☐

5.3.7. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration and consider a 2-arrow*

$$\begin{tikzcd} & e' \arrow[dr, "\psi"] & \\ e'' \arrow[ur, "\psi'"] \arrow[rr, "\psi''"'] & & e \end{tikzcd}$$

in $\mathsf{Fun}(X,E)$.

(i) If ψ and ψ' are p-cartesian 1-arrows, so is ψ''.
(ii) If ψ and ψ'' are p-cartesian 1-arrows, so is ψ'.
(iii) If ψ and ψ'' are p-cartesian 1-arrows and $p\psi'$ is invertible, then ψ' is invertible.

Proof. Via Lemma 5.3.5, (i) and (ii) are Lemmas 5.1.24(i) and (ii), while (iii) is Lemma 5.1.5. ☐

5.3.8. Lemma. *A 2-cell as below left*

$$\begin{tikzcd} Q \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "q"] & Q \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & \mathsf{Fun}(X,E) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "e'q"] \arrow[r, bend right, "eq"'] \arrow[r, phantom, "\Downarrow\chi q"] & E \end{tikzcd}$$

is p_*-cartesian if and only if each of its components χq is p-cartesian.

Proof. If χ is p_*-cartesian, then so is the restriction χq along any element $q\colon \mathbb{1} \to Q$. By Lemma 5.3.5 this tells us that χq defines a p-cartesian transformation.

Conversely, if χq is a p-cartesian transformation, then Lemma 5.3.5 tells us that χq is a p_*-cartesian transformation. Now consider the factorization $\chi = \chi_\chi \cdot \theta$ through p_*-cartesian lift χ_χ of $p_*\chi$. Because the components χq of χ are p_*-cartesian, the components θq of θ are isomorphisms. By Lemma 16.2.1, an arrow in an exponetial $\mathsf{Fun}(X, E)^Q$ is an isomorphism if and only if it is a pointwise isomorphism, so this implies that χ. By isomorphism stability of cartesian transformations, we thus conclude that χ is p_*-cartesian. ☐

Exercises.

5.3.i. Exercise. Prove Corollary 5.3.3.

5.4. Discrete cartesian fibrations

Recall from Definition 1.2.24 that an object E in an ∞-cosmos $\mathcal{K}$ is **discrete** if for all $X \in \mathcal{K}$, the functor-space $\mathsf{Fun}(X, E)$ is a Kan complex. Since a quasi-category is a Kan complex just when its homotopy category is a groupoid (see Corollary 1.1.15), equivalently, E is discrete if and only if every natural transformation with codomain E is invertible.

From this definition it follows that an isofibration $p\colon E \twoheadrightarrow B$, considered as an object of $\mathcal{K}_{/B}$, is discrete if and only if any 2-cell with codomain E that whiskers with p to an identity is invertible. In fact, the discrete objects are exactly those isofibrations that define conservative functors in $\mathfrak{h}\mathcal{K}$.

5.4.1. Lemma. *An isofibration $p\colon E \twoheadrightarrow B$ is a discrete object of $\mathcal{K}_{/B}$ if and only if $p\colon E \twoheadrightarrow B$ is a* ***conservative functor****: meaning any* $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & E \end{tikzcd}$ *for which $p\gamma$ is an isomorphism is invertible.*

Proof. Exercise 5.4.i. ☐

Our aim in this section is to study a special class of cartesian fibrations and cocartesian fibrations:

5.4.2. Definition. An isofibration $p\colon E \twoheadrightarrow B$ is a **discrete cartesian fibration** if it is a cartesian fibration and if it is discrete as an object of $\mathcal{K}_{/B}$. Dually, an isofibration $p\colon E \twoheadrightarrow B$ is a **discrete cocartesian fibration** if it is a cocartesian fibration and if it is discrete as an object of $\mathcal{K}_{/B}$

The fibers of a discrete object $p\colon E \twoheadrightarrow B$ in $\mathcal{K}_{/B}$ are discrete ∞-categories. In fact, in ∞-cosmoi $\mathcal{K}$ whose objects model $(\infty, 1)$-categories, an isofibration $p\colon E \twoheadrightarrow B$ defines a discrete object in $\mathcal{K}_{/B}$ if and only if its fibers are discrete ∞-categories, as will be proven in Proposition 16.2.3. Thus, in such ∞-cosmoi, the discrete cartesian fibrations and discrete cocartesian fibrations are "∞-groupoid-valued pseudofunctors."

There is also a direct 2-categorical characterization of the discrete cartesian fibrations, which reveals that, unlike the case for cartesian and cocartesian fibrations, for their discrete analogues, there are no special classes of p-cartesian or p-cocartesian cells.

5.4.3. Proposition.

(i) If $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, every natural transformation with codomain E is p-cartesian.

(ii) An isofibration $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration if and only if every 2-cell $\beta\colon b \Rightarrow pe$ has an essentially unique lift: given $\chi\colon e' \Rightarrow e$ and $\psi\colon e'' \Rightarrow e$ so that $p\chi = p\psi = \beta$, then there exists an isomorphism $\gamma\colon e'' \Rightarrow e'$ with $\chi \cdot \gamma = \psi$ and $p\gamma = \mathrm{id}$.

Note that (i) implies immediate that any commutative square

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

from a cartesian fibration q to a discrete cartesian fibration p defines a cartesian functor.

PROOF. By the definition of cartesian fibration, any 2-cell ψ with codomain E factors through a p-cartesian lift of $p\psi$ along a 2-cell γ so that $p\gamma = \mathrm{id}$. The discrete objects of $\mathcal{K}_{/B}$ are exactly those isofibrations with the property that any 2-cell with codomain E that whiskers with p to an identity is invertible. In particular, γ is an isomorphism, and now ψ is isomorphic to a p-cartesian transformation and hence by Lemma 5.1.4 itself p-cartesian.

By (i) and Lemma 5.1.3, it's now clear that if $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, then any 2-cell $\beta\colon b \Rightarrow pe$ has an essentially unique lift. For the converse, note first that any $p\colon E \twoheadrightarrow B$ satisfying this hypothesis is a discrete object: if $\psi\colon e' \Rightarrow e$ is so that $p\psi = \mathrm{id}$, then $\mathrm{id}\colon e \Rightarrow e$ is another lift of $p\psi$ and essential uniqueness provides an inverse isomorphism $\psi^{-1}\colon e \Rightarrow e'$.

To complete the proof, we now show that any 2-cell $\chi\colon e' \Rightarrow e$ is cartesian for p and to that end consider a pair $\tau\colon e'' \Rightarrow e$ and $\gamma\colon pe'' \Rightarrow pe'$ so that $p\tau = p\chi \cdot \gamma$. By the hypothesis that every 2-cell admits an essentially unique lift, we can construct a lift $\mu\colon \bar{e} \Rightarrow e'$ so that $p\mu = \gamma$. Now τ and $\chi \cdot \mu$ are two lifts of $p\tau$ with the same codomain, so there exists an isomorphism $\theta\colon e'' \Rightarrow \bar{e}$ with $p\theta = \mathrm{id}$. The composite $\mu \cdot \theta$ then defines the desired lift of γ to a cell so that $\tau = \chi \cdot \mu \cdot \theta$. ☐

5.4.4. EXAMPLE (domain projection from an element). For an element $b\colon 1 \to B$, the domain-projection functor $p_0\colon \mathsf{Hom}_B(B, b) \twoheadrightarrow B$ is a discrete cartesian fibration. Cartesianness was established in Proposition 5.1.23 and discreteness follows immediately from 2-cell conservativity. If $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Hom}_B(B, b) \end{tikzcd}$ is a natural transformation for which $p_0\gamma$ is an identity, then since $p_1\gamma$ is also an identity, γ must be invertible.

Dually, the codomain-projection functor $p_1\colon \mathsf{Hom}_B(b, B) \twoheadrightarrow B$ is a discrete cocartesian fibration.

5.4.5. LEMMA (pullback stability). *If*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square and p is a discrete cartesian fibration then q is a discrete cartesian fibration.

PROOF. In light of Proposition 5.1.20 it remains only to verify that q is discrete. Consider a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & F \end{tikzcd}$ so that $q\gamma$ is invertible. Then $fq\gamma = pg\gamma$ is invertible and conservativity of p implies that $g\gamma$ is invertible.

By Lemma 3.1.5, the pullback square of functor spaces

$$\begin{tikzcd} \mathsf{Fun}(X,F) \arrow[r, "g_*"] \arrow[d, two heads, "q_*"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(X,A) \arrow[r, "f_*"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

induces a smothering functor

$$\mathsf{hFun}(X,F) \to \mathsf{hFun}(X,E) \underset{\mathsf{hFun}(X,B)}{\times} \mathsf{hFun}(X,A)$$

We've just verified that the image of γ is an isomorphism, so conservativity implies that γ is also invertible. $\square$

In analogy with Theorem 5.1.11, there is an internal characterization of discrete cartesian fibrations, which in the discrete case takes a much simpler form. Recall any functor $p\colon E \to B$ induces functors $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ and $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ as in (5.1.10) by applying p to the generic arrow for E.

5.4.6. PROPOSITION (internal characterization of discrete fibrations). *An isofibration $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration if and only if the functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ is an equivalence and a discrete cocartesian fibration if and only if the functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ is an equivalence.*

Recall from Theorem 5.1.11(iii) that $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(p,B)$ admits a right adjoint with invertible counit. Proposition 5.4.6 asserts that p defines a discrete cartesian fibration if and only if the unit of that adjunction, a natural transformation that defines the factorization of any natural transformation with codomain E through the canonical p-cartesian lift of its image under p, is an isomorphism, in which case that adjunction defines an adjoint equivalence and all natural transformations with codomain E are p-cartesian.

PROOF. Assume first that $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration. By Theorem 5.1.11(i)⇒(iii), $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ then admits a right adjoint $\bar{r}$ with invertible counit $\bar{\epsilon}\colon k\bar{r} \cong \mathrm{id}$. We will show that in this case the unit $\bar{\eta}\colon \mathrm{id} \Rightarrow \bar{r}k$ is also invertible, proving that $k \dashv \bar{r}$ defines an adjoint equivalence.

Since the counit of $k \dashv \bar{r}$ is invertible, $k\bar{\eta}$ is an isomorphism. Thus $p_1 k\bar{\eta} = p_1\bar{\eta}$ and $p_0 k\bar{\eta} = pp_0\bar{\eta}$ are both isomorphisms. By conservativity of the discrete fibration $p\colon E \twoheadrightarrow B$ proven in Lemma 5.4.1, this implies that $p_0\bar{\eta}$ is invertible and now 2-cell conservativity for $E^{\mathbb{2}}$ reveals that $\bar{\eta}$ is an isomorphism.

Conversely, if $k\colon E^{\mathbb{2}} \xrightarrow{\sim} \mathsf{Hom}_B(B,p)$ is an equivalence, by Proposition 2.1.11, we may choose a right adjoint equivalence inverse $\bar{r}$. The counit of this adjoint equivalence is necessarily an isomorphism, so by Theorem 5.1.11(iii)⇒(i) we know that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. Since the unit of $k \dashv \bar{r}$ is also an isomorphism, Theorem 5.1.19(vi)⇒(iv) tells us that every natural transformation with codomain E is p-cartesian, and now the conservativity property for cartesian transformations of Lemma 5.1.5 tells us that $p\colon E \twoheadrightarrow B$ defines a conservative fun ctor, and in particular is discrete. $\square$

Since equivalences and simplicial limits in an ∞-cosmos are representably-defined notions, it follows immediately from Proposition 5.4.6 that:

5.4.7. PROPOSITION. *An isofibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ defines a discrete cartesian fibration if and only if for all $X \in \mathcal{K}$, the functor $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ defines a discrete cartesian fibration of quasi-categories.* $\square$

Using the internal characterization, it is straightforward to verify that discrete cartesian fibrations compose and cancel on the left:

5.4.8. Lemma.

(i) If $p\colon E \twoheadrightarrow B$ and $q\colon B \twoheadrightarrow A$ are discrete cartesian fibrations, so is $qp\colon E \twoheadrightarrow A$.
(ii) If $p\colon E \twoheadrightarrow B$ is an isofibration and $q\colon B \twoheadrightarrow A$ and $qp\colon E \twoheadrightarrow A$ are discrete cartesian fibrations, then so is $p\colon E \twoheadrightarrow B$.

Proof. By considering the defining pullback diagrams, the map $E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_A(A, qp)$ that tests whether $qp\colon E \twoheadrightarrow A$ is a discrete cartesian fibration factors as the map $E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$ that tests whether $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration followed by a pullback of the map $B^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_A(A, q)$ that tests whether $q\colon B \twoheadrightarrow A$ is a discrete cartesian fibration:

$$\begin{tikzcd}
E^{\mathbb{2}} \arrow[dr, dashed, two heads] \arrow[drrr, bend left, two heads] \arrow[dddr, bend right, two heads, "p_1"'] \\
& \mathsf{Hom}_B(B,p) \arrow[rr, two heads] \arrow[dr, dashed, two heads] \arrow[dd, two heads, "p_1"'] \arrow[drr, phantom, "\lrcorner" near start] & & B^{\mathbb{2}} \arrow[rr, two heads] \arrow[dr, dashed, two heads] \arrow[dd, two heads, "p_1"'] & & A^{\mathbb{2}} \arrow[dd, two heads, "p_1"] \\
& & \mathsf{Hom}_A(A,qp) \arrow[rr, two heads] \arrow[dl, two heads] \arrow[urrr, two heads] \arrow[dr, phantom, "\lrcorner" near start] & & \mathsf{Hom}_A(A,q) \arrow[dl, two heads] \arrow[ur, two heads] \arrow[dr, phantom, "\lrcorner" near start] \\
& E \arrow[rr, two heads, "p"'] & & B \arrow[rr, two heads, "q"'] & & A
\end{tikzcd}$$

Both parts now follow from the 2-of-3 property. ☐

The internal characterization of discrete cartesian fibrations is useful for establishing further examples.

5.4.9. Lemma. *A trivial fibration $p\colon E \xrightarrow{\sim\!\!\!\!\rightarrow} B$ is a discrete bifibration.*

Proof. Recall from Remark 5.1.13, that the canonical functors $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B, p)$ and $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(p, B)$ can be constructed as the Leibniz cotensor of the monomorphism $1\colon \mathbb{1} \hookrightarrow \mathbb{2}$ in the first case and $0\colon \mathbb{1} \hookrightarrow \mathbb{2}$ in the second with the trivial fibration $p\colon E \xrightarrow{\sim\!\!\!\!\rightarrow} B$. By Lemma 1.2.11, both maps are trivial fibrations and in particular equivalences. Now Proposition 5.4.6 proves that p is a discrete cartesian fibration and also a discrete cocartesian fibration. ☐

A final important family of examples of discrete cartesian fibrations are worth establishing. Proposition 5.1.23 proves that for any ∞-category A, the domain-projection functor $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$ defines a cartesian fibration. Thus functor does *not* define a discrete cartesian fibration in the ∞-cosmos $\mathcal{K}$, but recall that p_0-cartesian lifts can be constructed to project to identity arrows along $p_1\colon A^{\mathbb{2}} \twoheadrightarrow A$. This suggests that we might productively consider the domain-projection functor as a map over A, in which case we have the following result:

5.4.10. Proposition. *The functor*

$$\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & A \times A \arrow[dl, two heads, "\pi"] \\
& A
\end{tikzcd} \tag{5.4.11}$$

defines a discrete cartesian fibration in the slice ∞-cosmos $\mathcal{K}_{/A}$.

Proof. Note that 2-cell conservativity implies that (5.4.11) is a discrete object in $(\mathcal{K}_{/A})_{/\pi\colon A\times A\twoheadrightarrow A} \cong \mathcal{K}_{/A\times A}$, so it remains only to prove that this functor defines a cartesian fibration. We prove this using Theorem 5.1.11(i)⇔(ii). The first step is to compute the right representable comma object for the functor (5.4.11) by interpreting the formula (3.4.2) in the slice ∞-cosmos $\mathcal{K}_{/A}$ using Proposition 1.2.19. The $\mathbb{2}$-cotensor of the object $\pi\colon A\times A \twoheadrightarrow A$ is $\pi\colon A\times A^{\mathbb{2}} \twoheadrightarrow A$, so this right representable comma is computed by the left-hand pullback in $\mathcal{K}_{/A}$ below:

$$\begin{tikzcd}
\mathsf{Hom}_A(A,p_0) \arrow[rr] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddr, dotted, two heads, "p_2"] & & A\times A^{\mathbb{2}} \arrow[r, "\pi"] \arrow[d, two heads, "\mathrm{id}_A\times p_1"] \arrow[ddl, bend right] \arrow[dr, phantom, very near start, "\lrcorner"] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\
A^{\mathbb{2}} \arrow[rr, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & A\times A \arrow[r, "\pi"] \arrow[dl, two heads, "\pi"] & A \\
& A & &
\end{tikzcd}$$

Pasting with the right-hand pullback in $\mathcal{K}$, we recognize that the ∞-category so-constructed coincides with the right representable comma object for the functor $p_0\colon A^{\mathbb{2}} \twoheadrightarrow A$ considered as a map in $\mathcal{K}$. Under the equivalence $\mathsf{Hom}_A(A,p_0) \simeq A^{\mathbb{3}}$ established in the proof of Proposition 5.1.23, the isofibration $p_2\colon \mathsf{Hom}_A(A,p_0) \twoheadrightarrow A$ is evaluation at the final element $2 \in \mathbb{3}$ in the composable pair of arrows. Similarly, the canonical functor $i\colon A^{\mathbb{2}} \to \mathsf{Hom}_A(A,p_0)$ induced by id_{p_0} in $\mathcal{K}$ coincides with the canonical functor $i\colon A^{\mathbb{2}} \to \mathsf{Hom}_A(A,p_0)$ over A induced by $\mathrm{id}_{(p_1,p_0)}$ in $\mathcal{K}_{/A}$.

Now applying Proposition 5.1.23 and Theorem 5.1.11(i)⇒(ii) in $\mathcal{K}$, this functor i admits a right adjoint r over the *domain*-projection functor

$$\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "p_0"'] & & \mathsf{Hom}_A(A,p_0) \arrow[ll, bend left, dashed, "r"] \arrow[dl, two heads, "p_0"] \\
& A &
\end{tikzcd}$$

By the proofs of Theorem 5.1.11(iii)⇒(ii) and Proposition 5.1.23 this adjunction can be constructed by cotensoring $A^{(-)}$ the composite adjunction of categories

$$\begin{tikzcd}
\mathbb{2} \arrow[r, bend right, "\mathbb{2}\times 0"'] \arrow[r, phantom, "\top"] \arrow[rr, bend right=50, "\delta^1"'] & \mathbb{2}\times\mathbb{2} \arrow[l, bend right, "\mathbb{2}\times !"'] \arrow[r, bend right, "\ell"'] \arrow[r, phantom, "\top"] & \mathbb{3} \arrow[l, bend right, "k"'] \arrow[ll, bend right=50, "\sigma^0"'] \\
& \mathbb{1}+\mathbb{1} \arrow[ul, hook, "{(1,0)}"] \arrow[ur, hook, "{(2,0)}"'] &
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
A^{\mathbb{2}} \arrow[rr, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "{(p_1,p_0)}"'] & & A^{\mathbb{3}} \arrow[ll, bend left, "r"] \arrow[dl, two heads, "{(p_2,p_0)}"] \\
& A\times A &
\end{tikzcd}$$

where $\ell \dashv k$ is described in the proof of Proposition 5.1.23. The composite right adjoint is the functor $\sigma^0\colon \mathbb{3} \twoheadrightarrow \mathbb{2}$ that sends 0 and 1 to 0 and 2 to 1, while the composite left adjoint is the functor $\delta^1\colon \mathbb{2} \rightarrowtail \mathbb{3}$ that sends 0 to 0 and 1 to 2. In particular, this adjunction lies in the strict slice 2-category under the inclusion of the "endpoints" of $\mathbb{2}$ and $\mathbb{3}$.

It follows that upon cotensoring into A, we obtain a fibered adjunction over $A\times A$, which by Theorem 5.1.11(ii)⇒(i) implies that (5.4.11) is a cartesian fibration in $\mathcal{K}_{/A}$, completing the proof. □

Combining Propositions 5.1.23 and Proposition 5.4.10, we can now generalize both results to arbitrary comma ∞-categories.

5.4.12. Corollary. *For any functors* $C \xrightarrow{g} A \xleftarrow{f} B$ *between ∞-categories in an ∞-cosmos $\mathcal{K}$:*

(i) The domain-projection functor $p_0 \colon \mathrm{Hom}_A(f,g) \twoheadrightarrow B$ is a cartesian fibration. Moreover, a natural transformation χ with codomain $\mathrm{Hom}_A(f,g)$ is p_0-cartesian if and only if $p_1\chi$ is invertible.

(ii) The codomain-projection functor $p_1 \colon \mathrm{Hom}_A(f,g) \twoheadrightarrow C$ is a cocartesian fibration. Moreover, a natural transformation χ with codomain $\mathrm{Hom}_A(f,g)$ is p_1-cartesian if and only if $p_0\chi$ is invertible.

(iii) The functor

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a discrete cartesian fibration in $\mathcal{K}_{/C}$.

(iv) The functor

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_0"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

defines a discrete cocartesian fibration in $\mathcal{K}_{/B}$.

Proof. We prove (i) and (iii) and leave the dualizations to the reader. For (iii), we first use the cosmological functor $g^* \colon \mathcal{K}_{/A} \to \mathcal{K}_{/C}$, which preserves discrete cartesian fibrations, to establish that

$$\begin{tikzcd} \mathrm{Hom}_A(A,g) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & C \times A \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a discrete cartesian fibration in $\mathcal{K}_{/C}$; this argument works because $p_1 \colon \mathrm{Hom}_A(A,g) \twoheadrightarrow C$ is the pullback of $p_1 \colon A^{\mathbb{2}} \twoheadrightarrow A$ along g. Now

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[rr] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[ddr, two heads, "p_1"] \arrow[drr, phantom, "\lrcorner" very near start] & & \mathrm{Hom}_A(A,g) \arrow[d, two heads, "{(p_1,p_0)}"] \arrow[ddl, two heads, "p_1"'] \\ C \times B \arrow[rr, "C \times f"] \arrow[dr, two heads, "\pi"'] & & C \times A \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

is a pullback square in $\mathcal{K}_{/C}$, so Lemma 5.4.5 now implies that the pullback is also a discrete cartesian fibration.

Using (iii) we can now prove (i). This follows directly from a general claim that if

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & C & \end{tikzcd}$$

defines a cartesian fibration in $\mathcal{K}_{/C}$ then $p\colon E \twoheadrightarrow B$ defines a cartesian fibration in $\mathcal{K}$. By Theorem 5.1.11(i)⇔(ii), this functor defines a cartesian fibration in $\mathcal{K}_{/C}$ if and only if the functor i

$$\begin{tikzcd} E \arrow[rr, bend left, "i"] \arrow[dr, "(q,p)"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, "(qp_1,p_0)"] \\ & C \times B & \end{tikzcd}$$

admits a right adjoint r over $C \times B$. Composing with $\pi\colon C \times B \twoheadrightarrow B$, this fibered adjunction defines an adjunction over B, and Theorem 5.1.11(i)⇔(ii) applied this time in $\mathcal{K}$ allows us to conclude that $p\colon E \twoheadrightarrow B$ is a cartesian fibration. □

Note that the domain projection $p_0\colon \mathsf{Hom}_A(f,A) \twoheadrightarrow B$ is the pullback of $p_0\colon A^2 \twoheadrightarrow A$ along $f\colon B \to A$, so Proposition 5.1.20 proves directly from Proposition 5.1.23 that this functor is a cartesian fibration, but $p_0\colon \mathsf{Hom}_A(A,g) \twoheadrightarrow A$ is not similarly a pullback of $p_0\colon A^2 \twoheadrightarrow A$. This is why a more circuitous argument to the general result is needed.

Exercises.

5.4.i. Exercise. Prove Lemma 5.4.1.

5.5. The external Yoneda lemma

Let $b\colon 1 \to B$ be an element of an ∞-category B and consider its right representation $\mathsf{Hom}_B(B,b)$ as a comma ∞-category. In this case, there is no additional data given by the codomain-projection functor, but Example 5.4.4 observes that the domain-projection functor $p_0\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$ has a special property: it defines a discrete cartesian fibration. The fibers of this map over an element $a\colon 1 \to B$ are the internal mapping spaces $\mathsf{Hom}_B(a,b)$ of Definition 3.4.9. In this way, the right representation of the element b encodes the contravariant functor represented by b, which is why all along we've been referring to the comma ∞-categories $\mathsf{Hom}_B(B,b)$ as "representable."

Our aim in this section is to state and prove the Yoneda lemma in this setting, where contravariant representable functors are encoded as discrete cartesian fibrations. A dual statement applies to covariant representable functors encoded as discrete cocartesian fibration $p_1\colon \mathsf{Hom}_B(b,B) \twoheadrightarrow B$, but for ease of exposition we leave the dualization to the reader. Informally, the Yoneda lemma asserts that "evaluation at the identity defines an equivalence," so the first step towards the statement of the Yoneda lemma is to introduce this identity element, which in fact is something we've already encountered.

The identity arrow id_b induces an element $\ulcorner \mathrm{id}_b \urcorner\colon 1 \to \mathsf{Hom}_B(B,b)$ which Corollary 3.5.9 proves is terminal in the ∞-category $\mathsf{Hom}_B(B,b)$ of arrows in B with codomain b. The identity element inclusion defines a functor over B

$$\begin{tikzcd} 1 \arrow[rr, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[dr, "b"'] & & \mathsf{Hom}_B(B,b) \arrow[dl, "p_0"] \\ & B & \end{tikzcd} \tag{5.5.1}$$

Technically, this functor does not live in the sliced ∞-cosmos $\mathcal{K}_{/B}$ because the domain object $b\colon 1 \to B$ is not an isofibration but nevertheless for any isofibration $p\colon E \twoheadrightarrow B$, restriction along $\ulcorner \mathrm{id}_b \urcorner$ induces

a functor between sliced quasi-categorical functor spaces

$$\mathsf{Fun}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \xrightarrow{\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \overset{b}{\to} B, E \overset{p}{\twoheadrightarrow} B)$$

Here the codomain is the quasi-category defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_B(b,p) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(1,E) \arrow[d, two heads, "p_*"] \\ \mathbb{1} \arrow[r, "b"] & \mathsf{Fun}(1,B) \end{tikzcd}$$

which is isomorphic to $\mathsf{Fun}(1, E_b)$, the underlying quasi-category of the fiber E_b of $p\colon E \twoheadrightarrow B$ over b.

If a discrete cartesian fibration over B is thought of as a B-indexed discrete ∞-category valued contravariant functor, then maps of discrete cartesian fibrations over B are "natural transformations": the "naturality in B" arises because we only allow functors over B. This leads to our first statement of the fibrational Yoneda lemma:

5.5.2. Theorem (external Yoneda lemma, discrete case). *If $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration, then*

$$\mathsf{Fun}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \xrightarrow{\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \overset{b}{\to} B, E \overset{p}{\twoheadrightarrow} B) \cong \mathsf{Fun}(1, E_b)$$

is an equivalence of Kan complexes.

Theorem 5.5.2 is subsumed by a generalization that allows $p\colon E \twoheadrightarrow B$ to be any cartesian fibration, not necessarily discrete. In this case, p encodes an "∞-category-valued contravariant B-indexed functor," as does $p_0\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$. The correct notion of "natural transformation" between two such functors is now given by a cartesian functor over B; see Exercise 5.1.iv. To that end, for a pair of cartesian fibration $q\colon F \twoheadrightarrow B$ and $p\colon E \twoheadrightarrow B$, we write

$$\mathsf{Fun}^{\mathsf{cart}}_B(F \overset{q}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \subset \mathsf{Fun}_B(F \overset{q}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \tag{5.5.3}$$

for the sub quasi-category containing all those simplices whose vertices define *cartesian* functors from q to p.[7]

5.5.4. Theorem (external Yoneda lemma). *If $p\colon E \twoheadrightarrow B$ is a cartesian fibration, then*

$$\mathsf{Fun}^{\mathsf{cart}}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \xrightarrow{\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(1 \overset{b}{\to} B, E \overset{p}{\twoheadrightarrow} B) \cong \mathsf{Fun}(1, E_b)$$

is an equivalence of quasi-categories.

The proofs of these theorems overlap significantly and we develop them in parallel. The basic idea is to use the universal property of $\ulcorner \mathrm{id}_b \urcorner$ as a terminal element of $\mathsf{Hom}_B(B,b)$ to define a right adjoint to $\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}$ and prove that when $p\colon E \twoheadrightarrow B$ is discrete or when the domain is restricted to the sub-quasi-category of cartesian functors, this adjunction defines an adjoint equivalence. Note that the functor $\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}$ is the image of the functor $\ulcorner \mathrm{id}_b \urcorner$ under the 2-functor $\mathsf{Fun}_B(-,p)\colon \mathfrak{h}(\mathcal{K}_{/B})^{\mathrm{op}} \to \mathcal{QC}at$. If the adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ lived in the slice ∞-cosmos $\mathcal{K}_{/B}$, this would directly construct a right adjoint to $\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}$. The main technical difficulty in following the outline just given is that the adjunction that witnesses the terminality of $\ulcorner \mathrm{id}_b \urcorner$ does not live in the slice of the homotopy 2-category $\mathfrak{h}\mathcal{K}_{/B}$ but rather in a lax slice of the homotopy 2-category, that we now introduce.

[7]For any quasi-category Q and any subset S of its vertices, there is a "full" sub-quasi-category $Q_S \subset Q$ containing exactly those vertices and all the simplices of Q that they span.

5.5.5. DEFINITION. Consider a 2-category $\mathfrak{h}\mathcal{K}$ and an object $B \in \mathfrak{h}\mathcal{K}$. The **lax slice 2-category** $\mathfrak{h}\mathcal{K}_{/\!/B}$ is the strict 2-category whose

- objects are maps $f\colon X \to B$ in $\mathfrak{h}\mathcal{K}$ with codomain B;
- 1-cells are diagrams

$$\begin{tikzcd} X \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{5.5.6}$$

  in $\mathfrak{h}\mathcal{K}$; and
- 2-cells from the 1-cell displayed above to the 1-cell below-right are 2-cells $\theta\colon k \Rightarrow k'$ so that

$$\begin{tikzcd} X \arrow[rr, bend left, "k'"] \arrow[rr, "k"'] \arrow[rr, phantom, bend left=20, "\Uparrow\theta"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} = \begin{tikzcd} X \arrow[rr, "k'"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\alpha'}{\Rightarrow}"] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

5.5.7. LEMMA. *The identity functor* (5.5.1) *is right adjoint to the right comma cone*

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "!"] \arrow[dr, two heads, "p_0"'] & \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & 1 \arrow[dl, "b"] \\ & B & \end{tikzcd}$$

in $\mathfrak{h}\mathcal{K}_{/\!/B}$.

PROOF. Since 1 is the terminal ∞-category, we take the counit of the postulated adjunction to be the identity. By Definition 5.5.5 to define the unit, we must provide a 2-cell:

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, equal] \arrow[rr, bend left, "\ulcorner \mathrm{id}_b \urcorner !"] \arrow[rr, phantom, bend left=20, "\Uparrow\eta"] \arrow[dr, two heads, "p_0"'] & & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd} = \begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[r, "!"] \arrow[dr, two heads, "p_0"'] & 1 \arrow[r, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[d, "b"] \arrow[dl, phantom, "\overset{\phi}{\Rightarrow}"] & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

so that $p_0\eta = \phi$. This is the defining property of the unit in Lemma 3.5.8. The forgetful 2-functor $\mathfrak{h}\mathcal{K}_{/\!/B} \to \mathfrak{h}\mathcal{K}$ is faithful on 1- and 2-cells, so the verification of the triangle equalities in Lemma 3.5.8 proves that they also hold in $\mathfrak{h}\mathcal{K}_{/\!/B}$. $\square$

Using somewhat non-standard 2-categorical techniques, we will transfer the adjunction of Lemma 5.5.7 to an adjunction between the quasi-categories $\mathsf{Fun}_B(b,p)$ and $\mathsf{Fun}_B(p_0,p)$; see Proposition 5.5.14. Because our initial adjunction lives in the lax rather than the strict slice, the construction will be somewhat delicate, passing through a pair of auxiliary 2-categories that we now introduce.

5.5.8. DEFINITION. Let $\mathfrak{h}\mathcal{K}$ be the homotopy 2-category of an ∞-cosmos and write $\mathfrak{h}\mathcal{K}^{\lrcorner}$ for the strict 2-category whose

- objects are cospans

$$A \xrightarrow{f} B \twoheadleftarrow{p} E$$

  in which p is a cartesian fibration;

- 1-cells are diagrams of the form

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "a"'] \arrow[dr, phantom, "\Uparrow\phi"] & B' \arrow[d, "b"] & E' \arrow[l, two heads, "p'"'] \arrow[d, "e"] \\ A \arrow[r, "f"'] & B & E \arrow[l, two heads, "p"] \end{tikzcd} \tag{5.5.9}$$

- and whose 2-cells consist of triples $\alpha\colon a \Rightarrow \bar{a}$, $\beta\colon b \Rightarrow \bar{b}$, and $\epsilon\colon e \Rightarrow \bar{e}$ between the verticals of parallel 1-cell diagrams so that $p\epsilon = \beta p'$ and $\bar{\phi} \cdot f\alpha = \beta f' \cdot \phi$.

5.5.10. Definition. Let $\mathfrak{h}\mathcal{K}$ be the homotopy 2-category of an ∞-cosmos and write $\mathfrak{h}\mathcal{K}^{\boxdot}$ for the strict 2-category whose

- objects are pullback squares

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

whose verticals are cartesian fibrations;
- 1-cells are cubes

$$\begin{tikzcd} F' \arrow[rr, "g'"] \arrow[dr, "\ell"'] \arrow[dd, two heads, "q'"'] & & E' \arrow[dr, "e"] \arrow[dd, two heads, "p'" near start] \arrow[dl, phantom, "\Uparrow\chi"] & \\ & F \arrow[rr, "g" near start] \arrow[dd, two heads, "q" near start] & & E \arrow[dd, two heads, "p"] \\ A' \arrow[rr, "f'" near end] \arrow[dr, "a"'] & & B' \arrow[dr, "b"] \arrow[dl, phantom, "\Uparrow\phi"] & \\ & A \arrow[rr, "f"'] & & B \end{tikzcd} \tag{5.5.11}$$

whose vertical faces commute and in which $\chi\colon g\ell \Rightarrow eg'$ is a p-cartesian lift of $\phi q'$; and
- whose 2-cells are given by quadruples $\alpha\colon a \Rightarrow \bar{a}$, $\beta\colon b \Rightarrow \bar{b}$, $\epsilon\colon e \Rightarrow \bar{e}$, and $\lambda\colon \ell \Rightarrow \bar{\ell}$ in which ϵ and λ are, respectively, lifts of $\beta p'$ and $\alpha q'$ and so that $\phi \cdot f\alpha = \beta f' \cdot \phi$ and $\bar{\chi} \cdot g\lambda = \epsilon g' \cdot \chi$.

These definitions are arranged so that there is an evident forgetful 2-functor $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$.

5.5.12. Lemma. *The forgetful 2-functor $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is a smothering 2-functor.*

Proof. Proposition 5.1.20 tells us that $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is surjective on objects. To see that it is full on 1-cells, first form the pullbacks of the cospans in (5.5.9), then define χ to be any p-cartesian lift of $\phi q'$ with codomain eg'. By construction, the domain of χ lies strictly over faq' and so this functor factors uniquely through the pullback leg g defining the map ℓ of (5.5.11).

To prove that $\mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}^{\lrcorner}$ is full on 2-cells, consider a parallel pair of 1-cells in $\mathfrak{h}\mathcal{K}^{\boxdot}$. For one of these we use the notation of (5.5.11) and for the other we denote the diagonal functors by $\bar{a}, \bar{b}, \bar{e}$, and $\bar{\ell}$ and denote the 2-cells by $\bar{\phi}$ and $\bar{\chi}$; the requirement that these 1-cells be parallel implies that the pullback faces are necessarily the same. Now consider a triple $\alpha\colon a \Rightarrow \bar{a}$, $\beta\colon b \Rightarrow \bar{b}$, and $\epsilon\colon e \Rightarrow \bar{e}$ satisfying the conditions of Definition 5.5.8. Our task is to define a fourth 2-cell $\lambda\colon \ell \Rightarrow \bar{\ell}$ so that $q\lambda = \alpha q'$ and $\bar{\chi} \cdot g\lambda = \epsilon g' \cdot \chi$.

To achieve this, we first define a 2-cell $\gamma\colon g\ell \Rightarrow g\bar{\ell}$ using the induction property of the p-cartesian cell $\bar{\chi}\colon g\bar{\ell} \Rightarrow \bar{e}g'$ applied to the composite 2-cell $\epsilon g' \cdot \chi\colon g\ell \Rightarrow \bar{e}g'$ and the factorization $p\epsilon g' \cdot p\chi = \bar{\phi}q' \cdot f\alpha q'$. By construction $p\gamma = f\alpha q'$ so the pair $\alpha q'$ and γ induces a 2-cell $\lambda\colon \ell \Rightarrow \bar{\ell}$ so that $q\lambda = \alpha q'$ and $g\lambda = \gamma$. The quadruple $(\alpha, \beta, \epsilon, \lambda)$ now defines the required 2-cell in $\mathfrak{h}\mathcal{K}^{\boxdot}$.

Finally, for 2-cell conservativity, suppose α, β, and ϵ as above are isomorphisms. By the conservativity property for pullbacks to show that λ is an isomorphism, it suffices to prove that $q\lambda = \alpha q'$ is, which we know already, and that $g\lambda = \gamma$ is invertible. But γ was constructed as a factorization $\epsilon g' \cdot \chi = \bar{\chi} \cdot \gamma$ with $p\gamma = f\alpha q'$. Since ϵ is an isomorphism, $\epsilon g' \cdot \chi$ is p-cartesian, so Lemma 5.1.5 proves that γ is an isomorphism. ☐

5.5.13. Remark. While we cannot directly define a pullback 2-functor $\mathfrak{h}\mathcal{K}^{\lrcorner} \to \mathfrak{h}\mathcal{K}$ in the homotopy 2-category because the 2-categorical universal property of pullbacks in $\mathfrak{h}\mathcal{K}$ is weak and not strict, the zig zag of 2-functors $\mathfrak{h}\mathcal{K}^{\lrcorner} \leftarrow \mathfrak{h}\mathcal{K}^{\boxdot} \to \mathfrak{h}\mathcal{K}$, in which the backwards map is a smothering 2-functor and the forwards map evaluates at the pullback vertex, defines a reasonable replacement.

Using Lemma 5.5.12, we can now construct the desired adjunction:

5.5.14. Proposition. *For any element $b\colon 1 \to B$ and any cartesian fibration $p\colon E \twoheadrightarrow B$, the evaluation at the identity functor admits a right adjoint*

$$\begin{tikzcd} \mathrm{Fun}_B(p_0, p) \arrow[rr, bend left, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Fun}_B(b, p) \arrow[ll, bend left, dashed, "R"] \end{tikzcd}$$

defined by the domain-component of the p_*-cartesian lift of the right comma cone over b:

$$\begin{tikzcd} \mathrm{Fun}_B(b,p) \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "R"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathrm{Fun}(1,E) \arrow[dr, "!^*"] \arrow[dd, two heads, "p_*" near start] \arrow[dl, phantom, "\Uparrow\chi"] & \\ & \mathrm{Fun}_B(p_0,p) \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathrm{Fun}(\mathrm{Hom}_B(B,b),E) \arrow[dd, two heads, "p_*"] \\ \mathbb{1} \arrow[rr, "b" near start] \arrow[dr, equals] & & \mathrm{Fun}(1,B) \arrow[dr, "!^*"] \arrow[dl, phantom, "\Uparrow\phi"] & \\ & \mathbb{1} \arrow[rr, "p_0"'] & & \mathrm{Fun}(\mathrm{Hom}_B(B,b),B) \end{tikzcd} \tag{5.5.15}$$

The idea will be to transfer the adjunction of Lemma 5.5.7 through a sequence of 2-functors

$$\begin{tikzcd} & \mathfrak{h}\mathcal{QC}at^{\boxdot} \arrow[r, "\mathrm{ev}_{\top}"] \arrow[d] & \mathfrak{h}\mathcal{QC}at \\ \mathfrak{h}\mathcal{K}^{\mathrm{op}}_{/\!/B} \arrow[r] & \mathfrak{h}\mathcal{QC}at^{\lrcorner} & \end{tikzcd}$$

using Lemma 3.6.8 to lift along the middle smothering 2-functor.

Proof. Fixing a cartesian fibration $p\colon E \twoheadrightarrow B$ in $\mathcal{K}$, we define a 2-functor[8] $\mathcal{K}^{\mathrm{op}}_{/\!/B} \to \mathfrak{h}\mathcal{QC}at^{\lrcorner}$ that carries a 1-cell (5.5.6) to

$$\begin{tikzcd} \mathbb{1} \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\alpha"] & \mathsf{Fun}(Y,B) \arrow[d, "k^*"] & \mathsf{Fun}(Y,E) \arrow[l, two heads, "p_*"'] \arrow[d, "e"] \\ \mathbb{1} \arrow[r, "f"'] & \mathsf{Fun}(X,B) & \mathsf{Fun}(X,E) \arrow[l, two heads, "p_*"] \end{tikzcd}$$

and a 2-cell $\theta\colon k \Rightarrow k'$ to the 2-cell that acts via pre-whiskering with θ in its two non-identity components. By Corollary 5.1.16, the functors p_* are cartesian fibration of quasi-categories.

We now apply the 2-functor $\mathcal{K}^{\mathrm{op}}_{/\!/B} \to \mathfrak{h}\mathcal{QC}at^{\lrcorner}$ to the adjunction of Lemma 5.5.7 to obtain an adjunction in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$ and then use the smothering 2-functor of Lemma 5.5.12 and Lemma 3.6.8 to lift this to an adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$. As elaborated on in Exercise 3.6.ii, the lifted adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ can be constructed using any lifts of the objects, 1-cells, and either the unit or counit of the adjunction in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$.

In particular, we may take the left adjoint of the lifted adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ to be any lift of the image in $\mathfrak{h}\mathcal{QC}at^{\lrcorner}$ of the right adjoint of the adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ in $\mathfrak{h}\mathcal{K}_{/\!/B}$, and so our left adjoint is

$$\begin{tikzcd} \mathsf{Fun}_B(p_0,p) \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"'] & & \mathsf{Fun}(\mathrm{Hom}_B(B,b),E) \arrow[dd, two heads, "p_*" near start] \arrow[dr, "\ulcorner \mathrm{id}_b \urcorner^*"] & \\ & \mathsf{Fun}_B(b,p) \arrow[rr] \arrow[dd, two heads] & & \mathsf{Fun}(1,E) \arrow[dd, two heads, "p_*"] \\ \mathbb{1} \arrow[rr, "p_0" near start] \arrow[dr, equal] & & \mathsf{Fun}(\mathrm{Hom}_B(B,b),B) \arrow[dr, "\ulcorner \mathrm{id}_b \urcorner^*"] & \\ & \mathbb{1} \arrow[rr, "b"'] & & \mathsf{Fun}(1,B) \end{tikzcd}$$

We may also take the right adjoint to be any lift of the image of the right adjoint of the adjunction. This proves that the right adjoint is defined by (5.5.15). Since the counit of $! \dashv \ulcorner \mathrm{id}_b \urcorner$ is an identity, the counit of the lifted adjunction may also be taken to be an identity.

Finally, we compose with the forgetful 2-functor $\mathfrak{h}\mathcal{QC}at^{\boxdot} \to \mathfrak{h}\mathcal{QC}at$ that evaluates at the pullback vertex to project our adjunction in $\mathfrak{h}\mathcal{QC}at^{\boxdot}$ to the desired adjunction in $\mathfrak{h}\mathcal{QC}at$. □

The proof of the discrete case of the Yoneda lemma is now one line.

Proof of Theorem 5.5.2. If $p\colon E \twoheadrightarrow B$ is discrete, then $\mathsf{Fun}_B(p_0,p)$ and $\mathsf{Fun}_B(b,p)$ are Kan complexes, so the adjunction defined in Proposition 5.5.14 is an adjoint equivalence. □

Specializing to the case of two right representable discrete cartesian fibrations, we conclude that the Kan complex of natural transformations is equivalent to the underlying quasi-category of their internal mapping space.

5.5.16. Corollary (external Yoneda embedding). *For any elements $x, y\colon 1 \rightrightarrows A$ in an ∞-category A, evaluation at the identity of x induces an equivalence of Kan complexes*

$$\mathsf{Fun}_A(\mathrm{Hom}_A(A,x), \mathrm{Hom}_A(A,y)) \xrightarrow[\sim]{\mathrm{ev}_{\ulcorner \mathrm{id}_x \urcorner}} \mathsf{Fun}(1, \mathrm{Hom}_A(x,y)) \qquad \square$$

[8] To explain the variance, recall that the 2-functor $\mathsf{Fun}(-,B)\colon \mathfrak{h}\mathcal{K}^{\mathrm{op}} \to \mathfrak{h}\mathcal{QC}at$ is contravariant on 1-cells but covariant on 2-cells. Such 2-functors, like all 2-functors, preserve adjunctions, though in this case the left and right adjoints are interchanged, while the units and counits retain the same roles.

It remains to prove the general case of Theorem 5.5.4. The next step is to observe that the right adjoint in the adjunction of Proposition 5.5.14 lands in the sub quasi-category of cartesian functors from p_0 to p. Lemma 5.5.17 proves this after which it is short work to complete the proof of Theorem 5.5.4 by arguing that this restricted adjunction defines an adjoint equivalence.

5.5.17. Lemma. *For each vertex in $\mathsf{Fun}_B(b,p)$ below-left*

$$\mathsf{Fun}_B(b,p) \xrightarrow{\quad R \quad} \mathsf{Fun}_B(p_0,p)$$

$$\begin{tikzcd} 1 \arrow[rr, "e"] \arrow[dr, "b"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "Re"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

the map Re in $\mathsf{Fun}_B(p_0,p)$ above-right defines a cartesian functor in $\mathcal{K}_{/B}$.

Proof. From the definition of the right adjoint in (5.5.15) and Lemma 5.3.5, we see that Re is the domain component of a p-cartesian lift χ of the composite natural transformation below-left

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[r, two heads, "!"] \arrow[dr, two heads, bend right, "p_0"'] & 1 \arrow[r, "e"] \arrow[d, "b"] \arrow[dl, phantom, "\Uparrow\phi"] & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, bend left, "e!"] \arrow[rr, bend right, "Re"'] \arrow[rr, phantom, "\Uparrow\chi"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

Since $p_0 \colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$ is discrete, every natural transformation ψ with codomain $\mathsf{Hom}_B(B,b)$ is p_0-cartesian, so to prove that Re defines a cartesian functor, we must show that $Re\psi$ is p-cartesian. To that end, consider the horizontal composite

$$\begin{tikzcd} X \arrow[r, bend left, "y"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Uparrow\psi"] & \mathsf{Hom}_B(B,b) \arrow[r, bend left, "e!"] \arrow[r, bend right, "Re"'] \arrow[r, phantom, "\Uparrow\chi"] & E \end{tikzcd}$$

By naturality of whiskering, we have $\chi y \cdot Re\psi = e!\psi \cdot \chi x = \chi x$, since 1 is the terminal ∞-category and hence $e!\psi$ is an identity. Now Lemma 5.1.24(ii) implies that $Re\psi$ is p-cartesian. ☐

Proof of Theorem 5.5.4. By Lemma 5.5.17, the adjunction of Proposition 5.5.14 restricts to define an adjunction

$$\begin{tikzcd} \mathsf{Fun}_B^{\mathsf{cart}}(p_0,p) \arrow[r, bend left, "\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}_B(b,p) \arrow[l, bend left, dashed, "R"] \end{tikzcd}$$

Since the counit of the original adjunction $! \dashv \ulcorner \mathrm{id}_b \urcorner$ is an isomorphism and smothering 2-functors are conservative on 2-cells, the counit of the adjunction of Proposition 5.5.14 and hence also of the restricted adjunction is an isomorphism. As in the proof of Theorem 5.5.2, we will prove that $\mathrm{ev}_{\ulcorner \mathrm{id}_b \urcorner}$ is an equivalence by demonstrating that the unit of the restricted adjunction is also invertible. By

Lemma 16.2.1, it suffices to verify this elementwise, proving that the component of the unit indexed by a cartesian functor

$$\begin{tikzcd} \mathsf{Hom}_B(B,b) \arrow[rr, "f"] \arrow[dr, two heads, "p_0"'] & & E \arrow[dl, two heads, "p"] \\ & B & \end{tikzcd}$$

is an isomorphism.

Unpacking the proof of Proposition 5.5.14, the unit $\hat{\eta}$ of $\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner} \dashv R$ is defined to be a factorization

$$\begin{tikzcd} & \mathsf{Fun}_B(p_0,p) \arrow[r, hook] \arrow[d, phantom, "/\!/"] & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \arrow[r, "\ulcorner \mathrm{id}_b \urcorner^*"] \arrow[drr, equal] & \mathsf{Fun}(1,E) \arrow[d, "!^*"] \arrow[dl, phantom, "\Uparrow \mathsf{Fun}(\eta,E)"] & \\ \mathsf{Fun}_B(p_0,p) \arrow[r, "\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] \arrow[dr, equal] & \mathsf{Fun}_B(b,p) \arrow[rr, hook] \arrow[d, "R"] \arrow[dl, phantom, "\Uparrow \hat{\eta}"] & \arrow[d, phantom, "\Uparrow \chi"] & \mathsf{Fun}(1,E) \arrow[d, "!^*"] & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) \\ & \mathsf{Fun}_B(p_0,p) \arrow[rr, hook] & & \mathsf{Fun}(\mathsf{Hom}_B(B,b),E) & \end{tikzcd}$$

of the pre-whiskering 2-cell $\mathsf{Fun}(\eta,E)$ through the p_*-cartesian lift χ. The component of the pre-whiskering 2-cell $\mathsf{Fun}(\eta,E)$ at the cartesian functor f is $f\eta$. Since $p_0 \colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B$ is a discrete cartesian fibration, any 2-cell, such as η, which has codomain $\mathsf{Hom}_B(B,b)$ is p_0-cartesian, and since f is a cartesian functor, we then see that $f\eta$ is p-cartesian.

By Lemma 5.3.8, the components of the p_*-cartesian cell χ define p-cartesian natural transformations in $\mathcal{K}$. As $\hat{\eta}$ is a natural transformation with codomain $\mathsf{Fun}_B(p_0,p)$ its components project along p to the identity. In this way, we see that $\hat{\eta}f$ is a factorization of the p-cartesian transformation $f\eta$ through a p-cartesian lift of ϕ over an identity, and Lemma 5.1.5 proves that $\hat{\eta}f$ is an isomorphism, as desired. □

Theorem 5.5.4 implies the following generalization, replacing the elements $b \colon 1 \to B$ by a generalized element $b \colon X \to B$.

5.5.18. Proposition. *For any cartesian fibration $p \colon E \twoheadrightarrow B$ and map $b \colon X \to B$, restricting along the canonical induced functor*

$$\begin{tikzcd} X \arrow[rr, "\ulcorner \mathrm{id}_b \urcorner"] \arrow[dr, "b"'] & & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ & B & \end{tikzcd}$$

defines an equivalence of quasi-categories:

$$\mathsf{Fun}_B^{\mathrm{cart}}(\mathsf{Hom}_B(B,b) \stackrel{p_0}{\twoheadrightarrow} B, E \stackrel{p}{\twoheadrightarrow} B) \xrightarrow{\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}} \mathsf{Fun}_B(X \stackrel{b}{\to} B, E \stackrel{p}{\twoheadrightarrow} B).$$

Proof. Theorem 5.5.4 applies in $\mathcal{K}_{/X}$ to the cartesian fibration $p \times X \colon E \times X \twoheadrightarrow B \times X$ and the element $(b,X) \colon X \to B \times X$ to define an equivalence

$$\begin{tikzcd} \mathsf{Fun}_{B\times X}^{\mathrm{cart}}(\mathsf{Hom}_B(B,p) \stackrel{(p_1,p_0)}{\twoheadrightarrow} B \times X, p \times X) \arrow[r, "\mathsf{ev}_{\ulcorner \mathrm{id}_{(b,X)} \urcorner}"] \arrow[d, phantom, "\wr\Vert"] & \mathsf{Fun}_{B\times X}(X \xrightarrow{(b,X)} B \times X, p \times X) \arrow[d, phantom, "\wr\Vert"] \\ \mathsf{Fun}_B^{\mathrm{cart}}(\mathsf{Hom}_B(B,b) \stackrel{p_0}{\twoheadrightarrow} B, E \stackrel{p}{\twoheadrightarrow} B) \arrow[r, "\mathsf{ev}_{\ulcorner \mathrm{id}_b \urcorner}"] & \mathsf{Fun}_B(X \stackrel{b}{\to} B, E \stackrel{p}{\twoheadrightarrow} B) \end{tikzcd}$$

which transposes under the adjunction

$$\begin{tikzcd} \mathcal{K}_{/X} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathcal{K} \arrow[l, bend left, "-\times X"] \end{tikzcd}$$

to the equivalence of the statement. □

Later we will interpret the result of Proposition 5.5.18 as defining a left biadjoint to the inclusion $\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B}$ by the functor $(b\colon X \to B) \mapsto (p_0\colon \mathrm{Hom}_B(B, b) \twoheadrightarrow B)$.

Exercises.

5.5.i. Exercise. Given an element $f\colon 1 \to \mathrm{Hom}_A(x, y)$ in the internal mapping space between a pair of elements in an ∞-category A, use the explicit description of the inverse equivalence to the map of Corollary 5.5.16 to construct a map

$$\begin{tikzcd} \mathrm{Hom}_A(A, x) \arrow[rr, dashed, "f_*"] \arrow[dr, two heads, "p_0"'] & & \mathrm{Hom}_A(A, y) \arrow[dl, two heads, "p_0"] \\ & A & \end{tikzcd}$$

which represents the "natural transformation" defined by post-composing with f.[9]

[9]Hint: this construction is a special case of the construction given in the first half of the proof of Lemma 5.5.17.

Part II

Homotopy coherent category theory

CHAPTER 6

Homotopy coherent diagrams

Consider a diagram $d\colon 1 \to A^J$ in an ∞-category A indexed by a 1-category J. Via the isomorphism $\mathsf{Fun}(1, A^J) \cong \mathsf{Fun}(1, A)^J$, an element in the ∞-category of diagrams A^J equally defines a functor $d\colon J \to \mathsf{Fun}(1, A)$ valued in the underlying quasi-category of A. Applying $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$, this descends to a diagram $\mathsf{h}d\colon J \to \mathsf{h}A$ of shape J in the homotopy category of A; such diagrams are called *homotopy commutative*. But the original diagram d has a much richer property, defining what is called a *homotopy coherent* diagram of shape J in the quasi-category $\mathsf{Fun}(1, A)$.

To make the data involved in defining a homotopy coherent diagram most explicit, we first introduce a general notion of homotopy coherent diagram as a simplicial functor valued in a simplicial category whose hom-spaces are Kan complexes. What makes such diagrams "homotopy coherent" and not just "simplicially enriched" is that their domains are required to be "free" simplicial categories of a particular form that we refer to by the name *simplicial computads*. Because every quasi-category can be presented up to equivalence by a Kan-complex enriched category, it will follow that "all diagrams valued in quasi-categories are homotopy coherent."

To build intuition for the general notion of a homotopy coherent diagram, it is helpful to consider a special case of diagrams indexed by the category

$$\omega \coloneqq \qquad \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] \arrow[rrrr, bend right] & 1 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] & 2 \arrow[r] \arrow[rr, bend right] & 3 \arrow[r] & \cdots \end{tikzcd}$$

whose objects are finite ordinals and with a morphism $j \to k$ if and only if $j \leq k$ and valued in the Kan-complex enriched category of spaces $\mathcal{S}pace$. A ω-shaped graph in $\mathcal{S}pace$ is comprised of spaces X_k for each $k \in \omega$ together with continuous maps $f_{j,k}\colon X_j \to X_k$ whenever $j < k$.[1] This data defines a *homotopy commutative diagram* just when $f_{i,k} \simeq f_{j,k} \circ f_{i,j}$ whenever $i < j < k$.[2]

To extend this data to a *homotopy coherent diagram* $\omega \to \mathcal{S}pace$ requires:

- Chosen homotopies $h_{i,j,k}\colon f_{i,k} \simeq f_{j,k} \circ f_{i,j}$ whenever $i < j < k$. This amounts to specifying a path in $\mathsf{Map}(X_i, X_k)$ from the vertex $f_{i,k}$ to the vertex $f_{j,k} \circ f_{i,j}$, which is obtained as the composite of the two vertices $f_{i,j} \in \mathsf{Map}(X_i, X_j)$ and $f_{j,k} \in \mathsf{Map}(X_j, X_k)$.

[1]To simplify somewhat we adopt the convention that $f_{j,j}$ is the identity, making this data into a reflexive directed graph with implicitly designated identities.

[2]This data defines a strictly commutative diagram (aka a functor $\omega \to \mathcal{S}pace$) just when $f_{i,k} = f_{j,k} \circ f_{i,j}$ whenever $i < j < k$. Strictly commutative diagrams are certainly homotopy commutative. Homotopy coherent category theory arose from the search for conditions under which something like the converse implication held: a homotopy commutative diagram is realized by (i.e., naturally isomorphic to up to homotopy) a strictly commutative diagram, if and only if it extends to a homotopy coherent diagram [**34**, 2.5].

- For $i < j < k < \ell$, the chosen homotopies provide four paths in $\mathsf{Map}(X_i, X_\ell)$

$$\begin{tikzcd} f_{i,\ell} \arrow[r, dash, "h_{i,k,\ell}"] \arrow[d, dash, "h_{i,j,\ell}"'] & f_{k,\ell} \circ f_{i,k} \arrow[d, dash, "f_{k,\ell} \circ h_{i,j,k}"] \\ f_{j,\ell} \circ f_{i,j} \arrow[r, dash, "h_{j,k,\ell} \circ f_{i,j}"'] & f_{k,\ell} \circ f_{j,k} \circ f_{i,j} \end{tikzcd}$$

  We then specify a higher homotopy — a *2-homotopy* — filling in this square.
- For $i < j < k < \ell < m$, the previous choices provide 12 paths and six 2-homotopies in $\mathsf{Map}(X_i, X_m)$ that assemble into the boundary of a cube. We then specify a *3-homotopy*, a homotopy between homotopies between homotopies, filling in this cube.
- Etc.

Even in this simple case of the category $\boldsymbol{\omega}$, this data is a bit unwieldy. Our task is to define a category to index this homotopy coherent data arising from $\boldsymbol{\omega}$: the objects X_i, the functions $X_i \to X_j$, the 1-homotopies $h_{i,j,k}$, the 2-homotopies, and so on. This data will assemble into a simplicial category whose objects are the same as the objects of $\boldsymbol{\omega}$ but which will have n-morphisms in each dimension $n \geq 0$, to index the n-homotopies. Importantly, this simplicial category will be "freely generated" from a much smaller collection of data. We begin by studying such "freely generated" simplicial categories under the name **simplicial computads**.

6.1. Simplicial computads

6.1.1. Definition (free categories and atomic arrows). An arrow f in a 1-category is **atomic** if it is not an identity and if it admits no non-trivial factorizations: i.e., if whenever $f = g \circ h$, either g or h is an identity.

A 1-category is **free** if every arrow may be expressed uniquely as a composite of atomic arrows, with the convention that empty composites correspond to identity arrows.[3]

6.1.2. Digression (on free categories and reflexive directed graphs). The category of presheaves on the truncation

$$\boldsymbol{\Delta}_{\leq 1} \coloneqq \begin{tikzcd} \bullet \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \bullet \arrow[l, "i" description] \end{tikzcd} \qquad is = it = \mathrm{id}$$

defines the category $\mathcal{G}ph$ of (reflexive, directed) graphs: for us, a **graph** consists of a set of vertices, a set of edges each with a specified source and target vertex, and a distinguished "identity" endo-edge for each vertex. Any category has an underlying reflexive directed graph, and this forgetful functor admits a left adjoint, defining the free category whose atomic and identity arrows are the arrows in the given graph:

$$\begin{tikzcd} \mathcal{C}at \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{G}ph \arrow[l, bend right, "F"'] \end{tikzcd}$$

Recall, from Digression 1.2.3, that a simplicial category $\mathcal{A}$ may be presented by a family of 1-categories $\mathcal{A}_n$ of *n-arrows*, for $n \geq 0$, each with a common set of objects, that assemble into a diagram $\mathcal{A}_\bullet \colon \boldsymbol{\Delta}^{\mathrm{op}} \to$

[3]Alternatively, a 1-category is **free** if every non-identity arrow may be expressed uniquely as a non-empty composite of atomic arrows and if identity arrows admit no non-trivial factorizations.

$\mathcal{C}at$ comprised of identity-on-objects functors. The notion of "free" simplicial category was first introduced by Dwyer and Kan [**35**, 4.5].

6.1.3. Definition (simplicial computad). A simplicial category $\mathcal{A}$ is a **simplicial computad** if and only if

- Each category $\mathcal{A}_n$ of n-arrows is freely generated by the graph of atomic n-arrows.
- If f is an atomic n-arrow in $\mathcal{A}_n$ and $\sigma\colon [m] \twoheadrightarrow [n]$ is an epimorphism in $\mathbf{\Delta}$, then the degenerated m-arrow $f \cdot \sigma$ is atomic in $\mathcal{A}_m$.

By the Eilenberg-Zilber lemma[4], a simplicial category $\mathcal{A}$ is a simplicial computad if and only if all of its non-identity arrows can be expressed uniquely as a composite

$$f = (f_1 \cdot \alpha_1) \circ (f_2 \cdot \alpha_2) \circ \cdots \circ (f_\ell \cdot \alpha_\ell)$$

in which each f_i is non-degenerate and atomic and each α_i is a degeneracy operator in $\mathbf{\Delta}$.

6.1.4. Example. A 1-category A may be regarded as a simplicial category $A_\bullet$ in which $A_n := A$ for all n. In the construction, the hom-spaces of A coincide with the hom-sets of A. Such "constant" simplicial categories define simplicial computads if and only if the 1-category A is free.

6.1.5. Example. For any simplicial set U, let $\mathbb{2}[U]$ denote the simplicial category with two objects "$-$" and "$+$" and with hom-spaces defined by

$$\mathbb{2}[U](+,-) := \emptyset, \quad \mathbb{2}[U](+,+) := \mathbb{1} =: \mathbb{2}[U](-,-), \quad \mathbb{2}[U](-,+) := U.$$

This simplicial category is a simplicial computad because there are no composable sequences of arrows in $\mathbb{2}[U]$ containing more than one non-identity arrow. Every arrow from $-$ to $+$ is atomic.

6.1.6. Definition. A simplicial functor $G\colon \mathcal{A} \to \mathcal{B}$ between simplicial computads defines a **simplicial computad morphism** if it maps every atomic arrow f in $\mathcal{A}$ to an arrow Gf which is either atomic or an identity in $\mathcal{B}$. Write $s\mathcal{S}et\text{-}\mathcal{C}ptd \subset s\mathcal{S}et\text{-}\mathcal{C}at$ for the non-full subcategory of simplicial computads and their morphisms.

The axioms of Definition 6.1.3 assert that the atomic and identity n-arrows of a simplicial computad assemble into a diagram in $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ and a simplicial computad morphism restricts to define a natural transformation between the underlying $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}$-indexed graphs of atomic and identity arrows; in this way, restricting to the atomic or identity arrows defines a functor $\mathrm{atom}\colon s\mathcal{S}et\text{-}\mathcal{C}ptd \to \mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$. The next lemma tells us that the category $s\mathcal{S}et\text{-}\mathcal{C}ptd$ is canonically isomorphic to the intersection of $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ and $s\mathcal{S}et\text{-}\mathcal{C}at$ in $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$.

6.1.7. Lemma. *The functor that carries a simplicial computad to its underlying diagram of atomic and identity arrows and the inclusion of simplicial computads into simplicial categories define the legs of a pullback cone:*

$$\begin{tikzcd} s\mathcal{S}et\text{-}\mathcal{C}ptd \arrow[r, hook] \arrow[d, "\mathrm{atom}"'] \arrow[dr, phantom, "\lrcorner", very near start] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[d] \\ \mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}} \arrow[r, hook, "F^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}"] & \mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}} \end{tikzcd}$$

[4]The Eilbenberg-Zilber lemma asserts that any degenerate simplex in a simplicial set may be uniquely expressed as a degenerated image of a non-degenerate simplex; see [**38**, II.3.1, pp. 26-27].

Moreover since $s\mathcal{S}et\text{-}\mathcal{C}at$ and $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ have colimits, the functors to $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ preserve them, and $F^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ is an isofibration, it follows that $s\mathcal{S}et\text{-}\mathcal{C}ptd$ has colimits created by either of the functors to $s\mathcal{S}et\text{-}\mathcal{C}at$ or to $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$.

Proof. If $\mathcal{A}$ is a simplicial category, presented as a simplicial object $\mathcal{A}_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$, then $\mathcal{A}$ is a simplicial computad if and only if there exists a dotted lift as below-left

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}} \arrow[r, dashed, "\exists!"] \arrow[d, hook] & \mathcal{G}ph \arrow[d, "F"] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\mathcal{A}_\bullet"'] & \mathcal{C}at \end{tikzcd} \qquad \begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}} \arrow[r, dashed, bend left] \arrow[r, dashed, bend right] \arrow[r, phantom, "\Downarrow\exists!"] \arrow[d, hook] & \mathcal{G}ph \arrow[d, "F"] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{A}_\bullet"] \arrow[r, bend right, "\mathcal{B}_\bullet"'] \arrow[r, phantom, "\Downarrow G"] & \mathcal{C}at \end{tikzcd}$$

in which case this lift is necessarily unique. Correspondingly, as simplicial functor $G \colon \mathcal{A} \to \mathcal{B}$ defines a computad morphism if and only if the restricted natural transformation above-right also lifts through the free category functor, again necessarily uniquely. These facts verify that the category $s\mathcal{S}et\text{-}\mathcal{C}ptd$ is captured as the stated pullback.

Now consider a diagram $D \colon J \to s\mathcal{S}et\text{-}\mathcal{C}ptd$ and form its colimit cone in $s\mathcal{S}et\text{-}\mathcal{C}at$ and in $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$. The functors to $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ carry these to a pair of isomorphic colimit cones under the same diagram and since $F^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ is an isofibration (any category isomorphic to a free category is itself a free category and this isomorphism necessarily restricts to underlying graphs of atomic arrows), there exists a colimit cone under D in $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ whose image under $F^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ is equal to the image of the colimit cone under D in $s\mathcal{S}et\text{-}\mathcal{C}at$. Now the universal property of the pullback allows us to lift this cone to $s\mathcal{S}et\text{-}\mathcal{C}ptd$, and a similar argument using the 2-categorical universal property of the pullback diagram of categories demonstrates that the lifted cone is a colimit cone. □

6.1.8. Definition (simplicial subcomputads). A simplicial computad morphism $\mathcal{A} \hookrightarrow \mathcal{B}$ that is injective on objects and faithful displays $\mathcal{A}$ as a **simplicial subcomputad** of $\mathcal{B}$.

The simplicial subcomputad $\overline{S}$ **generated by** a set of arrows S in a simplicial computad $\mathcal{A}$ is the smallest simplicial subcomputad of $\mathcal{A}$ containing those arrows. The objects of $\overline{S}$ are those objects that appear as domains or codomains of arrows in S and its set of morphisms is the smallest subset of morphisms containing S that have the following closure properties:

- if $f \in \overline{S}$ and $\alpha \colon [n] \to [m] \in \mathbf{\Delta}$, then $f \cdot \alpha \in \overline{S}$.
- If $f, g \in \overline{S}$ are composable then $f \circ g \in \overline{S}$.

Lemma 6.1.7 proves that simplicial computads and computad morphisms are closed under colimits formed in the category of simplicial categories. For certain special colimit shapes, the property of being a simplicial subcomputad is also preserved:

6.1.9. Lemma. *A simplicial subcategory $\mathcal{A} \hookrightarrow \mathcal{B}$ of a simplicial computad $\mathcal{B}$ displays $\mathcal{A}$ as a simplicial subcomputad of $\mathcal{B}$ just when $\mathcal{A}$ is closed under factorizations: if f and g are composable arrows of $\mathcal{B}$ and $f \circ g \in \mathcal{A}$ then f and g are in $\mathcal{A}$.*

Proof. Exercise 6.1.i. □

6.1.10. Lemma. *Simplicial subcomputads are closed under coproduct, pushout, and colimit of countable sequences.*

Proof. Simplicial subcomputad inclusions are precisely those morphisms in $s\mathcal{S}et\text{-}\mathcal{C}ptd$ whose images in $\mathcal{G}ph^{\Delta^{\mathrm{op}}_{\mathrm{epi}}}$ are pointwise monomorphisms. Lemma 6.1.7 proves that colimits in $s\mathcal{S}et\text{-}\mathcal{C}ptd$ are created in $\mathcal{G}ph^{\Delta^{\mathrm{op}}_{\mathrm{epi}}}$. As colimits in this presheaf category are formed pointwise and as monomorphisms are stable under coproduct, pushout, and colimit of countable sequences, the result follows. □

6.1.11. Definition (relative simplicial computad). The class of all **relative simplicial computads** is the class of all simplicial functors that can be expressed as a countable composite of pushouts of coproducts of

- the unique simplicial functor $\emptyset \hookrightarrow \mathbb{1}$ and
- the simplicial subcomputad inclusion $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ for $n \geq 0$.

The next lemma reveals that relative simplicial computads differ from simplicial subcomputad inclusions only in the fact that the domains of relative simplicial computads need not be simplicial computads — but when they are, the codomain is also a simplicial computad and the map is a simplicial subcomputad inclusion.

6.1.12. Lemma. *If $\mathcal{A}$ is a simplicial computad, then an inclusion of simplicial categories $\mathcal{A} \hookrightarrow \mathcal{B}$ is a simplicial computad inclusion if and only if it is a relative simplicial computad. In particular, a simplicial category $\mathcal{B}$ is a simplicial computad if and only if $\emptyset \hookrightarrow \mathcal{B}$ is a relative simplicial computad.*

Proof. First note that $\emptyset \hookrightarrow \mathbb{1}$ and $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ are simplicial subcomputad inclusions: the first case is trivial and for the latter, all the non-identity arrows in $\mathbb{2}[U]$ atomic. By Lemma 6.1.10, to prove that any relative simplicial computad $\mathcal{A} \hookrightarrow \mathcal{B}$ whose domain is a simplicial computad is a subcomputad inclusion, it suffices to prove that for any simplicial computad $\mathcal{A}$ and any simplicial functor $f\colon \mathbb{2}[\partial\Delta[n]] \to \mathcal{A}$, not necessarily a computad morphism, the pushout of $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ along f is a simplicial computad containing $\mathcal{A}$ as a simplicial subcomputad.

The subcomputad inclusion $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ is full on r-arrows for $r < n$; for $r > n$, the codomain is constructed by adjoining one atomic arrow from $-$ to $+$ for each epimorphism $[r] \twoheadrightarrow [n]$. It follows that the pushout $\mathcal{A} \hookrightarrow \mathcal{A}'$ is similarly full on r-arrows for $r < n$, and for $r \geq n$, the category of r-arrows of $\mathcal{A}'$ is obtained from that of $\mathcal{A}$ by adjoining one atomic r-arrow for each degeneracy operator $[r] \twoheadrightarrow [n]$ with boundary specified by the attaching map f. If we adjoin an arrow to a free category along its boundary we get a free category with that arrow as an extra generator, so each of the categories of r-arrows of $\mathcal{A}'$ are freely generated, and it is clear from this description that on degenerating one of these extra adjoined arrows we just map it to one of the generating arrows we've adjoined at a higher dimension. This proves that $\mathcal{A} \hookrightarrow \mathcal{A}'$ is a simplicial subcomputad inclusion as claimed. It follows inductively that the codomain of a relative simplicial computad is a simplicial computad whenever its domain is, in which case the inclusion is a subcomputad inclusion.

For the converse, we inductively present any simplicial subcomputad inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ as a sequential composite of pushouts of coproducts of the maps $\emptyset \hookrightarrow \mathbb{1}$ and $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$. Note that $\mathbb{2}[\Delta[n]]$ contains a single non-degenerate atomic arrow not present in $\mathbb{2}[\partial\Delta[n]]$, namely the unique non-degenerate n-arrow representing the n-simplex.

At stage "-1," use the inclusion $\emptyset \hookrightarrow \mathbb{1}$ to attach any object in $\mathcal{B}\backslash\mathcal{A}$ to $\mathcal{A}$. At stage "0," use the inclusion $\mathbb{2}[\emptyset] \hookrightarrow \mathbb{2}[\Delta[0]]$ to attach each atomic 0-arrows of $\mathcal{B}$ that is not in $\mathcal{A}$. Iteratively, at stage "r," use the inclusion $\mathbb{2}[\partial\Delta[r]] \hookrightarrow \mathbb{2}[\Delta[r]]$ to attach each atomic r-arrows of $\mathcal{B}$ that is not in $\mathcal{A}$. There is a canonical map from the codomain of the relative simplicial computad defined in this way to $\mathcal{B}$, which by construction is bijective on objects and sends the unique atomic arrow attached by each cell to an atomic arrow in $\mathcal{B}$; in particular this comparison functor, which by construction lies

in $s\mathcal{S}et\text{-}\mathcal{C}ptd$ is in fact a simplicial subcomputad inclusion. The comparison is surjective on atomic n-arrows by construction and hence surjective on all arrows, because the unique factorization any arrow into non-degenerate atomics is present in the colimit at the stage corresponding to the dimension of that arrow. Thus, we have presented $\mathcal{A} \hookrightarrow \mathcal{B}$ as a relative simplicial computad inclusion. $\square$

The morphisms listed in Definition 6.1.11 are the generating cofibrations in the Bergner model structure on simplicial categories [**10**]. Hence, the relative simplicial computads are precisely the *cellular cofibrations*, those that are built as sequential composites of pushouts of coproducts of generating cofibrations (without closing under retracts). For non-cofibrant domains, the notion of Bergner cofibration is more general than the notion of relative simplicial computad. However:

6.1.13. Lemma. *Every retract of a simplicial computad is a simplicial computad.*

Proof. A simplicial category $\mathcal{A}$ is a retract of a simplicial computad $\mathcal{B}$ if there exist simplicial functors

$$\mathcal{A} \xhookrightarrow{S} \mathcal{B} \overset{R}{\twoheadrightarrow} \mathcal{A}$$

so that $RS = \mathrm{id}$. The category $\mathcal{A}$ is then recovered as the coequalizer (or the equalizer) of SR and the identity, so if we knew that this idempotent defined a computad morphism, we could appeal to Lemma 6.1.7 and be done; but we do not know this, so we must argue further.

First we demonstrate that every retract of a free category is free. To see this, we'll first show that the inclusion $\mathcal{A}_n \hookrightarrow \mathcal{B}_n$ satisfies the "2-of-3" property: of f and g are composable morphisms of $\mathcal{B}_n$ so that two of three of f, g, and $f \circ g$ lie in $\mathcal{A}_n$, then so does the third. This is clear in the case where $f, g \in \mathcal{A}_n$ so suppose that $f, f \circ g \in \mathcal{A}_n$. Then f and $f \circ g$ are fixed points for the idempotent SR and so we have $f \circ g = SR(f \circ g) = SR(f) \circ SR(g) = f \circ SR(g)$. In the free category $\mathcal{B}_n$, all morphisms are both monic and epic, so $g = SR(g)$ lies in $\mathcal{A}_n$ as well. Now by induction it is easy to verify that every arrow in $\mathcal{A}_n$ factors uniquely as a product of composites of atomic arrows in $\mathcal{B}_n$, with each of these composites defining an atomic arrow in $\mathcal{A}_n$.

This verifies the first condition of Definition 6.1.3. It remains only to verify that degenerate images of atomic arrows in $\mathcal{A}_n$ are atomic. To that end consider an epimorphism $\alpha\colon [m] \twoheadrightarrow [n]$ and an atomic n-arrow f in $\mathcal{A}_n$. If $f \cdot \alpha = g \circ h$ is a non-trivial factorization in $\mathcal{A}_m \hookrightarrow \mathcal{B}_m$ then since $\mathcal{B}$ is a simplicial computad we must have $S(f) = g' \circ h'$ with $g' \cdot \alpha = S(g)$ and $h' \cdot \alpha = S(h)$. Since $f = RS(f) = Rg' \circ Rh'$ is atomic, we must have Rg' or Rh' an identity, but then $Rg' \cdot \alpha = RS(g) = g$ or $Rh' \cdot \alpha = RS(h) = h$ is an identity, and so the factorization $f \cdot \alpha = g \circ h$ is trivial after all. $\square$

Consequently:

6.1.14. Corollary. *The simplicial computads are precisely the cofibrant simplicial categories in the Bergner model structure.* $\square$

6.1.15. Digression. The category of simplicial categories bears a model structure whose cofibrant objects are the simplicial computads, by Corollary 6.1.14, and whose fibrant objects are the Kan complex enriched categories that will feature prominently in §6.3. The weak equivalences, called **DK-equivalences** after Dwyer and Kan, are simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ whose action on homs

$$F_{x,y}\colon \mathcal{C}(x,y) \xrightarrow{\sim} \mathcal{D}(Fx, Fy)$$

is via weak homotopy equivalences of simplicial sets and which induce equivalences of homotopy categories.[5]

The fibrations are defined by a similar pair of conditions: these are simplicial functors $F\colon \mathcal{C} \to \mathcal{D}$ whose action on homs

$$F_{x,y}\colon \mathcal{C}(x,y) \twoheadrightarrow \mathcal{D}(Fx,Fy)$$

is via a Kan fibration of simplicial sets and which have the property that a 0-arrow $f\colon Fc \to d$ that defines an isomorphism in the homotopy category of $\mathcal{D}$ may be lifted to a 0-arrow $g\colon c \to c'$ that defines an isomorphism in the homotopy category of $\mathcal{C}$. More details can be found in the original **[10]** or in the survey text **[12]**.

Exercises.

6.1.i. Exercise. Prove Lemma 6.1.9.

6.1.ii. Exercise. Prove that the **r-skeleton** of a simplicial computad, defined by discarding all arrows of dimension[6] greater than r, is the simplicial subcomputad generated by the atomic arrows of dimension r.

6.2. Free resolutions and homotopy coherent simplices

The original example of a simplicial computad, also due to Dwyer and Kan **[35]**, is given by the *free resolution* of a 1-category C.

6.2.1. Definition (free resolutions). Write $F \dashv U$ for the free category and underlying graph functors of Digression 6.1.2. Note that the components of the counit and comultiplication of the comonad

$$(FU\colon \mathcal{C}at \to \mathcal{C}at, \epsilon\colon FU \Rightarrow \mathrm{id}, F\eta U\colon FU \Rightarrow FUFU)$$

define identity-on-objects functors.

For any 1-category C, we will define a simplicial category $FU_\bullet C$ with the same objects and with the category of n-arrows defined to be $(FU)^{n+1}C$. A 0-arrow is a sequence of composable arrows in C. A non-identity n-arrow is a sequence of composable arrows in C with each arrow in the sequence enclosed in exactly n pairs of well-formed parentheses.

The simplicial object $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ is formed by evaluating the comonad resolution at $C \in \mathcal{C}at$:

$$FUC \;\begin{smallmatrix}\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow\end{smallmatrix}\; (FU)^2C \;\begin{smallmatrix}\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow\end{smallmatrix}\; (FU)^3C \;\begin{smallmatrix}\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow\end{smallmatrix}\; (FU)^4C \qquad \cdots \tag{6.2.2}$$

For $j \geq 1$, the face maps

$$(FU)^k\epsilon(FU)^j\colon (FU)^{k+j+1}C \to (FU)^{k+j}C$$

remove the parentheses that are contained in exactly k others, while $FU \cdots FU\epsilon$ composes the morphisms inside the innermost parentheses. For $j \geq 1$, the degeneracy maps

$$F(UF)^k\eta(UF)^jU\colon (FU)^{k+j-1}C \to (FU)^{k+j}C$$

[5] The **homotopy category** of a simplicially enriched category is defined by applying the path components functor $\pi_0\colon s\mathcal{S}et \to \mathcal{S}et$ to the hom-spaces. In fact, by the local homotopically fully faithfulness property, it suffices to assume essential surjectivity on homotopy categories.

[6] An n-arrow f in a simplicial category $\mathcal{A}$ has **dimension** r if there exists a non-degenerate r-arrow g and an epimorphism $\alpha\colon [n] \twoheadrightarrow [r]$ so that $f = g \cdot \alpha$.

double up the parentheses that are contained in exactly k others, while $F \cdots UF\eta U$ inserts parentheses around each individual morphism.

6.2.3. Example (free resolution of a group). A classically important special case is given by the free resolution of a 1-object groupoid, whose automorphisms are the elements of a discrete group G. In this case, each category of n-arrows is again a 1-object groupoid. The category of 0-arrows is the group of words in the non-identity elements of G. The category of 1-arrows is the group of words of words, and so on.

We now explain the sense in which free resolutions are "resolutions" of the original 1-category. As discussed in Example 6.1.4, a 1-category C can be regarded as a constant simplicial category $C_\bullet$, whose hom-spaces coincide with the hom-sets of C. There is a canonical "augmentation" map $\epsilon\colon FU_\bullet C \to C$ in $s\mathcal{S}et\text{-}\mathcal{C}at$ that is determined by its degree zero component $\epsilon\colon FUC \to C$ which is just given by composition in C.

6.2.4. Proposition. *The functor $\epsilon\colon FU_\bullet C \to C$ is a local homotopy equivalence of simplicial sets. That is, for any pair of objects $x, y \in C$, the map $\epsilon\colon FU_\bullet C(x,y) \to C(x,y)$ is a simplicial homotopy equivalence: $FU_\bullet C(x,y)$ is homotopy equivalent to the discrete set $C(x,y)$ of arrows in C from x to y.*

Proof. The augmented simplicial object

$$\begin{tikzcd}[column sep=large]
C \arrow[r, two heads, leftarrow] \arrow[r, dashed, bend right] & (FU)C \arrow[r, shift left=2, two heads, leftarrow] \arrow[r, tail] \arrow[r, shift right=2, two heads, leftarrow] \arrow[r, dashed, bend right=40] & (FU)^2C \arrow[r, shift left=4, two heads, leftarrow] \arrow[r, shift left=2, tail] \arrow[r, two heads, leftarrow] \arrow[r, shift right=2, tail] \arrow[r, shift right=4, two heads, leftarrow] \arrow[r, dashed, bend right=50] & (FU)^3C \arrow[r, shift left=6, two heads, leftarrow] \arrow[r, shift left=4, tail] \arrow[r, shift left=2, two heads, leftarrow] \arrow[r, tail] \arrow[r, shift right=2, two heads, leftarrow] \arrow[r, shift right=4, tail] \arrow[r, shift right=6, two heads, leftarrow] \arrow[r, dashed, bend right=60] & (FU)^4C & \cdots
\end{tikzcd}$$

is split at the level of reflexive directed graphs (i.e., after applying U). These splittings are not functors, but that won't matter. These directed graph morphisms displayed here are all identity on objects, which means that for any $x, y \in C$ there is a split augmented simplicial set

$$\begin{tikzcd}
C(x,y) \arrow[r, two heads, leftarrow] \arrow[r, bend right] & (FU)C(x,y) \arrow[r, shift left=2, two heads, leftarrow] \arrow[r, tail] \arrow[r, shift right=2, two heads, leftarrow] \arrow[r, bend right=40] & (FU)^2C(x,y) \arrow[r, shift left=4, two heads, leftarrow] \arrow[r, shift left=2, tail] \arrow[r, two heads, leftarrow] \arrow[r, shift right=2, tail] \arrow[r, shift right=4, two heads, leftarrow] \arrow[r, bend right=50] & (FU)^3C(x,y) \arrow[r, shift left=6, two heads, leftarrow] \arrow[r, shift left=4, tail] \arrow[r, shift left=2, two heads, leftarrow] \arrow[r, tail] \arrow[r, shift right=2, two heads, leftarrow] \arrow[r, shift right=4, tail] \arrow[r, shift right=6, two heads, leftarrow] \arrow[r, bend right=60] & (FU)^4C(x,y) \cdots
\end{tikzcd}$$

and now some classical simplicial homotopy theory of Meyer **[67]** reviewed in Appendix C proves that $\epsilon\colon FU_\bullet C(x,y) \to C(x,y)$ is a simplicial homotopy equivalence. ☐

As the name suggests, free resolutions are simplicial computads:

6.2.5. Proposition (free resolutions are simplicial computads). *The free resolution of any 1-category defines a simplicial computad in which the atomic n-arrows are those enclosed in precisely one pair of outermost parentheses.*

Proof. In the free resolution of a 1-category C, the category of n-arrows is $(FU)^{n+1}C$. The category FUC is the free category on the underlying graph of C. Its arrows are sequences of composable non-identity arrows of C; the atomic 0-arrows are the non-identity arrows of C. An n-arrow is a sequence of composable arrows in C with each arrow in the sequence enclosed in exactly n pairs of

parentheses. The atomic n-arrows are those enclosed in precisely one pair of parentheses on the outside. Since composition in a free category is by concatenation, the unique factorization property is clear. Since degeneracy arrows "double up" on parentheses, these preserve atomics as required. $\square$

The atomic n-arrows in the free resolution of a 1-category index the generating n-homotopies in a homotopy coherent diagram, such as enumerated for the homotopy coherent ω-simplex at the start of this chapter.

6.2.6. Definition (the homotopy coherent ω-simplex). The **homotopy coherent ω-simplex $\mathfrak{C}\Delta[\omega]$** is a simplicial category defined to be the free resolution of the category

$$\boldsymbol{\omega} \coloneqq \qquad \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] \arrow[rrrr, bend right] & 1 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] & 2 \arrow[r] \arrow[rr, bend right] & 3 \arrow[r] & \cdots \end{tikzcd}$$

The objects of $\mathfrak{C}\Delta[\omega]$ are natural numbers $k \geq 0$. Unpacking Definition 6.2.1 we can completely describe its arrows:

- A non-identity 0-arrow from j to k is a sequence of non-identity composable morphisms from j to k, the data of which is uniquely determined by the objects being passed through. In particular, there are no 0-arrows from j to k if $j > k$, and the only 0-arrow from k to k is the identity. If $j < k$, the non-identity 0-arrows from j to k correspond to subsets
$$\{j,k\} \subset T^0 \subset [j,k]$$
of the closed interval $[j,k] = \{t \in \boldsymbol{\omega} \mid j \leq t \leq k\}$ containing both endpoints.
- A 1-arrow from j to k is a once bracketed sequence of non-identity composable morphisms from j to k. This data is specified by two nested subsets
$$\{j,k\} \subset T^0 \subset T^1 \subset [j,k]$$
the larger one T^1 specifying the underlying unbracketed sequence and the smaller one T^0 specifying the placement of the brackets.
- A r-arrow from j to k is an r times bracketed sequence of non-identity composable morphisms from j to k, the data of which is specified by nested subsets
$$\{j,k\} \subset T^0 \subset \cdots \subset T^r \subset [j,k] \tag{6.2.7}$$
indicating the locations of all of the parentheses.[7]

The face and degeneracy maps of (6.2.2) are the obvious ones, either duplicating or omitting one of the sets T^i. In particular, the r-arrows just enumerated are non-degenerate if and only if each of the inclusions $T^0 \subset \cdots \subset T^r$ is proper.

We now describe the geometry of the mapping spaces $\mathfrak{C}\Delta[\omega](j,k)$.

6.2.8. Lemma (homs in the homotopy coherent ω-simplex are cubes). *The mapping spaces of the homotopy coherent ω-simplex are defined for $j,k \in \boldsymbol{\omega}$ by*

$$\mathfrak{C}\Delta[\omega](j,k) \cong \begin{cases} \varnothing & j > k \\ \Delta[0] & k = j \text{ or } k = j+1 \\ \Delta[1]^{k-j-1} & j < k. \end{cases}$$

[7]The nesting is because parenthezations should be "well formed" with open brackets closed in the reverse order to that in which they were opened.

Proof. Because $\boldsymbol{\omega}$ has no arrows from j to k when $j > k$ these hom-spaces of $\mathfrak{C}\Delta[\omega]$ are similarly empty. When $k = j$ or $k = j+1$ we have $\{j,k\} = [j,k]$, using the notation of Definition 6.2.6, so $\mathfrak{C}\Delta[\omega](j,k) \cong \Delta[0]$ is comprised of a single point.

For $k > j$, there are $k-j-1$ elements of $[j,k]$ excluding the endpoints and so we see that $\mathfrak{C}\Delta[\omega](j,k)$ has 2^{k-j-1} vertices. The r-simplices of $\mathfrak{C}\Delta[\omega](j,k)$ are given by specifying $r+1$ vertices — each a subset $\{j,k\} \subset T^i \subset [j,k]$ — that respect the ordering of subsets relation. From this we see that

$$\mathfrak{C}\Delta[\omega](j,k) \cong \Delta[1]^{k-j-1}$$

is the nerve of the poset of subsets $\{j,k\} \subset T \subset [j,k]$ ordered by inclusion, as displayed for instance in the case $j = 0$ and $k = 4$:

$$\mathfrak{C}\Delta[\omega](0,4) := \left(\begin{tikzcd} & \{0,4\} \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dr, dotted] \arrow[ddrr, dotted] \arrow[dddl, dotted] \arrow[dddr, dotted] & & \{0,1,4\} \arrow[dl] \arrow[dd] \arrow[dddl, dotted] \\ \{0,3,4\} \arrow[rr] \arrow[dd] \arrow[ddrr, dotted] & & \{0,1,3,4\} \arrow[dd] & \\ & \{0,2,4\} \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & \{0,1,2,4\} \arrow[dl] \\ \{0,2,3,4\} \arrow[rr] & & \{0,1,2,3,4\} & \end{tikzcd}\right) \qquad \square$$

Proposition 6.2.5 proves that the homotopy coherent ω-simplex is a simplicial computad and its proof identifies its atomic arrows.

6.2.9. Lemma. *The simplicial category $\mathfrak{C}\Delta[\omega]$ is a simplicial computad whose atomic r-arrows are those with a single outermost parenthesis: i.e., those sequences of subsets*

$$\{j,k\} = T^0 \subset \cdots \subset T^r \subset [j,k]$$

for which $T^0 = \{j,k\}$. Geometrically, the atomic arrows from j to k are precisely the simplices in the hom-cube $\mathfrak{C}\Delta[\omega](j,k) \cong \Delta[1]^{k-j-1}$ that contain the initial vertex $\{j,k\}$. $\square$

The finite ordinals define full subcategories of $\boldsymbol{\omega}$. In this way, the homotopy coherent ω-simplex restricts to define homotopy coherent simplices in each finite dimension.

6.2.10. Definition (the homotopy coherent n-simplex). The **homotopy coherent n-simplex** $\mathfrak{C}\Delta[n]$ is the full subcategory of the homotopy coherent ω-simplex $\mathfrak{C}\Delta[\omega]$ spanned by the objects $0, \ldots, n$. Equivalently, it is the free resolution of the ordinal category with $n+1$ objects.

Explicitly the homotopy coherent n-simplex has mapping spaces given for $j,k \in [n]$ by cubes

$$\mathfrak{C}\Delta[n](j,k) \cong \begin{cases} \emptyset & j > k \\ \Delta[0] & k = j \text{ or } k = j+1 \\ \Delta[1]^{k-j-1} & j < k. \end{cases}$$

each of which may be understood as the nerve of the poset of subsets $\{j,k\} \subset T \subset [j,k]$ ordered by inclusion. An r-arrow may be represented as a nested sequence of subsets

$$\{j,k\} \subset T^0 \subset \cdots \subset T^r \subset [j,k]$$

The homotopy coherent n-simplex is a simplicial computad whose atomic r-arrows are those sequences for which $T^0 = \{j, k\}$ or those simplices that contain the initial vertex in the hom-cube.

Exercises.

6.2.i. Exercise. Compute the free resolution of the commutative square category $\mathbb{2} \times \mathbb{2}$ and compare it with the product $\mathfrak{C}\Delta[1] \times \mathfrak{C}\Delta[1]$ of two copies of the free resolution of $\mathbb{2}$. This computation implies that the functor $\mathfrak{C} \colon s\mathcal{S}et \to s\mathcal{S}et\text{-}\mathcal{C}ptd$ to be introduced in Definition 6.3.2 does not preserve products.

6.3. Homotopy coherent realization and the homotopy coherent nerve

Our aim now is to introduce the *homotopy coherent realization* of any simplicial set X, which will define a simplicial computad $\mathfrak{C}X$ whose objects are the vertices of X. The homotopy coherent realization of $\Delta[n]$ will be the homotopy coherent n-simplex $\mathfrak{C}\Delta[n]$ of Definition 6.2.10. The homotopy coherent realization of X will be defined by "gluing together" homotopy coherent n-simplices in a canonical way: succinctly, $\mathfrak{C}X$ is defined as a colimit in $s\mathcal{S}et\text{-}\mathcal{C}ptd$ of a diagram of homotopy coherent simplices indexed by the category of simplices of X. We will then leverage Lemma 6.2.9 into an explicit presentation of $\mathfrak{C}X$ as a simplicial computad stated as Theorem 6.3.10, recovering a result of Dugger and Spivak.

The homotopy coherent realization and homotopy coherent nerve functors are determined by the cosimplicial object

$$\begin{array}{ccc} \mathbf{\Delta} & \xrightarrow{\mathfrak{C}\Delta[\bullet]} & s\mathcal{S}et\text{-}\mathcal{C}at \\ [n] & \longmapsto & \mathfrak{C}\Delta[n] \end{array}$$

where a simplicial operator $\alpha \colon [n] \to [m]$ acts on an r-arrow from j to k, as described in Definition 6.2.10, by taking the direct image of the sequence of subsets (6.2.7).

We introduce these functors in turn.

6.3.1. Definition (homotopy coherent nerve). The **homotopy coherent nerve** of a simplicial category $\mathcal{A}$ is the simplicial set $\mathfrak{N}\mathcal{A}$ whose n-simplices

$$\mathfrak{N}\mathcal{A}_n := s\mathcal{S}et\text{-}\mathcal{C}at(\mathfrak{C}\Delta[n], \mathcal{A})$$

are defined to be diagrams $\mathfrak{C}\Delta[n] \to \mathcal{A}$; the simplicial operators act contravariantly on $\mathfrak{N}\mathcal{A}$ by precomposition.

Explicitly, a homotopy coherent n-simplex in $\mathcal{A}$ is given by:

- a sequence of objects $a_0, \ldots, a_n \in \mathcal{A}$ and
- a sequence of simplicial maps
$$a_{i,j} \colon \Delta[1]^{k-j-1} \to \mathcal{A}(a_j, a_k)$$
for each $0 \leq j < k \leq n$
- satisfying the simplicial functoriality condition:

$$\begin{tikzcd} \Delta[1]^{k-j-1} \times \Delta[1]^{j-i-1} \arrow[r, "\vee_j"] \arrow[d, "a_{j,k} \times a_{i,j}"'] & \Delta[1]^{k-i-1} \arrow[d, "a_{i,k}"] \\ \mathcal{A}(a_j, a_k) \times \mathcal{A}(a_i, a_j) \arrow[r, "\circ"] & \mathcal{A}(a_i, a_k) \end{tikzcd}$$

where

$$\begin{tikzcd}
\Delta[1]^{k-j-1} \times \Delta[1]^{j-i-1} \cong \Delta^{k-i-2} \arrow[r, hook, "\vee_j"] \arrow[d, phantom, "\wr\|"] & \Delta[1]^{k-i-1} \arrow[d, phantom, "\wr\|"] \\
\mathfrak{C}\Delta[n](j,k) \times \mathfrak{C}\Delta[n](i,j) \arrow[r, "\circ"'] & \mathfrak{C}\Delta[n](i,k)
\end{tikzcd}$$

is the map that sends a pair of r-simplices

$$\{i,j\} \subset S^0 \subset \cdots \subset S^r \subset [i,j] \qquad \text{and} \qquad \{j,k\} \subset T^0 \subset \cdots \subset T^r \subset [j,k]$$

to their union

$$\{i,k\} \subset S^0 \cup T^0 \subset \cdots \subset S^r \cup T^r \subset [i,k].$$

If the $\{0,1\}$-valued coordinates of the cube $\Delta[1]^{k-i-1}$ are indexed by integers $i < t < k$, then the image of $\vee_j$ is the $j = 1$ face of the cube.

6.3.2. Definition (homotopy coherent realization). The **homotopy coherent realization** functor $\mathfrak{C}$ is the pointwise left Kan extension of the cosimplicial object $\mathfrak{C}\Delta[\bullet]$ along the Yoneda embedding:

$$\begin{tikzcd}
& s\mathcal{S}et \arrow[dr, dashed, "\mathfrak{C}"] \arrow[d, phantom, "\Uparrow\cong"] & \\
\mathbf{\Delta} \arrow[ur, hook, "よ"] \arrow[rr, "\mathfrak{C}\Delta[\bullet]"'] & & s\mathcal{S}et\text{-}\mathcal{C}at
\end{tikzcd}$$

The value of a pointwise left Kan extension at an object $X \in s\mathcal{S}et$ can be computed as a colimit indexed by the comma category $\mathsf{Hom}_{s\mathcal{S}et}(よ, X)$ [**80**, 6.2.1]. This comma category is better know as the **category of simplices of** X, whose objects are simplices of X and in which a morphism from an n-simplex x to an m-simplex y is a simplicial operator $\alpha \colon [n] \to [m]$ so that $y \cdot \alpha = x$. In this case, the colimit formula gives

$$\mathfrak{C}X := \operatorname*{colim}_{[n]\in\mathbf{\Delta}, x\in X_n} \mathfrak{C}\Delta[n].$$

By general abstract nonsense:

6.3.3. Proposition. *The homotopy coherent realization functor is left adjoint to the homotopy coherent nerve:*

$$\begin{tikzcd}
s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}"]
\end{tikzcd}$$

Proof. The homotopy coherent nerve was defined so that this adjoint correspondence would hold for the standard simplices and the general result follows since every simplicial set is a colimit, indexed by its category of simplices, of standard simplices. See [**80**, 6.5.9] for more details. ☐

6.3.4. Lemma. *The homotopy coherent realization functor takes its values in the subcategory of simplicial computads and computad morphisms.*

$$\begin{tikzcd}
s\mathcal{S}et \arrow[rr, "\mathfrak{C}"] \arrow[dr, dashed, "\mathfrak{C}"'] & & s\mathcal{S}et\text{-}\mathcal{C}at \\
& s\mathcal{S}et\text{-}\mathcal{C}ptd \arrow[ur, hook] &
\end{tikzcd}$$

Proof. By Lemma 6.1.7, which proves that the category of simplicial computads is closed under colimits in the category of simplicial categories, it suffices to demonstrate that the cosimplicial object $\mathfrak{C}\Delta[\bullet]$ is valued in the subcategory of simplicial computads and simplicial computad morphisms. We know already that the homotopy coherent simplices are simplicial computads, so we need only verify that the simplicial operators act by computad morphisms.

A simplicial operator $\alpha\colon [n] \to [m]$ acts on the r-arrow from j to k described in Definition 6.2.10 by taking the direct image of the sequence of subsets (6.2.7). The condition that characterizes the atomic arrows, $\{j,k\} = T^0$, is preserved by direct images, so we see that α defines a computad morphism $\mathfrak{C}\alpha\colon \mathfrak{C}\Delta[n] \to \mathfrak{C}\Delta[m]$. As the subcategory $s\mathcal{S}et\text{-}\mathcal{C}ptd \hookrightarrow s\mathcal{S}et\text{-}\mathcal{C}at$ is closed under colimits, it follows that every homotopy coherent realization is a simplicial computad and any simplicial map $X \to Y$ induces a morphism of simplicial computads $\mathfrak{C}X \to \mathfrak{C}Y$. □

6.3.5. Lemma. *For any inclusion $X \hookrightarrow Y$ of simplicial sets, the morphism $\mathfrak{C}X \hookrightarrow \mathfrak{C}Y$ is a simplicial subcomputad inclusion. Moreover, in the case of $X \hookrightarrow \Delta[n]$, an atomic r-arrow*

$$\{j,k\} = T^0 \subset \cdots \subset T^r \subset [j,k]$$

from j to k of $\mathfrak{C}\Delta[n]$ lies in the subcomputad $\mathfrak{C}X$ if and only if the simplex spanned by the vertices of T^r lies in X.

Proof. Recall that every monomorphism of simplicial sets $X \hookrightarrow Y$ admits a canonical decomposition as a sequential composite of pushouts of coproducts of the simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$. Since all colimits are preserved by the left adjoint $\mathfrak{C}$ and Lemma 6.1.10 proves that simplicial subcomputads are closed under colimits of this form, it suffices to prove that $\mathfrak{C}\partial\Delta[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is a simplicial subcomputad inclusion. We'll argue more generally that for $X \subset \Delta[n]$, $\mathfrak{C}X \hookrightarrow \mathfrak{C}\Delta[n]$ is a simplicial subcomputad inclusion where the atomic arrows of $\mathfrak{C}X$ are as described in the second clause of the statement.

We argue using ideas from Reedy category theory reviewed in Appendix C; see in particular Lemma C.5.20. Our task is to show that the image of $X \hookrightarrow \Delta[n]$ under the functor

$$s\mathcal{S}et \xrightarrow{\mathfrak{C}} s\mathcal{S}et\text{-}\mathcal{C}ptd \xrightarrow{\text{atom}} \mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$$

that takes a simplicial set to its $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}$-indexed graph of atomic and identity arrows is a pointwise monomorphism.

The simplicial subset $X \subset \Delta[n]$ can be described as a colimit of certain faces of $\Delta[n]$ glued along their common faces; the functor just described preserves these colimits. Our claim asserts that the composite cosimplicial object $\mathfrak{C}\Delta[\bullet]\colon \mathbf{\Delta} \to s\mathcal{S}et \to s\mathcal{S}et\text{-}\mathcal{C}ptd \to \mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ is *Reedy monomorphic*: meaning that every atomic r-arrow $T^\bullet$ in the homotopy coherent n-simplex is uniquely expressible in the form $\alpha \cdot S^\bullet$ where $\alpha\colon [m] \rightarrowtail [n]$ is a monomorphism in $\mathbf{\Delta}$ and $S^\bullet$ is "non-co-degenerate," i.e., not in the image of any monomorphism. This is clear: take $m = |T^r| - 1$ and define $\alpha\colon [m] \rightarrowtail [n]$ to be the inclusion with image $T^r \subset [n]$. Then take $S^\bullet$ to be the atomic r-arrow from 0 to m in $\mathfrak{C}\Delta[m]$ whose direct image under α is $T^\bullet$. It is clear that $S^\bullet$ is not in the image of any smaller face map. This argument also reveals that the atomic r-arrows $T^\bullet$ of $\mathfrak{C}\Delta[n]$ that are present in the subcomputad $\mathfrak{C}X$ are exactly those for which the vertices of $T^\bullet$ are contained in one of its faces. □

Lemma 6.3.5 allows us to compute the following subcomputads of the homotopy coherent simplex. Before stating the results of these computations, we introduce notation that suggests the correct geometric intuition:

6.3.6. Notation (cubes, boundaries, and cubical horns). We introduce special notation for the following simplicial sets:

- Write $\square^k$ for the **simplicial cube** $\Delta[1]^k$.
- Write $\partial\square^k$ for the **boundary of the k-cube**. Formally, $\partial\square^k$ is the domain of the iterated Leibniz product $(\partial\Delta[1] \hookrightarrow \Delta[1])^{\hat{\times}k}$. If an r-simplex in $\square^k$ is represented by a k-tuple of maps $\rho_i \colon [r] \to [1]$, then that r-simplex lies in $\partial\square^k$ if and only if there is some i for which ρ_i is constant at either vertex of $[1]$.
- Write $\sqcap^{k,j}_e \subset \partial\square^k$ for the **cubical horn** containing only the face $e \in [1]$ in direction $1 \leq j \leq k$. Formally, $\sqcap^{k,j}_e$ is the domain of the iterated Leibniz product

  $$(\partial\Delta[1] \hookrightarrow \Delta[1])^{\hat{\times}j-1}\hat{\times}(\Delta[0] \xrightarrow{e} \Delta[1])\hat{\times}(\partial\Delta[1] \hookrightarrow \Delta[1])^{\hat{\times}k-j}.$$

  An r-simplex in $\square^k$ represented by a k-tuple of maps $\rho_i \colon [r] \to [1]$ lies in $\sqcap^{k,j}_e$ if and only if for some $i \neq j$ the map p_i is constant or p_j is the constant operator at $e \in [1]$.

6.3.7. Lemma (coherent subsimplices). *The homotopy coherent realizations of the simplicial sphere and inner and outer horns define subcomputads of the homotopy coherent n-simplex containing all of the objects and defined on homs by:*

(i) **spheres***: $\mathfrak{C}\partial\Delta[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is full on all arrows except those from 0 to n and has*

$$\begin{tikzcd} \mathfrak{C}\partial\Delta[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\|"] \\ \partial\square^{n-1} \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

(ii) **inner horns***: For $0 < k < n$, $\mathfrak{C}\Lambda^k[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is full on all arrows except those from 0 to n and has*

$$\begin{tikzcd} \mathfrak{C}\Lambda^k[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\|"] \\ \sqcap^{n-1,k}_1 \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

(iii) **outer horns***: $\mathfrak{C}\Lambda^n[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is full on all arrows except those from 0 to $n-1$ or n and has*

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n](0,n-1) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](0,n-1) \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Lambda^n[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\|"] \\ \partial\square^{n-2} \arrow[r, hook] & \square^{n-2} & \sqcap^{n-1,n-1}_0 \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

Similarly, $\mathfrak{C}\Lambda^0[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is full on all arrows except those from 0 or 1 to n and has

$$\begin{tikzcd} \mathfrak{C}\Lambda^0[n](1,n) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](1,n) \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Lambda^0[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\|"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\|"] \\ \partial\square^{n-2} \arrow[r, hook] & \square^{n-2} & \sqcap^{n-1,1}_0 \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

Proof. For (i), the only non-degenerate simplex of $\Delta[n]$ that is not present in $\partial\Delta[n]$ is the top dimensional n-simplex. Consequently, the only atomic r-arrows $T^\bullet$ that are not present in $\mathfrak{C}\partial\Delta[n]$ are those with $T^r = [0,n]$. The atomic r-arrows must also have $T^0 = \{0,n\}$ and consequently correspond precisely to those r-simplices of the cube $\square^{n-1}$ that contain both the first and last vertex. Thus, we see that $\mathfrak{C}\partial\Delta[n](0,n)$ is isomorphic to the cubical boundary $\partial\square^{n-1}$.

For (ii), the only non-degenerate simplex of $\partial\Delta[n]$ that is not present in an inner horn $\Lambda^k[n]$ is the k-th face of the top dimensional simplex. Consequently, the only atomic r-arrows $T^\bullet$ that are not present in $\mathfrak{C}\Lambda^k[n]$ but are present in $\mathfrak{C}\partial\Delta[n]$ are those with $T^r = [0,n]\backslash\{k\}$. The atomic r-arrows must also have $T^0 = \{0,n\}$ and consequently correspond precisely to those r-simplices of the cube $\square^{n-1}$ that contain both the first vertex and last vertex of the $k = 0$ face of the cube $\square^{n-1}$. Thus, we see that $\mathfrak{C}\Lambda^k[n](0,n)$ is isomorphic to the cubical horn $\sqcap_1^{n-1,k}$.

For (iii), the only non-degenerate simplex of $\partial\Delta[n]$ that is not present in the outer horn $\Lambda^n[n]$ is the n-th face of the top dimensional simplex. Consequently, the only atomic r-arrows $T^\bullet$ that are not present in $\mathfrak{C}\Lambda^n[n]$ but are present in $\mathfrak{C}\partial\Delta[n]$ are those with $T^r = [0,n-1]$; note that the source of such r-arrows is 0 and the target is $n-1$. The atomic r-arrows must also have $T^0 = \{0,n-1\}$, and so, as in the proof of (i), this identifies the inclusion $\mathfrak{C}\Lambda^n[n](0,n-1) \hookrightarrow \mathfrak{C}\Delta[n](0,n-1)$ as $\partial\square^{n-2} \hookrightarrow \square^{n-2}$. The subcomputad $\mathfrak{C}\Lambda^n[n]$ is also missing non-atomic r-arrows that are present in $\mathfrak{C}\partial\Delta[n]$, namely those composites of the unique arrow from $n-1$ to n with the atomic r-arrows from 0 to $n-1$ just described. Such non-atomic r-arrows are represented by sets of inclusions with $T^0 = \{0,n-1,n\}$ and $T^r = [0,n]$ and thus are found in the $n-1 = 1$ face of the cube $\square^{n-1}$. Thus, we see that $\mathfrak{C}\Lambda^n[n](0,n)$ is isomorphic to the cubical horn $\sqcap_0^{n-1,n-1}$. $\square$

6.3.8. Definition (bead shapes). We call the atomic r-arrows of $\mathfrak{C}\Delta[n](0,n)$ that are not present in $\mathfrak{C}\partial\Delta[n](0,n)$ **bead shapes**. By Lemma 6.3.7, an r-dimensional bead shape corresponds to a sequence of subsets

$$\{0,n\} = T^0 \subset T^1 \subset \cdots \subset T^{r-1} \subset T^r = [0,n] \tag{6.3.9}$$

and is non-degenerate if and only if each of the inclusions is proper.

Putting this together we can now explicitly present the simplicial computad structure on the homotopy coherent realization of any simplicial set.

6.3.10. Theorem (homotopy coherent realizations, explicitly). *For any simplicial set X, the homotopy coherent realization $\mathfrak{C}X$ is a simplicial computad in which:*

- *The objects of $\mathfrak{C}X$ are the vertices of the simplicial set X.*
- *The atomic 0-arrows are non-degenerate 1-simplices of X, with the initial vertex of the simplex defining the source of the 0-arrow and the final vertex of the simplex defining the target of the 0-arrow.*
- *The atomic 1-arrows are non-degenerate n-simplices of X, with the initial vertex of the simplex defining the source of the 0-arrow and the final vertex of the simplex defining the target of the 0-arrow.*
- *The atomic k-arrows are pairs comprised of a non-degenerate n-simplex in X together with a k-dimensional bead shape in $\mathfrak{C}\Delta[n]$. This source of this k-arrow is the initial vertex of the n-simplex, while the target is the final vertex of the n-simplex, and the arrow is non-degenerate if and only if the bead shape is non-degenerate.*

Note that the description of atomic k-arrows subsumes those of the atomic 0-arrows and atomic 1-arrows as there is a unique 1-dimensional bead shape in $\mathfrak{C}\Delta[n]$ and a 0-dimensional bead shape exists only in the case $n = 1$. The data of a non-degenerate atomic k-arrow from x to y in $\mathfrak{C}X$ is given by a "bead," that is a non-degenerate n-simplex in X from x to y, together with the additional data of a "bead shape": sequence of proper subset inclusions (6.3.9), which Dugger and Spivak refer to as a "flag of vertex data" [**32**]. Non-atomic k-arrows are then "necklaces," that is strings of beads in X joined head to tail, together with accompanying "vertex data" for each simplex.

Proof. As observed in the proof of Lemma 6.3.5, using the canonical skeletal decomposition of the simplicial set X

$$\begin{tikzcd}
& & & \coprod_{X_n \backslash L_n X} \partial\Delta[n] \arrow[r, hook] \arrow[d] & \coprod_{X_n \backslash L_n X} \Delta[n] \arrow[d] \arrow[dl, phantom, "\ulcorner"] & & \\
\varnothing \arrow[r, hook] & \mathrm{sk}_0 X \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{sk}_{n-1} X \arrow[r, hook] & \mathrm{sk}_n X \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{colim} = X
\end{tikzcd}$$

the homotopy coherent realization $\mathfrak{C}X$ is constructed iteratively by a process that adjoints one copy of $\mathfrak{C}\Delta[n]$ along a map of its boundary $\mathfrak{C}\partial\Delta[n]$ for each non-degenerate n-simplex of X. Thus, each atomic k-arrow arises from a unique pushout of this form, as the image of an atomic k-arrow in $\mathfrak{C}\Delta[n]$ that is not present in the subcomputad $\mathfrak{C}\partial\Delta[n]$. □

6.3.11. Remark. From the description of Theorem 6.3.10, the subcomputad inclusions $\mathfrak{C}X \hookrightarrow \mathfrak{C}Y$ induced by monomorphisms of simplicial sets $X \hookrightarrow Y$ are easily understood: an r-arrow in $\mathfrak{C}Y$ lies in $\mathfrak{C}X$ if and only if each bead in its representing necklace in Y lies in X. This generalizes the description given to subcomputads of homotopy coherent simplices in Lemma 6.3.5.

Our convention is to identify 1-categories with their nerves. The homotopy coherent realization of these simplicial sets then produces a simplicial computad. But we have already encountered a way to produce a simplicial computad from a 1-category: namely via the free resolution of Definition 6.2.1. This would lead to a potential source of ambiguity were it not for the happy coincidence that these two constructions are isomorphic:[8]

6.3.12. Proposition (free resolutions are homotopy coherent realizations). *For any 1-category, the free resolution is naturally isomorphic to the homotopy coherent realization of its nerve.*

Proof. Proposition 6.2.5 and Theorem 6.3.10 present both simplicial categories as simplicial computads. We will argue that they have the same objects and non-degenerate atomic k-arrows.

Both have the same set of objects, the objects of the 1-category coinciding with the vertices in its nerve. Atomic 0-arrows of the free resolution are morphisms in the category; while atomic 0-arrows in the coherent realization are non-degenerate 1-simplices of the nerve — these are the same thing. Atomic non-degenerate 1-arrows of the free resolution are sequences of at least two morphisms (enclosed in a single set of outer parentheses), while atomic 1-arrows of the coherent realization are non-degenerate simplices of dimension at least two — again these are the same. Finally a non-degenerate atomic k-arrow is a sequence of n composable morphisms with $(k-1)$ non-repeating bracketings; this non-degeneracy necessitates $n > k$. This data defines a n-simplex in the nerve together with a non-degenerate atomic k-arrow in $\mathfrak{C}\Delta[n](0,n)$, i.e., an atomic k-arrow in the coherent realization. □

We now exploit the description of the subcomputads of the homotopy coherent n-simplex in Lemma 6.3.7 to prove a key source of examples of quasi-categories, a result first stated in this form in **[26]**.

[8]Note the isomorphism between the homotopy coherent realization of the n-simplex and the free resolution of the ordinal category $[n]$ is tautologous The left Kan extension along the Yoneda embedding is defined so as to agree with $\mathfrak{C}\Delta[\bullet]\colon \mathbf{\Delta} \to \mathcal{sSet\text{-}Cptd}$ on the subcategory of representables. Many arguments involving simplicial sets can be reduced to a check on representables, with the extension to the general case following formally by "taking colimits." This result, however, is not one of them since we are trying to prove something for all *categories* and the embedding $\mathcal{Cat} \hookrightarrow \mathcal{sSet}$ does not preserve colimits.

6.3.13. Theorem. *The homotopy coherent nerve $\mathfrak{N}\mathcal{S}$ of a Kan-complex enriched category $\mathcal{S}$ is a quasi-category.*

Proof. By adjunction, to extend along an inner horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ mapping into the homotopy coherent nerve $\mathfrak{N}\mathcal{S}$ is to extend along simplicial subcomputad inclusions $\mathfrak{C}\Lambda^k[n] \hookrightarrow \mathfrak{C}\Delta[n]$ mapping into the Kan complex enriched category $\mathcal{S}$. By Lemma 6.3.7(ii), the only missing r-arrows are in the mapping space from 0 to n, so we are asked to solve a single lifting problem

$$\begin{tikzcd} \mathfrak{C}\Lambda^k[n](0,n) \cong \sqcap_1^{n-1,k} \arrow[r] \arrow[d, hook] & \mathsf{Map}(X_0, X_n) \\ \mathfrak{C}\Delta[n](0,n) \cong \square^{n-1} \arrow[ur, dashed] \end{tikzcd}$$

Cubical horn inclusions can be filled in the Kan complex $\mathsf{Map}(X_0, X_n)$, completing the proof. □

The same techniques can be used to prove a result along similar lines.

6.3.14. Lemma. *If $Q\colon \mathcal{S} \to \mathcal{T}$ is a simplicial functor between Kan complex enriched categories that is surjective on objects and defines a local trivial fibration then $\mathfrak{N}Q\colon \mathfrak{N}\mathcal{S} \xrightarrow{\sim\!\!\!\!\rightarrow} \mathfrak{N}\mathcal{T}$ is a trivial fibration of quasi-categories.*

Proof. Exercise 6.3.iv. □

6.3.15. Remark. More generally, the homotopy coherent nerve carries DK-equivalences between Kan complex enriched categories to equivalences of quasi-categories. This result follows from Lemma 6.3.14 by applying Ken Brown's lemma C.1.10 in the context of the Bergner model structure of Digression 6.1.15. An alternate direct proof of this fact appears as Proposition 16.2.18.

We conclude by giving a precise meaning to the notion that motivated this chapter.

6.3.16. Definition. Let X be a simplicial set. A **homotopy coherent diagram** of shape X in a Kan complex enriched category $\mathcal{S}$ is a simplicial functor $\mathfrak{C}X \to \mathcal{S}$ from the homotopy coherent realization of X to $\mathcal{S}$.

Exercise 6.2.i reveals that there are two possible notions of natural transformation between homotopy coherent diagram. We opt for the more structured of the two:

6.3.17. Definition. Let $F, G\colon \mathfrak{C}X \rightrightarrows \mathcal{S}$ be homotopy coherent diagrams of shape X in a Kan complex enriched category $\mathcal{S}$. A **homotopy coherent natural transformation** $\alpha\colon F \Rightarrow G$ is a homotopy coherent diagram of shape $X \times \mathbb{2}$

$$\begin{tikzcd} \mathfrak{C}X \arrow[d, hook, "0"'] \arrow[dr, "F"] & \\ \mathfrak{C}(X \times \mathbb{2}) \arrow[r, "\alpha"] & \mathcal{S} \\ \mathfrak{C}X \arrow[u, hook, "1"] \arrow[ur, "G"'] & \end{tikzcd}$$

that restricts to F and to G along the edges of the cylinder.

These notions are originally due to Boardman and Vogt in [**16**] who observed that homotopy coherent natural transformations do not define a the morphisms of a 1-category. Instead, as they observed, they define the 1-arrows of a quasi-category.

6.3.18. COROLLARY. *Homotopy coherent diagrams of shape X valued in a Kan complex enriched category $\mathcal{S}$ and homotopy coherent natural transformations between them define the objects and 1-arrows of a quasi-category $Coh(X, \mathcal{S})$ whose n-simplices are homotopy coherent diagram $\mathfrak{C}(X \times \Delta[n]) \to \mathcal{S}$.*

PROOF. By adjunction $\mathfrak{C} \dashv \mathfrak{N}$, the simplicial set $\mathrm{Coh}(X, \mathcal{S})$ is isomorphic to $(\mathfrak{N}\mathcal{S})^X$. As the quasi-categories form an exponential ideal in the category of simplicial sets, this follows immediately from Theorem 6.3.13. □

6.3.19. REMARK (all diagrams in homotopy coherent nerves are homotopy coherent). Corollary 6.3.18 explains that any homotopy coherent diagram $\mathfrak{C}X \to \mathcal{S}$ of shape X in a Kan complex enriched category $\mathcal{S}$ transposes to define a map of simplicial sets $X \to \mathfrak{N}\mathcal{S}$ valued in the quasi-category defined as the homotopy coherent nerve of $\mathcal{S}$. While not every quasi-category is isomorphic to a homotopy coherent nerve of a Kan complex enriched category, one consequence of the *internal* Yoneda lemma, to be proven much later, is that every quasi-category is equivalent to a homotopy coherent nerve. This explains the slogan introduced at the beginning of this chapter and the title for this part of the book: all diagrams in quasi-categories are homotopy coherent, thus quasi-category theory can be understood as "homotopy coherent category theory."

Exercises.

6.3.i. EXERCISE. Compare the simplicial computad structure of the homotopy coherent ω-simplex as given by Theorem 6.3.10 with the simplicial computad structure of Lemma 6.2.9.

6.3.ii. EXERCISE. Let J be a free 1-category on an underlying graph $G \hookrightarrow J$ of "atomic" arrows. Regarding the graph G as a 1-skeletal simplicial set, show that the homotopy coherent realization $\mathfrak{C}G$ is isomorphic to J, regarded as a simplicial category with discrete hom-sets.

6.3.iii. EXERCISE. Let $\mathcal{S}$ be a Kan complex enriched category. Show that the homotopy category of the quasi-category $\mathfrak{N}\mathcal{S}$, characterized by Lemma 1.1.12, is equivalent to the homotopy category of a Kan complex enriched category described in Digression 6.1.15.

6.3.iv. EXERCISE. Prove Lemma 6.3.14.

6.4. Homotopy coherent realizations of joins

We've just discovered that a simplicial set indexed diagram $X \to \mathfrak{N}\mathcal{S}$ in the quasi-category defined as the homotopy coherent nerve of a Kan complex enriched category $\mathcal{S}$ represents a homotopy coherent diagram $\mathfrak{C}X \to \mathcal{S}$. Similarly, by Proposition F.2.1, a cone $X \star \Delta[0] \to \mathfrak{N}\mathcal{S}$ under an X-shaped diagram corresponds to a homotopy coherent diagram $\mathfrak{C}(X \star \Delta[0]) \to \mathfrak{N}\mathcal{S}$. By Proposition 4.3.2, the cone defines a colimit cone in the quasi-category $\mathfrak{N}\mathcal{S}$ just when it represents an initial element in the quasi-category of cones. In the model of Proposition F.2.1(v), the Y-shaped generalized elements of the quasi-category of cones correspond to homotopy coherent diagrams $\mathfrak{C}(X \star Y) \to \mathcal{S}$. Thus, to calculate limits and colimits in homotopy coherent nerves, we are motivated to study homotopy coherent realizations of joins. In this section, we establish an alternate characterization of the simplicial computad $\mathfrak{C}(X \star Y)$ that we exploit in Theorems 7.4.2 and 16.4.20 to characterize limits and colimits in homotopy coherent nerves.

To explain the idea of this characterization, note that the join of the unique maps $! \colon X \to \Delta[0]$ and $! \colon Y \to \Delta[0]$ defines a canonical map $X \star Y \to \Delta[1]$ with the property that the fiber over the vertex 0 is X and the fiber over the vertex 1 is Y. Applying the homotopy coherent nerve we

may regard the simplicial category $\mathfrak{C}(X \star Y)$ as an object in the category $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$, the discrete simplicial category $\mathbb{2}$ arising as the homotopy coherent nerve of the 1-simplex. Since $\mathfrak{C}$ is a left adjoint, once again $\mathfrak{C}X$ is the fiber over the object 0 and $\mathfrak{C}Y$ is the fiber over 1 in $\mathbb{2}$. In Theorem 6.4.8 we will show that the simplicial computad $\mathfrak{C}(X \star Y)$ decomposes as a pullback of simplicial computads

$$\begin{tikzcd}[column sep=small] & \mathfrak{C}(X \star Y) \arrow[dl, "\mathfrak{C}(X\star !)"'] \arrow[dr, "\mathfrak{C}(!\star Y)"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ \mathfrak{C}(X \star \Delta[0]) \arrow[dr, "\mathfrak{C}(!\star\Delta[0])"'] & & \mathfrak{C}(\Delta[0] \star Y) \arrow[dl, "\mathfrak{C}(\Delta[0]\star !)"] \\ & \mathbb{2} & \end{tikzcd} \tag{6.4.1}$$

In the proof, it will be convenient to identify $\mathfrak{C}(X \star \Delta[0])$ and $\mathfrak{C}(\Delta[0] \star Y)$ as objects of categories we now introduce.

6.4.2. Definition. Let $s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/\mathbb{2}}$ and $s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/\mathbb{2}}$ denote the full subcategories of the slice category $s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$ spanned by those simplicial categories $\mathcal{A} \to \mathbb{2}$ whose fiber over 0 or 1, respectively, is isomorphic to $\mathbb{1}$. Define

$$s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/\mathbb{2}} := s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/\mathbb{2}} \cap s\mathcal{S}et\text{-}\mathcal{C}ptd \quad \text{and} \quad s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/\mathbb{2}} := s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/\mathbb{2}} \cap s\mathcal{S}et\text{-}\mathcal{C}ptd.$$

If $\mathcal{A} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/\mathbb{2}}$, we write "$\top$" for the unique object in the fiber over 1, and if $\mathcal{A} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/\mathbb{2}}$, we write "$\perp$" for the unique object in the fiber over 0.

6.4.3. Observation. For any $\mathcal{A} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/\mathbb{2}}$ and $\mathcal{B} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/\mathbb{2}}$, the fiber of $\mathcal{A} \times_{\mathbb{2}} \mathcal{B}$ over 0 is isomorphic to $\mathcal{A}$, while the fiber over 1 is isomorphic to $\mathcal{B}$. It follows that we can give the following simplified concrete description of $\mathcal{A} \times_{\mathbb{2}} \mathcal{B}$. Its

- objects are the disjoint union of the objects of $\mathcal{A}$ and $\mathcal{B}$,
- hom spaces are defined by

$$\mathcal{A} \times_{\mathbb{2}} \mathcal{B}(a', a) := \mathcal{A}(a', a) \qquad \mathcal{A} \times_{\mathbb{2}} \mathcal{B}(b, b') := \mathcal{B}(b, b')$$
$$\mathcal{A} \times_{\mathbb{2}} \mathcal{B}(a, b) := \mathcal{A}(a, \top) \times \mathcal{B}(\perp, b),$$

- with composition defined in the manifest way in $\mathcal{A}$ or in $\mathcal{B}$.

Over the next series of lemmas we give explicit descriptions of the functors that we will demonstrate are naturally isomorphic and establish some properties that will facilitate the proof.

6.4.4. Lemma.

(i) There is a canonical bifunctor

$$s\mathcal{S}et \times s\mathcal{S}et \xrightarrow{\mathfrak{C}(-\star-)} s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$$

which preserves connected colimits in each variable separately.

(ii) There is a canonical bifunctor

$$s\mathcal{S}et \times s\mathcal{S}et \xrightarrow{\mathfrak{C}(-\star\Delta[0]) \times \mathfrak{C}(\Delta[0]\star-)} s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/\mathbb{2}} \times s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/\mathbb{2}} \xrightarrow{\quad -\times_{\mathbb{2}}- \quad} s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$$

(iii) *The projection maps displayed in* (6.4.1) *induce a comparison map*

$$\Phi_{X,Y} \coloneqq (\mathfrak{C}(X\star !), \mathfrak{C}(! \star Y)) \colon \mathfrak{C}(X \star Y) \to \mathfrak{C}(X \star \Delta[0]) \times_{\mathbb{2}} \mathfrak{C}(\Delta[0] \star Y)$$

natural in X and Y, defining the components of a natural transformation

$$\begin{tikzcd} s\mathcal{S}et \times s\mathcal{S}et \arrow[rr, bend left, "\mathfrak{C}(-\star-)"] \arrow[rr, bend right, "\mathfrak{C}(-\star\Delta[0])\times_{\mathbb{2}}\mathfrak{C}(\Delta[0]\star-)"'] \arrow[rr, phantom, "\Downarrow\Phi"] & & s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}} \end{tikzcd}$$

Proof. To see the functoriality of the construction in (i) note that a pair of simplicial maps $f \colon X \to X'$ and $g \colon Y \to Y'$ gives rise to a commutative triangle of simplicial categories

$$\begin{tikzcd} \mathfrak{C}(X \star Y) \arrow[rr, "\mathfrak{C}(f\star g)"] \arrow[dr] & & \mathfrak{C}(X' \star Y') \arrow[dl] \\ & \mathbb{2} & \end{tikzcd}$$

By Lemma 6.3.4, this diagram lies in the full subcategory $s\mathcal{S}et\text{-}\mathcal{C}ptd \hookrightarrow s\mathcal{S}et\text{-}\mathcal{C}at$ of simplicial computads and computad morphisms. Finally, since the join preserves connected components in each variable, the left adjoint $\mathfrak{C}$ preserves all colimits, colimits in the slice category $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$ are created by the forgetful functor to $s\mathcal{S}et\text{-}\mathcal{C}ptd$, and colimits in $s\mathcal{S}et\text{-}\mathcal{C}ptd$ are created in $s\mathcal{S}et\text{-}\mathcal{C}at$ — this last fact by Lemma 6.1.7 — the bifunctor of (i) preserves connected components in each variable, as claimed.

The bifunctor in (ii) is constructed by fixing one of the variables in the bifunctor of (i), yielding simplicial categories $\mathfrak{C}(X \star \Delta[0]) \in s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/\mathbb{2}}$ and $\mathfrak{C}(\Delta[0] \star Y) \in s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/\mathbb{2}}$ respectively, and then composing with the product bifunctor $- \times_{\mathbb{2}} -$ in $s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$. Later, in Lemma 6.4.5, we will observe that this functor actually takes values in $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$ as well.

Finally, the components of the comparison natural transformation of (iii) are defined by applying the bifunctor $\mathfrak{C}(- \star -)$ to the unique maps $! \colon Y \to \Delta[0]$ and $! \colon X \to \Delta[0]$. Naturality of Φ follows easily from the bifunctoriality established in (i). ☐

The properties claimed for the domain functor of Φ in (i) also hold for the codomain functor constructed in (ii) as a consequence of the following lemma:

6.4.5. Lemma. *The (fibered) product bifunctor on $s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$ restricts to give a bifunctor*

$$s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/\mathbb{2}} \times s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/\mathbb{2}} \xrightarrow{\quad -\times_{\mathbb{2}}- \quad} s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$$

which preserves connected colimits in each variable separately.

Proof. We first show that for any $\mathcal{A} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/\mathbb{2}}$ and $\mathcal{B} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/\mathbb{2}}$ the fiber product $\mathcal{A} \times_{\mathbb{2}} \mathcal{B}$ in $s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$ lies in the subcategory $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}}$. Recall from Lemma 6.1.7 that the category $s\mathcal{S}et\text{-}\mathcal{C}ptd$ is defined as the intersection of $s\mathcal{S}et\text{-}\mathcal{C}at$ and $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ in $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$. The category $s\mathcal{S}et\text{-}\mathcal{C}at$ is closed in $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ under all limits and in particular under fibered products, and limits in $\mathcal{C}at^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ are constructed objectwise in $\mathcal{C}at$. So our task is to show that the fiber products being constructed here lie in the subcategory $\mathcal{G}ph \hookrightarrow \mathcal{C}at$.

If $\mathcal{A} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/2}$ this means that each of the graphs $\mathcal{A}_n$ in the corresponding diagram

$$\mathcal{A}_\bullet \colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}} \to \mathcal{G}ph \to \mathcal{C}at$$

of free categories has the property that the fiber of $\mathcal{A}_n \to \mathbb{2}$ over 1 is $\mathbb{1}$. Dually of course to say that $\mathcal{B} \in s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/2}$ means that each of the graphs $\mathcal{B}_n$ has the property that the fiber of $\mathcal{B}_n \to \mathbb{2}$ over 0 is $\mathbb{1}$. So dropping the subscripts to declutter notation our task is now to show that if $\mathcal{A} \to \mathbb{2}$ and $\mathcal{B} \to \mathbb{2}$ are objects of $\mathcal{G}ph_{/2}$ with the property that the fibers of the former over 1 and the latter over 0 are $\mathbb{1}$, then $\mathcal{A} \times_2 \mathcal{B} \in \mathcal{C}at_{/2}$ lies in the subcategory $\mathcal{G}ph_{/2}$.

To begin, recall from Observation 6.4.3 that the fiber of $\mathcal{A} \times_2 \mathcal{B}$ over 0 is $\mathcal{A}$ while the fiber over 1 is $\mathcal{B}$. These are both free categories and the inclusions $\mathcal{A} \sqcup \mathcal{B} \hookrightarrow \mathcal{A} \times_2 \mathcal{B}$ preserve atomic arrows. Next observe that the arrows in $\mathcal{A} \times_2 \mathcal{B}$ that are not in one of the fibers have the form $(f,g) \colon (a,\perp) \to (\top,b)$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and are easily seen to be atomic if and only if f is atomic in $\mathcal{A}$ and g is atomic in $\mathcal{B}$. This proves that the simplicial category $\mathcal{A} \times_2 \mathcal{B}$ is a simplicial computad: given an arrow $(f,g) \colon (a,\perp) \to (\top,b)$, factor $f = f_n \circ \cdots \circ f_1$ and $g = g_m \circ \cdots \circ g_1$ in the free categories generated by the graphs $\mathcal{A}$ and $\mathcal{B}$ to obtain the manifest atomic decomposition

$$(f,g) = (\mathrm{id}_\top, g_m) \circ \cdots \circ (\mathrm{id}_\top, g_2) \circ (f_n, g_1) \circ (f_{n-1}, \mathrm{id}_\perp) \circ \cdots \circ (f_1, \mathrm{id}_\perp).$$

This proves that the codomain of the fiber product bifunctor of the statement restricts to $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2}$.

It remains to argue that this functor preserves connected colimits in each variable. First note that the subcategories $s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/2}$ and $s\mathcal{S}et\text{-}\mathcal{C}at^{\perp}_{/2}$ introduced in Definition 6.4.2 are closed in $s\mathcal{S}et\text{-}\mathcal{C}at_{/2}$ under connected colimits. One way to see this is to observe that the functor $(-)^1 \colon s\mathcal{S}et\text{-}\mathcal{C}at_{/2} \to s\mathcal{S}et\text{-}\mathcal{C}at$ that carries a simplicial category over $\mathbb{2}$ to its fiber over 1 has a right adjoint. This colimit-preserving functor then carries a connected colimit of diagrams in $s\mathcal{S}et\text{-}\mathcal{C}at^{\top}_{/2}$ to a connected colimit of a diagram in $s\mathcal{S}et\text{-}\mathcal{C}at$ constant at $\mathbb{1}$, and this colimit is again $\mathbb{1}$. By Lemma 6.1.7 it follows that $s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/2}$ and $s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/2}$ are similarly closed in $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2}$ under connected colimits.

By Lemma 6.1.7 again, colimits in $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2}$ are computed in $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}_{/2}$. By the argument just made the same can be said for connected colimits in $s\mathcal{S}et\text{-}\mathcal{C}ptd^{\top}_{/2}$ and $s\mathcal{S}et\text{-}\mathcal{C}ptd^{\perp}_{/2}$. Since $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}$ is a presheaf category, its slice $\mathcal{G}ph^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{epi}}}_{/2}$ is cartesian closed, and thus the product functor $- \times_2 -$ is the left adjoint of a two-variable adjunction, and thus preserves colimits in each variable separately. $\square$

Arguing along the lines of Lemmas 6.4.4 and 6.4.5, we can show that the components of the comparison natural transformation Φ are computad morphisms.

6.4.6. Corollary. *The natural transformation introduced in Lemma 6.4.4 restricts along* $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2} \in s\mathcal{S}et\text{-}\mathcal{C}at_{/2}$ *to define a natural transformation*

$$\begin{tikzcd} s\mathcal{S}et \times s\mathcal{S}et \arrow[rr, bend left, "\mathfrak{C}(-\star-)"] \arrow[rr, bend right, "\mathfrak{C}(-\star\Delta[0])\times_2\mathfrak{C}(\Delta[0]\star-)"'] \arrow[rr, phantom, "\Downarrow\Phi"] & & s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2} \end{tikzcd}$$

Proof. The component $\Phi_{X,Y}$ of the comparison natural transformation is induced by a pair of simplicial computad morphisms $\mathfrak{C}(X\star!)$ and $\mathfrak{C}(!\star Y)$ under the universal property of the fibered product $\mathfrak{C}(X\star\Delta[0]) \times_2 \mathfrak{C}(\Delta[0]\star Y)$ in $s\mathcal{S}et\text{-}\mathcal{C}at_{/2}$. But since Lemma 6.4.5 tells us that $s\mathcal{S}et\text{-}\mathcal{C}ptd_{/2}$

is closed in $s\mathcal{S}et\text{-}\mathcal{C}at_{/\mathbb{2}}$ under this fibered product, we may infer that the induced morphism $\Phi_{X,Y}$ is actually a simplicial computad morphism as claimed. □

We're now prepared to show that the natural transformation defined in Corollary 6.4.6 is a natural isomorphism. We demonstrate this first in the case where X and Y are simplices and then take advantage of the cocontinuity of our two bifunctors to extend to the general case. To streamline our notation in what follows we let $\Delta[-1]$ denote the empty simplicial set.

6.4.7. Proposition. *For $n, m \geq -1$ the simplicial computad morphism*

$$\mathfrak{C}(\Delta[n] \star \Delta[m]) \xrightarrow{\Phi_{n,m}} \mathfrak{C}(\Delta[n] \star \Delta[0]) \times_{\mathbb{2}} \mathfrak{C}(\Delta[0] \star \Delta[m])$$

is an isomorphism.

Proof. Since $\Delta[n] \star \Delta[m] \cong \Delta[n+1+m]$, the domain of $\Phi_{n,m} := \Phi_{\Delta[n],\Delta[m]}$ is the homotopy coherent simplex $\mathfrak{C}\Delta[n+1+m]$, while the codomain is $\mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$. We shall simplify our notation in the following argument by identifying $\mathfrak{C}\Delta[m+1]$ with the full subcategory of $\mathfrak{C}\Delta[n+1+m]$ spanned by the objects $n, \ldots, n+1+m$. Under these identifications, $\mathfrak{C}\Delta[n+1+m]$ is fibered over $\mathbb{2}$ by the unique functor that acts on objects to map $0, \ldots, n \mapsto 0$ and $n+1, \ldots, n+1+m \mapsto 1$, and the full subcategories $\mathfrak{C}\Delta[n+1]$ and $\mathfrak{C}\Delta[m+1]$ are fibered by restricting that functor. In particular, $\Phi_{n,m}$ is bijective on objects.

With respect to these presentations, the component $\Phi_{n,m}$ may be described in terms of a pair of simplicial operators $\alpha, \beta \colon [n+1+m] \to [n+1+m]$ defined by

$$\alpha(i) = \begin{cases} i & i \leq n, \\ n+1 & i \geq n+1 \end{cases} \qquad \beta(j) = \begin{cases} n & j \leq n \\ j & j \geq n+1. \end{cases}$$

Note that the image of α lies in the full subcategory $\mathfrak{C}\Delta[n] \hookrightarrow \mathfrak{C}\Delta[n+1+m]$ while the image of β lies in the full subcategory $\mathfrak{C}\Delta[m] \hookrightarrow \mathfrak{C}\Delta[n+1+m]$. Hence the codomains of this pair of maps restrict to define a simplicial computad morphism

$$\Phi_{n,m} = (\mathfrak{C}\alpha, \mathfrak{C}\beta) \colon \mathfrak{C}\Delta[n+1+m] \longrightarrow \mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$$

and it's evident from the construction given in Lemma 6.4.4(iii) that the resulting morphism is $\Phi_{n,m}$.

The objects of $\mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$ are pairs of integers of the form (i, n) for $0 \leq i \leq n$ or $(n+1, j)$ for $n+1 \leq j \leq n+1+m$. Recall from Definition 6.2.6 that the 0-arrows in $\mathfrak{C}\Delta[n+1+m]$ from k to ℓ correspond to subsets $T \subset [k, \ell] = \{t \in \omega \mid k \leq t \leq \ell\}$ containing both elements. Consequently, the 0-arrows in $\mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$ correspond to pairs (S, T) of 0-arrows in $\mathfrak{C}\Delta[n+1+m]$ of one of the following three forms:

- $(S, T) \colon (i, n) \to (i', n)$ in which case $S \colon i \to i'$ is a 0-arrow and $T = \{n\} \colon n \to n$ is the identity,
- $(S, T) \colon (n+1, j) \to (n+1, j')$ in which case $T \colon j \to j'$ is a 0-arrow and $S = \{n+1\} \colon n+1 \to n+1$ is the identity, or
- $(S, T) \colon (i, n) \to (n+1, j)$, in which case $S \colon i \to n+1$ is a 0-arrow and $T \colon n \to j$ is a 0-arrow.

Given such a 0-arrow (S, T) in $\mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$, it is easily checked for each of the cases catalogued above, that the set

$$S \oplus T := (S \backslash \{n+1\}) \cup (T \backslash \{n\})$$

defines a 0-arrow in $\mathfrak{C}\Delta[n+1+m]$, which goes from i to i' in the first case, from j to j' in the second, and from i to j in the third.

Note further, by construction, that $\Phi_{n,m}(S \oplus T) = (\alpha(S), \beta(T)) = (S, T)$. In fact, recalling that the simplicial operators $\alpha, \beta\colon [n+1+m] \to [n+1+m]$ act on 0-arrows in $\mathfrak{C}\Delta[n+1+m]$ by taking their direct image, it is easy to verify that every 0-arrow U in $\mathfrak{C}\Delta[n+1+m]$ coincides with $U = \alpha(U) \oplus \beta(T)$. In this way, we see that $S \oplus T$ is the unique 0-arrow of $\mathfrak{C}\Delta[n+1+m]$ that is mapped to the 0-arrow (S, T) in $\mathfrak{C}\Delta[n+1] \times_{\mathbb{2}} \mathfrak{C}\Delta[m+1]$; thus $\Phi_{n,m}$ acts bijectively on 0-arrows.

Lemma 6.2.8 reveals that the hom-spaces in the homotopy coherent simplices are nerves of posets; in particular, the r-arrows in these simplicial categories are uniquely determined by the ordered sequences of 0-arrows that comprise their vertices. Since α, β, and $\oplus$ all preserve the subset inclusion ordering between 0-arrows, we may infer from the fact that $\Phi_{n,m}$ acts bijectively on 0-arrows that it acts bijectively on r-arrows as well. Thus, we conclude that $\Phi_{n,m}$ defines an isomorphism of simplicial computads, as required. □

6.4.8. Theorem. *For any simplicial sets X and Y, the simplicial computad morphism*

$$\Phi_{X,Y}\colon \mathfrak{C}(X \star Y) \xrightarrow{\cong} \mathfrak{C}(X \star \Delta[0]) \underset{\mathbb{2}}{\times} \mathfrak{C}(\Delta[0] \star Y)$$

is an isomorphism.

Proof. For any simplicial set Z, let $\mathrm{el}(Z)$ denote its **category of elements**, the comma category associated to the cospan

$$\begin{tikzcd} & \mathrm{el}(Z) \arrow[dl, "!"'] \arrow[dr, "\pi"] & \\ \mathbb{1} \arrow[dr, "Z"'] & \Leftarrow & \mathbf{\Delta} \arrow[dl, "よ"] \\ & s\mathcal{S}et & \end{tikzcd}$$

It follows from the coYoneda lemma that we may canonically present Z as a colimit of the diagram

$$d_Z\colon \mathrm{el}(Z) \xrightarrow{\pi} \mathbf{\Delta} \xrightarrow{よ} s\mathcal{S}et$$

of standard simplices; see [**80**, 6.5.7]. Indeed, we may express Z as a *connected* component of standard simplices by means of the following trick. First, we form a connected category $\mathrm{el}(Z)_\perp$ by adjoining an initial object $\perp$ to $\mathrm{el}(Z)$ and extend d_Z to a diagram on $\mathrm{el}(Z)_\perp$ by mapping $\perp$ to the initial object $\Delta[-1] = \emptyset$ in $s\mathcal{S}et$. Since $\Delta[-1]$ is initial, any cocone under the original diagram extends uniquely to a cocone under the extended diagram, and consequently the colimits of the original and extended diagrams coincide.

In this way, the simplicial sets X and Y may be expressed as connected colimits of diagrams $d_X\colon \mathrm{el}(X)_\perp \to s\mathcal{S}et$ and $d_Y\colon \mathrm{el}(Y)_\perp \to s\mathcal{S}et$ of standard simplices. Composing these diagrams with the natural transformation Φ we obtain a natural transformation

$$\begin{tikzcd} \mathrm{el}(X)_\perp \times \mathrm{el}(Y)_\perp \arrow[r, "d_X \times d_Y"] & s\mathcal{S}et \times s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}(-\star-)"] \arrow[r, bend right, "\mathfrak{C}(-\star\Delta[0])\times_{\mathbb{2}}\mathfrak{C}(\Delta[0]\star-)"'] \arrow[r, phantom, "\Downarrow\Phi"] & s\mathcal{S}et\text{-}\mathcal{C}ptd_{/\mathbb{2}} \end{tikzcd} \tag{6.4.9}$$

whose components are instances of the morphisms that Proposition 6.4.7 demonstrates are isomorphisms.

From Lemma 6.4.4 and Lemma 6.4.5 we know that the domain and codomain functors of Φ preserve connected colimits in each variable. Consequently, the colimit of the domain diagram of

(6.4.9) is isomorphic to $\mathfrak{C}(X \star Y)$, while the colimit of the codomain diagram is isomorphic to $\mathfrak{C}(X \star \Delta[0]) \times_{\mathbb{2}} \mathfrak{C}(\Delta[0] \star Y)$. Further, as a consequence of naturality of Φ, the morphism induced by (6.4.9) between these colimits is isomorphic to the component $\Phi_{X,Y}$, which is therefore an isomorphism as postulated. $\square$

Exercises.

6.4.i. Exercise. Use Theorem 6.4.8 to enumerate the data required to define a simplicial functor $F\colon \mathfrak{C}(X \star Y) \to \mathcal{S}$ restricting to a given pair of homotopy coherent diagrams $F_X\colon \mathfrak{C}X \to \mathcal{S}$ and $F_Y\colon \mathfrak{C}Y \to \mathcal{S}$. The exercise heralds applications of Theorem 6.4.8 that appear as Proposition 7.4.4.

6.5. Hom spaces in quasi-categories

Recall from Definition 3.4.9 that the **internal mapping space** between a pair of elements $a, b\colon 1 \rightrightarrows A$ in an ∞-category is the comma ∞-category associated to this cospan, which may be constructed by the following pullback diagram

$$\begin{tikzcd} \mathsf{Hom}_A(a,b) \arrow[r, "\phi"] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ 1 \arrow[r, "{(b,a)}"'] & A \times A \end{tikzcd}$$

Proposition 3.4.10 reveals that internal mapping spaces are always discrete ∞-categories.

In the case where $A = \mathfrak{N}\mathcal{A}$ is a quasi-category defined, via Theorem 6.3.13, as the homotopy coherent nerve of a Kan complex enriched category $\mathcal{A}$, it is natural to ask whether the Kan complex $\mathsf{Hom}_{\mathfrak{N}\mathcal{A}}(a,b)$ coincides with the Kan complex $\mathcal{A}(a,b)$, which we refer to as the **external mapping space** between the objects $a, b \in \mathcal{A}$. The aim in this section, achieved in Corollary 6.5.12, is to prove that these Kan complexes are equivalent. This result shows that the internal mapping space "has the correct homotopy type" to capture the mapping spaces in a Kan complex enriched category.

Our proof of this equivalence relates these hom-spaces to two additional models for the hom-space between a pair of elements in a quasi-category that we now introduce.

6.5.1. Definition (right and left hom-spaces). Given elements a, b in a quasi-category A, define the **right hom-space** $\mathsf{Hom}^R_A(a,b)$ from a to b to be the simplicial set with n-simplices given by $n+1$-simplices σ in A with the property that the final vertex is b and the initial face is degenerate at a.

$$\Delta[n] \xrightarrow{\sigma} \mathsf{Hom}^R_A(a,b) \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \Delta[n] \arrow[r, "!"] \arrow[d, "\delta^{n+1}"'] & \Delta[0] \arrow[d, "a"] \\ \Delta[n+1] \arrow[r, dashed, "\sigma"] & A \\ \Delta[0] \arrow[u, "\{n+1\}"] \arrow[ur, "b"'] & \end{tikzcd}$$

A simplicial operator $\alpha\colon [m] \to [n]$ acts by restriction along the map $\alpha \star \Delta[0]\colon \Delta[m+1] \to \Delta[n+1]$.

Dually, define the **left hom-space** $\mathsf{Hom}^L_A(a,b)$ from a to b to be the simplicial set with n-simplices given by $n+1$-simplices σ in A with the property that the initial vertex is a and the final face is

degenerate at b.

$$\Delta[n] \xrightarrow{\sigma} \mathsf{Hom}^L_A(a,b) \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \Delta[0] \arrow[d, "\{0\}"'] \arrow[dr, "a"] \\ \Delta[n+1] \arrow[r, "\sigma"] & A \\ \Delta[n] \arrow[u, "\delta^0"] \arrow[r, "!"'] & \Delta[0] \arrow[u, "b"'] \end{tikzcd}$$

A simplicial operator $\alpha\colon [m] \to [n]$ acts by restriction along the map $\Delta[0]\star\alpha\colon \Delta[m+1] \to \Delta[n+1]$.

Alternatively, the left and right hom-spaces can be constructed as pullbacks of the slice quasi-categories defined in Proposition 4.2.5.

6.5.2. Lemma. *Given elements a, b in a quasi-category A, the left and right hom-spaces arise as fibers of the canonical projections from the slice quasi-categories:*

$$\begin{tikzcd} \mathsf{Hom}^L_A(a,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & {}^{a/}A \arrow[d, two heads, "\pi"] \\ \Delta[0] \arrow[r, "b"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Hom}^R_A(a,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A_{/b} \arrow[d, two heads, "\pi"] \\ \Delta[0] \arrow[r, "a"'] & A \end{tikzcd}$$

Consequently, when A is a quasi-category, both $\mathsf{Hom}^L_A(a,b)$ and $\mathsf{Hom}^R_A(a,b)$ are Kan complexes.

Proof. The characterization of $\mathsf{Hom}^L_A(a,b)$ and $\mathsf{Hom}^R_A(a,b)$ as fibers follows directly from transposing Definition 6.5.1 across the adjunction of Proposition 4.2.5, and is left to the reader.

To see that $\mathsf{Hom}^L_A(a,b)$ is a Kan complex when A is a quasi-category, we first show that the projection $\pi\colon {}^{a/}A \twoheadrightarrow A$ is a **left fibration**, a map with the right lifting property against the left horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k < n$. This lifting property follows by adjunction; the lifting problem below-left transposes to the lifting problem below-right

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "\sigma"] \arrow[d, hook] & {}^{a/}A \arrow[d, two heads, "\pi"] \\ \Delta[n] \arrow[r, "\tau"'] \arrow[ur, dashed] & A \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \Delta[0] \star \Lambda^k[n] \underset{\Delta[n+1]}{\cup} \Delta[n] \arrow[r, "\sigma\cup\tau"] \arrow[d, hook] & A \\ \Delta[n+1] \arrow[ur, dashed] \end{tikzcd}$$

under the "Leibniz adjunction" associated to the join-slice adjunction of Proposition 4.2.5; see Proposition C.2.9(ii). Here $\Delta[0] \star \Lambda^k[n] \cup_{\Delta[n+1]} \Delta[n]$ is the simplicial subset of $\Delta[n+1]$ formed by gluing the final δ^0-face $\Delta[n]$ to $\Delta[0] \star \Lambda^k[n]$. As observed in Lemma D.2.11, this produces the inner horn inclusion $\Lambda^{k+1}[n+1] \to \Delta[n+1]$. Since A is a quasi-category, the right-hand lifting problem can be solved, and hence so can the left-hand one.

By pullback stability of lifting properties (see Lemma C.2.3), the map $\mathsf{Hom}^L_A(a,b) \twoheadrightarrow \Delta[0]$ is also a left fibration; in particular $\mathsf{Hom}^L_A(a,b)$. Additionally, this is enough to show that the homotopy category of the quasi-category $\mathsf{Hom}^L_A(a,b)$, characterized in Lemma 1.1.12, is a groupoid: lifting against left horn inclusions $\Lambda^0[2] \hookrightarrow \Delta[2]$ shows that every arrow admits a left inverse, which then admits a further left inverse. By Corollary 1.1.15 we conclude that $\mathsf{Hom}^L_A(a,b)$ is a Kan complex.

Dually, the projection $\pi\colon A_{/b} \twoheadrightarrow A$ is a right fibration, and so $\mathrm{Hom}^R_A(a,b)$ is a Kan complex as well. ☐

We now show that the left and right hom-spaces are equivalent to the internal mapping spaces of a quasi-category.

6.5.3. Proposition (relating hom-spaces). *Suppose a and b are elements of a quasi-category A. Then there exist canonical equivalences of Kan complexes*

$$\mathrm{Hom}^L_A(a,b) \xrightarrow{\sim} \mathrm{Hom}_A(a,b) \xleftarrow{\sim} \mathrm{Hom}^R_A(a,b)$$

One way to see this is to outsource the work to Appendix D. Corollary D.6.6 proves for any element $x\colon 1 \to A$ of a quasi-category A, that there are canonical equivalences $A_{/x} \rightleftarrows \mathrm{Hom}_A(A,x)$ and ${}^{x/}A \rightleftarrows \mathrm{Hom}_A(x,A)$ over A. These then pull back over another element $y\colon 1 \to A$ to define the canonical equivalences postulated in Proposition 6.5.3. However, the Corollary D.6.6 relies upon the general equivalence between slice and comma quasi-categories of cones proven as Proposition D.6.5, which in turn relies upon the equivalence between the join and fat join constructions sketched in Proposition 4.2.7 and proven in Proposition D.6.4. Rather than take such a circuitous route, we give a direct proof, via a mild reinterpretation of an argument first developed by Dugger and Spivak [**31**, §4.3].

Proof. The two equivalences are dual under the involution that reverses the ordering of the vertices in every simplex in a simplicial set, so we focus on the latter. For a simplicial set U, a map $U \to \mathrm{Hom}_A(a,b)$ corresponds to a cylinder $U \times \Delta[1] \to A$ whose endpoints are constant at a and b, while a map $U \to \mathrm{Hom}^R_A(a,b)$ corresponds to a map $U \star \Delta[0] \to A$ where the subcomplex U is mapped to the vertex a and the final vertex is mapped to b. This observation leads us to consider the following pushouts

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta[0] \arrow[d, hook] \\ \Delta[0] \star U \arrow[r] & \Sigma^L U \end{tikzcd} \qquad \begin{tikzcd} U + U \arrow[r, "!+!"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta[0]+\Delta[0] \arrow[d, hook] \\ U \times \Delta[1] \arrow[r] & \Sigma U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "!"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta[0] \arrow[d, hook] \\ U \star \Delta[0] \arrow[r] & \Sigma^R U \end{tikzcd} \tag{6.5.4}$$

Note that all three simplicial sets $\Sigma^L U$, ΣU, and $\Sigma^R U$ have exactly two vertices, which we call $-$ and $+$, and moreover every non-degenerate 1-simplex has source $-$ and target $+$, so we may regard these constructions as defining objects in the category $s\mathcal{S}et_{*,*} := {}^{\Delta[0]+\Delta[0]/}s\mathcal{S}et$ of doubly-pointed simplicial sets.

As noted above, the simplicial sets $\Sigma^L U$, ΣU, and $\Sigma^R U$ describe the "shapes" in a doubly-pointed simplicial set A that correspond to U-shaped diagrams in $\mathrm{Hom}^L_A(a,b)$, $\mathrm{Hom}_A(a,b)$, and $\mathrm{Hom}^R_A(a,b)$, a relationship that is codified by the following of adjunctions:

$$\begin{tikzcd} s\mathcal{S}et_{*,*} \arrow[r, bend right, "\mathrm{Hom}^L"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\Sigma^L"'] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et_{*,*} \arrow[r, bend right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\Sigma"'] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et_{*,*} \arrow[r, bend right, "\mathrm{Hom}^R"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\Sigma^R"'] \end{tikzcd} \tag{6.5.5}$$

It follows that we can define the natural comparison $\mathsf{Hom}^L_A(a,b) \to \mathsf{Hom}_A(a,b) \leftarrow \mathsf{Hom}^R_A(a,b)$ by taking the mates of the natural transformations

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left=50, "\Sigma^L"] \arrow[r, "\Sigma" description] \arrow[r, bend right=50, "\Sigma^R"'] \arrow[r, phantom, shift left=3, "\Uparrow v_L"] \arrow[r, phantom, shift right=3, "\Downarrow v_R"] & s\mathcal{S}et_{*,*} \end{tikzcd}$$

under these adjunctions. As left adjoints, both of the functors Σ and Σ^R are left Kan extensions of their restrictions along the Yoneda embedding よ: $\mathbf{\Delta} \hookrightarrow s\mathcal{S}et$, so it suffices to specify the natural transformation $v_R \colon \Sigma \Rightarrow \Sigma^R$ on the standard simplices and then left Kan extend up to the category of simplicial sets.

With this in mind, observe that there exist order preserving maps

$$\begin{array}{ccc} [n]\times[1] & \xrightarrow{v^n_L} & [n+1] \\ (i,0) & \longmapsto & 0 \\ (i,1) & \longmapsto & i+1 \end{array} \qquad \begin{array}{ccc} [n]\times[1] & \xrightarrow{v^n_R} & [n+1] \\ (i,0) & \longmapsto & i \\ (i,1) & \longmapsto & n+1 \end{array}$$

Taking nerves, these define families of maps, natural in $[n] \in \Delta$, which induce the desired natural transformation components upon passing to the quotients:

$$\begin{tikzcd} \Delta[n+1] \arrow[d, two heads] & \Delta[n]\times\Delta[1] \arrow[l, two heads, "v^n_L"'] \arrow[r, two heads, "v^n_R"] \arrow[d, two heads] & \Delta[n+1] \arrow[d, two heads] \\ \Sigma^L\Delta[n] & \Sigma\Delta[n] \arrow[l, dashed, two heads, "v^n_L"'] \arrow[r, dashed, two heads, "v^n_R"] & \Sigma^R\Delta[n] \end{tikzcd}$$

By definition, n-simplex in $\mathsf{Hom}_A(a,b)$, $\mathsf{Hom}^L_A(a,b)$, or $\mathsf{Hom}^R_A(a,b)$ correspond to maps from $\Sigma\Delta[n]$, $\Sigma^L\Delta[n]$, or $\Sigma^R\Delta[n]$, respectively, to $\langle a,b\rangle \colon \Delta[0]+\Delta[0] \to A$ in $s\mathcal{S}et_{*,*}$. The corresponding component of the mates $\hat{v_L} \colon \mathsf{Hom}^L \to \mathsf{Hom}$ and $\hat{v_R} \colon \mathsf{Hom}^R \to \mathsf{Hom}$ acts on n-simplices by precomposing with the maps $v^n_L \colon \Sigma[n] \to \Sigma^L\Delta[n]$ and $v^n_R \colon \Sigma\Delta[n] \to \Sigma^R\Delta[n]$. Since these maps are epimorphisms, the comparisons $\mathsf{Hom}^L_A(a,b) \hookrightarrow \mathsf{Hom}_A(a,b) \hookleftarrow \mathsf{Hom}^R_A(a,b)$ are injective.

The technical heart of the proof is the following lemma, which shows that $v_R \colon \Sigma\Delta[\bullet] \to \Sigma^L\Delta[\bullet]$ and $v_R \colon \Sigma\Delta[\bullet] \to \Sigma^R\Delta[\bullet]$ define pointwise Joyal weak equivalences between Reedy cofibrant cosimplicial objects. Then Lemma C.5.22, which encodes a general Reedy category theory trick, tells us that when we then map from such a natural transformation into a fibrant object $\langle a,b\rangle \colon \Delta[0]+\Delta[0] \to A$ in the slice Joyal model structure on $s\mathcal{S}et_{*,*}$, the resulting map of simplicial sets, in this case $\hat{v}_L \colon \mathsf{Hom}^L_A(a,b) \to \mathsf{Hom}_A(a,b)$ or $\hat{v}_R \colon \mathsf{Hom}^R_A(a,b) \to \mathsf{Hom}_A(a,b)$, is then an equivalence of quasi-categories. $\square$

6.5.6. Lemma. *The natural transformations*

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend left=50, "\Sigma^L\Delta[\bullet]"] \arrow[r, "\Sigma\Delta[\bullet]" description] \arrow[r, bend right=50, "\Sigma^R\Delta[\bullet]"'] \arrow[r, phantom, shift left=3, "\Uparrow v_L"] \arrow[r, phantom, shift right=3, "\Downarrow v_R"] & s\mathcal{S}et_{*,*} \end{tikzcd}$$

are pointwise Joyal weak equivalences between Reedy cofibrant cosimplicial objects.

PROOF. Recall the category $s\mathcal{S}et_{*,*}$ is defined as a slice category ${}^{\Delta[0]+\Delta[0]}/s\mathcal{S}et$, and consequently inherits a sliced version of the Joyal model structure of Digression 1.1.29; a bisimplicial set is cofibrant just when its two chosen basepoints are distinct. To prove that $\Sigma^L\Delta[\bullet]$, $\Sigma^R\Delta[\bullet]$, and $\Sigma\Delta[\bullet]$ are Reedy cofibrant, we appeal to Lemma C.5.20, which tells that we need only verify that the equalizer of the face maps $\delta^0, \delta^1 \colon \Sigma\Delta[0] \rightrightarrows \Sigma\Delta[1]$, $\delta^0, \delta^1 \colon \Sigma^R\Delta[0] \rightrightarrows \Sigma^L\Delta[1]$, and $\delta^0, \delta^1 \colon \Sigma^R\Delta[0] \rightrightarrows \Sigma^R\Delta[1]$ are the initial object, and in each case we compute that these equalizers are isomorphic to the simplicial set $\partial\Delta[1]$, as desired.

Now to see that the components $v^n_L \colon \Sigma^L\Delta[n] \to \Sigma\Delta[n]$ and $v^n_R \colon \Sigma\Delta[n] \to \Sigma^R\Delta[n]$ are weak Joyal equivalences, we appeal to the 2-of-3 property and verify that the canonical endpoint-preserving maps $\Sigma^L\Delta[n] \to \Delta[1]$, $\Sigma^R\Delta[n] \to \Delta[1]$, and $\Sigma\Delta[n] \to \Delta[1]$, which commute with these maps, are weak Joyal equivalences. Hence, $\Sigma^L\Delta[\bullet]$, $\Sigma^R\Delta[\bullet]$, and $\Sigma\Delta[\bullet]$ can be understood as three cosimplicial resolutions of $\Delta[1] \in s\mathcal{S}et_{*,*}$. These arguments are originally due to Dugger and Spivak in [**31**, 9.3-4].

We start by demonstrating that $\Sigma^L\Delta[n] \to \Delta[1]$ is a weak categorical equivalence; the argument for $\Sigma^R\Delta[n] \to \Delta[1]$ is dual. Write $\Delta[n]_{i|i+1}$ for the quotient of $\Delta[n]$ which collapses the face spanned by the vertices $\{0 \cdots i\}$ to a point and the face spanned by the vertices $\{i+1 \cdots n\}$ to a point. This simplicial set has two vertices and has a non-degenerate k-simplex for each non-degenerate k-simplex of $\Delta[n]$ whose image surjects onto $\Delta[1]$. As usual, we identify non-degenerate simplices in $\Delta[n]$ with their vertices $\{v_0 \ldots v_k\}$. Simplices with $v_0 \le i$ and $v_k > i$ correspond bijectively to the non-degenerate simplices of $\Delta[n]_{i|i+1}$ so we assign the same labeling to the latter. Note that $\Sigma^L\Delta[n] \coloneqq \Delta[n+1]_{0|1}$.

To prove that $v^n_L \colon \Sigma^L\Delta[n] \to \Delta[1]$ is a weak Joyal equivalence, we demonstrate that a section $\Delta[1] \xrightarrow{[01]} \Delta[n+1]_{0|1}$ is inner anodyne by factoring this inclusion as

$$\Delta[1] = X^1 \hookrightarrow X^2 \hookrightarrow \cdots \hookrightarrow X^{n+1} = \Delta[n+1]_{0|1} \tag{6.5.7}$$

and showing that each $X^j \hookrightarrow X^{j+1}$ is inner anodyne. In this filtration, X^j is defined to be the simplicial subset of $\Delta[n+1]_{0|1}$ that contains all non-degenerate j-simplices that contain the edge $\{0,1\}$. That is, these simplicial subsets can be constructed by forming the unions

$$X^2 = \bigcup_{1<i\le n+1} \{01i\}, \quad X^3 = \bigcup_{1<i<j\le n+1} \{01ij\}, \quad X^4 = \bigcup_{1<i<j<k\le n+1} \{01ijk\}$$

and so on. Now each stage in this filtration is canonically a pushout of a coproduct of inner horns

$$\begin{tikzcd} \coprod\limits_{1<i_1<\cdots<i_{j-1}\le n+1} \Lambda^1[j] \arrow[r, tail, "\sim"] \arrow[d] & \coprod\limits_{1<i_1<\cdots<i_{j-1}\le n+1} \Delta[j] \arrow[d] \\ X^{j-1} \arrow[r, tail, "\sim"] & X^j \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which demonstrates that the composite inclusion $\Delta[1] \hookrightarrow \Delta[n+1]_{0|1}$ is inner anodyne as claimed. Thus the retraction $\Sigma^L\Delta[n] \coloneqq \Delta[n+1]_{0|1} \to \Delta[1]$ is a weak categorical equivalence. A dual argument establishes the same result for $\Sigma^r\Delta[n] \coloneqq \Delta[n+1]_{n|n+1} \to \Delta[1]$.

To establish the weak equivalence $\Sigma\Delta[n] \to \Delta[1]$, we first characterize the simplices of the domain in terms of the quotient map $\Delta[n] \times \Delta[1] \to \Sigma\Delta[n]$. To condense notation, write $\{0, 1, \ldots, n\}$ and $\{0', 1', \ldots, n'\}$ for the vertices spanned by $\Delta[n] \times \{0\}$ and $\Delta[n] \times \{1\}$ in $\Delta[n] \times \Delta[1]$, respectively, using the prime as shorthand for the second coordinate. In this notation, non-degenerate simplices in

$\Delta[n]\times\Delta[1]$ correspond to sequences $\{v_0 \cdots v_j v'_{j+1} \cdots v'_k\}$ of elements of $[n]$ such that the v_i and v'_i are strictly increasing and with $v_j \le v'_{j+1}$. Non-degenerate simplices containing at least one primed vertex and one unprimed one correspond bijectively to non-degenerate simplices in the quotient $\Sigma\Delta[n]$.

Let σ^i be the $(n+1)$-simplex $\{01 \cdots (i-1)ii'(i+1)' \cdots n'\}$, the "$i$th shuffle" of the cylinder. Its quotient in $\Sigma\Delta[n]$ is a simplex of shape $\Delta[n+1]_{i|i+1}$. We will show that the weak categorical equivalence $\Sigma^L\Delta[n] \to \Delta^1$ factors through $\Sigma\Delta[n] \to \Delta[1]$ along a filtration

$$\Sigma^L\Delta[n] = \Delta^{n+1}_{0|1} = Y^0 \hookrightarrow Y^1 \hookrightarrow \cdots \hookrightarrow Y^n = \Sigma\Delta[n],$$

in which Y^i is defined to be the union of Y^{i-1} with the simplex σ^i in $\Sigma\Delta[n]$. Note that the intersection of Y^{i-1} and σ^i in $\Sigma\Delta[n]$ is the n-simplex $\{01 \cdots (i-1)i' \cdots n'\}$, which has shape $\Delta[n]_{i-1|i}$. The map $\delta^0\colon \Delta[n] \to \Delta[n+1]$ induces a map $\Delta[n]_{i-1|i} \to \Delta[n+1]_{i|i+1}$ over $\Delta[1]$. In this way we see that the inclusion $Y^{i-1} \hookrightarrow Y^i$ may be described via a pushout

$$\begin{tikzcd}
\Delta[n]_{i-1|i} \arrow[r, hook] \arrow[d] & \Delta[n+1]_{i|i+1} \arrow[d] \\
Y^{i-1} \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & Y^i
\end{tikzcd}$$

Our next task is to argue that the horizontal inclusions are inner anodyne, so that we can conclude that the pushouts $Y^{i-1} \hookrightarrow Y^i$ are as well. As before, we factor this map as

$$\Delta[n]_{i-1|i} = Z^1 \hookrightarrow Z^2 \hookrightarrow \cdots \hookrightarrow Z^{n+1} = \Delta[n+1]_{i|i+1}$$

and show each step is inner anodyne. Let

$$Z^2 = Z^1 \cup \bigcup_{i<j\le n+1} \{01j\}, \quad Z^3 = Z^2 \cup \bigcup_{1<j<k\le n+1, i<k} \{01jk\}, \quad Z^4 = Z^3 \cup \bigcup_{1<j<k<l\le n+1, i<l} \{01jkl\}, \ldots$$

Each Z^j is obtained by attaching the non-degenerate j-simplices that contain the edge $[01]$ to Z^{j-1}. As above, the attaching maps are inner horn inclusions $\Lambda^1[j] \hookrightarrow \Delta[j]$, and consequently $Z^{j-1} \hookrightarrow Z^j$ may be expressed as a pushout of a coproduct of maps of this form. Note that in this case if some of the intermediate vertices are less than or equal to i, the last few faces of the $\Lambda^1[j]$-horn will be degenerate at 0.

This shows that the inclusion $\Delta[n]_{i-1|i} \hookrightarrow \Delta[n+1]_{i|i+1}$ is inner anodyne, and hence that the inclusion $\Sigma^L\Delta[n] \hookrightarrow \Sigma\Delta[n]$ is inner anodyne. A dual construction proves that $\Sigma^R\Delta[n] \hookrightarrow \Sigma\Delta[n]$ is inner anodyne as well. If we like, we can conclude now that the canonical map $\Sigma\Delta[n] \to \Delta[1]$ is a weak Joyal equivalence, but in fact these maps are sections of v^n_L and v^n_R so the conclusion follows by the 2-of-3 property. $\square$

6.5.8. Remark. As just observed in the proof of Lemma 6.5.6, the natural maps

$$\Sigma^L\Delta[\bullet] \overset{v^\bullet_L}{\twoheadleftarrow} \Sigma\Delta[\bullet] \overset{v^\bullet_R}{\twoheadrightarrow} \Sigma^R\Delta[\bullet]$$

defined in the proof of Proposition 6.5.3 have pointwise-defined sections

$$\Sigma^L\Delta[n] \overset{\sim}{\rightarrowtail} \Sigma\Delta[n] \overset{\sim}{\leftarrowtail} \Sigma^R\Delta[n]$$

that including the $n+1$-simplices whose quotients define $\Sigma^L\Delta[n]$ and $\Sigma^R\Delta[n]$ as the first and last shuffles of the cylinder $\Delta[n]\times\Delta[1]$ whose quotient defines $\Sigma\Delta[n]$. These sections are inner anodyne

extensions, which explain why their retractions are weak Joyal equivalences. However, these sections do not assemble into a cosimplicial object in $s\mathcal{S}et_{*,*}$.

6.5.9. Proposition (relating hom-spaces and function complexes). *For any pair of objects a, b in a Kan complex enriched category $\mathcal{A}$, there exists a canonical trivial cofibration*

$$\mathcal{A}(a,b) \overset{\sim}{\rightarrowtail} \mathsf{Hom}^R_{\mathfrak{N}\mathcal{A}}(a,b)$$

between the external mapping space from a to b and the right hom space from a to b in the homotopy coherent nerve $\mathfrak{N}\mathcal{A}$.

Proof. Recall from Example 6.1.5 the construction of a bipointed simplicial category $\langle -,+\rangle \colon \mathbb{1}+\mathbb{1} \to \mathbb{2}[U]$ from a simplicial set U. Maps out of $\mathbb{2}[U]$ in the category $s\mathcal{S}et\text{-}\mathcal{C}at_{*,*} := {}^{\mathbb{1}+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$ correspond to diagrams of shape U in a specified mapping space, which is to say there is an adjunction

$$\begin{tikzcd} s\mathcal{S}et\text{-}\mathcal{C}at_{*,*} \arrow[r, bend right, "\mathsf{Map}"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathbb{2}[-]"'] \end{tikzcd}$$

whose right adjoint sends a bipointed simplicial category $\langle a,b\rangle \colon \mathbb{1}+\mathbb{1} \to \mathcal{A}$ to the mapping space $\mathcal{A}(a,b)$.

We obtain a second adjunction between the same pair of categories by composing the right-hand adjunction of (6.5.5) with the adjunction of Proposition 6.3.3:

$$\begin{tikzcd} s\mathcal{S}et\text{-}\mathcal{C}at_{*,*} \arrow[r, bend right, "\mathfrak{N}"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{*,*} \arrow[l, bend right, "\mathfrak{C}"'] \arrow[r, bend right, "\mathsf{Hom}^R"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\Sigma^R"'] \end{tikzcd}$$

Now as in the proof of Proposition 6.5.3, we may define the natural comparison map $\mathcal{A}(a,b) \to \mathsf{Hom}^R_{\mathfrak{N}\mathcal{A}}(a,b)$ as the mate of a corresponding natural transformation $\omega \colon \mathfrak{C}\Sigma^R \Rightarrow \mathbb{2}[-]$.

To that end, given a simplicial set U form the following diagram of simplicial categories

$$\begin{tikzcd} \mathfrak{C}(U) \arrow[r] \arrow[d, hook] & \mathbb{1} \arrow[d] \arrow[ddr, bend left, "-"] \\ \mathfrak{C}(U \star \Delta[0]) \arrow[r] \arrow[drr, bend right, "\tau^U"'] & \mathfrak{C}\Sigma^R U \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\omega^U"] \\ & & \mathbb{2}[U] \end{tikzcd}$$

The pushout square is obtained by applying the left adjoint $\mathfrak{C}$ to the pushout (6.5.4) that defines $\Sigma^R U$. The natural map $\tau^U \colon \mathfrak{C}(U \star \Delta[0]) \to \mathbb{2}[U]$, defined in Lemma 6.5.10 below sends every object of $\mathfrak{C}(U \star \Delta[0])$ except the cocone vertex to $- \in \mathbb{2}[U]$ and carries the cone vertex to $+ \in \mathbb{2}[U]$. This is enough to imply commutativity of the outer square, inducing the desired simplicial functor $\omega^U \colon \mathfrak{C}\Sigma^R U \to \mathbb{2}[U]$, which inherits naturality in U from the naturality of $\tau \colon \mathfrak{C}(-)^{\triangleright} \Rightarrow \mathbb{2}[-]$.

By adjunction, n-simplices in $\mathcal{A}(a,b)$ and in $\mathsf{Hom}^R_{\mathfrak{N}\mathcal{A}}(a,b)$ correspond, respectively, to simplicial functors $\mathbb{2}[\Delta[n]] \to \mathcal{A}$ and $\mathfrak{C}\Sigma^R\Delta[n] \to \mathcal{A}$ in $s\mathcal{S}et\text{-}\mathcal{C}at_{*,*}$. Furthermore, the corresponding

component $\hat{\omega}\colon \mathcal{A}(a,b) \to \mathsf{Hom}^R_{\mathfrak{N}\mathcal{A}}(a,b)$ of the mate of $\omega\colon \mathfrak{C}\Sigma^R \to \mathbb{2}[-]$ acts on n-simplices by pre-composition with $\omega^n\colon \mathfrak{C}\Sigma^R\Delta[n] \to \mathbb{2}[\Delta[n]]$. Since the maps ω^n are epimorphisms, the comparison $\hat{\omega}\colon \mathcal{A}(a,b) \hookrightarrow \mathsf{Hom}^R_{\mathfrak{N}\mathcal{A}}(a,b)$ is injective.

Finally, to see that $\hat{\omega}$ is an equivalence, it suffices by Lemma C.5.22 to show that the simplicial functors $\omega^n\colon \mathfrak{C}\Sigma^R\Delta[n] \to \mathbb{2}[\Delta[n]]$ define pointwise weak equivalences between Reedy cofibrant cosimplicial objects in $s\mathcal{S}et\text{-}\mathcal{C}at_{*,*}$. Furthermore, by Remark C.5.23, we need not establish the full Bergner model structure of Digression 6.1.15 to make this argument. Indeed, all of the objects under consideration are two-object simplicial categories with only one non-trivial hom-space, pointing in the same direction, so the notions of cofibration, fibrant, and weak equivalence, all reduce to the corresponding notions in Quillen's model structure for Kan complexes on simplicial sets.

Lemma 6.5.6 shows that the cosimplicial object $\Sigma^R\Delta[\bullet] \in s\mathcal{S}et^{\mathbf{\Delta}}_{*,*}$ is Reedy cofibrant. By Lemma 6.3.5, the same is then true of $\mathfrak{C}\Sigma^R\Delta[\bullet] \in s\mathcal{S}et\text{-}\mathcal{C}at_{*,*}$. Similarly, Example C.5.21 observes that the Yoneda embedding $\Delta[\bullet] \in s\mathcal{S}et^{\mathbf{\Delta}}$ is Reedy cofibrant. Since $\mathbb{2}[-]\colon s\mathcal{S}et \to s\mathcal{S}et\text{-}\mathcal{C}at_{*,*}$ preserves monomorphisms and colimits, it follows that $\mathbb{2}[\Delta[\bullet]] \in s\mathcal{S}et\text{-}\mathcal{C}at^{\mathbf{\Delta}}_{*,*}$ is too.

It remains only to to show that $\omega^n\colon \mathfrak{C}\Sigma^R\Delta[n] \to \mathbb{2}[\Delta[n]]$ is a DK-equivalence of simplicial categories, as defined in Digression 6.1.15. The DK-equivalences satisfy the 2-of-3 property, so we prove this by showing that two canonical quotient maps

$$\begin{tikzcd}
\mathfrak{C}\Sigma^R\Delta[n] \arrow[rr, "\omega^n"] \arrow[dr, two heads] & & \mathbb{2}[\Delta[n]] \arrow[dl, two heads] \\
& \mathbb{2} &
\end{tikzcd}$$

to the discrete simplicial category $\mathbb{2}$ are DK-equivalences.

The map $\mathbb{2}[\Delta[n]] \to \mathbb{2}$ is clearly a DK-equivalence since $\Delta[n] \to \Delta[0]$ is a weak homotopy equivalence of simplicial sets and the functor acts as the identity on homotopy categories of these simplicial categories.

For the remaining case, observe that the quotient map $\mathfrak{C}\Sigma^R\Delta[n] \to \mathbb{2}$ is defined by applying the homotopy coherent realization functor $\mathfrak{C}$ to the quotient map $\Sigma^R\Delta[n] \to \Delta[1]$ considered in the proof of Lemma 6.5.6. There we constructed a section, displayed in (6.5.7) in the dual case of $\Sigma^L\Delta[n] \to \Delta[1]$ and showed that this map is an inner anodyne extension, a sequential of pushouts of inner horn inclusions. The homotopy coherent realization functor carries this diagram to a sequential composite of pushouts of simplicial subcomputad inclusions $\mathfrak{C}\Lambda^1[j] \hookrightarrow \mathfrak{C}\Delta[j]$ for $2 \le j \le n+1$, described explicitly in Lemma 6.3.7(ii). These maps are evidently DK-equivalences, since they act bijectively on objects and act via anodyne extensions on homs; in fact, recall that only one of these maps on homs is not an identity. And since the simplicial sets in the filtration (6.5.7) have only two objects and all non-degenerate 1-simplices pointing in the same direction, the transfinite composite $\mathbb{2} \to \mathfrak{C}\Sigma^R\Delta[n]$ again acts bijectively on objects and acts via an anodyne extension on its only non-trivial hom. Since its section is a DK-equivalence, the simplicial functor $\mathfrak{C}\Sigma^R\Delta[n] \to \mathbb{2}$ is as well. This completes the proof. $\square$

Recall from Example 6.1.5 the simplicial category $\mathbb{2}[X]$ with with two objects "$-$" and "$+$" and with hom-spaces defined by

$$\mathbb{2}[X](+,-) \coloneqq \emptyset, \quad \mathbb{2}[X](+,+) \coloneqq \mathbb{1} =: \mathbb{2}[X](-,-), \quad \mathbb{2}[X](-,+) \coloneqq X.$$

6.5.10. LEMMA. *There is a natural transformation*

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}(-)^{\triangleright}"] \arrow[r, bend right, "2[-]"'] \arrow[r, phantom, "\Downarrow\tau"] & s\mathcal{S}et\text{-}\mathcal{C}ptd \end{tikzcd}$$

whose components $\tau^U \colon \mathfrak{C}U^{\triangleright} \to 2[U]$ *act on objects by mapping every vertex of* U *to* $-$ *and mapping the cocone point of* $U^{\triangleright} := U \star \Delta[0]$ *to* $+$.

PROOF. We actually define τ as a natural transformation valued in pointed simplicial computads:

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}(-)^{\triangleright}"] \arrow[r, bend right, "2[-]"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{1}/s\mathcal{S}et\text{-}\mathcal{C}ptd \end{tikzcd} \tag{6.5.11}$$

First to explain the codomain of (6.5.11), recall from Theorem 6.3.10 and Example 6.1.5 that both homotopy coherent realizations and directed suspensions define simplicial computads, which we regard as pointed simplicial computads under the cone point $\top \colon \mathbb{1} \to \mathfrak{C}X^{\triangleright}$ and $+ \colon \mathbb{1} \to 2[X]$. The advantage of fixing one object is this makes both functors $\mathfrak{C}(-)^{\triangleright}$ and $2[-]$ cocontinuous, and consequently they may be recovered as the left Kan extensions of their restrictions to the full subcategory of representables. Accordingly, to define the natural transformation (6.5.11), it suffices to define a natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend left, "\mathfrak{C}(\Delta[\bullet]^{\triangleright})"] \arrow[r, bend right, "2[\Delta[\bullet]]"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{1}/s\mathcal{S}et\text{-}\mathcal{C}ptd \end{tikzcd}$$

and then take the left Kan extension along the Yoneda embedding よ: $\mathbf{\Delta} \hookrightarrow s\mathcal{S}et$.

For $[n] \in \mathbf{\Delta}$, the component $\tau^n \colon \mathfrak{C}\Delta[n+1] \to 2[\Delta[n]]$ maps the objects $0, \ldots, n$ to $-$ and maps $n+1$ to $+$. Its action on arrows is easily described upon recalling from Lemma 6.2.8 that the hom-spaces in $\mathfrak{C}\Delta[n+1]$ and $2[\Delta[n]]$ are both nerves of posets; thus, all that is required is to define an order-preserving mapping on atomic 0-arrows. Indeed, since only one of the mapping spaces of $2[\Delta[n]]$ is non-trivial, we need only specify the images of the atomic 0-arrows in $\mathfrak{C}\Delta[n+1]$ from i to $n+1$ for each $0 \le i \le n$. With this in mind, we define τ^n to map the unique atomic 0-arrow in $\mathfrak{C}\Delta[n+1](i, n+1)$ to the 0-arrow i in $2[\Delta[n]](-,+) = \Delta[n]$.

Note this construction is natural in the morphisms of $\mathbf{\Delta}$, furnishing the claimed natural transformation. □

Finally, combining Propositions 6.5.3 and 6.5.9 we conclude:

6.5.12. COROLLARY. *For any pair of objects* a, b *in a Kan complex enriched category* $\mathcal{A}$, *there exists a canonical trivial cofibration in the Joyal model structure*

$$\mathcal{A}(a,b) \overset{\sim}{\rightarrowtail} \mathsf{Hom}_{\mathfrak{N}\mathcal{A}}(a,b)$$

between the external mapping space from a *to* b *and the hom space from* a *to* b *in the homotopy coherent nerve* $\mathfrak{N}\mathcal{A}$. *Consequently, the external mapping space from* a *to* b *in* $\mathcal{A}$ *is equivalent to the internal mapping space from* a *to* b *in* $\mathfrak{N}\mathcal{A}$. □

Exercises.

6.5.i. EXERCISE. Verify the first claim made in Lemma 6.5.2, that $\mathrm{Hom}^L_A(a,b)$ and $\mathrm{Hom}^R_A(a,b)$ arise as fibers of the projections $\pi\colon {}^{a/}A \twoheadrightarrow A$ and $\pi\colon A_{/b} \twoheadrightarrow A$ from the slice quasi-categories.

6.5.ii. EXERCISE. Prove that the canonical anodyne extension $\mathcal{A}(a,b) \to \mathrm{Hom}_{\mathfrak{N}\mathcal{A}}(a,b)$ of Corollary 6.5.12 is natural in the action of a simplicial functor $F\colon \mathcal{A} \to \mathcal{B}$ between Kan complex enriched categories, when the action of F on the internal homs is defined by the strict universal property of the simplicial pullbacks the square

$$\begin{tikzcd}
\mathcal{A}(a,b) \arrow[r, "F_{a,b}"] \arrow[d, tail, "\wr"] & \mathcal{B}(Fa,Fb) \arrow[d, tail, "\wr"] \\
\mathrm{Hom}_{\mathfrak{N}\mathcal{A}}(a,b) \arrow[r, "\ulcorner F \urcorner"] & \mathrm{Hom}_{\mathfrak{N}\mathcal{B}}(Fa,Fb)
\end{tikzcd}$$

commutes.

CHAPTER 7

Weighted limits in ∞-cosmoi

7.1. Weighted limits and colimits

Let $(\mathcal{V}, \times, \mathbb{1})$ denote a complete and cocomplete cartesian closed monoidal category. The examples we have in mind are $(s\mathcal{S}et, \times, \mathbb{1})$, its cartesian closed subcategory $(\mathcal{C}at, \times, \mathbb{1})$, or its further cartesian closed subcategory $(\mathcal{S}et, \times, \mathbb{1})$.

Ordinary limits and colimits are objects representing the functor of cones with a given apex over or under a fixed diagram. *Weighted* limits and colimits are defined analogously, except that the cones over or under a diagram might have exotic "shapes, " which are allowed to vary with the objects indexing the diagram. More formally, in the $\mathcal{V}$-enriched context, the *weight*, defining the "shape" of a cone over a diagram indexed by $\mathcal{A}$ or under a diagram indexed by $\mathcal{A}^{\mathrm{op}}$, takes the form of a functor in $\mathcal{V}^{\mathcal{A}}$.

Before introducing the general notion of weighted limit and colimit, we first reacquaint ourselves an example that we have seen already in Digression 1.2.5.

7.1.1. Example (tensors and cotensors). A diagram indexed by the category $\mathbb{1}$ valued in a $\mathcal{V}$-enriched category $\mathcal{M}$ is just an object A in $\mathcal{M}$. In this case, the weight is just an object U of $\mathcal{V}$. The U-weighted limit of the diagram A is an object of $\mathcal{M}$ denoted A^U — or denoted $\{U, A\}$ whenever superscripts are inconvenient — called the **cotensor** of $A \in \mathcal{M}$ with $U \in \mathcal{V}$ defined by the universal property

$$\mathcal{M}(X, A^U) \cong \mathcal{V}(U, \mathcal{M}(X, A)),$$

this isomorphism between mapping spaces in $\mathcal{V}$. Dually, the U-weighted colimit of A is an object $U \otimes A \in \mathcal{M}$ called the **tensor** of $A \in \mathcal{M}$ with $U \in \mathcal{V}$ defined by the universal property

$$\mathcal{M}(U \otimes A, X) \cong \mathcal{V}(U, \mathcal{M}(A, X)),$$

this isomorphism again between mapping spaces in $\mathcal{V}$. Assuming such objects exist, the cotensor and tensor define $\mathcal{V}$-enriched bifunctors

$$\mathcal{V}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{-^-} \mathcal{M} \qquad\qquad \mathcal{V} \times \mathcal{M} \xrightarrow{-\otimes-} \mathcal{M}$$

in a unique way making the defining isomorphisms natural in U and A as well.

Since $\mathcal{V}$ is cartesian closed, it is tensored and cotensored over itself, with $U \otimes V := U \times V$ and $V^U := \mathcal{V}(U, V)$. In particular, the defining natural isomorphisms characterizing tensors and cotensors in $\mathcal{M}$ can be rewritten as

$$\mathcal{M}(X, A^U) \cong \mathcal{M}(X, A)^U \qquad \text{and} \qquad \mathcal{M}(U \otimes A, X) \cong \mathcal{M}(A, X)^U.$$

The fact that the natural isomorphisms defining tensors and cotensors are required to exist in $\mathcal{V}$ (and not merely in $\mathcal{S}et$) has the following consequence:

7.1.2. Lemma (associativity of tensors and cotensors). *If $\mathcal{M}$ is a $\mathcal{V}$-category with tensors and cotensors then for any $U, V \in \mathcal{V}$ and $A \in \mathcal{M}$, there exist natural isomorphisms*

$$U \otimes (V \otimes A) \cong (U \times V) \otimes A \qquad \textit{and} \qquad (A^V)^U \cong A^{U \times V}.$$

Proof. By the defining universal property

$$\mathcal{M}(X, (A^V)^U) \cong \mathcal{V}(U, \mathcal{M}(X, A^V)) \cong \mathcal{V}(U, \mathcal{V}(V, \mathcal{M}(X, A)))$$
$$\cong \mathcal{V}(U \times V, \mathcal{M}(X, A)) \cong \mathcal{M}(X, A^{U \times V})$$

for all $X \in \mathcal{M}$. By the Yoneda lemma $(A^V)^U \cong A^{U \times V}$. The case for tensors is similar. □

We now introduce the general notions of weighted limit and weighted colimit from three different viewpoints. We introduce these perspectives in the reverse of the logical order, because we find this route to be the most intuitive. We first describe the axioms that characterize the weighted limit and colimit bifunctors, whenever they exist. We then explain how weighted limits and colimits can be constructed, again assuming these exist. We then finally introduce the general universal property that defines a particular weighted limit or colimit, which tells us when the notions just introduced do in fact exist.

7.1.3. Definition (weighted limits and colimits, axiomatically). For a small $\mathcal{V}$-enriched category $\mathcal{A}$ and a large $\mathcal{V}$-enriched category $\mathcal{M}$, the **weighted limit** and **weighted colimit bifunctors**

$$\lim^{\mathcal{A}}_{-} - : (\mathcal{V}^{\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M} \qquad \text{and} \qquad \mathrm{colim}^{\mathcal{A}}_{-} - : \mathcal{V}^{\mathcal{A}} \times \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M}$$

are characterized by the following pair of axioms whenever they exist:

(i) Weighted (co)limits with representable weights evaluate at the representing object:
$$\lim^{\mathcal{A}}_{\mathcal{A}(a,-)} F \cong F(a) \qquad \text{and} \qquad \mathrm{colim}^{\mathcal{A}}_{\mathcal{A}(-,a)} G \cong G(a).$$

(ii) The weighted (co)limit bifunctors are cocontinuous in the weight: for any diagrams $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$, the functor $\mathrm{colim}^{\mathcal{A}}_{-} G$ preserves colimits, while the functor $\lim^{\mathcal{A}}_{-} F$ carries colimits to limits.[1]

We interpret axiom (ii) to mean that weights can be "made-to-order": a weight constructed as a colimit of representables — as all $\mathcal{V}$-valued functors are — will stipulate the expected universal property.

7.1.4. Definition (weighted limits and colimits, constructively). The **limit** of $F \in \mathcal{M}^{\mathcal{A}}$ **weighted by** $W \in \mathcal{V}^{\mathcal{A}}$ is computed by the functor cotensor product:

$$\lim^{\mathcal{A}}_W F := \int_{a \in \mathcal{A}} F(a)^{W(a)} := \mathrm{eq}\left(\prod_{a \in \mathcal{A}} F(a)^{W(a)} \rightrightarrows \prod_{a,b \in \mathcal{A}} F(b)^{\mathcal{A}(a,b) \times W(a)} \right), \tag{7.1.5}$$

where the product and equalizer should be interpreted as *conical limits*; see Digression 1.2.5 or Definition 7.1.14 below. The maps in the equalizer diagram are induced by the actions $\mathcal{A}(a,b) \times W(a) \to W(b)$ and $F(a) \to F(b)^{\mathcal{A}(a,b)}$ of the hom-object $\mathcal{A}(a,b)$ on the $\mathcal{V}$-functors W and F; the latter case makes use of the natural isomorphism $F(b)^{\mathcal{A}(a,b) \times W(a)} \cong (F(b)^{\mathcal{A}(a,b)})^{W(a)}$ of Lemma 7.1.2.

Dually, the **colimit** of $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ **weighted by** $W \in \mathcal{V}^{\mathcal{A}}$ is computed by the functor tensor product:

$$\mathrm{colim}^{\mathcal{A}}_W G := \int^{a \in \mathcal{A}} W(a) \otimes G(a) := \mathrm{coeq}\left(\coprod_{a,b \in \mathcal{A}} (W(a) \times \mathcal{A}(a,b)) \otimes G(b) \rightrightarrows \coprod_{a \in \mathcal{A}} W(a) \otimes G(a) \right), \tag{7.1.6}$$

[1] More precisely, as will be proven in Proposition 7.1.12, the weighted colimit functor $\mathrm{colim}^{\mathcal{A}}_{-} G$ preserves weighted colimits, while the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries weighted colimits to weighted limits.

where the coproduct and coequalizer should be interpreted as *conical colimits*. One of the maps in the coequalizer diagram is induced by the action $\mathcal{A}(a,b) \otimes G(b) \to G(a)$ of $\mathcal{A}(a,b)$ on the contravariant $\mathcal{V}$-functor G and the natural isomorphism $(W(a) \times \mathcal{A}(a,b)) \otimes G(b) \cong W(a) \otimes (\mathcal{A}(a,b) \otimes G(b))$ of Lemma 7.1.2; the other uses the covariant action of $\mathcal{A}(a,b)$ on W as before.

7.1.7. Definition (weighed limits and colimits, the universal property). The **limit** $\lim^{\mathcal{A}}_W F$ of the diagram $F \in \mathcal{M}^{\mathcal{A}}$ weighted by $W \in \mathcal{V}^{\mathcal{A}}$ and the **colimit** $\operatorname{colim}^{\mathcal{A}}_W G$ of $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ weighted by $W \in \mathcal{V}^{\mathcal{A}}$ are characterized by the universal properties:

$$\mathcal{M}(X, \lim^{\mathcal{A}}_W F) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{M}(X,F)) \qquad \text{and} \qquad \mathcal{M}(\operatorname{colim}^{\mathcal{A}}_W G, X) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{M}(G,X)), \tag{7.1.8}$$

each of these defining an isomorphism between objects of $\mathcal{V}$.

When the indexing category $\mathcal{A}$ is clear from context, as is typically the case, we frequently drop it from the notation for the weighted limit and weighted colimit. We now argue that these three definitions characterize the same objects. Along the way, we obtain results of interest in their own right, that we record separately.

7.1.9. Lemma. *The category $\mathcal{V}$ admits all weighted limits, as defined by the formula of* (7.1.5) *satisfying the natural isomorphism of* (7.1.8). *Explicitly, for a weight $W \colon \mathcal{A} \to \mathcal{V}$ and a diagram $F \colon \mathcal{A} \to \mathcal{V}$, the weighted limit*

$$\lim^{\mathcal{A}}_W F \coloneqq \mathcal{V}^{\mathcal{A}}(W,F),$$

is the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations from W to V.

Proof. The $\mathcal{V}$-functor $\mathcal{V}(\mathbb{1}, -) \colon \mathcal{V} \to \mathcal{V}$ represented by the monoidal unit is naturally isomorphic to the identity functor. So taking $X = \mathbb{1}$ in the universal property of (7.1.8) in the case where the diagram $F \in \mathcal{V}^{\mathcal{A}}$ is valued in the $\mathcal{V}$-category $\mathcal{V}$, we have

$$\lim^{\mathcal{A}}_W F \cong \mathcal{V}^{\mathcal{A}}(W,F).$$

Simultaneously, the formula (7.1.5) computes the $\mathcal{V}$-object $\mathcal{V}^{\mathcal{A}}(W,F)$ of $\mathcal{V}$-natural transformations from W to F defined in Definition A.3.8. $\square$

The $\mathcal{V}$-object of $\mathcal{V}$-natural transformations satisfies the natural isomorphism $\mathcal{V}(V, \mathcal{V}^{\mathcal{A}}(W,F)) \cong \mathcal{V}^{\mathcal{A}}(W, \mathcal{V}(V,F))$ for any $V \in \mathcal{V}$. Applying the observation that W-weighted limits of $\mathcal{V}$-valued functors are $\mathcal{V}$-objects of natural transformations to the functor $\mathcal{M}(X, F-)$ and $\mathcal{M}(G-, X)$ in the case of $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$, we may re-express the natural isomorphism (7.1.8) as:

7.1.10. Corollary. *The weighted limits and weighted colimits of* (7.1.8) *are representably defined as weighted limits in $\mathcal{V}$: for $W \in \mathcal{V}^{\mathcal{A}}$ and $F \in \mathcal{M}^{\mathcal{A}}$ and $G \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}$ the weighted limit and colimit are characterized by isomorphisms*

$$\mathcal{M}(X, \lim_W F) \cong \lim_W \mathcal{M}(X,F) \qquad \textit{and} \qquad \mathcal{M}(\operatorname{colim}_W G, X) \cong \lim_W \mathcal{M}(G,X) \tag{7.1.11}$$

natural in X in $\mathcal{V}$. $\square$

We now unify the Definitions 7.1.3, 7.1.4, and 7.1.7.

7.1.12. Proposition. *When the limits and colimits of* (7.1.5) *and* (7.1.6) *exist they define objects satisfying the universal properties* (7.1.8) *or equivalently* (7.1.11) *and bifunctors satisfying the axioms of Definition 7.1.3.*

Proof. The proofs are dual, so we confine our attention to the limit case. The general case of the implication Definition 7.1.4 ⇒ 7.1.7 — for either weighted limits or weighted colimits — is a direct consequence of the special case of this implication for weighted limits valued in $\mathcal{M} = \mathcal{V}$ proven as Lemma 7.1.9 and Corollary 7.1.10. The limits of (7.1.5) in $\mathcal{M}$ are also defined representably in terms of the analogous limits in $\mathcal{V}$. So the objected defined by (7.1.5) represents the $\mathcal{V}$-functor $\lim_W \mathcal{M}(-, F)$ that defines the weighted limit $\lim_W F$.

It remains to prove that the weighted limits of Definitions 7.1.4 and 7.1.7 satisfy the axioms of Definition 7.1.3. In the case of a $\mathcal{V}$-valued diagram $F \in \mathcal{V}^{\mathcal{A}}$, axiom (i) is the $\mathcal{V}$-Yoneda lemma:

$$\mathcal{V}^{\mathcal{A}}(\mathcal{A}(a, -), F) \cong F(a)$$

proven in Theorem A.3.11. Once again, the general case for $F \in \mathcal{M}^{\mathcal{A}}$ follows from the special case for $\mathcal{V}$-valued diagrams, for to demonstrate an isomorphism $\lim_{\mathcal{A}(a,-)} F \cong F(a)$ in $\mathcal{M}$ it suffices to demonstrate an isomorphism $\mathcal{M}(X, \lim_{\mathcal{A}(a,-)} F) \cong \mathcal{M}(X, F(a))$ in $\mathcal{V}$ for all $X \in \mathcal{M}$ and have such a natural isomorphism by applying (7.1.11) and the observation just made to the functor $\mathcal{M}(X, F-) \in \mathcal{V}^{\mathcal{A}}$.

For the axiom (ii), consider a diagram $K \colon \mathcal{J}^{\mathrm{op}} \to \mathcal{V}^{\mathcal{A}}$ of weights and a weight $V \in \mathcal{V}^{\mathcal{J}}$ so that $\operatorname{colim}_V^{\mathcal{J}} K \cong W$. For any $F \in \mathcal{M}^{\mathcal{A}}$, we will show that the $\mathcal{V}$-functor $\lim_{-}^{\mathcal{A}} F \colon (\mathcal{V}^{\mathcal{A}})^{\mathrm{op}} \to \mathcal{M}$ carries the V-weighted colimit of K to the V-weighted limit of the composite diagram $\lim_K^{\mathcal{A}} F \colon \mathcal{J} \to \mathcal{M}$.

The universal property (7.1.8), applied first to the $\operatorname{colim}_V^{\mathcal{J}} K$-weighted limit of the diagram F and the object X, and then to the V-weighted colimit of the diagram K and the object $\mathcal{M}(X, F)$, supplies isomorphisms:

$$\mathcal{M}(X, \lim^{\mathcal{A}}_{\operatorname{colim}_V^{\mathcal{J}} K} F) \cong \mathcal{V}^{\mathcal{A}}(\operatorname{colim}_V^{\mathcal{J}} K, \mathcal{M}(X, F)) \cong \mathcal{V}^{\mathcal{J}}(V, \mathcal{V}^{\mathcal{A}}(K, \mathcal{M}(X, F))).$$

Applying (7.1.8) twice more, first for the weights Kj for each $j \in \mathcal{J}$ and then for the weight V and the diagram $\lim_K^{\mathcal{A}} F \colon \mathcal{J} \to \mathcal{M}$, we have

$$\cong \mathcal{V}^{\mathcal{J}}(V, \mathcal{M}(X, \lim_K^{\mathcal{A}} F)) \cong \mathcal{M}(X, \lim_V^{\mathcal{J}} \lim_K^{\mathcal{A}} F).$$

By the Yoneda lemma, this proves that

$$\lim^{\mathcal{A}}_{\operatorname{colim}_V^{\mathcal{J}} K} F \cong \lim_V^{\mathcal{J}} \lim_H^{\mathcal{A}} F,$$

i.e., that the weighted limit functor $\lim_{-}^{\mathcal{A}} F$ is carries a weighted colimit of weights to the analogous weighted limit of weights. □

7.1.13. Remark (for unenriched indexing categories). When the indexing category is unenriched, the limit and colimit formulas from Definition 7.1.4 simplify

$$\lim_W^{\mathcal{A}} F \cong \mathrm{eq}\left(\prod_{a \in \mathcal{A}} F(a)^{W(a)} \rightrightarrows \prod_{\mathcal{A}(a,b)} F(b)^{W(a)} \right)$$

$$\operatorname{colim}_W^{\mathcal{A}} G \cong \mathrm{coeq}\left(\coprod_{\mathcal{A}(a,b)} (W(a) \otimes G(b) \rightrightarrows \coprod_{a \in \mathcal{A}} W(a) \otimes G(a) \right)$$

and in fact, it suffices to consider only non-identity arrows or even just atomic arrows.

7.1.14. Definition (conical limits and colimits). The unit for the cartesian product defines a terminal object $\mathbb{1} \in \mathcal{V}$. The constant diagram at the terminal object then defines a terminal object $\mathbb{1} \in \mathcal{V}^{\mathcal{A}}$. A

limit weighted by the terminal weight is called a **conical limit** and a colimit weighted by the terminal weight is called a **conical colimit**. It is common to use the simplified notation $\lim F \coloneqq \lim^{\mathcal{A}}_{\mathbb{1}} F$ and $\operatorname{colim} G \coloneqq \operatorname{colim}^{\mathcal{A}}_{\mathbb{1}} G$.

Conical limits and colimits satisfy the defining universal properties

$$\mathcal{M}(X, \lim F) \cong \mathcal{V}^{\mathcal{A}}(\mathbb{1}, \mathcal{M}(X, F)) \qquad \text{and} \qquad \mathcal{M}(\operatorname{colim} G, X) \cong \mathcal{V}^{\mathcal{A}}(\mathbb{1}, \mathcal{M}(G, X)),$$

which say that $\lim F$ and $\operatorname{colim} G$ represent the functors of $\mathcal{V}$-enriched conical cones over F or under G, respectively.

We can now properly understand the formulae for weighted limits and colimits given in Definition 7.1.4. In particular, these formulae give criteria under which weighted limits or colimits are guaranteed to exist.

7.1.15. Corollary. *If $\mathcal{M}$ is a $\mathcal{V}$-enriched category that admits cotensors and conical limits of all unenriched diagram shapes, then $\mathcal{M}$ admits all weighted limits. Dually, if $\mathcal{M}$ admits tensors and conical colimits of all unenriched diagram shapes, then $\mathcal{M}$ admits all weighted colimits.*

7.1.16. Remark. If $\mathcal{M}$ is a $\mathcal{V}$-category whose underlying unenriched category admits all small limits, then if $\mathcal{M}$ admits cotensors *and tensors* over $\mathcal{V}$, then $\mathcal{M}$ admits all weighted limits. Via the Yoneda lemma, the presence of tensors suffices to internalize the isomorphism of sets expressing the unenriched universal property of limits to an isomorphism in $\mathcal{V}$ that expresses the universal property of conical limits. See Exercise 7.1.i.

7.1.17. Example (commas). The comma ∞-category is the limit in the ∞-cosmos $\mathcal{K}$ of the diagram $\lrcorner \to \mathcal{K}$ given by the cospan

$$C \xrightarrow{g} A \xleftarrow{f} B$$

weighted by the diagram $\lrcorner \to s\mathcal{S}et$ given by the cospan

$$\mathbb{1} \xhookrightarrow{1} \mathbb{2} \xhookleftarrow{0} \mathbb{1}$$

Under the simplification of Remark 7.1.13, the formula for the weighted limit reduces to the equalizer

$$\operatorname{eq}\left(\begin{tikzcd} & A^{\mathbb{2}} \arrow[dr, "{(p_1,p_0)}"] & \\ C \times A^{\mathbb{2}} \times B \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & & A \times A \\ & C \times B \arrow[ur, "g \times f"'] & \end{tikzcd}\right)$$

which computes the pullback of (3.4.2). The universal property (7.1.8) says that functors $X \to \operatorname{Hom}_A(f, g)$ in $\mathcal{K}$ correspond to simplicial natural transformations, the data of which is given by the three dashed vertical maps that fit into two commutative squares:

$$\begin{tikzcd} \mathbb{1} \arrow[r, hook, "1"] \arrow[d, dashed, "c"'] & \mathbb{2} \arrow[d, dashed, "\alpha"] & \mathbb{1} \arrow[l, hook', "0"'] \arrow[d, dashed, "b"] \\ \operatorname{Fun}(X, C) \arrow[r, "g_*"'] & \operatorname{Fun}(X, A) & \operatorname{Fun}(X, B) \arrow[l, "f_*"] \end{tikzcd}$$

7.1.18. Example (Bousfield-Kan homotopy limits). In their classic book on homotopy limits and colimits [19], Bousfield and Kan define the **homotopy limit** of a diagram indexed by a 1-category A and valued in a Kan-complex enriched category $\mathcal{M}$ to be the limit weighted by the functor

$$\begin{aligned} A &\longrightarrow s\mathcal{S}et \\ a &\longmapsto A_{/a} \end{aligned}$$

which carries each object $a \in A$ the nerve of the slice category over a.

7.1.19. Example (Kan extensions as weighted co/limits). The usual colimit or limit formula that computes the value of a pointwise left or right Kan extension of an unenriched functor $F\colon C \to E$ along $K\colon C \to D$ at an object $d \in D$ can be succinctly expressed by the weighted colimit or weighted limit

$$\mathrm{lan}_K F(d) := \mathrm{colim}_{D(K-,d)} F \qquad \text{and} \qquad \mathrm{ran}_K F(d) := \lim_{D(d,K-)} F.$$

We conclude with a few results from the general theory of weighted limits and colimits. Immediately from their defining universal properties, it can be verified that:

7.1.20. Lemma (weighted limits of restricted diagrams). *Suppose given a $\mathcal{V}$-functor $K\colon \mathcal{A} \to \mathcal{B}$, a weight $W\colon \mathcal{A} \to \mathcal{V}$, and diagrams $F\colon \mathcal{B} \to \mathcal{M}$ and $G\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{M}$. Then the W-weighted limit or colimit of the restricted diagram is isomorphic to the* lan_K *W-weighted limit or colimit of the original diagram:*

$$\lim{}^{\mathcal{A}}_W (F \circ K) \cong \lim{}^{\mathcal{B}}_{\mathrm{lan}_K W} F \qquad \textit{and} \qquad \mathrm{colim}^{\mathcal{A}}_W (G \circ K) \cong \mathrm{colim}^{\mathcal{B}}_{\mathrm{lan}_K W} G.$$

Proof. Exercise 7.1.iii. □

An enriched adjunction is comprised of a pair of $\mathcal{V}$-functors $F\colon \mathcal{B} \to \mathcal{A}$ and $U\colon \mathcal{A} \to \mathcal{B}$ together with a family of isomoprphisms $\mathcal{A}(Fb, a) \cong \mathcal{B}(b, Ua)$ that are $\mathcal{V}$-natural in both variables; see Definition A.3.16. The usual Yoneda-style argument enriches to show:

7.1.21. Proposition (weighted RAPL/LAPC). *A $\mathcal{V}$-enriched right adjoint functor $U\colon \mathcal{A} \to \mathcal{B}$ preserves all weighted limits that exist in $\mathcal{A}$, while it's $\mathcal{V}$-enriched left adjoint $F\colon \mathcal{B} \to \mathcal{A}$ preserves all weighted colimits that exist in $\mathcal{B}$.*

Proof. Exercise 7.1.iv. □

By the axioms of Definition 1.2.1, ∞-cosmoi will admit a large class of simplicially-enriched weighted limits built from the simplicial cotensors and conical simplicial limits enumerated in 1.2.1(i). In practice, ∞-cosmoi often arise as subcategories (of "fibrant objects") in a larger category that is also admits simplicially-enriched weighted colimits, which can then be reflected back into the ∞-cosmos to defined *weighted bicolimits*. This is a story for much later, so we will confine our attention to the case of weighted limits for the rest of this chapter.

Exercises.

7.1.i. Exercise. Suppose $\mathcal{M}$ is a tensored and $\mathcal{V}$-enriched category whose underlying unenriched category admits limits of all unenriched diagram shapes. Show that $\mathcal{M}$ admits conical limits of all unenriched diagram shapes, proving the extension of Corollary 7.1.15 described in Remark 7.1.16.

7.1.ii. Exercise. Taking the base for enrichment $\mathcal{V}$ to be $\mathcal{S}et$, compute the following weighted limits of a simplicial set X, regarded as a diagram in $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$, weighted by:

(i) the standard n-simplex $\Delta[n] \in \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$,

(ii) the **spine** of the n-simplex, the simplicial subset $\Gamma[n] \hookrightarrow \Delta[n]$ obtained by gluing together the n edges from i to $i+1$ into a composable path,

(iii) the n-simplex boundary $\partial\Delta[n] \in \mathcal{S}et^{\Delta^{\mathrm{op}}}$.[2]

7.1.iii. Exercise. Prove Lemma 7.1.20.

7.1.iv. Exercise. Prove Proposition 7.1.21.

7.2. Flexible weighted limits and the collage construction

Our aim in this section is to introduce a special class of $s\mathcal{S}et$-valued weights whose associated weighted limit notions are homotopically well-behaved. Borrowing a term from 2-category theory, we refer to these weights as *flexible*. All of the limits enumerated in 1.2.1(i) are flexible limits. In fact, we will prove that ∞-cosmoi admit all flexible weighted limits because these can be built out of the limits enumerated in 1.2.1(i). In §8.2, we will use this observation to help us verify the limit axiom for newly constructed ∞-cosmoi in a more systematic way.

In this section, we will characterize the class of flexible weights as precisely those whose associated *collages* define relative simplicial computads, which will allow us to readily produce examples. In §7.3, we will establish the homotopical properties of flexible weighted limits and make precise the relationships between this class of the limits and limits assumed present in any ∞-cosmos by axiom 1.2.1(i).

7.2.1. Definition (flexible weights and projective cell complexes). Let $\mathcal{A}$ be a simplicial category.

- A simplicial natural transformation of the form
$$\partial\Delta[n] \times \mathcal{A}(a,-) \hookrightarrow \Delta[n] \times \mathcal{A}(a,-)$$
is called a **projective n-cell at $a \in \mathcal{A}$**.
- A natural transformation $\alpha \colon V \hookrightarrow W$ in $s\mathcal{S}et^{\mathcal{A}}$ that can be expressed as a countable composite of pushouts of coproducts of projective cells is called a **projective cell complex**.
- A weight $W \in s\mathcal{S}et^{\mathcal{A}}$ is **flexible** just when $\varnothing \hookrightarrow W$ is a projective cell complex.

7.2.2. Remark (on generalized projective cells). Since any monomorphism of simplicial sets $U \hookrightarrow V$ can be decomposed as a sequential composite of pushouts of coproducts of boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, the class of projective cell complexes may be also be described as the class of maps in $s\mathcal{S}et^{\mathcal{A}}$ that can be expressed as a countable composite of pushouts of coproducts of monomorphisms $U \times \mathcal{A}(a,-) \hookrightarrow V \times \mathcal{A}(a,-)$ for some $a \in \mathcal{A}$.

7.2.3. Example. Since any simplicial set can be decomposed as a sequential composite of pushouts of coproducts of boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, simplicial cotensors are flexible weights.

7.2.4. Example. Conical products also define flexible weighted limits, built by attaching one projective 0-cell for each object in the indexing set.

7.2.5. Non-Example. Conical limits indexed by any 1-category that contains non-identity arrows are *not* flexible because the legs of a conical cone over the domain and codomain of each arrow in the diagram are required to define a strictly commutative triangle of 0-arrows. The specifications for a flexible weight allow us to freely attach n-arrows of any dimension, but do not provide a mechanism for demanding strict commutativity of any diagram of n-arrows — only commutativity up to the presence of a higher cell.

[2] The limit of a simplicial object weighted by $\partial\Delta[n]$ is called the **nth-matching object**; see Appendix C.

7.2.6. Digression (on flexible limits in 2-category theory). Simplicial limits weighted by flexible weights should be thought of as analogous to **flexible 2-limits**, i.e., 2-limits built out of products, inserters, equifiers, and retracts (splittings of idempotents) [**14**]. More precisely, simplicial limits weighted by flexible weights are analogous to the **PIE limits**, those built just from products, inserters, and equifiers, but we choose to adopt the moniker from the slight larger class of weights because we find it to be more evocative. The PIE limits also include iso-inserters, descent objects, comma objects, and Eilenberg-Moore objects, as well as all pseudo, lax, and oplax limits. Many important 2-categories, such as the 2-category of accessible categories and accessible functors, fail to admit all 2-categorical limits, but do admit all PIE-limits [**65**].

The weights for flexible limits are the cofibrant objects in a model structure on the diagram 2-category $\mathcal{C}at^{\mathcal{A}}$ that is enriched over the folk model structure on $\mathcal{C}at$; the PIE weights are exactly the *cellular* cofibrant objects. Correspondingly, the projective cell complexes of Definition 7.2.1 are exactly the cellular cofibrations in the projective model structure on $s\mathcal{S}et^{\mathcal{A}}$.

Recall that a weight is intended to describe the "shape" of cones over diagrams indexed by a particular category. In the case a weight $W\colon \mathcal{A} \to s\mathcal{S}et$ valued in simplicial sets, we can describe the shape of W-cones directly as a simplicial category called the *collage* of the weight.

7.2.7. Definition (collage construction). The **collage** of a weight $W\colon \mathcal{A} \to s\mathcal{S}et$ is a simplicial category $\mathrm{coll}W$ which contains $\mathcal{A}$ as a full subcategory together with one additional object $\perp$ whose endomorphism space is the point. The mapping spaces from any object $a \in \mathcal{A}$ to $\perp$ are empty, while the mapping spaces from $\perp$ to the image of $\mathcal{A} \hookrightarrow \mathrm{coll}W$ are defined by:

$$\mathrm{coll}W(\perp, a) \coloneqq W(a)$$

with the action maps $\mathcal{A}(a,b) \times W(a) \to W(b)$ from the simplicial functor W used to define composition. This defines a simplicial category together with a canonical inclusion $\mathbb{1} + \mathcal{A} \hookrightarrow \mathrm{coll}W$ that is bijective on objects and fully faithful on $\mathbb{1}$ and $\mathcal{A}$ separately.

7.2.8. Proposition (collage adjunction).

(i) The collage construction defines a fully faithful functor

$$s\mathcal{S}et^{\mathcal{A}} \xrightarrow{\mathrm{coll}} {}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at$$

from the category of $\mathcal{A}$-indexed weights to the category of simplicial categories under $\mathbb{1} + \mathcal{A}$ whose essential image is comprised of those $\langle e, F\rangle\colon \mathbb{1} + \mathcal{A} \to \mathcal{E}$ that are bijective on objects, fully faithful when restricted to $\mathbb{1}$ and $\mathcal{A}$, and have the property that there are no arrows in $\mathcal{E}$ from the image of F to e.

(ii) The collage functor admits a right adjoint

$$\begin{tikzcd} s\mathcal{S}et^{\mathcal{A}} \arrow[r, bend left, "\mathrm{coll}"] \arrow[r, phantom, "\perp"] & {}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend left, "\mathrm{wgt}"] \end{tikzcd}$$

which carries a pair $\langle e, F\rangle\colon \mathbb{1} + \mathcal{A} \to \mathcal{E}$ to the weight $\mathcal{E}(e, F-)\colon \mathcal{A} \to s\mathcal{S}et$.

Proof. The construction of the collage functor is straightforward and left to the reader. The characterization of its essential image follows from the observation that to define a simplicial functor $W\colon \mathcal{A} \to s\mathcal{S}et$ requires no more and no less than

- the specification of simplicial sets $W(a)$ for each $a \in \mathcal{A}$ and
- and the specification of simplicial maps $\mathcal{A}(a,b) \times W(a) \to W(b)$ for each $a, b \in \mathcal{A}$

so that this action is associative in sense that the diagram

$$\begin{tikzcd} \mathcal{A}(b,c) \times \mathcal{A}(a,b) \times W(a) \arrow[r, "\circ"] \arrow[d] & \mathcal{A}(a,c) \times W(a) \arrow[d] \\ \mathcal{A}(b,c) \times W(b) \arrow[r] & W(c) \end{tikzcd}$$

commutes. This is the same as what is required to extend $\mathbb{1} + \mathcal{A}$ to a simplicial category in which all of the additional maps start at $\perp$ and end at an object in $\mathcal{A}$.

The adjunction asserts that simplicial functors

$$\begin{tikzcd} & \mathbb{1} + \mathcal{A} \arrow[dl] \arrow[dr, "\langle e,F \rangle"] & \\ \mathrm{coll} W \arrow[rr, "G"'] & & \mathcal{E} \end{tikzcd}$$

from $\mathrm{coll}W$ to $\mathcal{E}$ under $\mathbb{1} + \mathcal{A}$ stand in natural bijective correspondence with simplicial natural transformations $\gamma\colon W \Rightarrow \mathcal{E}(e, F-)$. Since the inclusion $1 + \mathcal{A} \hookrightarrow \mathrm{coll}W$ is bijective on objects and full on most homs, the data of the simplicial functor requires only the specification of the maps $\mathrm{coll}W(\perp, a) = W(a) \to \mathcal{E}(e, Fa)$. These define the components of the simplicial natural transformation γ and functoriality of G corresponds to naturality of γ. □

The collage adjunction has a useful and important interpretation.

7.2.9. Corollary. *The collage of a weight $W\colon \mathcal{A} \to s\mathcal{S}et$ realizes the shape of W-weighted cones in the sense that simplicial functors $G\colon \mathrm{coll}W \to \mathcal{E}$ with domain $\mathrm{coll}W$ stand in bijection to W-cones over the diagram $G|_{\mathcal{A}}$ with summit $G(\perp)$.* □

On account of Lemma 7.1.20, we'll be interested in computing left Kan extensions of weights encoded as collages.

7.2.10. Lemma. *For any weight $W\colon \mathcal{A} \to s\mathcal{S}et$ and simplicial functor $K\colon \mathcal{A} \to \mathcal{B}$, the pushout of simplicial categories*

$$\begin{tikzcd} \mathbb{1} + \mathcal{A} \arrow[r, "\mathbb{1}+K"] \arrow[d, hook] & \mathbb{1} + \mathcal{B} \arrow[d, hook] \\ \mathrm{coll}W \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{coll}(\mathrm{lan}_K W) \end{tikzcd}$$

computes the collage of the weight $\mathrm{lan}_K W\colon \mathcal{B} \to s\mathcal{S}et$.

Proof. By the defining universal property, a simplicial functor out of the pushout is given by a pair of functors $\langle e, F \rangle\colon \mathbb{1} + \mathcal{B} \to \mathcal{E}$ and $G\colon \mathrm{coll}W \to \mathcal{E}$ so that

$$\begin{tikzcd} & \mathbb{1} + \mathcal{A} \arrow[dl] \arrow[dr, "\langle e,FK \rangle"] & \\ \mathrm{coll} W \arrow[rr, "G"'] & & \mathcal{E} \end{tikzcd}$$

By Corollary 7.2.9, this data defines a W-cone with summit e over the diagram $FK\colon \mathcal{A} \to \mathcal{E}$. By Lemma 7.1.20 such data equivalently describes a $\mathrm{lan}_K W$-cone with summit e over the diagram $F\colon \mathcal{B} \to$

$\mathcal{E}$. Applying Corollary 7.2.9 again, we conclude that the pushout is given by the simplicial category $\mathrm{coll}(\mathrm{lan}_K W)$ as claimed. $\square$

In analogy with Corollary 4.3.5, we can encode simplicial W-weighted limits as a right Kan extension from the indexing simplicial category to the simplicial category that describes the shape of W-cones.

7.2.11. LEMMA. *For any simplicial functor $F\colon \mathcal{A} \to \mathcal{E}$ and any weight $W\colon \mathcal{A} \to s\mathcal{S}et$, the weighted limit $\lim_W F$ exists if and only if the pointwise right Kan extension of F along $\mathcal{A} \hookrightarrow \mathrm{coll}W$ exists, in which case $\mathrm{lan}\, F(\perp) \cong \lim_W F$.*

PROOF. Since $\mathcal{A} \hookrightarrow \mathrm{coll}W$ is fully faithful, pointwise right Kan extensions may be chosen to define genuine extensions

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] & \\ \mathrm{coll}W \arrow[rr, dashed, "\mathrm{ran}\,F"'] & & \mathcal{E} \end{tikzcd}$$

By Corollary 7.2.9, this data defines a W-cone over F with summit $\mathrm{ran}\,F(\perp)$ that we denote by $\lambda\colon W \Rightarrow \mathcal{E}(\mathrm{ran}\,F(\perp), F(-))$.

By Corollary 7.2.9 again, the data of a cone over the right Kan extension diagram displayed below-left

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow\alpha"] & \\ \mathrm{coll}W \arrow[rr, bend right, "G"'] & {} & \mathcal{E} \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathcal{A} \arrow[dl, hook'] \arrow[dr, "F"] & \\ \mathrm{coll}W \arrow[rr, "\mathrm{ran}\,F"] \arrow[rr, bend right=50, "G"'] \arrow[rr, phantom, bend right=25, "\exists!\Uparrow\eta"] & & \mathcal{E} \end{tikzcd}$$

defines a W-cone

$$W \xRightarrow{\gamma} \mathcal{E}(G(\perp), G|_{\mathcal{A}}(-)) \xRightarrow{\alpha_*} \mathcal{E}(G(\perp), F(-))$$

over F. The universal property of the right Kan extension depicted above-right says this cone factors uniquely through the W-cone λ along a map $\eta_\perp\colon G(\perp) \to \mathrm{ran}\,F(\perp)$. Thus, the right Kan extension of F along $\mathcal{A} \hookrightarrow \mathrm{coll}W$ equips the resulting W-cone with the universal property of the W-weighted limit. $\square$

A particularly convenient aspect of the collage construction is that it allows us to detect the class of flexible weights.

7.2.12. THEOREM (flexible weights and collages). *A natural transformation $\alpha\colon V \hookrightarrow W$ between weights in $s\mathcal{S}et^{\mathcal{A}}$ is a projective cell complex if and only if $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a relative simplicial computad. In particular, W is a flexible weight if and only if $\mathbb{1} + \mathcal{A} \hookrightarrow \mathrm{coll}W$ is a relative simplicial computad.*

PROOF. If $\alpha\colon V \hookrightarrow W$ is a projective cell complex, then it can be presented as a countable composite of pushouts of coproducts of projective cells of varying dimensions indexed by the objects $a \in \mathcal{A}$. Since the collage construction is a left adjoint, it preserves these colimits, and hence the map $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ as a transfinite composite of pushouts of coproducts of simplicial functors $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ in ${}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at$. This composite colimit diagram is connected — note $\mathrm{coll}\varnothing = \mathbb{1} + \mathcal{A}$, so this cell complex presentation is also preserved by the forgetful functor ${}^{\mathbb{1}+\mathcal{A}/}s\mathcal{S}et\text{-}\mathcal{C}at \to s\mathcal{S}et\text{-}\mathcal{C}at$ and the simplicial functor $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$

can be understood as a transfinite composite of pushouts of coproducts of $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ in $s\mathcal{S}et\text{-}\mathcal{C}at$.

This is advantageous because there is a pushout square in $s\mathcal{S}et\text{-}\mathcal{C}at$

$$\begin{tikzcd} \mathbb{2}[\partial\Delta[n]] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \arrow[d, hook] \\ \mathbb{2}[\Delta[n]] \arrow[r] & \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-)) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\mathrm{id}_a"] \arrow[d, hook] & \partial\Delta[n] \times \mathcal{A}(a,a) \arrow[d, hook] \\ \Delta[n] \arrow[r, hook, "\mathrm{id}_a"] & \Delta[n] \times \mathcal{A}(a,a) \end{tikzcd} \tag{7.2.13}$$

whose horizontals send the two objects $-$ and $+$ of the simplicial computads defined in Example 6.1.5 to $\perp$ and a and act on the non-trivial hom-spaces via the inclusions displayed above left whose component in $\mathcal{A}(a,a)$ is constant at the identity element at a. The fact that $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$ is a pushout of $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ can be verified by transposing across the adjunction of Proposition 7.2.8 and applying the Yoneda lemma. From this we see that $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a transfinite composite of pushouts of coproducts of simplicial functors $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$, which proves that this map is a relative simplicial computad.

Conversely, if $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ is a relative simplicial computad, then it can be presented as a countable composite of pushouts of coproducts of simplicial functors $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$; since this inclusion is bijective on objects the inclusion $\varnothing \hookrightarrow \mathbb{1}$ is not needed. Since the only arrows of $\mathrm{coll}W$ that are not present in $\mathrm{coll}V$ have domain $\perp$ and codomain $a \in \mathcal{A}$, the characterization of the essential image of the collage functor of Proposition 7.2.8(i) allows us to identify each stage of the countable composite

$$\mathrm{coll}V \hookrightarrow \mathrm{coll}(W^1) \hookrightarrow \cdots \hookrightarrow \mathrm{coll}(W^i) \hookrightarrow \mathrm{coll}(W^{i+1}) \hookrightarrow \cdots \hookrightarrow \mathrm{coll}W$$

as the collage of some weight $W^i\colon \mathcal{A} \to s\mathcal{S}et$. Each attaching map $\mathbb{2}[\partial\Delta[n]] \to \mathrm{coll}W^i$ in the cell complex presentation acts on objects by mapping $-$ and $+$ to $\perp$ and a for some $a \in \mathcal{A}$, and hence factors through the top horizontal of the pushout square (7.2.13). Hence, the inclusion $\mathrm{coll}(W^i) \hookrightarrow \mathrm{coll}(W^{i+1})$ is a pushout of a coproduct of the maps $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$, one for each cell $\mathbb{2}[\partial\Delta[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ whose attaching map sends $+$ to $a \in \mathrm{coll}(W^i)$. As the collage functor is fully faithful, we have now expressed $\mathrm{coll}(\alpha)\colon \mathrm{coll}V \hookrightarrow \mathrm{coll}W$ as a countable composite of pushouts of coproducts of simplicial functors $\mathrm{coll}(\partial\Delta[n] \times \mathcal{A}(a,-)) \hookrightarrow \mathrm{coll}(\Delta[n] \times \mathcal{A}(a,-))$. A fully faithful functor that preserves colimits also reflects them, so in this way we see that $\alpha\colon V \hookrightarrow W$ is a countable composite of pushouts of coproducts of projective cells, proving that it is a projective cell complex as claimed. ☐

As the first of many applications, we introduce the weights for pseudo limits by constructing their collages and observe immediately that this class of weights is flexible.

7.2.14. Lemma. *For any simplicial set X, the coherent realization of the canonical inclusion $\mathbb{1} + X \hookrightarrow X^{\triangleleft}$ defines a collage and relative simplicial computad $\mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleleft})$ and hence a flexible weight for homotopy coherent diagrams of shape X.*

Proof. By Lemma 6.3.5, $\mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleleft})$ is a simplicial subcomputad inclusion and hence by Lemma 6.1.12 a relative simplicial computad. Thus, Theorem 7.2.12 tells us that the collage $\mathbb{1} + \mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleleft})$ encodes a flexible weight $W_X\colon \mathfrak{C}X \to s\mathcal{S}et$. ☐

7.2.15. Definition (weights for pseudo limits). We call the weight $W_X\colon \mathfrak{C}X \to s\mathcal{S}et$ of Lemma 7.2.14 the **weight for the pseudo limit of a homotopy coherent diagram of shape** X. The W_X-weighted limit

of a homotopy coherent diagram of shape X is then referred to as the **pseudo weighted limit** or simply the **pseudo limit** of that diagram.[3] Since the left adjoint of the collage adjunction is fully faithful, its unit is an isomorphism, and this permits us to define the weight W_X explicitly: for a vertex $x \in X$,

$$W_X(x) \coloneqq \mathfrak{C}(X^{\triangleleft})(\perp, x).$$

7.2.16. Remark. We call W_X the weight for a "pseudo" limit because we anticipate considering homotopy cohererent diagrams valued in Kan-complex enriched categories, in which all arrows in positive dimension are automatically invertible. By Corollary 7.3.3, W_X-weighted limits also exist for diagrams valued in ∞-cosmoi, which are only quasi-categorically enriched. In such contexts, it would be more appropriate to refer to W_X as the weight for *oplax* limits, since in that context the 1-arrows of $\mathfrak{C}(X^{\triangleleft})$ will likely map to non-invertible morphisms.[4]

Exercises.

7.2.i. Exercise. Compute the collage of the weight $U \colon \mathbb{1} \to s\mathcal{S}et$ and use Theorem 7.2.12 to give a second proof that simplicial cotensors are flexible weighted limits. Compare this argument with that given in Example 7.2.3.

7.2.ii. Exercise. The inclusion into the join $\mathbb{1} + X \hookrightarrow X^{\triangleleft} \cong \mathbb{1} \star X$ is bijective on vertices. The complement of the image contains a non-degenerate $(n+1)$-simplex for each non-degenerate n-simplex of X whose initial vertex is $\perp$ and whose 0th face is isomorphic to the image of that simplex. Use Theorem 6.3.10 to describe the atomic arrows of the simplicial computad $\mathfrak{C}(X^{\triangleleft})$ that are not in the image of $\mathbb{1} + \mathfrak{C}X$.

7.2.iii. Exercise. Verify — either directly from Definition 7.2.1 or by applying Theorem 7.2.12 — that conical products are flexible weighted limits.

7.3. Homotopical properties of flexible weighed limits

In a $\mathcal{V}$-model category $\mathcal{M}$, the fibrant objects are closed under weighted limits whose weights are projective cofibrant; see Corollary C.3.13. For instance, the fibrant objects in a $\mathcal{C}at$-enriched model structure are closed under flexible weighted limits [**58**, 5.4] in the sense of [**14**]. Specializing this argument to the case of ∞-cosmoi, we obtain the following result:

7.3.1. Proposition (flexible weights are homotopical). *Let $W \colon \mathcal{A} \to s\mathcal{S}et$ be a flexible weight and let $\mathcal{K}$ be an ∞-cosmos.*

(i) The weighted limit $\lim_W^{\mathcal{A}} F$ of any diagram $F \colon \mathcal{A} \to \mathcal{K}$ may be expressed as a countable inverse limit of pullbacks of products of isofibrations

$$Fa^{\Delta[n]} \twoheadrightarrow Fa^{\partial\Delta[n]} \tag{7.3.2}$$

one for each projective n-cell at a in the given projective cell complex presentation of W.

(ii) If $V \hookrightarrow W \in s\mathcal{S}et^{\mathcal{A}}$ is a projective cell complex between flexible weights, then for any diagram $F \colon \mathcal{A} \to \mathcal{K}$, the induced map between weighted limits

$$\lim_W F \twoheadrightarrow \lim_V F$$

is an isofibration.

[3] Despite the wishy-washy name, pseudo limits are strict limit notions, satisfying their defining universal property of Definition 7.1.7 up to isomorphism.

[4] The distinct between "lax" and "oplax" limits in this context is explained in Exercise 17.1.i.

(iii) *If $\alpha\colon F \Rightarrow G$ is a simplicial natural transformation between a pair of diagrams $F, G\colon \mathcal{A} \rightrightarrows \mathcal{K}$ whose components $\alpha_a\colon Fa \xrightarrow{\sim} Ga$ are equivalences, then the induced map*

$$\lim_W F \xrightarrow[\sim]{\alpha} \lim_W G$$

is an equivalence.

Proof. To begin, observe that the axioms of Definition 7.1.3 imply that the limit of F weighted by the weight $U \times \mathcal{A}(a, -)$, for $U \in s\mathcal{S}et$ and $a \in \mathcal{A}$, is the cotensor Fa^U. Consequently, the map of weighted limits induced by the projective n-cell at a is the isofibration (7.3.2). By definition, any flexible weight is built as a countable composite of pushouts of coproducts of these projective cells and the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries each of these conical colimits to the corresponding limit notion. So it follows that $\lim^{\mathcal{A}}_W F$ may be expressed as a countable inverse limit of pullbacks of products of the maps (7.3.2). This proves (i).

The same argument proves (ii). By definition, a relative cell complex $V \hookrightarrow W$ is built as a countable composite of pushouts of coproducts of these projective cells and the weighted limit functor $\lim^{\mathcal{A}}_{-} F$ carries each of these conical colimits to the corresponding limit notion. So it follows that $\lim^{\mathcal{A}}_W F$ is the limit of a countable tower of isofibrations whose base in $\lim^{\mathcal{A}}_V F$, where each of these isofibrations is the pullback of products of the maps (7.3.2) appearing in the projective cell complex decomposition of $V \hookrightarrow W$. As products, pullbacks, and limits of towers of isofibrations are isofibrations, (ii) follows.

In (i), we have decomposed each weighted limit $\lim^{\mathcal{A}}_W F$ as the limit of a tower of isofibrations, in which each of these isofibrations is the pullback of a product of the isofibrations (7.3.2). We argue inductively that if $\alpha\colon F \Rightarrow G$ is a componentwise equivalence, then the map induced between the towers of isofibrations for F and for G by the projective cell complex presentation of W is a levelwise equivalence. It follows from the standard argument in abstract homotopy theory reviewed in Appendix C that the inverse limit is then an equivalence, proving (iii).

The bottom of the tower of isofibrations is $\lim_\varnothing F \cong 1 \cong \lim_\varnothing G$, which is certainly an equivalence. For the inductive step, observe that upon taking the map of weighted limits induced by each projective n-cell at a in W, we obtain a commutative square

$$\begin{tikzcd} Fa^{\Delta[n]} \arrow[r, "\alpha_a^{\Delta[n]}", "\sim"'] \arrow[d, two heads] & Ga^{\Delta[n]} \arrow[d, two heads] \\ Fa^{\partial\Delta[n]} \arrow[r, "\alpha_a^{\partial\Delta[n]}", "\sim"'] & Ga^{\partial\Delta[n]} \end{tikzcd}$$

defining a pointwise equivalence between the isofibrations; the simplicial cotensor, as cosmological functor, preserves equivalences. Now the product of squares of this form gives a commutative square whose horizontals are isofibrations and whose verticals are equivalences. A pullback of this square forms the next layer in the tower of isofibrations; by the inductive hypothesis, the map between the codomains of the pulled back isofibrations is already known to be an equivalence. Now the equivalence-invariance of pullbacks of isofibrations established in Appendix C completes the proof. ☐

Immediately from the construction of Proposition 7.3.1(i):

7.3.3. Corollary (∞-cosmoi admit all flexible weighted limits). *∞-cosmoi admit all flexible weighted limits and cosmological functors preserve them.* ☐

Our aim is now to describe a converse of sorts to Proposition 7.3.1(i), which proves that the flexible weighted limit of any diagram in an ∞-cosmos can be constructed out of the limits of diagrams of isofibrations axiomatized in 1.2.1(i). Over a series of lemmas, we will construct each of the limits listed there as instances of flexible weighted limits. It will follow that any quasi-categorically enriched category equipped with a class of representably-defined isofibrations that possesses flexible weighted limits will admit all of the simplicial limits of 1.2.1(i). This will help us identify new examples of ∞-cosmoi.

To start, simplicial cotensors are flexible weighted limits. For any simplicial set U, the collage of $U\colon \mathbb{1} \to s\mathcal{S}et$ is the simplicial computad $\mathbb{2}[U]$. As $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{2}[U]$ is a simplicial subcomputad inclusion, Theorem 7.2.12 tells us that $U\colon \mathbb{1} \to s\mathcal{S}et$ is a flexible weight; this solves Exercise 7.2.i.

This leaves only the conical limits. The weights for products are easily seen to be flexible directly from Definition 7.2.1. However, the weights for conical pullbacks or limits of towers of isofibrations are not flexible because the definition of a cone over either diagram shape imposes composition relations on 0-arrows.

7.3.4. Example (the collage of the conical pullback). Let $\lrcorner$ denote the 1-category $c \to a \leftarrow b$. Its collage is the 1-category with four objects and five non-identity 0-arrows as displayed

$$\begin{tikzcd} \perp \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\ c \arrow[r] & a \end{tikzcd}$$

regarded as a constant simplicial category as in Example 6.1.4. Because the square commutes, this category is not free and hence does *not* define a simplicial computad, though the subcategory $\mathbb{1}+ \lrcorner$ is free and hence is a simplicial computad. Lemma 6.1.11 tells us that the inclusion is not a relative simplicial computad and so by Theorem 7.2.12, the weight for the conical pullback is not flexible.

Our strategy is to modify the weights for pullbacks and for limits of countable towers so that each composition equation involved in defining cones over such diagrams is replaced by the insertion of an "invertible" arrow of one dimension up, where we must also take care to define this "invertibility" without specifying any equations between arrows in the next dimension. We have a device for specifying just this sort of isomorphism: recall from Exercise 1.1.iv(i) a diagram $\mathbb{I} \to \mathsf{Fun}(A, B)$ specifies a "homotopy coherent isomorphism" between a pair of 0-arrows f and g from A to B, given by:

- a pair of 1-arrows $\alpha\colon f \to g$ and $\beta\colon g \to f$
- a pair of 2-arrows

$$\begin{tikzcd}[column sep=small] & g \arrow[dr, "\beta"] \arrow[d, phantom, "\Phi"] & \\ f \arrow[ur, "\alpha"] \arrow[rr, equal] & {} & f \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & f \arrow[dr, "\alpha"] \arrow[d, phantom, "\Psi"] & \\ g \arrow[ur, "\beta"] \arrow[rr, equal] & {} & g \end{tikzcd}$$

- a pair of 3-arrows whose outer faces are Φ and Ψ and whose inner faces are degenerate
- etc

We now introduce the weight for pullback diagrams whose cone shapes are given by squares inhabited by a homotopy coherent isomorphism.

7.3.5. Definition (iso-commas). The **iso-comma object** $C \underset{A}{\dot{\times}} B$ of a cospan

$$C \xrightarrow{g} A \xleftarrow{f} B$$

in a simplicially-enriched and cotensored category $\mathcal{M}$ is the limit weighted by a weight $W_{\dot\times}\colon \lrcorner \to s\mathcal{S}et$ defined by the cospan

$$\mathbb{1} \xhookrightarrow{1} \mathbb{I} \xhookleftarrow{0} \mathbb{1}$$

Under the simplification of Remark 7.1.13, the formula for the weighted limit reduces to the equalizer

$$\mathrm{eq}\left(\begin{tikzcd} & A^{\mathbb{I}} \arrow[dr, "{(q_1,q_0)}"] & \\ C \times A^{\mathbb{I}} \times B \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & & A \times A \\ & C \times B \arrow[ur, "g\times f"'] & \end{tikzcd}\right)$$

where the maps $(q_1, q_0)\colon A^{\mathbb{I}} \to A \times A$ are defined by restricting along the endpoint inclusion $\mathbb{1}+\mathbb{1} = \partial\mathbb{I} \hookrightarrow \mathbb{I}$. In an ∞-cosmos, this map is an isofibration and the equalizer defining the iso-comma object is computed by the pullback

$$\begin{tikzcd} C \underset{A}{\dot\times} B \arrow[r] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd} \tag{7.3.6}$$

7.3.7. Lemma. *Iso-comma objects are flexible weighted limits and in particular exist in any ∞-cosmos.*

Proof. Reprising the notation for the category $\lrcorner$ used in Example 7.3.4, the weight $W_{\dot\times}$ is constructed by the pushout

$$\begin{tikzcd} \partial\mathbb{I} \times \lrcorner(a,-) \arrow[r] \arrow[d, hook] & \lrcorner(b,-) \sqcup \lrcorner(c,-) \arrow[d, hook] \\ \mathbb{I} \times \lrcorner(a,-) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & W_{\dot\times} \end{tikzcd}$$

where the attaching map picks out the two arrows in the cospan $\lrcorner$. As a projective cell complex, $W_{\dot\times}$ is built from a project 0-cell at b, a projective 0-cell at c, and two projective k-cells at a for each $k > 0$, correspnoding to the non-degenerate simplices of $\mathbb{I}$. As described by Remark 7.2.2, these may be attached all at once. In this way, we see that $W_{\dot\times}$ is a flexible weight, so Corollary 7.3.3 tells us that iso-comma objects exist in any ∞-cosmos, a fact that is also evident from the pullback (7.3.6). ☐

7.3.8. Remark. In the homotopy 2-category of an ∞-cosmos, there is a canonical invertible 2-cell defining the **iso-comma cone**:

$$\begin{tikzcd} & C \underset{A}{\dot\times} B \arrow[dl, "q_1"'] \arrow[dr, "q_0"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\cong}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

that has a weak universal property analogous to that of the comma cone presented in Proposition 3.4.6. The proof, like the proof of that result, makes use of the fact that $A^{\mathbb{I}}$ is the weak $\mathbb{I}$-cotensor in the homotopy 2-category. The proof of this fact is somewhat delicate, making use of marked simplicial sets as appeared already in the proof of Corollary 1.1.16, which gives 1-cell induction.

Our notation for iso-commas is deliberately similar to the usual notation for pullbacks. In an ∞-cosmos, iso-commas can be used to compute "homotopy pullbacks" of diagrams in which neither map is an isofibration. When at least one map of the cospan is an isofibration, these constructions are equivalent.

7.3.9. Lemma (iso-commas and pullbacks). *In an ∞-cosmos $\mathcal{K}$, pullbacks and iso-commas of cospans in which at least one map is an isofibration are equivalent. More precisely, given a pullback square as below-left and an iso-comma square as below-right*

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, two heads, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \qquad \begin{tikzcd} C \underset{A}{\dot{\times}} B \arrow[r, two heads, "q_0"] \arrow[d, two heads, "q_1"'] \arrow[dr, phantom, "\phi\,\cong"] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

$P \simeq C \underset{A}{\dot{\times}} B$ over C and up to isomorphism over B.

Proof. Applying Lemma 1.2.13 to the functor $b\colon P \to B$, we can replace the span $(c, b)\colon P \to C \times B$ by a span $(cq, p)\colon Pb \twoheadrightarrow C \times B$ whose legs are both isofibrations that is related via an equivalence $s\colon P \xrightarrow{\sim} Pb$ that lies over C on the nose and over B up to isomorphism. We will show that under the hypothesis that f is an isofibration, this new span is equivalent to the iso-comma span.

To see this, note that the factorization constructed in (1.2.14) is in fact defined using an iso-comma, constructed via the pullback in the top square of the diagram below-left. Since the map b is itself defined by a pullback, the bottom square of the diagram below-left is also a pullback, defining the left-hand pullback rectangle:

$$\begin{tikzcd} Pb \arrow[r] \arrow[d, two heads, "{(q,p)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ P \times B \arrow[r, "b \times B"] \arrow[d, two heads, "c \times B"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \times B \arrow[d, two heads, "f \times B"] \\ C \times B \arrow[r, "g \times B"'] & A \times B \end{tikzcd} \qquad \begin{tikzcd} C \underset{A}{\dot{\times}} B \arrow[r] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \underset{A}{\dot{\times}} B \arrow[r, two heads] \arrow[d, two heads, "{(q_1,q_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{I}} \arrow[d, two heads, "{(q_1,q_0)}"] \\ C \times B \arrow[r, "g \times B"'] & A \times B \arrow[r, two heads, "A \times f"'] & A \times A \end{tikzcd}$$

Now the iso-comma is constructed by a similar pullback rectangle, displayed above-right. And because f is an isofibration, Lemma 1.2.11 tells us that the Leibniz tensor $i_0 \widehat{\pitchfork} f\colon B^{\mathbb{I}} \xrightarrow{\sim} A \underset{A}{\dot{\times}} B$ of $i_0\colon \mathbb{1} \hookrightarrow \mathbb{I}$ with $f\colon B \twoheadrightarrow A$ is a trivial fibration. This equivalence commutes with the projections to $A \times B$ and hence the maps $(cq, p)\colon Pb \twoheadrightarrow C \times B$ and $(q_1, q_0)\colon C \underset{A}{\dot{\times}} B \twoheadrightarrow C \times B$, defined as pullbacks of an equivalence pair of isofibrations along $g \times B$, are equivalent as claimed. $\square$

We now introduce a flexible weight diagrams given by a countable tower of 0-arrows whose cone shapes will have a homotopy coherent isomorphism in the triangle over each generating arrow in the diagram.

7.3.10. Definition (iso-towers). Recall the category $\boldsymbol{\omega}$ whose objects are natural numbers and whose morphisms are freely generated by maps $\iota_{n,n+1}\colon n \to n+1$ for each n.

The **iso-tower** of a diagram $F\colon \boldsymbol{\omega}^{\mathrm{op}} \to \mathcal{M}$ in a simplicially enriched and cotensored category $\mathcal{M}$ is the limit weighted by the diagram $W_{\leftarrow}\colon \boldsymbol{\omega}^{\mathrm{op}} \to s\mathcal{S}et$ defined by the pushout

$$\begin{tikzcd} \coprod_{n\in\omega} \partial\mathbb{I}\times\omega(-,n) \arrow[r, "{(\mathrm{id}_n,\iota_{n,n+1})}"] \arrow[d, hook] & \coprod_{n\in\omega} \omega(-,n) \arrow[d, hook] \\ \coprod_{n\in\omega} \mathbb{I}\times\omega(-,n) \arrow[r] & W_{\leftarrow} \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd} \tag{7.3.11}$$

in $s\mathcal{S}et^{\omega^{\mathrm{op}}}$.

By Definition 7.1.3(ii), in an ∞-cosmos the iso-tower of a diagram

$$F := \qquad \cdots \xrightarrow{f_{n+2,n+1}} F_{n+1} \xrightarrow{f_{n+1,n}} F_n \xrightarrow{f_{n,n-1}} \cdots \xrightarrow{f_{2,1}} F_1 \xrightarrow{f_{1,0}} F_0$$

is constructed by the pullback

$$\begin{tikzcd} \lim_{W_{\leftarrow}} F \arrow[r, "\phi"] \arrow[d, two heads, "\rho"'] \arrow[dr, phantom, "\lrcorner" very near start] & \prod_{n\in\omega} F_n^{\mathbb{I}} \arrow[d, two heads, "\Pi(q_1,q_0)"] \\ \prod_{n\in\omega} F_n \arrow[r, "{(f_{n+1,n},\mathrm{id}_{F_n})}"'] & \prod_{n\in\omega} F_n\times F_n \end{tikzcd} \tag{7.3.12}$$

The limit cone is generated by a 0-arrow $\rho_n\colon \lim_{W_{\leftarrow}} F \to F_n$ for each $n\in\omega$ together with a homotopy coherent isomorphism ϕ_n in each triangle over a generating arrow $F_{n+1}\to F_n$ in the ω^{op}-indexed diagram.

7.3.13. Lemma. *Iso-towers are flexible weighted limits and in particular exist in any ∞-cosmos.*

Proof. The weight $W_{\leftarrow}$ is a projective cell complex built by attaching one projective 0-cell at each $n\in\omega$ — forming the coproduct appearing in the upper right-hand corner of (7.3.11) — and then by attaching a projective k-cell at each $n\in\omega$ for each non-degenerate k-simplex of $\mathbb{I}$. Rather than attach each projective k-cell for fixed $n\in\omega$ in sequence, by Remark 7.2.2 these can all be attached at once by taking a single pushout of the "generalized projective cell at n" defined by the map $\partial\mathbb{I}\times\omega(-,n)\hookrightarrow\mathbb{I}\times\omega(-,n)$. These are the maps appearing as the left-hand vertical of (7.3.11). Now Corollary 7.3.3 or the formula (7.3.12) make it clear that such objects exist in any ∞-cosmos. $\square$

7.3.14. Lemma (iso-towers and inverse limits). *In an ∞-cosmos $\mathcal{K}$, the inverse limit of a countable tower of isofibrations is equivalent to the iso-pullback of that tower.*

Proof. We will rearrange the formula (7.3.12) to construct the iso-tower $\lim_{W_{\leftarrow}} F$ as an inverse limit of a countable tower of isofibrations $P\colon \omega^{\mathrm{op}}\to\mathcal{K}$ that is pointwise equivalent to the diagram $F\colon\omega^{\mathrm{op}}\to\mathcal{K}$. In the case where the diagram F is also given by a tower of isofibrations

$$\begin{tikzcd} \lim P \arrow[d, dashed, "\wr"'] & \cong & \cdots \arrow[r, two heads, "p_{n+2,n+1}"] & P_{n+1} \arrow[r, two heads, "p_{n+1,n}"] \arrow[d, "\wr"', "e_n"] & P_n \arrow[r, two heads, "p_{n,n-1}"] \arrow[d, "\wr"', "e_n"] & \cdots \arrow[r, two heads, "p_{2,1}"] & P_1 \arrow[r, two heads, "p_{1,0}"] \arrow[d, "\wr"', "e_1"] & P_0 \arrow[d, "\wr"', "e_0"] \\ \lim F & \cong & \cdots \arrow[r, two heads, "f_{n+2,n+1}"'] & F_{n+1} \arrow[r, two heads, "f_{n+1,n}"'] & F_n \arrow[r, two heads, "f_{n,n-1}"'] & \cdots \arrow[r, two heads, "f_{2,1}"'] & F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0 \end{tikzcd} \tag{7.3.15}$$

the equivalence invariance of the inverse limit of a diagram of isofibrations will imply that the limits $\lim_{W_\leftarrow} F \cong \lim P$ and $\lim F$ are equivalent as claimed.

The ∞-categories P_n will be defined as conical limits of truncated versions of the diagram (7.3.12). To start define $P_0 \coloneqq F_0$ and e_0 to be the identity, then define P_1, $p_{1,0}$, and e_1 via the pullback

$$\begin{tikzcd}
P_1 \arrow[r, two heads] \arrow[rr, bend left, two heads, "p_{1,0}"] \arrow[d, two heads, "e_1"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & F_0^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_0 \\
F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0
\end{tikzcd}$$

Note that $P_1 \cong F_1 \underset{F_0}{\dot{\times}} F_0$ computes the iso-comma objects of the cospan given by id_{F_0} and $f_{1,0}$.

Now define P_2, $p_{2,1}$, and e_2 using the composite pullback

$$\begin{tikzcd}
P_2 \arrow[r, two heads] \arrow[rr, bend left, two heads, "p_{2,1}"] \arrow[d, two heads, "\wr"] \arrow[dd, bend right, two heads, "e_2"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[r, two heads, "\sim"] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & P_1 \arrow[r, two heads] \arrow[d, two heads, "e_1"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & F_0^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_0 \\
\bullet \arrow[r, two heads] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & F_1^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_1 \arrow[r, two heads, "f_{1,0}"'] & F_0 \\
F_2 \arrow[r, two heads, "f_{2,1}"'] & F_1
\end{tikzcd}$$

Continuing inductively, P_n, $p_{n,n-1}$, and e_n are defined by appending the diagram

$$\begin{tikzcd}
& F_{n-1}^{\mathbb{I}} \arrow[r, two heads, "\sim", "q_0"'] \arrow[d, two heads, "\wr"', "q_1"] & F_{n-1} \\
F_n \arrow[r, two heads, "f_{n,n-1}"'] & F_{n-1}
\end{tikzcd}$$

to the limit cone defining P_{n-1} and taking the limit of this composite diagram.

There is one small problem with the construction just given: it defines a diagram (7.3.15) in which each square commutes up to isomorphism — the isomorphism encoded by the map $P_n \to F_{n-1}^{\mathbb{I}}$ — not on the nose. But because the maps $f_{n+1,n}$ are isofibrations this is no problem. The isomorphism inhabiting the square $e_0 p_{1,0} \cong f_{1,0} e_1$ can be lifted along $f_{1,0}$ to define a new map $e_1' \colon P_1 \xrightarrow{\sim\!\!\!\!\!\!\!\to} F_1$ isomorphic to e_1; as observed in the proof of Theorem 1.4.7 this e_1' is then also an equivalence, so we replace e_1 with e_1', and then continue inductively to lift away the isomorphisms in the square $e_1' p_{2,1} \cong f_{2,1} e_2$.

Since inverse limits of towers of isofibrations are equivalence-invariant, it follows that $\lim P \simeq \lim F$. By construction $\lim P \cong \lim_{W_\leftarrow} F$, so it follows that $\lim_{W_\leftarrow} F \simeq \lim F$, which is what we wanted to show. $\square$

Exercises.

7.3.i. EXERCISE ([**86**, 2.2.2]). Show that ∞-cosmoi admit **wide pullbacks**: limits of finite or countable diagrams of the following form

$$\begin{tikzcd}[column sep=small] \cdots & A_n \arrow[dr, two heads, "p_{n-1}"'] & & A_{n-1} \arrow[dl, "f_{n-1}"] \arrow[dr, two heads, "p_{n-2}"'] & & \cdots \arrow[dl, "f_{n-2}"] \arrow[dr, two heads, "p_1"'] & & A_1 \arrow[dl, "f_1"] \arrow[dr, two heads, "p_0"'] & & A_0 \arrow[dl, "f_0"] \\ & \cdots & B_{n-1} & & B_{n-2} & \cdots & B_1 & & B_0 & \end{tikzcd}$$

and that their construction is invariant under pointwise equivalence between diagrams.

7.4. Flexible weighted homotopy limits model ∞-categorical limits

When working in a category $\mathcal{M}$ that is enriched over quasi-categories or Kan complexes we will have occasion for interest in a notion of weighted limit and colimit satisfying a relaxed version of the universal property expressed in Definition 7.1.7, in which the defining isomorphism is replaced by an equivalence of quasi-categories or Kan complexes. Borrowing the standard terminology in use in algebraic topology, we refer to these constructions as “homotopy limits” and “homotopy colimits.” The weights that appear in the examples we consider are flexible weights, so we confine ourselves to the flexibly-weighted case, where the notion of “homotopy limit” is easier to define.

7.4.1. DEFINITION (flexible weighted homotopy limits). Given a flexible weight $W \in s\mathcal{S}et^{\mathcal{A}}$ and a diagram $F \in \mathcal{M}^{\mathcal{A}}$ valued in a quasi-categorically enriched category $\mathcal{M}$, a W-cone $\Lambda\colon W \to \mathcal{M}(L, -)$ displays the object $L \in \mathcal{M}$ as a **flexible weighted homotopy limit** of F weighted by W if for all objects $X \in \mathcal{M}$ the map

$$\mathcal{M}(X, L) \xrightarrow[\sim]{\Lambda\circ-} \lim^{\mathcal{A}}_W \mathcal{M}(X, F-) \cong s\mathcal{S}et^{\mathcal{A}}(W, \mathcal{M}(X, F-))$$

induced by post-composition with Λ is an equivalence of quasi-categories,[5] in which case we denote object L by $\lim^{\simeq}_W F$.

For homotopy coherent diagrams of shape Y, there is a flexible weight of particular interest, the weight W_Y for pseudo limits introduced in Definition 7.2.15. We refer to flexible weighted homotopy limits of this shape as **pseudo homotopy limits** for short. By Corollary 7.2.9, simplicial functors $\mathfrak{C}(\Delta[0] \star Y) \to \mathcal{M}$ stand in bijection to W_Y-shaped cones in $\mathcal{M}$ so we refer to homotopy coherent diagrams of shape $\Delta[0] \star Y$ as **pseudo weighted cones**. Our present aim, which will occupy the remainder of this section, is to prove the following result.

7.4.2. THEOREM. *For any Kan complex enriched category $\mathcal{S}$, simplicial set Y, and homotopy coherent diagram $F\colon \mathfrak{C}Y \to \mathcal{S}$, if F admits a pseudo homotopy limit in $\mathcal{S}$, then the pseudo homotopy limit cone $\Lambda\colon \mathfrak{C}(Y^{\triangleleft}) \to \mathcal{S}$ transposes to define a limit cone $\lambda\colon Y^{\triangleleft} \to \mathfrak{N}\mathcal{S}$ over the corresponding diagram $f\colon Y \to \mathfrak{N}\mathcal{S}$ in the homotopy coherent nerve of $\mathcal{S}$. Consequently, if $\mathcal{S}$ admits pseudo homotopy limits for all simplicial sets Y, then the quasi-category $\mathfrak{N}\mathcal{S}$ admits all limits.*

Dually, pseudo homotopy colimits in $\mathcal{S}$ define colimits in the quasi-category $\mathfrak{N}\mathcal{S}$. In fact, the converse to Theorem 7.4.2 holds as well and will appear as will appear as Theorem 16.4.20: a cone over a diagram $Y \to \mathfrak{N}\mathcal{S}$ is a limit cone in the quasi-category $\mathfrak{N}\mathcal{S}$ if and only if the transposed cone defines a pseudo homotopy limit cone for the homotopy coherent diagram diagram $\mathfrak{C}Y \to \mathcal{S}$ in the Kan complex enriched category $\mathcal{S}$.

[5]Since W is a flexible weight, Corollary 7.3.3 proves that the W-weighted limit of the diagram of quasi-categories $\mathcal{M}(X, F-)\colon \mathcal{A} \to \mathcal{Q}Cat$ is a quasi-category.

The proof of Theorem 7.4.2 proceeds directly from the characterization of a limit cone over a quasi-categorically valued diagram as a terminal element in the quasi-category of cones given in Proposition 4.3.2. It will be most convenient to make use for the slice quasi-category model $\mathfrak{N}\mathcal{S}_{/f}$ for the quasi-category of cones established by Proposition D.6.5 and Proposition F.2.1. By that result, a diagram $f\colon Y \to \mathfrak{N}\mathcal{S}$ admits a limit in $\mathfrak{N}\mathcal{S}$ if and only if there exists some cone λ over f as depicted below-left, which enjoys the lifting property below-right

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, hook] & \mathfrak{N}\mathcal{S} \\ \Delta[0] \star Y \arrow[ur, dashed, "\lambda"'] \end{tikzcd} \qquad \begin{tikzcd} \Delta[0] \star Y \arrow[r, hook, "\{n\}\star Y"'] \arrow[rr, bend left, "\lambda"] & \partial\Delta[n] \star Y \arrow[r] \arrow[d, hook] & \mathfrak{N}\mathcal{S} \\ & \Delta[n] \star Y \arrow[ur, dashed] \end{tikzcd}$$

for all $n \geq 1$. Taking transposes under the homotopy coherent nerve $\dashv$ homotopy coherent realization adjunction, this may equivalently be encoded by a cone as below-left with the lifting-property below-right:

$$\begin{tikzcd} \mathfrak{C}Y \arrow[r, "F"] \arrow[d, hook] & \mathcal{S} \\ \mathfrak{C}(\Delta[0] \star Y) \arrow[ur, dashed, "\Lambda"'] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}(\Delta[0] \star Y) \arrow[r, hook, "\mathfrak{C}(\{n\}\star Y)"'] \arrow[rr, bend left, "\Lambda"] & \mathfrak{C}(\partial\Delta[n] \star Y) \arrow[r] \arrow[d, hook] & \mathcal{S} \\ & \mathfrak{C}(\Delta[n] \star Y) \arrow[ur, dashed] \end{tikzcd} \tag{7.4.3}$$

To prove Theorem 7.4.2, we show that these extensions can be constructed whenever Λ is a pseudo homotopy limit cone, satisfying the universal property of Definition 7.4.1 for the weight W_Y for pseudo limits. The first step towards this proof applies the analysis of §6.4 to characterize the homotopy coherent diagrams appearing in (7.4.3).

7.4.4. Proposition. *If $\mathcal{S}$ admits the pseudo weighted limit of a diagram $F_Y\colon \mathfrak{C}Y \to \mathcal{S}$, then simplicial functors $F\colon \mathfrak{C}(X \star Y) \to \mathcal{S}$ that extend $F_Y\colon \mathfrak{C}Y \to \mathcal{S}$ stand in bijective correspondence to simplicial functors $F_X\colon \mathfrak{C}(X \star \Delta[0]) \to \mathcal{S}$ that map the cone point $\top$ to the pseudo weighted limit $\lim_{W_Y} F_Y$.*

Note the hypothesis of Proposition 7.4.4 is for a *strict* pseudo weighted limit rather than a pseudo *homotopy* limit. We will apply this result in the ∞-cosmos of Kan complexes, which admits such limits by Corollary 7.3.3.

Proof. We start by solving Exercise 6.4.i. Under the isomorphism $\mathfrak{C}(X \star Y) \cong \mathfrak{C}(X \star \Delta[0]) \times_2 \mathfrak{C}(\Delta[0] \star Y)$ of Theorem 6.4.8 and the description of $\mathfrak{C}(X \star \Delta[0]) \times_2 \mathfrak{C}(\Delta[0] \star Y)$ given in Observation 6.4.3, we find that a simplicial functor $F\colon \mathfrak{C}(X \star Y) \to \mathcal{S}$ is uniquely determined by the following data:

(i) a pair of simplicial functors $F_X\colon \mathfrak{C}X \to \mathcal{S}$ and $F_Y\colon \mathfrak{C}Y \to \mathcal{S}$ and
(ii) a family of simplicial maps

$$\mathfrak{C}(X \star \Delta[0])(x, \top) \times \mathfrak{C}(\Delta[0] \star Y)(\perp, y) \xrightarrow{F_{x,y}} \mathcal{S}(F_X x, F_Y y)$$

that is simplicially natural in $x \in \mathfrak{C}X$ and $y \in \mathfrak{C}Y$.

Transposing this family of simplicial maps and taking the manifest enriched end to encode the naturality in $y \in \mathfrak{C}Y$ this reduces to a family of simplicial maps

$$\mathfrak{C}(X \star \Delta[0])(x, \top) \xrightarrow{F_{x,-}} \int_{y \in \mathfrak{C}Y} \mathcal{S}(F_X x, F_Y y)^{\mathfrak{C}(\Delta[0]\star Y)(\perp, y)} =: \lim_{W_Y} \mathcal{S}(F_X x, F_Y -). \tag{7.4.5}$$

which is simplicially natural in $x \in \mathfrak{C}X$. By Definition 7.1.4 and 7.2.15 this end computes the pseudo weighted limit of the diagram $\mathcal{S}(F_X x, F_Y -) \colon \mathfrak{C}Y \to \mathcal{K}an$. If $\mathcal{S}$ admits a pseudo-weighted limit of the diagram $F_Y \colon \mathfrak{C}Y \to \mathcal{S}$, then the codomain of (7.4.5) is, by Definition 7.1.7, naturally isomorphic to $\mathcal{S}(F_X x, \lim_{W_Y} F_Y)$. So then simplicial functors $F \colon \mathfrak{C}(X \star Y) \to \mathcal{S}$ that extend $F_Y \colon \mathfrak{C}Y \to \mathcal{S}$ stand in bijective correspondence to simplicial functors $F_X \colon \mathfrak{C}(X \star \Delta[0]) \to \mathcal{S}$ that map $\top$ to the limit $\lim_{W_Y} F_Y$. □

With this result in hand, we return to analyzing the lifting problem (7.4.3). The next result characterizes solutions to lifting problems of that general form.

7.4.6. Proposition. *Solutions to lifting problems of the form*

$$\begin{tikzcd}
\mathfrak{C}(\partial\Delta[n] \star Y) \arrow[r, "F"] \arrow[d, hook] & \mathcal{S} \arrow[d, "P"] \\
\mathfrak{C}(\Delta[n] \star Y) \arrow[r, "G"'] \arrow[ur, dashed, "L"] & \mathcal{T}
\end{tikzcd} \tag{7.4.7}$$

correspond bijectively to pairs of lifts making the following diagram commute:

$$\begin{tikzcd}
\mathfrak{C}\Lambda^{n+1}[n+1](0,n) \arrow[rr] \arrow[dr, hook] \arrow[dd, hook] & & \mathcal{S}(F0, Fn) \arrow[dr] \arrow[dd] & \\
& \mathfrak{C}\Lambda^{n+1}[n+1](0,n+1) \arrow[rr] \arrow[dd, hook] & & \lim_{W_Y} \mathcal{S}(F0, F-) \arrow[dd] \\
\mathfrak{C}\Delta[n+1](0,n) \arrow[rr] \arrow[dr, hook] \arrow[uurr, dashed] & & \mathcal{T}(G0, Gn) \arrow[dr] & \\
& \mathfrak{C}\Delta[n+1](0,n+1) \arrow[rr] \arrow[uurr, dashed] & & \lim_{W_Y} \mathcal{T}(G0, G-)
\end{tikzcd}$$

Proof. Under the isomorphism of Theorem 6.4.8, the left-hand inclusion in the lifting problem (7.4.7) is isomorphic to

$$\mathfrak{C}\Lambda^{n+1}[n+1] \times_2 \mathfrak{C}(\Delta[0] \star Y) \hookrightarrow \mathfrak{C}\Delta[n+1] \times_2 \mathfrak{C}(\Delta[0] \star Y)$$

which is obtained by applying the functor $- \times_2 \mathfrak{C}(\Delta[0] \star Y)$ to the inclusion $\mathfrak{C}\Lambda^{n+1}[n+1] \hookrightarrow \mathfrak{C}\Delta[n+1]$. By Lemma 6.3.7, the simplicial subcategory $\mathfrak{C}\Lambda^{n+1}[n+1]$ differs from $\mathfrak{C}\Delta[n+1]$ only on account of the proper inclusions

$$\mathfrak{C}\Lambda^{n+1}[n+1](0,n) \hookrightarrow \mathfrak{C}\Delta[n+1](0,n) \qquad \text{and} \mathfrak{C}\Lambda^{n+1}[n+1](0,n+1) \hookrightarrow \mathfrak{C}\Delta[n+1](0,n+1).$$

Combining this information with the description of simplicial functors with domain $\mathfrak{C}(X \star Y)$ given in Proposition 7.4.4, we find that each lift in (7.4.7) corresponds to a pair of solutions to the lifting

problems

$$
\begin{tikzcd}
\mathfrak{C}\Lambda^{n+1}[n+1](0,n) \arrow[r, "F_{0,n}"] \arrow[d, hook] & \mathcal{S}(F0,Fn) \arrow[d, "P_{F0,Fn}"] \\
\mathfrak{C}\Delta[n+1](0,n) \arrow[r, "G_{0,n}"'] \arrow[ur, dashed, "L_{0,n}"'] & \mathcal{T}(G0,Gn)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C}\Lambda^{n+1}[n+1](0,n+1) \arrow[r, "F_{0,-}"] \arrow[d, hook] & \lim_{W_Y}\mathcal{S}(F0,F-) \arrow[d, "\lim_{W_Y} P"] \\
\mathfrak{C}\Delta[n+1](0,n+1) \arrow[r, "G_{0,-}"'] \arrow[ur, dashed, "L_{0,-}"'] & \lim_{W_Y}\mathcal{T}(G0,G-)
\end{tikzcd}
$$

together satisfying the functoriality condition expressed by the commutative square

$$
\begin{tikzcd}
\mathfrak{C}\Delta[n+1](0,n) \arrow[r, "L_{0,n}"] \arrow[d, "\{n,n+1\}\circ-"'] & \mathcal{S}(F0,Fn) \arrow[d, "\Lambda_F\circ-"] \\
\mathfrak{C}\Delta[n+1](0,n+1) \arrow[r, "L_{0,-}"'] & \lim_{W_Y}\mathcal{S}(F0,F-)
\end{tikzcd}
$$

in which the right-hand map is induced by post-composition with the W_Y-weighted cone $\Lambda_F\colon W_Y \to \mathcal{S}(Fn, F-)$ encoded by the composite simplicial functor

$$\mathfrak{C}(\Delta[0]\star Y) \xrightarrow{\mathfrak{C}(\{n\}\star Y)} \mathfrak{C}(\partial\Delta[n]\star Y) \xrightarrow{F} \mathcal{S} \qquad \square$$

The unpacked lifting property of Proposition 7.4.6 can be simplified still further via the geometric descriptions of the mapping spaces in homotopy coherent simplices and horns given in Lemma 6.3.7, which supplies isomorphisms:

$$
\begin{tikzcd}
\mathfrak{C}\Lambda^{n+1}[n+1](0,n) \arrow[r, hook, "\{n,n+1\}\circ-"] \arrow[d, hook] & \mathfrak{C}\Lambda^{n+1}[n+1](0,n+1) \arrow[d, hook] \\
\mathfrak{C}\Delta[n+1](0,n) \arrow[r, hook, "\{n,n+1\}\circ-"'] & \mathfrak{C}\Delta[n+1](0,n)
\end{tikzcd}
\quad\cong\quad
\begin{tikzcd}
\partial\square^{n-1} \arrow[r, hook] \arrow[d, hook] & \sqcap_0^{n,n} \arrow[d, hook] \\
\square^{n-1} \arrow[r, hook, "\square^{n-1}\times\{1\}"'] & \square^n
\end{tikzcd}
$$

This leads us to the following technical lemma.

7.4.8. Lemma. *For a commutative square of simplicial maps $qh = kp$, there exists a bijective correspondence between lifting problems and their solutions of the forms displayed below-left and below-right:*

$$
\begin{tikzcd}
\partial\square^{n-1} \arrow[rrr] \arrow[dr] \arrow[dd, hook] & & & A \arrow[dr, "h"] \arrow[dd, "p"' near start] & \\
& \sqcap_0^{n,n} \arrow[rrr] \arrow[dd, hook] & & & B \arrow[dd, "q"] \\
\square^{n-1} \arrow[rrr] \arrow[dr, "\square^{n-1}\times\{1\}"'] \arrow[urrr, dashed] & & & C \arrow[dr, "k"] & \\
& \square^n \arrow[rrr] \arrow[urrr, dashed] & & & D
\end{tikzcd}
\quad\leftrightsquigarrow\quad
\begin{tikzcd}
\partial\square^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Hom}_B(B,h) \arrow[d, "\mathrm{Hom}_q(B,p)"] \\
\square^{n-1} \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_D(q,k)
\end{tikzcd}
$$

Proof. The isomorphism $\square^n \cong \square^{n-1}\times\Delta[1]$ restricts to subspaces to give an isomorphism

$$\sqcap_0^{n,n} \cong \partial\square^{n-1}\times\Delta[1] \underset{\partial\square^{n-1}\times\{0\}}{\cup} \square^{n-1}\times\{0\}.$$

Put another way, the inclusion of this cubical horn is isomorphic to a Leibniz product inclusion

$$(\sqcap_0^{n,n} \hookrightarrow \square^n) \cong (\partial\square^{n-1} \hookrightarrow \square^{n-1}) \mathbin{\widehat{\times}} (\{0\} \hookrightarrow \Delta[1]).$$

Transposing across the Leibniz product ⊣ Leibniz exponential two-variable adjunction of Proposition C.2.9 provides a bijection between solutions to lifting problems

$$\begin{tikzcd} \sqcap^{n,n}_0 \arrow[r] \arrow[d, hook] & B \arrow[d, "q"] \\ \square^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\square^{n-1} \arrow[r] \arrow[d, hook] & B^{\mathbb{2}} \arrow[d, "\mathsf{Hom}_q(B,q)"] \\ \square^{n-1} \arrow[r] \arrow[ur, dashed] & \mathsf{Hom}_D(q, D) \end{tikzcd} \tag{7.4.9}$$

Under this transposition, the compatibility condition between the two lifts in the commutative cube displayed in the commutative square below-left, transposes to the compatibility condition displayed in the commutative square below-right

$$\begin{tikzcd} \square^{n-1} \arrow[r, dashed] \arrow[d, hook, "\square^{n-1}\times\{1\}"'] & A \arrow[d, "h"] \\ \square^n \arrow[r, dashed] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \square^{n-1} \arrow[r, dashed] \arrow[d, dashed] & A \arrow[d, "h"] \\ B^{\mathbb{2}} \arrow[r, "p_1"'] & B \end{tikzcd}$$

which can be expressed by requiring the lift to define a functor $\square^{n-1} \dashrightarrow \mathsf{Hom}_B(B, h)$. Similarly, the commutative squares displayed on the top and bottom of the commutative cube transpose to the commutative diagrams

$$\begin{tikzcd} \square^{n-1} \arrow[d] & \partial\square^{n-1} \arrow[l, hook'] \arrow[r] \arrow[d] & A \arrow[d, "h"] \\ B & B^{\mathbb{2}} \arrow[l, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd} \qquad \begin{tikzcd} \square^{n-1} \arrow[r] \arrow[d] & C \arrow[d, "k"] \\ D^{\mathbb{2}} \arrow[r, "p_1"'] & D \end{tikzcd}$$

The left-hand square and right-hand square tell us that the codomain of the lifting problem (7.4.9) may be pulled back to define a functor $\square^{n-1} \to \mathsf{Hom}_D(q, k)$, while the middle square tells us that the domain of this lifting problem can be pulled back to define a functor $\partial\square^{n-1} \to \mathsf{Hom}_B(B, h)$. Thus, this information combines to define the transposed lifting problem

$$\begin{tikzcd} \partial\square^{n-1} \arrow[r] \arrow[d, hook] & \mathsf{Hom}_B(B,h) \arrow[d, "\mathsf{Hom}_q(B,p)"] \\ \square^{n-1} \arrow[r] \arrow[ur, dashed] & \mathsf{Hom}_D(q,k) \end{tikzcd}$$

of the statement. □

Cubical boundaries lift against trivial fibrations of simplicial sets. The next lemma gives us a criterion under which the map $\mathsf{Hom}_q(B, p) \colon \mathsf{Hom}_B(B, h) \to \mathsf{Hom}_D(q, k)$ is a trivial fibration.

7.4.10. Lemma. *Given a commutative square between Kan complexes whose vertical arrows are Kan fibrations*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, two heads, "p"'] & B \arrow[d, two heads, "q"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

then the induced map $\mathsf{Hom}_q(B, p) \colon \mathsf{Hom}_B(B, h) \twoheadrightarrow \mathsf{Hom}_D(q, k)$ *is a trivial fibration if and only if the square is a homotopy pullback, i.e., if and only if the map* $A \xrightarrow{\sim} B \times_D C$ *is an equivalence.*

PROOF. First note that Proposition 3.4.5, applied in the ∞-cosmos of Kan complexes, tells us that the map $\mathsf{Hom}_q(B,p)\colon \mathsf{Hom}_B(B,h) \twoheadrightarrow \mathsf{Hom}_D(q,k)$ induced by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, two heads, "p"'] & B \arrow[r, equal] \arrow[d, two heads, "q"] & B \arrow[d, equal] \\ C \arrow[r, "k"'] & D & B \arrow[l, two heads, "q"] \end{tikzcd}$$

is again a Kan fibration. The commutative square gives rise to a commutative square

$$\begin{tikzcd} A \arrow[r, "\ulcorner \mathrm{id} \urcorner"] \arrow[d, "\wr"'] & \mathsf{Hom}_B(B,h) \arrow[d, two heads, "\mathsf{Hom}_q(B,p)"] \\ B \times_D C \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathsf{Hom}_D(q,k) \end{tikzcd}$$

Since B and D are Kan complexes, the comma objects $\mathsf{Hom}_B(B,h)$ and $\mathsf{Hom}_D(q,k)$ are equivalent to the iso-comma objects introduced in Definition 7.3.5. Lemma 7.3.9 then tells us that the horizontal maps in the above commutative square are equivalences. By the 2-of-3 property, we conclude that $\mathsf{Hom}_q(B,p)\colon \mathsf{Hom}_B(B,h) \xrightarrow{\sim\!\!\!\!\rightarrow} \mathsf{Hom}_D(q,k)$ is a trivial fibration, as desired. □

Combining these results, we conclude:

7.4.11. COROLLARY. *If $P\colon \mathcal{S} \to \mathcal{T}$ is a simplicial functor between Kan complex enriched categories and levelwise Kan fibration, then a lifting problem*

$$\begin{tikzcd} \mathfrak{C}(\partial\Delta[n] \star Y) \arrow[r, "F"] \arrow[d, hook] & \mathcal{S} \arrow[d, "P"] \\ \mathfrak{C}(\Delta[n] \star Y) \arrow[r, "G"'] \arrow[ur, dashed, "L"] & \mathcal{T} \end{tikzcd}$$

has a solution whenever the square

$$\begin{tikzcd} \mathcal{S}(F0, Fn) \arrow[r, "\Lambda_{F\circ -}"] \arrow[d, two heads, "P_{F0,Fn}"'] & \lim_{W_Y} \mathcal{S}(F0, F-) \arrow[d, two heads, "\lim_{W_Y} P"] \\ \mathcal{T}(G0, Gn) \arrow[r, "\Lambda_{G\circ -}"'] & \lim_{W_Y} \mathcal{T}(G0, G-) \end{tikzcd}$$

is a homotopy pullback of Kan complexes.

PROOF. Since the weights for pseudo limits are flexible, applying Proposition 7.3.1 in the ∞-cosmos of Kan complexes, we see that the pseudo weighted limits $\lim_{W_Y} \mathcal{S}(F0, F-)$ and $\lim_{W_Y} \mathcal{T}(G0, G-)$ of the diagrams of Kan complexes are again Kan complexes and the map between them is again a Kan fibration. So Lemma 7.4.10 applies to tell us that if this square is a homotopy pullback, then a certain induced map between comma objects is a trivial fibration. If this map is a trivial fibration, then it clearly admits extensions along cubical boundary inclusions $\partial\square^{n-1} \hookrightarrow \square^{n-1}$. Lemma 7.4.8 translates this lifting property into an equivalent lifting property, which Proposition 7.4.6 reveals is exactly what is needed to solve the lifting problem of the statement. □

Corollary 7.4.11 specializes to a proof of Theorem 7.4.2.

Proof of Theorem 7.4.2. Consider a homotopy coherent diagram $F\colon \mathfrak{C}Y \to \mathcal{S}$ in a Kan complex enriched category that admits a pseudo homotopy limit cone $\Lambda\colon \mathfrak{C}\Delta[0] \star Y \to \mathcal{S}$. To see that the transposed cone $\lambda\colon \Delta[0] \star Y \to \mathfrak{N}\mathcal{S}$ defines a limit for the quasi-category valued diagram $f\colon Y \to \mathfrak{N}\mathcal{S}$, we must solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}Y \arrow[r, "F"] \arrow[d, hook] & \mathcal{S} \\ \mathfrak{C}(\Delta[0] \star Y) \arrow[ur, dashed, "\Lambda"'] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}(\Delta[0] \star Y) \arrow[r, hook, "\mathfrak{C}(\{n\}\star Y)"'] \arrow[rr, bend left, "\Lambda"] & \mathfrak{C}(\partial\Delta[n] \star Y) \arrow[r] \arrow[d, hook] & \mathcal{S} \\ & \mathfrak{C}(\Delta[n] \star Y) \arrow[ur, dashed] \end{tikzcd}$$

for all $n \geq 1$. Let $K\colon \mathfrak{C}(\partial\Delta[n] \star Y) \to \mathcal{S}$ be a simplicial functor that restricts along $\mathfrak{C}(\{n\} \star Y)$ to the cone Λ over F. By Corollary 7.4.11, this diagram can be extended along $\mathfrak{C}(\partial\Delta[n] \star Y) \hookrightarrow \mathfrak{C}(\Delta[n] \star Y)$ whenever the diagram

$$\begin{tikzcd} \mathcal{S}(K0, Kn) \arrow[r, "\Lambda\circ-"] \arrow[d, two heads, "!"'] & \lim_{W_Y} \mathcal{S}(K0, K-) \arrow[d, two heads, "!"] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

is a homotopy pullback of Kan complexes. This is the case just when the map

$$\begin{tikzcd} \mathcal{S}(K0, Kn) \arrow[r, "\Lambda\circ-"] \arrow[d, phantom, "\wr\Vert"] & \lim_{W_Y} \mathcal{S}(K0, K-) \arrow[d, phantom, "\wr\Vert"] \\ \mathcal{S}(K0, \Lambda(\perp)) \arrow[r, "\Lambda\circ-"'] & \lim_{W_Y} \mathcal{S}(K0, F-) \end{tikzcd}$$

is an equivalence, which encodes the universal property of the pseudo homotopy limit cone Λ with summit $Kn \coloneqq \Lambda(\perp)$. $\square$

To assist with later calculations, we now constructed some explicit models of pseudo homotopy limit cones of homotopy coherent diagrams for simple but important diagram shapes. In particular, we'll consider as an indexing 1-category J either

- a discrete category, indexing product diagrams,
- the cospan category $\lrcorner$, indexing pullback diagrams, or
- the category ω^{op}, indexing inverse limits of composable sequences.

In each case, J is a free category on an underlying graph $G \hookrightarrow J$ of "atomic" arrows; we regard G as a 1-skeletal simplicial set. As the following lemma indicates, such diagrams $J \to \mathcal{S}$ are automatically "homotopy coherent."

7.4.12. Lemma. *Let J be a 1-category freely generated by the graph $G \hookrightarrow J$ of atomic arrows.*

(i) The homotopy coherent realization $\mathfrak{C}G$ is isomorphic to J, regarded as a simplicial category with discrete hom-sets. Hence diagrams $J \to \mathcal{S}$ in a Kan complex enriched category correspond bijectively to diagrams $G \to \mathfrak{N}\mathcal{S}$ in the homotopy coherent nerve.

(ii) For any Kan complex enriched category $\mathcal{S}$, the quasi-categories $\mathfrak{N}\mathcal{S}^J$ and $\mathfrak{N}\mathcal{S}^G$ are equivalent. Hence, up to equivalence, we can represent a quasi-categorical diagram $J \to \mathfrak{N}\mathcal{S}$ by a point-set diagram $J \to \mathcal{S}$.

PROOF. The isomorphism $\mathfrak{C}G \cong J$ was observed in Exercise 6.3.ii. Hence diagrams $\mathfrak{C}G \cong J \to \mathcal{S}$ in a Kan complex enriched category correspond bijectively to diagrams $G \to \mathfrak{N}\mathcal{S}$ in the homotopy coherent nerve.

Note further that when J is a 1-category freely generated by a graph G, the inclusion $G \hookrightarrow J$ is inner anodyne when considered as a monomorphism of simplicial sets. Since $\mathfrak{N}\mathcal{S}$ is a quasi-category, restriction along this map induces an equivalence $\mathfrak{N}\mathcal{S}^J \xrightarrow{\sim} \mathfrak{N}\mathcal{S}^G$. So, up to equivalence, we can represent any diagram $J \to \mathfrak{N}\mathcal{S}$ by its restriction $G \hookrightarrow J \to \mathfrak{N}\mathcal{S}$, which transposes to a strictly commuting diagram $\mathfrak{C}G \cong J \to \mathcal{S}$ by (i) $\square$

Theorem 7.4.2 shows that limits of homotopy coherent diagrams in Kan complex enriched categories weighted by the weight for pseudo limits model limits in the underlying quasi-category. In the following definition, we construct cones of the appropriate shape over the strictly commuting homotopy coherent diagrams considered in Lemma 7.4.12.

7.4.13. DEFINITION. When J is the free 1-category generated by a graph G, a **strictly commuting pseudo cone** over a diagram $F\colon J \cong \mathfrak{C}G \to \mathcal{S}$ is formed by restricting a strict cone $\alpha\colon L \Rightarrow F(-)$ along the unique map $W_G \to 1$ of weights:

$$W_G \xrightarrow{!} 1 \xrightarrow{\alpha} \mathcal{S}(L, F-)$$

Such strictly commuting pseudo cones are of interest on account of the following result which consider diagrams $F\colon J \to \mathcal{S}$ in a Kan complex enriched category $\mathcal{S}$ of one of the types enumerated above in which certain maps $Fj \to Fj'$ are **representable Kan fibrations**, inducing Kan fibrations $\mathcal{S}(X, Fj) \twoheadrightarrow \mathcal{S}(X, Fj')$ for all $X \in \mathcal{S}$.

7.4.14. PROPOSITION. *Suppose J is a discrete category, the pullback shape $\lrcorner$, or ω^{op}, and $F\colon J \to \mathcal{S}$ is a diagram in a Kan complex enriched category comprised of representable Kan fibrations, in the case $J = \omega^{\mathrm{op}}$, and in which one of the maps is a representable Kan fibration, in the case $J = \lrcorner$. Then the strictly commuting pseudo cone formed from the limit cone $\pi\colon \lim F \Rightarrow F(-)$ over a diagram $F\colon J \to \mathcal{S}$ valued in a Kan complex enriched category presents $\lim F$ as a pseudo homotopy limit of F in $\mathcal{S}$.*

PROOF. For all $X \in \mathcal{S}$, postcomposition with the weighted cone $\pi!\colon W_G \to \mathcal{S}(\lim F, F-)$ determines a map of Kan complexes

$$\begin{tikzcd} \mathcal{S}(X, \lim F) \cong \lim_J \mathcal{S}(X, F-) \cong \lim_1 \mathcal{S}(X, F-) \arrow[r, "!_*"'] & \lim_{W_G} \mathcal{S}(X, F-) \end{tikzcd}$$

with the composite $(\pi!)_*\colon \mathcal{S}(X, \lim F) \to \lim_{W_G} \mathcal{S}(X, F-)$,

which factors as indicated through the map induced by the map $!\colon W_G \to 1$ of weights upon taking weighted limits of the diagram $\mathcal{S}(X, F-)\colon G \to \mathcal{K}an$. Our task is to show that this map is an equivalence.

In this way we see that it suffices to show that for a diagram $F\colon J \to \mathcal{K}an$ of one of the types enumerated in the statement, that the induced map from the strict limit of F to the pseudo limit of F is an equivalence, which is achieved by the next three lemmas. $\square$

7.4.15. LEMMA. *For any family of objects $\{A_j\}_{j \in J}$ in a simplicial category with products, the strict limit cone $\pi\colon \prod_j A_j \to A_j$ defines a pseudo homotopy limit cone.*

PROOF. In this case the result is trivial because the weight for the pseudo limit of a discrete diagram is isomorphic to the terminal weight. $\square$

Lemma 7.4.15 applies in particular to the ∞-cosmos $\mathcal{K}an$ of Kan complexes.

7.4.16. Lemma. *The strict pseudo cone formed from the pullback cone over a diagram of Kan complexes and Kan fibrations*

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d, two heads, "p"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

defines a pseudo homotopy limit cone in $\mathcal{K}an$.

Proof. Unpacking Definition 7.2.15, the weight $W_{\lrcorner} \colon \mathfrak{C}(\lrcorner) \cong \lrcorner \to s\mathcal{S}et$ for pseudo limits over the pullback shape is given by the simplicial functor which maps the outer objects b and c of $\lrcorner$ to $\Delta[0]$ and the middle object a to $\lrcorner^{\mathrm{op}} \cong \Lambda^0[2]$. From the pullback diagram of the statement, we derive the following commutative diagram

$$\begin{tikzcd} C \arrow[r, two heads, "p"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "\Delta", "\wr"'] & A \arrow[d, equal] & B \arrow[l, "f"'] \arrow[d, equal] \\ C \arrow[r, two heads, "p"'] & A & A^{\lrcorner^{\mathrm{op}}} \arrow[l, two heads, "\sim", "\mathrm{ev}_c"'] \arrow[r, two heads, "\sim", "\mathrm{ev}_b"'] & A & B \arrow[l, "f"] \end{tikzcd}$$

Here the upper row is a wide pullback diagram whose limit is simply the pullback P of the original diagram. The lower row is the wide pullback diagram whose limit is the end that computes the limit of the original diagram weighted by $W_{\lrcorner}$. Since $\lrcorner$ is a contractible simplicial set and A is a Kan complex, the indicated maps are equivalences. It follows from Proposition 7.3.1 that the induced map of limits is an equivalence, as elaborated upon in Exercise 7.4.i. □

7.4.17. Lemma. *The strict pseudo cone formed from the limit cone over a sequence of Kan fibrations between Kan complexes*

$$\cdots \xrightarrow{p_n} A_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} A_1 \xrightarrow{p_0} A_0$$

defines a pseudo homotopy limit cone.

Proof. The diagram shape of the statement, the graph underlying the free category $\boldsymbol{\omega}^{\mathrm{op}}$, is the ordered set $\mathbb{N}^{\mathrm{op}}$ with objects $n \in \boldsymbol{\omega}$ and non-identity edges $n+1 \to n$. The weight $W_{\mathbb{N}^{\mathrm{op}}} \colon \mathfrak{C}\mathbb{N}^{\mathrm{op}} \cong \boldsymbol{\omega}^{\mathrm{op}} \to s\mathcal{S}et$ maps each object n to the 1-skeletal simplicial set $\mathbb{N}$ with connecting map from one integer to its predecessor given by the successor map $s \colon \mathbb{N} \to \mathbb{N}$. From the given sequence of Kan fibrations we may derive the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r, equal] & A_n \arrow[r, two heads, "p_{n-1}"] \arrow[d, "\Delta", "\wr"'] & \cdots \arrow[r, equal] & A_2 \arrow[r, two heads, "p_1"] \arrow[d, "\Delta", "\wr"'] & A_1 \arrow[r, equal] \arrow[d, "\Delta", "\wr"'] & A_1 \arrow[r, two heads, "p_0"] \arrow[d, "\Delta", "\wr"'] & A_0 \arrow[r, equal] \arrow[d, "\Delta", "\wr"'] & A_0 \arrow[d, "\Delta", "\wr"'] \\ \cdots & A_n^{\mathbb{N}} \arrow[l, two heads, "A_n^s"] \arrow[r, two heads, "p_{n-1}^{\mathbb{N}}"'] & \cdots & A_2^{\mathbb{N}} \arrow[l, two heads, "A_2^s"] \arrow[r, two heads, "p_1^{\mathbb{N}}"'] & A_1^{\mathbb{N}} & A_1^{\mathbb{N}} \arrow[l, two heads, "A_1^s"] \arrow[r, two heads, "p_0^{\mathbb{N}}"'] & A_0^{\mathbb{N}} & A_0^{\mathbb{N}} \arrow[l, two heads, "A_0^s"] \end{tikzcd}$$

Here the upper row is a wide pullback diagram whose limit is simply the limit of the original digram. The lower row is the wide pullback diagram whose limit is the end that computes the limit weighted by $W_{\mathbb{N}^{\mathrm{op}}}$. The components of the transformation from top to bottom are equivalences because $\mathbb{N}$ is a contractible simplicial set and each of the A_n are Kan complexes. It follows from Proposition 7.3.1 that the induced map of limits is an equivalence, as elaborated upon in Exercise 7.4.i. □

Exercises.

7.4.i. Exercise.

(i) Show that ∞-cosmoi admit **wide pullbacks**: limits of a finite or countable diagram of the following form

$$\begin{tikzcd}
\cdots & B_n \arrow[dr, two heads, "p_{n-1}"'] & & B_{n-1} \arrow[dl, "f_{n-1}"] \arrow[dr, two heads, "p_{n-2}"'] & & \cdots \arrow[dl, "f_{n-2}"] \arrow[dr, two heads, "p_1"'] & & B_1 \arrow[dl, "f_1"] \arrow[dr, two heads, "p_0"'] & & B_0 \arrow[dl, "f_0"] \\
& \cdots & A_{n-1} & & A_{n-2} & \cdots & A_1 & & A_0 &
\end{tikzcd}$$

in which the right-facing arrows are isofibrations.

(ii) Show that the limits constructed in (i) are invariant under pointwise equivalence between diagrams.

7.5. Weak 2-limits revisited

To wrap up this chapter on weighted limits, we briefly switch the base of enrichment from simplicial sets to categories and reconsider the weak 2-limits introduced in Chapter 3. We give a general definition that unifies the weak 2-limits introduced in special cases there and prove their essential uniqueness in a uniform manner.

Before turning our attention to weak 2-limits we describe the explicit construction of the weighted limit of any $\mathcal{C}at$-valued diagram. Let $\mathcal{A}$ be a small 2-category and consider any pair of 2-functors $F, W \colon \mathcal{A} \rightrightarrows \mathcal{C}at$, the first regarded as the diagram and the second as the weight.

7.5.1. Lemma (on the construction of weighted limits in $\mathcal{C}at$). *For any diagram $F \colon \mathcal{A} \to \mathcal{C}at$ and weight $W \colon \mathcal{A} \to \mathcal{C}at$, the weighted limit $\lim_W F \in \mathcal{C}at$ exists, defined to be the category whose*

- *objects are 2-natural transformations $\alpha \colon W \Rightarrow F$ and*
- *morphisms are modifications.*

Proof. By Definition 7.1.7, the W-weighted limit of F is a category $\lim_W F \in \mathcal{C}at$ characterized by a natural isomorphism of categories

$$\mathcal{C}at(X, \lim_W F) \cong \mathcal{C}at^{\mathcal{A}}(W, \mathcal{C}at(X, F-))$$

for any category X. The 2-functor $\mathcal{C}at(\mathbb{1}, -) \colon \mathcal{C}at \to \mathcal{C}at$ is the identity, so taking $X = \mathbb{1}$ this tells us that

$$\lim_W F \cong \mathcal{C}at^{\mathcal{A}}(W, F),$$

the category of 2-natural transformations and modifications from W to F. $\square$

7.5.2. Definition (weak 2-limits in a 2-category). Consider a 2-functor $F \colon \mathcal{A} \to \mathcal{C}$ indexed by a small 2-category $\mathcal{A}$ and a weight $W \colon \mathcal{A} \to \mathcal{C}at$. A W-cone with summit $P \in \mathcal{C}$

$$W \xRightarrow{\lambda} \mathcal{C}(P, F-)$$

displays P as a weak 2-limit of F if and only if for all $X \in \mathcal{C}$ the functor induced by composition with λ

$$\mathcal{C}(X, P) \xrightarrow{\lambda_*} \lim_W \mathcal{C}(X, F-)$$

to the W-weighted limit of the diagram $\mathcal{C}(X, F-) \colon \mathcal{A} \to \mathcal{C}at$ is smothering, in the sense of Definition 3.1.2.

The weak universal property encoded by a smothering functor is sufficiently strong to characterize the limit objects up to equivalence in the ambient 2-category:

7.5.3. Proposition (uniqueness of weak 2-limits). *For any fixed diagram and fixed weight, any pair of weak 2-limits are equivalent via an equivalence that commutes with the legs of the limit cones.*

Proof. If

$$W \xRightarrow{\lambda} \mathcal{C}(P, F-) \qquad \text{and} \qquad W \xRightarrow{\lambda'} \mathcal{C}(P', F-)$$

both define weak 2-limit W-weighted cones over a 2-functor $F \colon \mathcal{A} \to \mathcal{C}$, then for any $X \in \mathcal{C}$, we have a pair of smothering functors

$$\mathcal{C}(X, P) \xrightarrow{\lambda_*} \lim_W \mathcal{C}(X, F-) \xleftarrow{\lambda'_*} \mathcal{C}(X, P')$$

Taking $X = P$, the identity $\mathrm{id}_P \in \mathcal{C}(P, P)$ maps to the cone $\lambda \in \lim_W \mathcal{C}(P, F-)$, which then lifts along the right-hand smothering functor to define a 1-cell $u \colon P \to P'$; a 1-cell $v \colon P' \to P$ is defined similarly as the lift of $\lambda' \in \lim_W \mathcal{C}(P', F-)$ along λ_*. By construction, both u and v commute with the legs of the limit cones λ and λ'.

Now λ_* carries the composite $vu \in \mathcal{C}(P, P)$ to the cone $W \Rightarrow \mathcal{C}(P, F-)$ whose component at $a \in \mathcal{A}$ is the composite

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[rrr, bend right, "\lambda_a"'] & P' \arrow[r, "v"] \arrow[rr, bend right, "\lambda'_a"'] & P \arrow[r, "\lambda_a"] & Fa \end{tikzcd}$$

which equals simply the cone leg λ_a. Thus id_P and vu lie in the same fiber of the smothering functor λ_* and so by Lemma 3.1.3 must be isomorphic via an isomorphism that whiskers to identities along the legs of the limit cone. Similarly, $uv \cong \mathrm{id}_{P'}$, proving that $P \simeq P'$ as claimed. □

Exercises.

7.5.i. Exercise. An **inserter** is a limit of a diagram indexed by the parallel pair category $\bullet \rightrightarrows \bullet$ weighted by the weight

$$\mathbb{1} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{2} \qquad \in \mathcal{C}at$$

Prove that the homotopy 2-category of an ∞-cosmos has weak inserters of any parallel pair of functors $f, g \colon A \rightrightarrows B$ constructed by the pullback

$$\begin{tikzcd} \mathrm{Ins}(f,g) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ A \arrow[r, "{(g,f)}"'] & B \times B \end{tikzcd}$$

CHAPTER 8

More ∞-cosmoi

Our aim in this section is to introduce further examples of ∞-cosmoi.

8.1. The ∞-cosmos of isofibrations

Our first example is a special case of a more general result that will appear in Appendix E that we nonetheless spell out in detail to illustrate the ideas involved in this sort of argument. The walking arrow category $\mathbb{2}$ is an *inverse Reedy category*, where the domain of the non-identity arrow is assigned "degree 1" and the codomain is assigned "degree zero." This Reedy structure motivates the definitions in the ∞-**cosmos of isofibrations** that we now introduce:

8.1.1. Proposition (∞-cosmoi of isofibrations). *For any ∞-cosmos $\mathcal{K}$ there is an ∞-cosmos $\mathcal{K}^{\mathbb{2}}$ whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ in $\mathcal{K}$

(ii) functor-spaces, say from $q\colon F \twoheadrightarrow A$ to $p\colon E \twoheadrightarrow B$, are defined by pullback

$$\begin{tikzcd} \mathsf{Fun}(F \overset{q}{\twoheadrightarrow} A, E \overset{p}{\twoheadrightarrow} B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(F,E) \arrow[d, two heads, "p_*"] \\ \mathsf{Fun}(A,B) \arrow[r, "q^*"'] & \mathsf{Fun}(F,B) \end{tikzcd}$$

(iii) isofibrations from q to p are commutative squares

$$\begin{tikzcd} F \arrow[rr, two heads, "g"] \arrow[dr, two heads] \arrow[dd, two heads, "q"'] & & E \arrow[dd, two heads, "p"] \\ & \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \\ A \arrow[rr, two heads, "f"'] & & B \end{tikzcd}$$

in which the horizontals and the induced map from the initial vertex to the pullback of the cospan are isofibrations in $\mathcal{K}$

(iv) limits are defined pointwise in $\mathcal{K}$

(v) and in which a map

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

is an equivalence in the ∞-cosmos $\mathcal{K}^{\mathbb{2}}$ if and only if g and f are equivalences in $\mathcal{K}$.

Relative to these definitions, the domain, codomain, and identity functors

$$\mathcal{K}^2 \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} \xleftarrow{\ \text{id}\ } \mathcal{K}$$

are all cosmological.

Proof. The diagram category $\mathcal{K}^2$ inherits its simplicially enriched limits, defined pointwise, from $\mathcal{K}$. The functor-spaces described in (ii) are the usual ones for an enriched category of diagrams. This verifies 1.2.1(i).

For axiom 1.2.1(ii) note that the product and simplicial cotensor functors carry pointwise isofibrations to isofibrations. The pullback of an isofibration as in (iii) along a commutatative square from an isofibration r to p may be formed in $\mathcal{K}$. Our task is to show that the induced map t is an isofibration and also that the square from t to r is an isofibration in the sense of (iii):

$$\begin{tikzcd}
G \times_E F \arrow[rrr] \arrow[drr] \arrow[ddd, dashed, "t"'] \arrow[ddr, dotted, two heads] & & & F \arrow[drr, two heads] \arrow[ddd, two heads, "q"' near start] \arrow[ddr, two heads, "z"] & & \\
& & G \arrow[rrr, two heads] \arrow[ddd, two heads, "r" near end] & & & E \arrow[ddd, two heads, "p"] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[rrr, dotted] & & & \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] & \\
C \times_B A \arrow[rrr] \arrow[drr, two heads] & & & A \arrow[drr, two heads] & & \\
& & C \arrow[rrr, two heads] & & & B
\end{tikzcd} \tag{8.1.2}$$

The map t factors as a pullback of z followed by a pullback of r as displayed above, and is thus an isofibration, as claimed. This observation also verifies that the square from t to r defines an isofibration. A similar argument verifies the Leibniz stability of the isofibrations and that the limit of a tower of isofibration is an isofibartion. This proves that $\mathcal{K}^2$ defines an ∞-cosmos in such a way so that the domain, codomain, and identity functors are cosmological.

Finally, since pullbacks of isofibrations in $\mathcal{QCat}$ are invariant under equivalences, a pair of equivalences (g, f) induces an equivalence between the functor-spaces defined in (ii). The converse, that an equivalence in $\mathcal{K}^2$ defines a pair of equivalences in $\mathcal{K}$ follows from the fact that domain and codomain-projection functors are cosmological and Lemma 1.3.2. ☐

In close analogy with Proposition 3.6.3 we have a smothering 2-functor that relates the homotopy 2-category of $\mathcal{K}^2$ to the 2-category of isofibrations, commutative squares, and parallel natural transformations in the homotopy 2-category of $\mathcal{K}$.

8.1.3. Lemma. *There is an identity on objects and 1-cells smothering 2-functor* $\mathfrak{h}(\mathcal{K}^2) \to (\mathfrak{h}\mathcal{K})^2$ *whose codomain is the 2-category whose*

- *objects are isofibrations in* $\mathcal{K}$,
- *1-cells are commutative squares between such,*

- *2-cells are pairs of 2-cells in* $\mathfrak{h}\mathcal{K}$

$$\begin{tikzcd} F \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

PROOF. Exercise 8.1.i. □

Any ∞-cosmos admits an **∞-cosmos of trivial fibrations**, defined similarly. We leave the details to the reader.

8.1.4. PROPOSITION. *Let $\mathcal{K}$ be an ∞-cosmos.*

(i) For any ∞-category B in $\mathcal{K}$, the full subcategory $\mathcal{K}^{\simeq}_{/B} \hookrightarrow \mathcal{K}_{/B}$ spanned by the trivial fibrations with codomain B defines an ∞-cosmos, with limits, isofibrations, equivalences, and trivial fibrations created by the inclusion.

(ii) The full subcategory $\mathcal{K}^{\mathbb{2}}_{\simeq} \hookrightarrow \mathcal{K}^{\mathbb{2}}$ spanned by the trivial fibrations defines an ∞-cosmos, with limits, isofibrations, equivalences, and trivial fibrations created by the inclusion.

PROOF. The details, which are similar to Propositions 1.2.19 and 8.1.1, are left to Exercise 8.1.ii. □

Note that the sliced ∞-cosmoi $\mathcal{K}^{\simeq}_{/B}$ of trivial fibrations are weakly contractible in the following sense sense: the functor spaces are contractible Kan complexes, and consequently and each functor in $\mathcal{K}^{\simeq}_{/B}$ is an equivalence. Hence the unique functor $\mathcal{K}^{\simeq}_{/B} \to \mathbb{1}$ to the terminal ∞-cosmos is a *cosmological biequivalence*, a notion whose general properties are explored in Chapter 14.

Exercises.

8.1.i. EXERCISE. Prove Lemma 8.1.3.

8.1.ii. EXERCISE. Prove Proposition 8.1.4.

8.2. Replete sub ∞-cosmoi

For any ∞-cosmos $\mathcal{K}$ and any subcategory of its underlying 1-category — that is for any subset of its objects and subcategory of its 0-arrows — one can form a quasi-categorically enriched subcategory $\mathcal{L} \subset \mathcal{K}$ that contains exactly those objects and 0-arrows and all higher dimensional arrows that they span. We call such subcategories $\mathcal{L}$ **full on positive-dimensional arrows**; note the functor spaces of $\mathcal{L}$ are quasi-categories because all inner horn inclusions are bijective on vertices. We will take particular interest in subcategories that satisfy a further "repleteness" condition.

8.2.1. DEFINITION. Let $\mathcal{K}$ be an ∞-cosmos. A subcategory $\mathcal{L} \subset \mathcal{K}$ is **replete** in $\mathcal{K}$ if it is full on positive-dimensional arrows and moreover:

(i) Every ∞-category in $\mathcal{K}$ that is equivalent to an object in $\mathcal{L}$ lies in $\mathcal{L}$.
(ii) Any equivalence in $\mathcal{K}$ between objects in $\mathcal{L}$ lies in $\mathcal{L}$.
(iii) Any arrow in $\mathcal{K}$ that is isomorphic in $\mathcal{K}$ to an arrow in $\mathcal{L}$ lies in $\mathcal{L}$.

8.2.2. LEMMA. *Suppose $\mathcal{L} \subset \mathcal{K}$ is a replete subcategory of an ∞-cosmos. Then any map $p \colon E \to B$ in $\mathcal{L}$ that defines an isofibration in $\mathcal{K}$ is a representably-defined isofibration in $\mathcal{L}$: that is for all $X \in \mathcal{L}$, $p_* \colon \mathsf{Fun}_{\mathcal{L}}(X, E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{L}}(X, B)$ is an isofibration of quasi-categories.*

PROOF. Since $\mathcal{K}$ is an ∞-cosmos, axiom 1.2.1(ii) requires that $p_* \colon \mathsf{Fun}_{\mathcal{K}}(X, E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{K}}(X, B)$ is an isofibration of quasi-categories. Because the inner horn inclusions are bijective on vertices and $\mathsf{Fun}_{\mathcal{L}}(X, E) \hookrightarrow \mathsf{Fun}_{\mathcal{K}}(X, E)$ is full on positive-dimensional arrows, it follows immediately that the restricted map $p_* \colon \mathsf{Fun}_{\mathcal{L}}(X, E) \twoheadrightarrow \mathsf{Fun}_{\mathcal{L}}(X, B)$ lifts against the inner horn inclusions. Thus it remains only to solve lifting problems of the form displayed below-left

$$\begin{tikzcd} \mathbb{1} \arrow[r, "e"] \arrow[d, hook] & \mathsf{Fun}_{\mathcal{L}}(X,E) \arrow[r, hook] \arrow[d, two heads, "p_*"'] & \mathsf{Fun}_{\mathcal{K}}(X,E) \arrow[d, two heads, "p_*"] \\ \mathbb{I} \arrow[r, "\beta"'] \arrow[urr, dashed] & \mathsf{Fun}_{\mathcal{L}}(X,B) \arrow[r, hook] & \mathsf{Fun}_{\mathcal{K}}(X,B) \end{tikzcd}$$

The lifting problem defines a 0-arrow $e \colon X \to E$ in $\mathcal{L}$ and an isomorphism $\beta \colon b \cong pe$ in $\mathcal{L}$. Its solution in $\mathcal{K}$ defines a 0-arrow $e' \colon X \to E$ in $\mathcal{K}$ so that $pe' = b$ together with an isomorphism $e \cong e'$ in $\mathcal{K}$. By fullness on positive-dimensional arrows, to show that this lift factors through the inclusion $\mathsf{Fun}_{\mathcal{L}}(X, E) \hookrightarrow \mathsf{Fun}_{\mathcal{K}}(X, E)$, we need only argue that the map e' lies in $\mathcal{L}$, but this is the case by condition (iii) of Definition 8.2.1. ☐

The following result describes a condition under which a replete subcategory $\mathcal{L} \subset \mathcal{K}$ inherits an ∞-cosmos structure created from $\mathcal{K}$.

8.2.3. PROPOSITION. *Suppose $\mathcal{L} \subset \mathcal{K}$ is a replete subcategory of an ∞-cosmos. If $\mathcal{L}$ is closed under flexible weighted limits in $\mathcal{K}$, then $\mathcal{L}$ defines an ∞-cosmos with isofibrations, equivalences, trivial fibrations and simplicial limits created by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$, which then defines a cosmological functor.*

When these conditions hold, we refer to $\mathcal{L}$ as a **replete sub ∞-cosmos** of $\mathcal{K}$ and $\mathcal{L} \hookrightarrow \mathcal{K}$ as a **cosmological embedding**.

PROOF. To say that a replete subcategory $\mathcal{L} \hookrightarrow \mathcal{K}$ is closed under flexible weighted limits means that for any diagram in $\mathcal{L}$ and any limit cone in $\mathcal{K}$ that limit cone lies in $\mathcal{L}$ and satisfies appropriate simplicially-enriched universal property of Definition 7.1.7 in there. We must verify that each of the limits of axiom 1.2.1(i) exist in $\mathcal{L}$. Immediately, $\mathcal{L}$ has a terminal object, products, and simplicial cotensors, since all of these are flexible weighted limits. By Lemmas 7.3.7 and 7.3.13, $\mathcal{L}$ also admits the construction of iso-comma objects and of iso-towers.

Define the class of isofibrations in $\mathcal{L}$ to be those maps in $\mathcal{L}$ that define isofibrations in $\mathcal{K}$. By Lemmas 7.3.9 and 7.3.14, pullbacks and limits of towers of isofibrations are equivalent in $\mathcal{K}$ to the iso-commas and iso-towers formed over the same diagrams. Since these latter limit cones lie in $\mathcal{L}$ by hypothesis, so do the equivalence former cones by repleteness of $\mathcal{L}$ in $\mathcal{K}$.

There is a little more still to verify: namely that pullbacks and limits of towers of isofibrations satisfy the simplicially-enriched universal property as conical limits in $\mathcal{L}$. In the case of a pullback diagram

$$\begin{tikzcd} P \arrow[r, "b"] \arrow[d, two heads, "c"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

in $\mathcal{L}$ we must show that for each $X \in \mathcal{L}$, the functor-space $\mathsf{Fun}_{\mathcal{L}}(X, P)$ is isomorphic to the pullback $\mathsf{Fun}_{\mathcal{L}}(X, C) \times_{\mathsf{Fun}_{\mathcal{L}}(X,A)} \mathsf{Fun}_{\mathcal{L}}(X, B)$ of functor spaces. We have such an isomorphism for functor spaces in $\mathcal{K}$ and on account of the commutative diagram

$$\begin{tikzcd} \mathsf{Fun}_{\mathcal{L}}(X,P) \arrow[r, dashed] \arrow[d, hook] & \mathsf{Fun}_{\mathcal{L}}(X,C) \underset{\mathsf{Fun}_{\mathcal{L}}(X,A)}{\times} \mathsf{Fun}_{\mathcal{L}}(X,B) \arrow[d, hook] \\ \mathsf{Fun}_{\mathcal{K}}(X,P) \arrow[r, "\cong"] & \mathsf{Fun}_{\mathcal{K}}(X,C) \underset{\mathsf{Fun}_{\mathcal{K}}(X,A)}{\times} \mathsf{Fun}_{\mathcal{K}}(X,B) \end{tikzcd}$$

and fullness on positive-dimensional arrows, we need only verify surjectivity of the dotted map on 0-arrows. So consider a cone $(h: X \to B, k: X \to C)$ over the pullback diagram in $\mathcal{L}$. By the universal property of the isocomma $A \underset{B}{\dot{\times}} E$, there exists a factorization $y: X \to C \underset{A}{\dot{\times}} B$ in $\mathcal{L}$. Composing with the equivalence $C \underset{A}{\dot{\times}} B \simeq P$, this map is equivalent to the factorization $z: X \to P$ of the cone (h, k) through the limit cone (b, c) in $\mathcal{K}$ that exists on account of the strict universal property of the pullback in there. By repleteness, the isomorphism between z and the composite of y with the equivalence suffices to show that z lies in $\mathcal{L}$. Hence, the functor spaces in $\mathcal{L}$ are isomorphic. A similar argument invoking Lemma 7.3.14 proves that inverse limits of towers of isofibrations define conical limits in $\mathcal{L}$. This completes the proof of the limit axiom 1.2.1(i).

Since the isofibrations in $\mathcal{L}$ are a subset of the isofibrations in $\mathcal{K}$ and the limit constructions in both contexts coincide, most of the closure properties of 1.2.1(ii) are inherited from the closure properties in $\mathcal{K}$. The one exception is the requirement that the isofibrations in $\mathcal{L}$ define isofibrations of quasi-categories representably, which was proven for any replete subcategory in Lemma 8.2.2. This proves that $\mathcal{L}$ defines an ∞-cosmos.

Finally, we argue that the equivalences in $\mathcal{L}$ coincide with those of $\mathcal{K}$, which will imply that the trivial fibrations in $\mathcal{L}$ coincide with those of $\mathcal{K}$ as well. Condition (ii) of Definition 8.2.1 implies that for any arrow in $\mathcal{L}$ that defines an equivalence in $\mathcal{K}$, its equivalence inverse and witnessing homotopies of Lemma 1.2.15 lie in $\mathcal{L}$. Because we have already shown that $\mathcal{L}$ admits cotensors with $\mathbb{I}$ preserved by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$, Lemma 1.2.15 implies that this data defines an equivalence in $\mathcal{L}$. Conversely, any equivalence in $\mathcal{L}$ extends to the data of (1.2.16) and since $\mathcal{L} \hookrightarrow \mathcal{K}$ preserves $\mathbb{I}$-cotensors, this data defines an equivalence in $\mathcal{K}$. Thus, by construction, the ∞-cosmos structure of $\mathcal{L}$ is preserved and reflected by the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$ as claimed. ☐

In practice the repleteness condition of Definition 8.2.1 is satisfied by any subcategory of objects and 0-arrows that is determined by some ∞-categorical property, so the main task in verifying that a subcategory defines an ∞-cosmos is verifying the closure under flexible weighted limits.

8.2.4. Proposition. *For any ∞-cosmos $\mathcal{K}$, let $\mathcal{K}_\top$ denote the quasi-categorically enriched category whose*

(i) objects are ∞-categories in $\mathcal{K}$ that possess a terminal object
(ii) functor spaces $\mathsf{Fun}_\top(A, B) \subset \mathsf{Fun}(A, B)$ are the sub-quasi-categories whose 0-arrows preserve terminal objects and containing all n-arrows they span

Then the inclusion $\mathcal{K}_\top \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure on $\mathcal{K}_\top$ from $\mathcal{K}$, and moreover for each object of $\mathcal{K}_\top$ defined as a flexible weighted limit of some diagram in $\mathcal{K}_\top$, its terminal element is created by the 0-arrow legs of the limit cone.

Proof. We apply Proposition 8.2.3. Lemma 2.2.6 and Proposition 2.1.10 verify the repleteness condition, so it remains only to prove closure under flexible weighted limits, which we do by induction

over the tower of isofibrations constructed in Proposition 7.3.1(i), which expresses a flexible weighted limit $\lim_W F$ as the inverse limit of a tower of isofibrations

$$\lim_W F \longrightarrow \cdots \longrightarrow\!\!\!\!\!\rightarrow \lim_{W_{k+1}} F \longrightarrow\!\!\!\!\!\rightarrow \lim_{W_k} F \longrightarrow\!\!\!\!\!\rightarrow \cdots \longrightarrow\!\!\!\!\!\rightarrow \lim_{W_0} F \longrightarrow\!\!\!\!\!\rightarrow 1$$

each of which is a pullback of products of maps of the form (7.3.2) indexed by the projective cells of the flexible weight W. We'll argue inductively that each ∞-category in this tower possesses a terminal element that's created by the legs of the tower of isofibrations.

For the base case, note that if $(A_i)_{i\in I}$ is a family of ∞-categories possessing terminal elements $t_i \colon 1 \to A_i$, then the product of the adjunctions $! \dashv t_i$ defines an adjunction

$$\begin{tikzcd} 1 \cong \prod_{i\in I} 1 \arrow[r, bend right, "(t_i)_{i\in I} \cong \prod_i t_i"'] \arrow[r, phantom, "\perp"] & \prod_{i\in I} A_i \arrow[l, bend right, "!"'] \end{tikzcd}$$

exhibiting $(t_i)_{i\in I}$ as a terminal element of $\prod_{i\in I} A_i$. By construction, this terminal element is jointly created by the legs of the limit cone. Note that by construction the product-projection functors preserve this terminal element and the map into the product ∞-category $\prod_{i\in I} A_i$ induced by any family of terminal element preserving functors $(f \colon X \to A_i)_{i\in I}$ will preserve terminal elements. This verifies that the subcategory $\mathcal{K}_\top$ is closed under products.

For the inductive step consider a pullback diagram

$$\begin{tikzcd} \lim_{W_{k+1}} F \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\Delta[n]} \arrow[d, two heads] \\ \lim_{W_k} F \arrow[r, "\ell"'] & A^{\partial\Delta[n]} \end{tikzcd}$$

that arises from the attaching map for a projective n-cell. The inductive hypothesis tells us that $\lim_{W_k} F$ admits a terminal element t_k and for each vertex of $i \in \partial\Delta[n]$, the corresponding component $\ell_i \colon \lim_{W_k} F \to A$ of the limit cone preserves it. Since F is a diagram valued in $\mathcal{K}_\top$ and A is an ∞-category in its image, we know that A must possess a terminal element $t \colon 1 \to A$. By Proposition 2.1.7(iii), the constant diagram at t then defines a terminal element in $A^{\partial\Delta[n]}$ and $A^{\Delta[n]}$, which we also denote by t. By terminality, there is a 1-arrow $\alpha \colon \ell(t_k) \to t \in A^{\partial\Delta[n]}$ whose components at each $i \in \partial\Delta[n]$ are isomorphisms in A. By Lemma 16.2.1, it follows that α is also an isomorphism, which tells us that $\ell(t_k)$ is also a terminal element of $A^{\partial\Delta[n]}$. The same argument demonstrates that terminal elements in simplicial cotensors, in this case by $\partial\Delta[n]$ are jointly created by the 0-arrow components of the limit cone, namely by evaluation on each of the vertices of the cotensoring simplicial set. The proof is completed now by the following lemma. $\square$

8.2.5. Lemma. *Consider a pullback diagram*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in which the ∞-categories A, B, and E possess a terminal element and the functors f and p preserve them. Then F possesses a terminal element that is created by the legs of the pullback cone q and g.

Proof. If $e\colon 1 \to E$ and $a\colon 1 \to A$ are terminal, then this implies that $f(a) \cong p(e) \in B$. Using the fact that p is an isofibration, there is a lift $e' \cong e$ of this isomorphism along p that then defines another terminal element of E. The pair (a, e') now induces an element t of F that we claim is terminal.

To see this we'll apply Proposition 4.3.10, which proves that t is a terminal element of F if and only if the domain-projection functor $p_0\colon \mathsf{Hom}_F(F,t) \twoheadrightarrow F$ is a trivial fibration. By construction of t, we know that the domain-projection functors for the elements gt, qt, and $pgt = fqt$ are all trivial fibrations and moreover the top and bottom faces of the cube

$$\begin{tikzcd}
& \mathsf{Hom}_F(F,t) \arrow[rr] \arrow[dl, two heads] \arrow[d, two heads, "p_0"] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathsf{Hom}_E(E,gt) \arrow[dl, two heads] \arrow[d, two heads, "\wr"', "p_0"] \\
\mathsf{Hom}_A(A,qt) \arrow[rr] \arrow[dd, two heads, "p_0"', "\wr"] & F \arrow[dl, two heads, "q"'] \arrow[rr, "g"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Hom}_B(B,fqt) \arrow[dd, two heads, "\wr"', "p_0"] & E \arrow[dl, two heads, "p"] \\
& & & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

are pullbacks. Since homotopy pullbacks are homotopical, the fact that the three maps between the cospans are equivalences implies that the map between their pullbacks is also an equivalence, as required. ☐

Applying the result of Proposition 8.2.4 to $\mathcal{K}^{\mathrm{co}}$ constructs an ∞-cosmos $\mathcal{K}_\perp$ whose objects are ∞-categories in $\mathcal{K}$ that possess an initial object and 0-arrows are initial-element-preserving functors. Combining these, we get an ∞-cosmos for the pointed ∞-categories of Definition 4.4.1, those that possess a zero element.

8.2.6. Proposition. *For any ∞-cosmos $\mathcal{K}$, let $\mathcal{K}_*$ denote the quasi-categorically enriched category of pointed ∞-categories, i.e., ∞-categories that possess a zero element, and functors that preserve them. Then the inclusion $\mathcal{K}_* \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure on $\mathcal{K}_*$ from $\mathcal{K}$.*

Proof. This result follows directly from Proposition 8.2.4, since the ∞-cosmos $\mathcal{K}_*$ of ∞-categories in $\mathcal{K}$ possessing a zero object is isomorphic to $(\mathcal{K}_\top)_\perp \cong (\mathcal{K}_\perp)_\top$, the idea being that an initial element $a\colon 1 \to A$ in $\mathcal{K}_\top$ is encoded by a terminal-object preserving functor, which says exactly that a is a zero element. ☐

Applying the result of Proposition 8.2.4 or its dual to the ∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over $B \in \mathcal{K}$, we obtain two new ∞-cosmoi of interest.

8.2.7. Corollary. *For any ∞-category B in an ∞-cosmos $\mathcal{K}$, the sliced ∞-cosmos $\mathcal{K}_{/B}$ admits sub ∞-cosmoi*

$$\mathcal{R}ari(\mathcal{K})_{/B} \longrightarrow \mathcal{K}_{/B} \longleftarrow \mathcal{L}ari(\mathcal{K})_{/B}$$

whose

- *objects are isofibrations over B admitting a right adjoint right inverse or left adjoint right inverse, respectively, and*
- *0-arrows are functors over B that commute with the respective right or left adjoints up to fibered isomorphism*

with the ∞-cosmos structures created by the inclusions.

Proof. These ∞-cosmoi are defined by $\mathcal{R}ari(\mathcal{K})_{/B} \coloneqq (\mathcal{K}_{/B})_{\top}$ and $\mathcal{L}ari(\mathcal{K})_{/B} \coloneqq (\mathcal{K}_{/B})_{\perp}$. □

Leveraging Corollary 8.2.7, we can establish similar cosmological embeddings

$$\mathcal{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{2} \hookleftarrow \mathcal{L}ari(\mathcal{K})$$

The quasi-categorically enriched subcategories $\mathcal{R}ari(\mathcal{K})$ and $\mathcal{L}ari(\mathcal{K})$ are replete in $\mathcal{K}^2$, so by Proposition 8.2.3 we need only check closure under flexible weighted limits. We argue separately for cotensors, which are easy, and for the conical limits, which are harder. For this, we make use of a general 1-categorical result making use of the fact that the codomain-projection functor $\mathrm{cod}\colon \mathcal{R}ari(\mathcal{K}) \to \mathcal{K}$ is a Grothendieck fibration of underlying 1-categories, defined by restricting the Grothendieck fibration $\mathrm{cod}\colon \mathcal{K}^2 \to \mathcal{K}$.

8.2.8. Lemma. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration between 1-categories. Suppose that $\mathcal{J}$ is a small category, that $D\colon \mathcal{J} \to \mathcal{E}$ is a diagram, and that*

(i) the diagram $PD\colon \mathcal{J} \to \mathcal{B}$ has a limit L in $\mathcal{B}$ with limit cone $\lambda\colon \Delta L \Rightarrow PD$,

(ii) the diagram $\lambda^*D\colon \mathcal{J} \to \mathcal{E}_L$

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[dr, "\Delta L"'] \arrow[dr, phantom, shift left=3, "\Uparrow\lambda"] & \mathcal{E} \arrow[d, "P"] \\ & \mathcal{B} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{J} \arrow[r, bend left, "D"] \arrow[r, bend right, "\lambda^*D"'] \arrow[r, phantom, "\Uparrow\chi"] & \mathcal{E} \arrow[d, "P"] \\ & \mathcal{B} \end{tikzcd}$$

constructed by lifting the cone λ to a cartesian natural transformation $\chi\colon \lambda^*D \Rightarrow D$ has a limit M in the fibre $\mathcal{E}_L$ with limiting cone $\mu\colon \Delta M \Rightarrow \lambda^*D$, and

(iii) the limit $\mu\colon \Delta M \Rightarrow \lambda^*D$ is preserved by the re-indexing functor $u^*\colon \mathcal{E}_L \to \mathcal{E}_B$ associated with any arrow $u\colon B \to L$ in $\mathcal{B}$.

Then the composite cone

$$\Delta M \xRightarrow{\mu} \lambda^*D \xRightarrow{\chi} D$$

displays M as a limit of the diagram D in $\mathcal{E}$.

Proof. Any arrow $f\colon E \to E'$ in the domain of a Grothendieck fibration $P\colon \mathcal{E} \to \mathcal{B}$ factors uniquely up to isomorphism through a “vertical” arrow in the fiber $\mathcal{E}_{PE}$ followed by a “horizontal” cartesian lift of Pf with codomain E'.

Given a cone $\alpha\colon \Delta E \Rightarrow D$ with summit $E \in \mathcal{E}$ over D, by (i) its image $P\alpha\colon \Delta PE \Rightarrow PD$ factors uniquely through the limit cone $\lambda\colon \Delta L \Rightarrow D$ via a map $b\colon PE \to L \in \mathcal{B}$. By the universal property of the cartesian lift χ of λ constructed in (ii), it follows that α factors uniquely through χ via a natural transformation $\beta\colon \Delta E \Rightarrow \lambda^*D$ so that $P\beta = \Delta b$. This arrow factors uniquely up to isomorphism via “vertical” natural transformation $\gamma\colon \Delta E \to \alpha^*D \cong b^*\gamma^*D$ followed by a “horizontal” cartesian lift of b. By (iii), the limit cone $\mu\colon \Delta M \Rightarrow \lambda^*D$ in $\mathcal{E}_L$ pulls back along b to a limit cone in $\mathcal{E}_{PE}$ through which the pullback of β factors via a map $k\colon E \to b^*M$. Thus, β itself factors uniquely through μ via the composite of this map $k\colon E \to b^*M$ with the cartesian arrow $b^*M \to M$ lifting $b\colon PE \to L$. □

8.2.9. Proposition. *For any ∞-cosmos $\mathcal{K}$, the ∞-cosmos of isofibrations admits sub ∞-cosmoi*

$$\mathcal{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{2} \hookleftarrow \mathcal{L}ari(\mathcal{K})$$

whose

- *objects are isofibrations admitting a right adjoint right inverse or left adjoint right inverse, respectively, and*
- *0-arrows are commutative squares between the right or left adjoints, respectively, whose mates are isomorphisms*

with the ∞-cosmos structures created by the inclusions.

We refer to a commutative square between right adjoints whose mate is an isomorphism as an **exact square**.

Proof. The quasi-categorically enriched subcategories $\mathscr{R}ari(\mathcal{K})$ and $\mathcal{L}ari(\mathcal{K})$ are replete in $\mathcal{K}^2$, so by Proposition 8.2.3 we need only check that $\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ is closed under flexible weighted limits; the argument for $\mathcal{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ is dual. We argue separately for cotensors and for the conical limits.

If $p\colon E \twoheadrightarrow B$ is an isofibration admitting a right adjoint right inverse in $\mathcal{K}$ and U is a simplicial set, then the cosmological functor $(-)^U\colon \mathcal{K} \to \mathcal{K}$ carries this data to a right adjoint right inverse to $p^U\colon E^U \twoheadrightarrow B^U$, which proves that the simplicial cotensor in $\mathcal{K}^2$ of an object in $\mathscr{R}ari(\mathcal{K})$ lies in $\mathscr{R}ari(\mathcal{K})$. The limit cone for the cotensor is given by the canonical map of simplicial sets $U \to \mathsf{Fun}(E^U \xrightarrow{p^U} B^U, E \xrightarrow{p} B)$ defined on each vertex $u\colon \mathbb{1} \to U$ by the commutative square

$$\begin{tikzcd} E^U \arrow[r, "u^*"] \arrow[d, two heads, "p^U"'] & E \arrow[d, two heads, "p"] \\ B^U \arrow[r, "u^*"'] & B \end{tikzcd} \tag{8.2.10}$$

The maps u^* define the components of a simplicial natural transformation from $(-)^U$ to the identity functor and thus the mate of this commutative square is an identity, so the limit cone for the U-cotensor lies in $\mathscr{R}ari(\mathcal{K})$. Finally, to verify the universal property of the cotensor in $\mathscr{R}ari(\mathcal{K})$, we must show that for any commutative square whose domain is an isofibration admitting a right adjoint right inverse

$$\begin{tikzcd} F \arrow[r] \arrow[d, two heads, "q"] & E^U \arrow[d, two heads, "p^U"'] \\ A \arrow[r] & B^U \end{tikzcd}$$

that composes with each of the squares (8.2.10) to an exact square is itself exact. To see this take the mate to define a 1-arrow in $\mathsf{Fun}(A, E^U) \cong \mathsf{Fun}(A, E)^U$ and note that the hypothesis says that the components of this 1-arrow are invertible for each vertex of U. Lemma 16.2.1 then tells us that this 1-arrow is invertible as required.

Taking U to be a set, the argument just given proves also that $\mathscr{R}ari(\mathcal{K})$ is closed in $\mathcal{K}^2$ under products. It remains only to show that it is closed under the remaining conical limits. By pullback stability of fibered adjunctions, the Grothendieck fibration of 1-categories $\mathsf{cod}\colon \mathcal{K}^2 \to \mathcal{K}$ restricts to $\mathsf{cod}\colon \mathscr{R}ari(\mathcal{K}) \to \mathcal{K}$, so we may appeal to Lemma 8.2.8 to calculate 1-categorical limit cones in $\mathscr{R}ari(\mathcal{K}) \subset \mathcal{K}^2$ as composites of cartesian cells with limit cones of fiberwise diagrams. By Corollary 8.2.7, these fiberwise limits in $\mathcal{K}_{/B}$ of diagrams in $\mathscr{R}ari(\mathcal{K})_{/B}$ lie in $\mathscr{R}ari(\mathcal{K})_{/B} \hookrightarrow \mathscr{R}ari(\mathcal{K})$. Moreover, these 1-categorical limits are preserved by the simplicial cotensor, which by Proposition A.5.6 implies that their universal property enriches to define conical limits. In this way we see that

$\mathcal{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ is closed under flexible weighted limits and thus defines a cosmological embedding, as claimed. □

Proposition 8.2.9 allows us to construct further ∞-cosmoi of interest.

8.2.11. PROPOSITION. *For any ∞-cosmos $\mathcal{K}$ and simplicial set J, there exist sub ∞-cosmoi*

$$\mathcal{K}_{\top,J} \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_{\perp,J}$$

whose

- *objects are ∞-categories in $\mathcal{K}$ that admit all limits of shape J or all colimits of shape J, respectively,*
- *0-arrows are the functors that preserve them*

with the ∞-cosmos structures created by the inclusions. Moreover for each object of $\mathcal{K}_{\top,J}$ or $\mathcal{K}_{\perp,J}$ defined as a flexible weighted limit of some diagram in that ∞-cosmos, its J-shaped limits or colimits arecreated by the 0-arrow legs of the limit or colimits cones respectively.

PROOF. First note that the quasi-categorically enriched subcategories $\mathcal{K}_{\top,J}$ and $\mathcal{K}_{\perp,J}$ are replete in $\mathcal{K}$, so by Proposition 8.2.3 we need only confirm that the inclusions are closed under flexible weighted limits. We prove this in the case of colimits, the other case being dual.

For any fixed simplicial set J, there is a cosmological functor $F_J\colon \mathcal{K} \to \mathcal{K}^2$ defined on objects by mapping an ∞-category A to the isofibration $A^{J^\triangleright} \twoheadrightarrow A^J$ in the notation of 4.2.6 and a functor $f\colon A \to B$ to the commutative square

$$\begin{tikzcd} A^{J^\triangleright} \arrow[r, "f^{J^\triangleright}"] \arrow[d, two heads] & B^{J^\triangleright} \arrow[d, two heads] \\ A^J \arrow[r, "f^J"'] & B^J \end{tikzcd}$$

By Corollary 4.3.5, A admits colimits of shape J if and only if this isofibration admits a left adjoint right inverse, and now it is clear that $f\colon A \to B$ preserves these colimits if and only if the square displayed above is exact. In summary, the quasi-categorically enriched subcategory $\mathcal{K}_{\perp,J}$ is defined by the pullback

$$\begin{tikzcd} \mathcal{K}_{\perp,J} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}ari(\mathcal{K}) \arrow[d, hook] \\ \mathcal{K} \arrow[r, "F_J"] & \mathcal{K}^2 \end{tikzcd}$$

Proposition 8.2.9 proves that $\mathcal{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ is closed under flexible weighted limits and $F_J\colon \mathcal{K} \to \mathcal{K}^2$ preserves them, so it follows, as in the proof of Lemma 6.1.7, that $\mathcal{K}_{\perp,J}$ is closed in $\mathcal{K}$ under flexible weighted limits. Now Proposition 8.2.3 proves that the inclusion $\mathcal{K}_{\perp,J} \hookrightarrow \mathcal{K}$ creates an ∞-cosmos structure. □

8.2.12. PROPOSITION. *The ∞-cosmos of isofibrations admits sub ∞-cosmoi*

$$\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^2 \hookleftarrow co\mathcal{C}art(\mathcal{K})$$

whose objects are cartesian or cocartesian fibrations, respectively, and whose 0-arrows are cartesian functors, with the ∞-cosmos structures created by the inclusions. Similarly, for any ∞-category B in an ∞-cosmos $\mathcal{K}$,

the sliced ∞-cosmos $\mathcal{K}_{/B}$ admits sub ∞-cosmoi

$$\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow co\mathcal{C}art(\mathcal{K})_{/B}$$

whose objects are cartesian or cocartesian fibrations over B, respectively, and whose 0-arrows are cartesian functors, with the ∞-cosmos structures created by the inclusions.

Proof. The quasi-categorically enriched subcategories $\mathcal{C}art(\mathcal{K})$ and $co\mathcal{C}art(\mathcal{K})$ are replete in $\mathcal{K}^2$ by Corollary 5.1.17. By Theorems 5.1.11 and 5.1.19, the quasi-categorically enriched category $\mathcal{C}art(\mathcal{K})$ is defined by the pullback

$$\begin{tikzcd} \mathcal{C}art(\mathcal{K}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner" very near start] & \mathscr{R}ari(\mathcal{K}) \arrow[d, hook] \\ \mathcal{K}^2 \arrow[r, "K"] & \mathcal{K}^2 \end{tikzcd}$$

along the simplicial functor that sends an isofibration $p\colon E \twoheadrightarrow B$ to the isofibration $k\colon E^2 \twoheadrightarrow \mathsf{Hom}_B(B,p)$ defined by Remark 5.1.13. The simplicial functor K is constructed out of weighted limits and thus preserves all weighted limits, and the replete subcategory inclusion $\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ creates flexible weighted limits by Proposition 8.2.9. Hence, as in the proof of Lemma 6.1.7, $\mathcal{C}art(\mathcal{K})$ is closed in $\mathcal{K}^2$ under flexible weighted limits, and now Proposition 8.2.3 proves that the inclusion $\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^2$ creates an ∞-cosmos structure.

The result for cartesian fibrations with a fixed base can be proven directly by a similar argument or deduced by considering $\mathcal{K}_{/B} \subset \mathcal{K}^2$ as the (non-replete!) subcategory whose n-arrows have id_B as their codomain components. □

Exercises.

8.2.i. Exercise. Use an argument similar to that given in the proof of Proposition 8.2.6 to prove that any ∞-cosmos $\mathcal{K}$ admits a sub ∞-cosmos $\mathcal{S}tab(\mathcal{K}) \hookrightarrow \mathcal{K}$ whose objects are the stable ∞-categories of Definition 4.4.5 and whose morphisms are the **exact functors**, which preserve the zero elements and the exact triangles.

8.2.ii. Exercise. Consider a functor between isofibrations

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"', "\vdash"] & & F \arrow[dl, "q", "\dashv"'] \\ & B \arrow[ul, bend left, dashed, "r"] \arrow[ur, bend right, dashed, "s"'] & \end{tikzcd}$$

in which p admits a right adjoint right inverse r and q admits a right adjoint right inverse s. Prove that if $gr \cong s$ over B, then the mate of the identity $qg = p$ is an isomorphism. This proves that the 0-arrows in the ∞-cosmos $\mathscr{R}ari(\mathcal{K})_{/B}$ are *exact transformations* between right adjoint right inverse adjunctions.

8.2.iii. Exercise. Prove that $\mathscr{L}ari(\mathcal{K}) \cong \mathscr{R}ari(\mathcal{K}^{\mathrm{co}})^{\mathrm{co}}$.

8.2.iv. Exercise. Use Exercise 1.2.iv to show that if $\mathcal{L} \hookrightarrow \mathcal{K}$ is a replete sub ∞-cosmos, then an object $A \in \mathcal{L}$ is discrete if and only if A is discrete as an object of $\mathcal{K}$.

8.2.v. EXERCISE ([**40**, 4.1.5]). Given a diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] & E'' \arrow[l] \arrow[d, two heads] \\ B' \arrow[r] & B & B'' \arrow[l] \end{tikzcd}$$

where the vertical maps are cocartesian fibrations and the squares define cartesian functors, verify that the induced functor

$$E' \times_E E'' \to B' \times_B B''$$

is a cocartesian fibration and the projections define cartesian functors. Show also that if the vertical maps are discrete cocartesian fibrations so is this induced functor.

Homotopy coherent adjunctions and monads

Bar and cobar resolutions are ubiquitous in modern homotopy theory, defining for instances various completions of spaces and spectra **[19]** and free resolutions such as given in Definition 6.2.1. Formally, these bar or cobar constructions are associated to the monad or comonad of an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \qquad\qquad \eta\colon \mathrm{id}_B \Rightarrow uf\ ,\ \epsilon\colon fu \Rightarrow \mathrm{id}_A$$

between ∞-categories. The monad and comonad resolutions associated to an adjunction are dual. By the triangle equalities, the unit and counit maps give rise to a coaugmented cosimplicial object in $\mathsf{hFun}(B,B)$, the "monad resolution"

$$\begin{tikzcd} \mathrm{id}_B \arrow[r, "\eta"] & uf \arrow[r, shift left=2, "\eta uf"] \arrow[r, shift right=2, "uf\eta"'] & ufuf \arrow[l, "u\epsilon f" description] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & ufufuf \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd} \tag{9.0.1}$$

an augmented simplicial object in $\mathsf{hFun}(A,A)$, the "comonad resolution"

$$\begin{tikzcd} \mathrm{id}_A & fu \arrow[l, "\epsilon"'] \arrow[r, "f\eta u" description] & fufu \arrow[l, shift right=2, "\epsilon fu"'] \arrow[l, shift left=2, "fu\epsilon"] \arrow[r, shift left=2] \arrow[r, shift right=2] & fufufu \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots \end{tikzcd} \tag{9.0.2}$$

and augmented simplicial objects in $\mathsf{hFun}(A,B)$ and $\mathsf{hFun}(B,A)$ admitting forwards and backwards contracting homotopies

$$\begin{tikzcd} u \arrow[r, shift left, "\eta u"] & ufu \arrow[l, shift left, "u\epsilon"] \arrow[r, shift left=4, "\eta ufu"] \arrow[r, shift right=1.5, "uf\eta u" description] & ufufu \arrow[l, shift right=1.5, "u\epsilon fu" description] \arrow[l, shift left=4, "ufu\epsilon"] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & ufufufu \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots \\ f \arrow[r, shift right, "f\eta"'] & fuf \arrow[l, shift right, "\epsilon f"'] \arrow[r, shift left=1.5, "f\eta uf" description] \arrow[r, shift right=4, "fuf\eta"'] & fufuf \arrow[l, shift right=4, "\epsilon fuf"'] \arrow[l, shift left=1.5, "fu\epsilon f" description] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & fufufuf \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] & \cdots \end{tikzcd} \tag{9.0.3}$$

A classical categorical observation tells a richer story, relating the four resolutions displayed above. There is a strict 2-category $\mathcal{A}dj$ containing two objects and an adjunction between them — the free 2-category containing an adjunction — and collectively these four diagrams display the image of a 2-functor whose domain is $\mathcal{A}dj$ **[87]**. More precisely, each diagram is the image of one of the four hom-categories of this two object 2-category: a 2-functor $\mathcal{A}dj \to \mathfrak{h}\mathcal{K}$ extending the adjunction $f \dashv u$ is defined by a pair of objects $A, B \in \mathfrak{h}\mathcal{K}$, the monad and comonad resolutions in the functor categories $\mathsf{hFun}(B,B)$ and $\mathsf{hFun}(A,A)$, and the dual pair of split augmented simplicial objects in $\mathsf{hFun}(A,B)$ and $\mathsf{hFun}(B,A)$. The fact that these resolutions assemble into a 2-functor says that, e.g., that the image of

the comonad resolution under u is an augmented simplicial object in $\mathsf{hFun}(A, B)$ that admits "extra degeneracies."

In this chapter, we will prove that any adjunction in the homotopy 2-category of an ∞-cosmos — that is, any adjunction between ∞-categories — can be lifted to a *homotopy coherent adjunction* in the ∞-cosmos. The data of a homotopy coherent adjunction is indexed by a simplicial computad that is uncannily closely related to the free adjunction $\mathcal{A}dj$. In fact, we define the *free homotopy coherent adjunction* to be the 2-category $\mathcal{A}dj$ regarded as a simplicial category by identifying its hom-categories with their nerves. Section 9.1 is spent justifying this definition by introducing a graphical calculus that allows us to precisely understand homotopy coherent adjunction data and prove that $\mathcal{A}dj$ is a simplicial computad.

In a homotopy coherent adjunction, the resolutions (9.0.1), (9.0.2), and (9.0.3) lift to homotopy coherent diagrams

$$\mathbf{\Delta}_+ \to \mathsf{Fun}(B, B)\,, \quad \mathbf{\Delta}_+^{\mathrm{op}} \to \mathsf{Fun}(A, A)\,, \quad \mathbf{\Delta}_\top \to \mathsf{Fun}(A, B)\,, \quad \mathbf{\Delta}_\perp \to \mathsf{Fun}(B, A)$$

indexed by the 1-categories introduced in Definition 2.3.9 and valued in the functor quasi-categories of the ∞-cosmos. In the case of the split augmented simplicial objects, the contracting homotopies, also called "splittings" or "extra degeneracies," are given by the bottom and top η's respectively. Applying Proposition 2.3.11, it follows that the geometric realization or homotopy invariant realization of the simplicial objects spanned by maps in the image of fuf and ufu are simplicial homotopy equivalent to f and u. Dual results apply to the (homotopy invariant) totalization of the cosimplicial object spanned by these same objects; in this case the "extra codegeneracies" are given by the top and bottom ϵ's.

The main theorem of this chapter proves that homotopy coherent adjunctions are abundant: indeed *any* adjunction of ∞-categories extends to a homotopy coherent adjunction. Homotopy coherent adjunctions extending a subcomputad of generating adjunction data are not unique on the nose. However, whenever the subcomputad of generating adjunction is "parental" — loosely, generating from the universal property of either the right adjoint or the left adjoint exclusively — then extensions to a full homotopy coherent adjunction define the vertices of a contractible Kan complex, proving appropriately generated extensions are "homotopically unique."

All of the results in this chapter apply to any adjunction defined in (the homotopy 2-category of) a quasi-categorically enriched category $\mathcal{K}$. Yet since we'll typically apply these results to ∞-cosmoi, we retain the usual notation $\mathsf{Fun}(A, B)$ for the hom quasi-categories of $\mathcal{K}$ to trigger the correct intuition in these contexts.

9.1. The free homotopy coherent adjunction

In this section, we present a strict 2-category $\mathcal{A}dj$ introduced by Schanuel and Street under the name "the free adjunction" [**87**], which has the universal property that it is the free 2-category containing an adjunction. Immediately after introducing this classical object, we take the unorthodox step of reconsidering it as a simplicial category via a mechanism that we shall describe. We develop a new presentation of $\mathcal{A}dj$ by introducing a graphical calculus that allows us to prove the surprising fact that this simplicial category is a simplicial computad. This justifies referring to it as the *free homotopy coherent adjunction*. The remainder of this chapter will explore the consequences of this definition.

9.1.1. Definition (the free adjunction). Let $\mathcal{A}dj$ denote the 2-category with two objects $+$ and $-$ and the four hom-categories

$$\mathcal{A}dj(+,+) \coloneqq \mathbf{\Delta}_+ , \quad \mathcal{A}dj(-,-) \coloneqq \mathbf{\Delta}_+^{\mathrm{op}} , \quad \mathcal{A}dj(-,+) \coloneqq \mathbf{\Delta}_\top , \quad \mathcal{A}dj(+,-) \coloneqq \mathbf{\Delta}_\bot$$

displayed in the following cartoon:

$$\begin{tikzcd} + \arrow[loop left, "\mathbf{\Delta}_+"] \arrow[r, bend left, "\mathbf{\Delta}_\bot \cong \mathbf{\Delta}_\top^{\mathrm{op}}"] \arrow[r, phantom, "\bot"] & - \arrow[l, bend left, "\mathbf{\Delta}_\top \cong \mathbf{\Delta}_\bot^{\mathrm{op}}"] \arrow[loop right, "\mathbf{\Delta}_+^{\mathrm{op}}"] \end{tikzcd}$$

Here $\mathbf{\Delta}_\top, \mathbf{\Delta}_\bot \subset \mathbf{\Delta} \subset \mathbf{\Delta}_+$ are the subcategories of order-preserving maps that preserve the top or bottom elements, respectively, in each ordinal, as described in Definition 2.3.9. Their intersection

$$\mathbf{\Delta}_{\bot,\top} \coloneqq \mathbf{\Delta}_\bot \cap \mathbf{\Delta}_\top \cong \mathbf{\Delta}_+^{\mathrm{op}}$$

is the subcategory of order-preserving maps that preserve both the top and bottom elements in each ordinal. This identifies $\mathbf{\Delta}_+^{\mathrm{op}}$ with the subcategory $\mathbf{\Delta}_{\bot,\top} \subset \mathbf{\Delta}_+$ of "intervals," as is elaborated upon in Digression 9.1.8.

The horizontal composition maps in $\mathcal{A}dj$ are defined in $\mathcal{A}dj(-,-)^{\mathrm{op}} \cong \mathcal{A}dj(+,+) \cong \mathbf{\Delta}_+$ by the ordinal sum operation:

$$\begin{tikzcd} \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \arrow[r, "\oplus"] & \mathbf{\Delta}_+ \\ {[n],[m]} \arrow[r, mapsto] \arrow[d, "\alpha,\beta"'] \arrow[dr, phantom, "\mapsto"] & {[n+m+1]} \arrow[d, "\alpha\oplus\beta"] \\ {[n'],[m']} \arrow[r, mapsto] & {[n'+m'+1]} \end{tikzcd} \qquad \alpha \oplus \beta(i) \coloneqq \begin{cases} \alpha(i) & i \leq n \\ \beta(i-n-1)+n'+1 & i > n \end{cases}$$

The object $[-1] \in \mathbf{\Delta}_+$ serves as the identity for ordinal sum, and thus represents the identity 1-cells on $-$ and $+$ in $\mathcal{A}dj$. Ordinal sum restricts to the subcategories $\mathbf{\Delta}_\bot, \mathbf{\Delta}_\top \subset \mathbf{\Delta}_+$ to give bifunctors

$$\begin{tikzcd} \mathbf{\Delta}_+ \times \mathbf{\Delta}_\top \arrow[r, "\oplus"] \arrow[d, phantom, "\wr\Vert"] & \mathbf{\Delta}_\top \arrow[d, phantom, "\wr\Vert"] \\ \mathcal{A}dj(+,+) \times \mathcal{A}dj(-,+) \arrow[r, "\circ"] & \mathcal{A}dj(-,+) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{\Delta}_\bot \times \mathbf{\Delta}_+ \arrow[r, "\oplus"] \arrow[d, phantom, "\wr\Vert"] & \mathbf{\Delta}_\bot \arrow[d, phantom, "\wr\Vert"] \\ \mathcal{A}dj(+,-) \times \mathcal{A}dj(+,+) \arrow[r, "\circ"] & \mathcal{A}dj(+,-) \end{tikzcd}$$

defining these horizontal composition operations that we'll later come to think of as "actions" of $\mathcal{A}dj(+,+)$ on the left and right of $\mathcal{A}dj(-,+)$ and $\mathcal{A}dj(+,-)$. The opposites of these functors defines the action of $\mathcal{A}dj(-,-)$ on the right and left of $\mathcal{A}dj(-,+)$ and $\mathcal{A}dj(+,-)$.

9.1.2. Lemma. *The 2-category $\mathcal{A}dj$ contains a distinguished adjunction*

$$\begin{tikzcd} + \arrow[r, bend left, "{[0]}"] \arrow[r, phantom, "\bot"] & - \arrow[l, bend left, "{[0]}"] \end{tikzcd}$$

with unit $! \colon [-1] \to [0] \in \mathbf{\Delta}_+ \cong \mathcal{A}dj(+,+)$ *and counit given by the same map in* $\mathbf{\Delta}_+^{\mathrm{op}} \cong \mathcal{A}dj(-,-)$.

Proof. We must verify the triangle equalities in $\mathcal{A}dj(-,+)$ and $\mathcal{A}dj(+,-)$; these categories are opposites and the calculation in each case is dual, so we focus on the case of $\mathcal{A}dj(-,+)$. The whiskered composite of the unit $! \colon [-1] \to [0] \in \mathcal{A}dj(+,+)$ with the right adjoint $[0] \in \mathcal{A}dj(-,+)$ is the map $\delta^0 \colon [0] \to [1] \in \mathcal{A}dj(-,+)$, which is indeed top-element preserving. The whiskered composite of the counit in $\mathcal{A}dj(-,-)$ with the right adjoint in $\mathcal{A}dj(-,+)$ is defined by whiskering the opposite map

$!\colon [-1] \to [0] \in \mathcal{A}dj(+,+) \cong \mathcal{A}dj(-,-)^{\mathrm{op}}$ with $[0] \in \mathcal{A}dj(+,-) \cong \mathcal{A}dj(-,+)^{\mathrm{op}}$. This composite is the map $\delta^1\colon [0] \to [1] \in \mathcal{A}dj(+,-) \cong \mathcal{A}dj(-,+)^{\mathrm{op}}$, which is indeed top-element preserving. Under the isomorphism $\mathcal{A}dj(-,+) \cong \mathcal{A}dj(+,-)^{\mathrm{op}}$, this corresponds to the map $\sigma^0\colon [1] \to [0] \in \mathcal{A}dj(-,+)$. Now the composite $\sigma^0 \cdot \delta^0\colon [0] \to [0] \in \mathcal{A}dj(-,+)$ is the identity as required. □

The 2-categorical universal property of $\mathcal{A}dj$ — that 2-functors $\mathcal{A}dj \to \mathcal{C}$ correspond to adjunctions in the 2-category $\mathcal{C}$ — is stated without proof in [**87**]. We will take a roundabout route to verifying it in Proposition 9.1.13 that uses the simplicial computads of Chapter 6.

Throughout this text we have found it convenient to identify 1-categories with their nerves, which define simplicial sets. The 1-categories can be characterized as those simplicial sets that admit unique extensions along any inner horn inclusion or spine inclusion, or as those 2-coskeletal simplicial sets that admit unique extensions along inner horn or spine inclusions in dimensions 2 and 3; see Remark 1.1.5. Similarly, in developing homotopy coherent category theory it will be convenient to identity 2-categories with the simplicial categories obtained by identifying each of the hom-categories with its nerve — a categorification of the previous construction. As a corollary of the characterization of nerves of 1-categories, we obtain a characterization of the simplicially enriched categories that arise in this way.

9.1.3. Lemma (2-categories as simplicial categories). *A 2-category $\mathcal{A}$ may be regarded as a quasi-categorically enriched category whose*

- *objects are the objects of $\mathcal{A}$*
- *0-arrows in $\mathcal{A}(x,y)$ are the 1-cells of $\mathcal{A}$ from x to y*
- *1-arrows in $\mathcal{A}(x,y)$ from f to g are the 2-cells of $\mathcal{A}$ of the form* $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$
- *and in which there exists a 2-arrow σ in $\mathcal{A}(x,y)$ whose faces $\sigma^i \coloneqq \sigma \cdot \delta^i$ are 1-arrows in $\mathcal{A}(x,y)$*

$$\begin{tikzcd} & g \arrow[dr, "\sigma^0"] \arrow[d, phantom, "\sigma"] & \\ f \arrow[ur, "\sigma^2"] \arrow[rr, "\sigma^1"'] & {} & h \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} x \arrow[r, "g" description] \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, bend left=30, "\Downarrow\sigma^2"] \arrow[r, phantom, bend right=30, "\Downarrow\sigma^0"] & y \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\sigma^1"] & y \end{tikzcd}$$

if and only if σ^1 is the vertical composite $\sigma^0 \cdot \sigma^2$

with the higher-dimensional arrows determined by the property that each of the hom-spaces is 2-coskeletal. Conversely, a simplicially-enriched category $\mathcal{A}$ is isomorphic to a 2-category if and only if each of its hom-spaces are 2-coskeletal simplicial sets that admit unique extensions along the spine inclusions in dimensions 2 and 3. □

We now give the first of two presentations of the free homotopy coherent adjunction. Since we use the same notation for 1-categories and their nerves, we also adopt the same notation for a 2-category and its corresponding simplicial category under the embedding of Lemma 9.1.3.

9.1.4. Definition (the free homotopy coherent adjunction, as a 2-category). The **free homotopy coherent adjunction** to be the free adjunction $\mathcal{A}dj$, regarded as a simplicial category. Explicitly $\mathcal{A}dj$ has two objects $+$ and $-$ and the four hom quasi-categories defined by

$$\mathcal{A}dj(+,+) \coloneqq \mathbb{\Delta}_+\,, \quad \mathcal{A}dj(-,-) \coloneqq \mathbb{\Delta}_+^{\mathrm{op}}\,, \quad \mathcal{A}dj(-,+) \coloneqq \mathbb{\Delta}_\top\,, \quad \mathcal{A}dj(+,-) \coloneqq \mathbb{\Delta}_\perp$$

with the composition maps defined in 9.1.1.

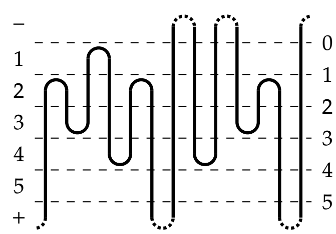
This presentation of the free homotopy coherent adjunction is not particularly enlightening. Via Lemma 9.1.3 and Definition 9.1.1 we could in principle describe the n-arrows in $\mathcal{A}dj$ but it's tricky to get a real feel for them. We will now reintroduce this simplicial category in a different guise that achieves just this. Before doing so, note:

9.1.5. Observation. To specify a simplicial category, thought of as an identity-on-objects simplicial object in $\mathcal{C}at$, it suffices to specify:

- a set of objects
- for each $n \geq 0$, a set of n-arrows whose domains and codomains are among the specified object set,
- a right action of the morphisms in $\mathbf{\Delta}$ on this graded set of arrows that preserves domains and codomains,
- a "horizontal" composition operation for the n-arrows with compatible (co)domains that preserves the simplicial action.

We will now reintroduce $\mathcal{A}dj$ following this outline, by exhibiting its graded set of n-arrows between the objects $+$ and $-$.

9.1.6. Definition (strictly undulating squiggles). Define a graded set of arrows between objects $-$ and $+$ whose n-arrows are strictly undulating squiggles on $n+1$ lines, such as displayed below in the case $n = 5$:



+,2,3,1,4,2,+,−,4,−,3,2,+,−

The lines are labeled $0, 1, \ldots, n$ and the gaps between them are labeled $-, 1, \ldots, n, +$. A squiggle must start, on the right-hand side, and end, on the left-hand side, in either the gap $-$ or $+$. The right-hand starting gap becomes the domain of the squiggle and the left-hand ending gap becomes its codomain, these conventions chosen to follow the usual composition order. Each turning point of the squiggle must lie entirely within a single gap. The qualifier "strict undulation" refers to the requirement that adjacent turning points should be distinct and that they should oscillate up and down as we proceed from right to left.

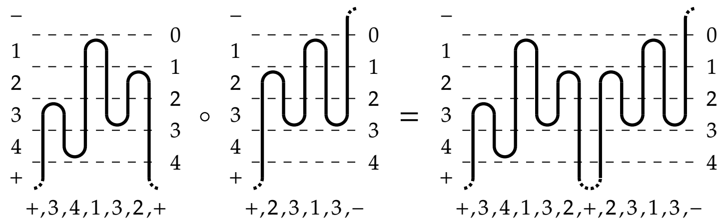
Formally, the data of a **strictly undulating squiggle** on $n+1$ lines can be encoded by a string $\underline{a} = (a_0, a_1, \ldots, a_{r-1}, a_r)$ of letters in the set $\{-, 1, 2, \ldots, n, +\}$ corresponding to the gaps in which each successive turning point occurs, whose **width** is the integer $w(\underline{a}) \coloneqq r$, subject to the following conditions:

(i) The **domain** $a_{w(\underline{a})}$ and **codomain** a_0 of $\underline{a}$ are both in $\{-, +\}$.
(ii) If $a_0 = -$ then for all $0 \leq i < w(\underline{a})$ we have $a_i < a_{i+1}$ whenever i is even and $a_i > a_{i+1}$ whenever i is odd, and if $a_0 = +$ then for all $0 \leq i < w(\underline{a})$ we have $a_i > a_{i+1}$ whenever i is even and $a_i < a_{i+1}$ whenever i is odd.

9.1.7. Lemma. *The graded set of strictly undulating squiggles admits a right action of the morphisms in $\mathbf{\Delta}$ that preserves domains and codomains and a "horizontal" composition operation for arrows in the same degree with compatible (co)domains that preserves the simplicial action. Relative to the horizontal composition, an n-arrow*

is atomic if and only if there are no instances of $+$ or $-$ occurring in its interior. While the faces of an atomic arrow need not be atomic, the degeneracies of an atomic arrow always are.

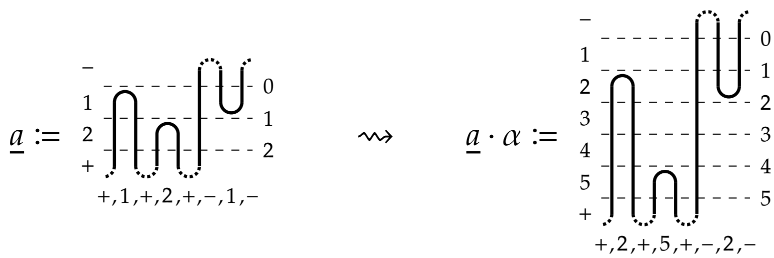
Proof. The horizontal composition of n-arrows is given by horizontal juxtaposition of strictly undulating squiggles on $n+1$ lines, which produces a well-formed squiggle just when the codomain of the right-hand squiggle matches the domain of the left-hand squiggle:



This operation is clearly associative. Moreover, any strictly undulating squiggle admits a unique decomposition into squiggles that do not contain $+$ or $-$ in their interior sequences of gaps, which proves that any squiggle $\underline{a} = (a_0, a_1, \ldots, a_{r-1}, a_r)$ with the property that $a_1, \ldots, a_{r-1} \in \{1, \ldots, n\}$ is atomic.

The right action of the simplicial operators on strictly undulating squiggles is best described in two cases. If $\alpha\colon [m] \twoheadrightarrow [n]$ is an epimorphism, then a strictly undulating squiggle on lines $0, \ldots, n$ becomes a strictly undulating squiggle on lines $0, \ldots, m$ by replacing each line labeled $i \in [n]$ with lines labeled by each element of the fiber $\alpha^{-1}(i)$ and then "pulling these lines apart" to create new gaps:

$$\begin{aligned} [5] &\xrightarrow{\ \alpha\ }\!\!\!\!\rightarrow [2] \\ 0,1 &\longmapsto 0 \\ 2,3,4 &\longmapsto 1 \\ 5 &\longmapsto 2 \end{aligned}$$

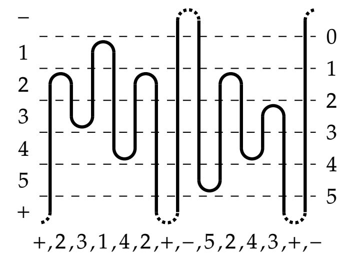


Note that "pulling apart lines" does not create instances of $+$ or $-$ in the interior sequence of gaps, so the action by degeneracy operators preserves atomic arrows.

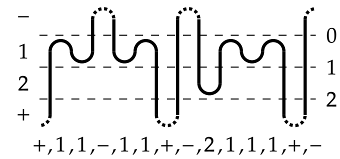
If $\alpha\colon [m] \rightarrowtail [n]$ is a monomorphism, then a strictly undulating squiggle on lines $0, \ldots, n$ becomes a strictly undulating squiggle on lines $0, \ldots, m$ by removing the lines labelled by elements $i \in [n]$ not in the image of α and renumbering the lines that are in the image in sequence. The original squiggle will still undulate between the new lines, but may not do so "strictly" — it is possible for the squiggle to turn around mutliple times in the same gap — but this is easily corrected by "pulling the string

taut":

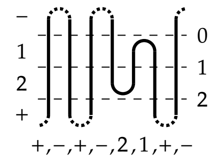
$$\begin{array}{ccc} [2] & \overset{\alpha}{\rightarrowtail} & [5] \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 4 \\ 2 & \longmapsto & 5 \end{array} \qquad \underline{a} :=$$



$$\rightsquigarrow \qquad \underline{a} \cdot \alpha :=$$



$$:=$$



Note that in both of these cases, the actions by epimorphisms and monomorphisms preserve domains and codomains of squiggles and respect horizontal concatenations.

Now in general, a simplicial operator $\alpha\colon [m] \to [n]$ can be factored uniquely in $\mathbf{\Delta}$ as an epimorphism followed by a monomorphism, so it acts on a strictly undulating squiggle on $n+1$ lines by first "removing lines and pulling taut" and then by "duplicating lines and pulling apart." $\square$

We leave the formalization of these geometric descriptions of the actions of the simplicial operators on strictly undulating squiggles presented as sequences satisfying the axioms of Definition 9.1.6(i) and (ii) to Exercise 9.1.iii, with the hint that a combinatorial description of this action can easily be defined using the "interval representation" of $\mathbf{\Delta}^{\mathrm{op}}$.

9.1.8. Digression (the interval representation). There is a faithful **interval representation** of the category $\mathbf{\Delta}_+^{\mathrm{op}}$ as a subcategory of $\mathbf{\Delta}$ in the form of a functor

$$\begin{tikzcd} \mathbf{\Delta}_+^{\mathrm{op}} \arrow[r, hook, "\mathrm{ir}"] & \mathbf{\Delta} \\ {[n-1]} \arrow[r, mapsto] \arrow[d, "\delta^i"'] & {[n]} \\ {[n]} \arrow[r, mapsto] & {[n+1]} \arrow[u, "\sigma^i"'] \arrow[d, "\delta^{i+1}"] \\ {[n+1]} \arrow[u, "\sigma^i"] \arrow[r, mapsto] & {[n+2]} \end{tikzcd}$$

whose image is the subcategory $\mathbf{\Delta}_{\perp,\top} := \mathbf{\Delta}_\perp \cap \mathbf{\Delta}_\top \subset \mathbf{\Delta}$ of simplicial operators that preserve both the top and bottom elements in each ordinal.

If we think of the elements of $[n]$ as labelling the lines in the graphical representation of an n-arrow, then the elements of $\mathbf{ir}[n] := [n+1]$ label the gaps, with the bottom element 0 relabeled as $-$ and the top element $n+1$ relabeled as $+$. If the elementary face operators δ^i and elementary degeneracy operators σ^i act by removing and duplicating the lines in a squiggle diagram, then the functors $\mathbf{ir}\,\delta^i$ and $\mathbf{ir}\,\sigma^i$ describe the corresponding actions on the gaps.

The graphical description of its n-arrows of $\mathcal{A}dj$ clearly exhibits a simplicial computad structure on a simplicial category we now more formally introduce following the outline of Observation 9.1.5:

9.1.9. Definition (the free homotopy coherent adjunction, as a simplicial computad). The **free homotopy coherent adjunction** $\mathcal{A}dj$ is the simplicial computad with

- two objects $+$ and $-$,
- whose n-arrows are strictly undulating squiggles on $n+1$ lines of Definition 9.1.6, oriented from right to left, with the right-hand starting position defining the domain object and the left-hand ending position defining the codomain object,
- with the simplicial operators acting as described in Lemma 9.1.7, and
- with horizontal composition defined by horizontal juxtaposition of squiggle diagrams,

which means that the atomic n-arrows are those squiggles each of whose interior undulations occurs between the lines 0 and n.

We now reconcile our two descriptions of the free homotopy coherent adjunction, comparing Definition 9.1.9 with Definition 9.1.4. Zaganidis discovered a similar proof in his PhD thesis [**106**, §2.3.3] that improves upon some aspects of the authors' original argument.

9.1.10. Proposition. *The simplicial computad $\mathcal{A}dj$ is isomorphic to the 2-category $\mathcal{A}dj$.*

Proof. We explain how the n-arrows in each of the four homs of the simplicial category $\mathcal{A}dj$ correspond to sequences of n composable arrows in the corresponding hom-categories $\mathbf{\Delta}_+$, $\mathbf{\Delta}_\perp$, $\mathbf{\Delta}_\top$, and $\mathbf{\Delta}_{\perp,\top}$ of the 2-category $\mathcal{A}dj$. Here it is most convenient to think of $\mathbf{\Delta}_\perp$, $\mathbf{\Delta}_\top$, and $\mathbf{\Delta}_{\perp,\top}$ as subcategories of $\mathbf{\Delta}$, the latter via the interval representation of Digression 9.1.8.

To easily visualize the isomorphism, shade the region under a strictly undulating squiggle on $n+1$ lines to define a topological space S embedded in the plane. In the case of a squiggle from $-$ to $-$, this shaded region includes both the left and right boundary regions of the squiggle diagram. For each $i \in [n]$, the connected components of the intersection S_i of the shaded region with the region above the labelled i define a finite linearly ordered set, ordered from left to right by the order in which the intervals defined as the intersection of each component with the line i appear on that line. These ordinals define the objects in the composable sequence of arrows in $\mathbf{\Delta}_+$ or one of its subcategory $\mathbf{\Delta}_{\perp,\top}$, $\mathbf{\Delta}_\perp$, or $\mathbf{\Delta}_\top$ corresponding to the squiggle diagram.[1] Formally, the ordinal representing the ith object in the sequence of n arrows counts the number of "maximal convex subsequences" of the sequence $\underline{a} = (a_0, \ldots, a_{w(\underline{a})})$ that encodes the squiggle; a **maximal convex subsequence** is a maximal sequence of consecutive entries a_j satisfying the condition $a_j \leq i$.

The intersections of the shaded region S with the regions above the lines $i = 0, \ldots, n$ define a nested sequence of subspaces

$$S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_{n-1} \hookrightarrow S_n.$$

Taking path components yields a composable sequence

$$\pi_0 S_0 \to \pi_0 S_1 \to \cdots \to \pi_0 S_{n-1} \to \pi_0 S_n.$$

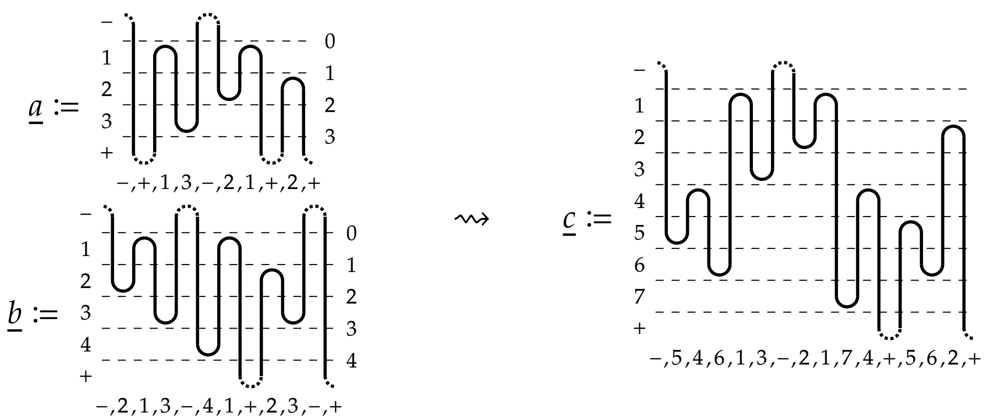
As explained above, each of the sets $\pi_0 S_i$ is linearly ordered, from left to right, by the positions in which each component of S_i intersects the line labeled i and the functions induced by the inclusions are order-preserving. This defines the composable sequence of arrows in $\mathbf{\Delta}_+$, $\mathbf{\Delta}_\perp$, $\mathbf{\Delta}_\top$, or $\mathbf{\Delta}_{\perp,\top}$ corresponding to a strictly undulating squiggle on $n+1$-lines.

For instance, suppose the shaded region under the squiggle diagram intersects the line labelled i in $m+1$ components, corresponding to the ordinal $[m]$ and similarly suppose the shaded region

[1] In the case of a squiggle from $+$ to $+$, it is possible that the squiggle diagram does not intersect the line labeled 0, in which case the subspace S_0 is empty; if this is the case, the squiggle may also fail to cross the next several lines. In each of the other three hom-categories, the ordinals defined by taking the connected components of the intersection of each line with the shaded region are non-empty because either the left or right boundary of the squiggle diagram is also shaded.

under the squiggle diagram intersects the line labelled $i+1$ in $k+1$ components, corresponding to the ordinal $[k]$. We identify the corresponding simplicial operator $\alpha\colon [m] \to [k]$ by taking the fiber over $t \in [k]$ to be the subset of shaded regions $s \in [m]$ above the ith line that belong to the same connected component as t in the shaded region above the line $i+1$. Formally, the element $t \in [k]$ represents a maximal convex subsequence of $\underline{a} = (a_0, \ldots, a_{w(\underline{a})})$ comprised of those $a_j \leq i+1$. This convex subsequence is partitioned into possibly smaller maximally convex subsequences satisfying the more restrictive condition $a_j \leq i$ and the elements $s \in [m]$ indexing these subsequences form the fiber of α over t. Note that if the domain of the squiggle is $-$, then the top element of each ordinal is necessarily preserved because the right-hand boundary of the squiggle diagram defines a connected shaded region; similarly, if the codomain of the squiggle is $-$, then the bottom element of each ordinal is preserved. The constructs a map from the simplicial computad $\mathcal{Adj}$ of Definition 9.1.9 to the 2-category $\mathcal{Adj}$ of Definition 9.1.4.

The converse map can be constructed by iterating a splicing operation that we now introduce. This splicing operation proves that each of the hom-spaces of the simplicial computad $\mathcal{Adj}$ satisfies a "strict Segal condition" that says that the set of $n+m$-arrows is isomorphic to the set of pairs of n-arrows and m-arrows whose last and first vertices coincide. In more detail, if $\underline{a}$ and $\underline{b}$ are strictly undulating squiggles on $n+1$ and $m+1$ lines that lie in the same hom-space with the property that the nth vertex of $\underline{a}$ coincides with the 0th vertex of $\underline{b}$, then these squiggle diagrams can be uniquely spliced to form a strictly undulating squiggle $\underline{c}$ on $n+m+1$ lines whose face spanned by the vertices $0, \ldots, n$ is $\underline{a}$ and whose face spanned by the vertices $n, \ldots, m+n$ is $\underline{b}$.



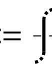
This splicing operation is defined graphically by separating the squiggle diagrams for $\underline{a}$ and for $\underline{b}$ into an ordered sequence of components by cutting the squiggle $\underline{a}$ each time it enters or leaves the gap marked "+," removing the dotted arcs at the bottom of the squiggle diagram, and cutting the squiggle $\underline{b}$ each time it enters or leaves the gap marked "−," removing the dotted arcs at the top of the depicted squiggle diagram. This requires two cuts for each occurrence of "+" or "−" in the interior of $\underline{a}$ or $\underline{b}$ respectively and one cut for each occurrence of "+" or "−" as the source or target of $\underline{a}$ or $\underline{b}$ respectively. The condition that the nth vertex of $\underline{a}$ coincides with the 0th vertex of $\underline{b}$ ensures that the numbers of cuts made to $\underline{a}$ (the number of times $\underline{a}$ intersects the nth line) and $\underline{b}$ (the number of times $\underline{b}$ intersects the 0th line) are the same. Now form $\underline{c}$ by sewing together these squiggle components in order to form the crossings of the line labeled "n."

By iterating the splicing operation, we can construct a strictly undulating squiggle on $n+1$ lines from a sequence of n composable arrows once we know how to encode a single 1-arrow of $\mathcal{Adj}$ as a strictly undulating squiggle on 2 lines. We give full details in the case of a 1-arrow from $+$ to $+$. A

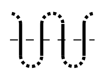
simplicial operator $\alpha \colon [m] \to [k]$ defines a strictly undulating squiggle on 2 lines from $+$ to $+$ by a sequence $\underline{a}$ with "$-$" occurring $m+1$ times (each occurrence of which corresponds to an intersection of the shaded region and the 0th line) and "$+$" occurring $k+2$ times (each consecutive pair bounding an intersection of the shaded region and the 1st line). The strings between each consecutive pair of $+$s correspond to the elements $i \in [k]$. If the fiber over i is empty, the sequence is "$+1+$." If the fiber contains a single element, the sequence is "$+-1-+$," and if the fiber contains $s > 1$ elements, the sequence is "$+-(1-)^{s-1}+$." ☐

Note that the planar orientation of a strictly undulating squiggle diagram makes the numerical labels for the lines, gaps, and the sequence of undulation points redundant. Going forward, we typically omit them.

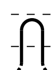
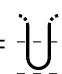
9.1.11. Example (adjunction data in $\mathcal{A}dj$). For later use we name some of the low-dimensional non-degenerate atomic arrows in $\mathcal{A}dj$. There exist just two atomic 0-arrows, which we call

$\underline{f} :=$  and $\underline{u} :=$ .

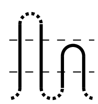
Since $\mathcal{A}dj$ is a simplicial computad, all of its other 0-arrows may be obtained as a unique alternating composite of these two, for example:

$\underline{fufuf} :=$ .

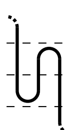
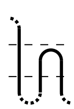
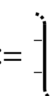
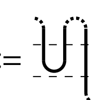
There are exactly two non-degenerate atomic 1-arrows in $\mathcal{A}dj$, these being:

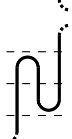
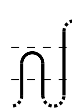
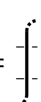
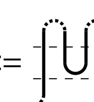
$\underline{\eta} :=$  and $\underline{\epsilon} :=$ .

Again, since $\mathcal{A}dj$ is a simplicial computad, all of its other 1-arrows are uniquely expressible as horizontal composites of the 1-arrows $\underline{\eta}$ and $\underline{\epsilon}$ with degenerated 1-arrows obtained from $\underline{f}$ and $\underline{u}$, such as for example

$\underline{uf\eta} :=$  and $\underline{\epsilon\epsilon} :=$ .

There exist two non-degenerate atomic 2-arrows with minimal width, whose faces are easily computed:

$\underline{\alpha} :=$  $\underline{\alpha} \cdot \delta^2 := \underline{f\eta} :=$  $\underline{\alpha} \cdot \delta^1 := \underline{f} :=$  $\underline{\alpha} \cdot \delta^0 := \underline{\epsilon f} :=$ 

$\underline{\beta} :=$  $\underline{\beta} \cdot \delta^2 := \underline{\eta u} :=$  $\underline{\beta} \cdot \delta^1 := \underline{u} :=$  $\underline{\beta} \cdot \delta^0 := \underline{u\epsilon} :=$ 

For each width $w \geq 3$, there exist exactly two non-degenerate atomic 2-arrows, the description of which we leave to Exercise 9.1.v. There are countably many atomic non-degenerate n-arrows in each dimension $n \geq 2$, which we shall partially enumerate in the proof of Proposition 9.2.11.

In the "homotopy 2-category" of the simplicial category $\mathcal{A}dj$, which is isomorphic to the 2-category $\mathcal{A}dj$, the atomic 2-arrows $\underline{\alpha}$ and $\underline{\beta}$ witness the triangle equalities, proving that this 2-category contains an adjunction $(\underline{f}, \underline{u}, \underline{\eta} \colon \mathrm{id}_+ \Rightarrow \underline{uf}, \underline{\epsilon} \colon \underline{fu} \Rightarrow \mathrm{id}_-)$.

9.1.12. Lemma. *The simplicial computad $\mathcal{A}dj$ contains a distinguished adjunction*

$$\begin{tikzcd} + \arrow[r, bend left, "\underline{f}"] \arrow[r, phantom, "\perp"] & - \arrow[l, bend left, "\underline{u}"] \end{tikzcd}$$

with unit $\underline{\eta}\colon \mathrm{id}_+ \to \underline{uf} \in \mathcal{A}dj(+,+)$ *and counit* $\underline{\epsilon}\colon \underline{fu} \to \mathrm{id}_- \in \mathcal{A}dj(-,-)$.

Proof. By Proposition 9.1.10 this follows from Lemma 9.1.2 though the reader may prefer to prove this result directly. ☐

We can now prove the universal property of the free adjunction $\mathcal{A}dj$ claimed by Schanuel and Street.

9.1.13. Proposition. *For any 2-category $\mathcal{C}$, 2-functors $\mathcal{A}dj \to \mathcal{C}$ correspond to adjunctions in the 2-category $\mathcal{C}$.*

Proof. Lemma 9.1.2 identifies an adjunction in $\mathcal{A}dj$ that we denote in the notation of Lemma 9.1.12 by $(\underline{f},\underline{u},\underline{\eta}\colon \mathrm{id}_+ \Rightarrow \underline{uf}, \underline{\epsilon}\colon \underline{fu} \Rightarrow \mathrm{id}_-)$. A 2-functor $\mathcal{A}dj \to \mathcal{C}$ carries this data to an adjunction in the 2-category $\mathcal{C}$.

Conversely, we suppose that $\mathcal{C}$ is equipped with an adjunction $(f,u,\eta\colon \mathrm{id}_B \Rightarrow uf, \epsilon\colon fu \Rightarrow \mathrm{id}_A)$. To construct a 2-functor $\mathcal{A}dj \to \mathcal{C}$ it suffices, because the embedding $2\text{-}\mathcal{C}at \hookrightarrow s\mathcal{S}et\text{-}\mathcal{C}at$ is fully faithful, to consider both 2-categories as simplicial categories via Lemma 9.1.3 and instead define a simplicial functor $\mathcal{A}dj \to \mathcal{C}$; see Exercise 9.1.i. Now since $\mathcal{A}dj$ is a simplicial computad and the homs of $\mathcal{C}$ are 2-coskeletal, it suffices to define a simplicial functor $\mathrm{sk}_2\,\mathcal{A}dj \to \mathcal{C}$ from the subcomputad of $\mathcal{A}dj$ generated by its atomic 0-, 1-, and 2-arrows. This functor is given by the mapping

- $+ \mapsto B$ and $- \mapsto A$ on objects,
- $\underline{f} \mapsto f$ and $\underline{u} \mapsto u$ on atomic 0-arrows,
- $\underline{\eta} \mapsto \eta$ and $\underline{\epsilon} \mapsto \epsilon$ on atomic non-degenerate 1-arrows,

and for each of the atomic non-degenerate 2-arrows of Exercise 9.1.v we must, by Lemma 9.1.3, verify that the 2-cells in $\mathcal{C}$ defined by their boundaries compose vertically as indicated.

For $\underline{\alpha}$ and $\underline{\beta}$, the unique non-degenerate atomic 2-arrows of minimal width 3, the required composition relations are

$$\epsilon f \cdot f\eta = \mathrm{id}_f \qquad \text{and} \qquad u\epsilon \cdot \eta u = \mathrm{id}_u,$$

which hold by the triangle equalities for the adjunction in $\mathcal{C}$. For the atomic 2-arrows of odd width $2r+1$, the required composition relations are the "higher order triangle equalities"

$$\epsilon^r f \cdot f\eta^r = \mathrm{id}_f \qquad \text{and} \qquad u\epsilon^r \cdot \eta^r u = \mathrm{id}_u,$$

which can easily be seen to hold by depicting the left-hand expressions as pasting diagrams.[2] For the atomic 2-arrows of even with $2r$, the required composites are closely related "higher order triangle equalties"

$$\epsilon^{r+1} \cdot f\eta^r u = \epsilon \qquad \text{and} \qquad u\epsilon^r \cdot \eta^{r+1} = \eta,$$

which again can easily be seen to hold by depicting the left-hand expressions as pasting diagrams. ☐

[2]Here η^r refers to the horizontal composite of r copies of the unit and the "$\cdot$" expresses a vertical composite of a whiskered copy of this with a whiskered copy of ϵ^r, the horizontal composite of r-copies of the counit. The 1-cell codomain of the former and 1-cell domain of the latter fit together like a cartoon depiction of a closed mouth of pointy teeth.

Motivated by this result and the fact that the simplicial category $\mathcal{A}dj$ is a simplicial computad, we use it to define a notion of *homotopy coherent adjunction* in any quasi-categorically enriched category, and in particular in any ∞-cosmos.

9.1.14. DEFINITION. A **homotopy coherent adjunction** in a quasi-categorically enriched category $\mathcal{K}$ is a simplicial functor $\mathcal{A}dj \to \mathcal{K}$. Explicitly, it picks out:

- a pair of objects $A, B \in \mathcal{K}$
- together with four homotopy coherent diagrams

$$\mathbf{\Delta}_+ \to \mathsf{Fun}(B,B), \quad \mathbf{\Delta}_+^{\mathrm{op}} \to \mathsf{Fun}(A,A), \quad \mathbf{\Delta}_\top \to \mathsf{Fun}(A,B), \quad \mathbf{\Delta}_\perp \to \mathsf{Fun}(B,A) \tag{9.1.15}$$

  that are functorial with respect to the composition action of $\mathcal{A}dj$.

The 0- and 1-dimensional data of the these diagrams has the form displayed in (9.0.1), (9.0.2), and (9.0.3). We interpret the homotopy coherent diagrams (9.1.15) as defining homotopy coherent versions of the bar and cobar resolutions of the adjunction $(f \dashv u, \eta, \epsilon)$.

Exercises.

9.1.i. EXERCISE. Prove that the embedding $2\text{-}\mathcal{C}at \hookrightarrow s\mathcal{S}et\text{-}\mathcal{C}at$ defined by Lemma 9.1.3 is fully faithful: prove that simplicial functors $\mathcal{A} \to \mathcal{B}$ between 2-categories define 2-functors.

9.1.ii. EXERCISE. Use Lemma 9.1.3 to prove that the homotopy coherent ω-simplex $\mathfrak{C}\Delta[\omega]$ is a 2-category.[3]

9.1.iii. EXERCISE. Describe the action of a general simplicial operator $\alpha\colon [m] \to [n]$ on a strictly undulating squiggle on $n+1$ lines represented by a sequence $\underline{a} = (a_0, a_1, \ldots, a_{r-1}, a_r)$ of "gaps" $a_i \in \{-, 1, \ldots, n, +\}$.

9.1.iv. EXERCISE. Give a graphical interpretation of the dualities

$$\mathcal{A}dj(-,-) \cong \mathcal{A}dj(+,+)^{\mathrm{op}} \qquad \text{and} \qquad \mathcal{A}dj(-,+) \cong \mathcal{A}dj(+,-)^{\mathrm{op}}$$

of Definition 9.1.4.

9.1.v. EXERCISE.

(i) Describe the non-degenerate atomic 2-arrows of $\mathcal{A}dj$ and compute their faces and compare your results with the composition relations appearing in the last paragraph of the proof of Proposition 9.1.13.

(ii) Describe the degenerate atomic 2-arrows of $\mathcal{A}dj$ and "compute" their faces.

9.1.vi. EXERCISE. Use the graphical calculus presented in Definition 9.1.9 to verify the following observation of Karol Szumiło: the simplicial category $\mathcal{A}dj$ is isomorphic to the Dwyer-Kan hammock localization [33] of the category consisting of two objects $+$ and $-$ and a single non-identity arrow $+ \xrightarrow{\sim} -$ that is a weak equivalence.

9.1.vii. EXERCISE. For any homotopy coherent adjunction as in Definition 9.1.14, define internal versions of monad resolution, the comonad resolution, the bar resolution, and the comonad resolution

$$B \xrightarrow{(uf)^\bullet} B^{\mathbf{\Delta}_+}, \quad A \xrightarrow{(fu)_\bullet} A^{\mathbf{\Delta}_+^{\mathrm{op}}}, \quad A \xrightarrow{\text{bar}} B^{\mathbf{\Delta}_\top}, \quad B \xrightarrow{\text{cobar}} A^{\mathbf{\Delta}_\perp},$$

[3] In light of Lemma 9.1.3, Theorem 6.4 of [78] proves more generally that the simplicial computads defined as free resolutions of strict 1-categories are always 2-categories.

and explore the relationships between these functors.[4]

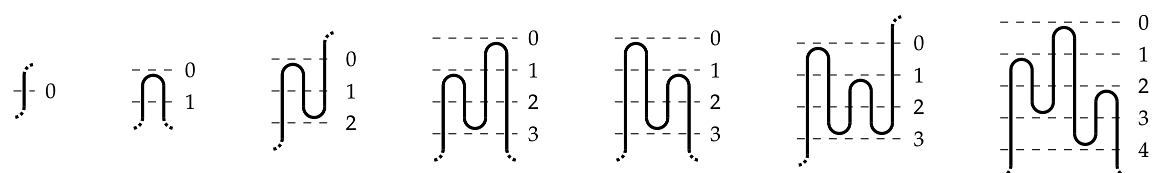
9.2. Homotopy coherent adjunction data

Any homotopy coherent adjunction in an ∞-cosmos or general quasi-categorically enriched category has an underlying adjunction in its homotopy 2-category. Remarkably, this low-dimensional adjunction data may always be extended to give a full homotopy coherent adjunction by repeated invoking the universal property of the unit, as expressed in Proposition 9.3.2. In this section, we filter the free homotopy coherent adjunction $\mathcal{A}dj$ by a sequence of "parental" subcomputads, which must contain, for each atomic n-arrow with codomain "$-$," its "fillable parent," the atomic $n+1$-arrow with codomain "$+$" obtained by whiskering with $\underline{u}$ and the "precomposing" with $\underline{\eta}$.

In §9.3 we then use this filtration to prove that any adjunction in an ∞-cosmos — or more precisely any diagram indexed by a parental subcomputad — extends to a homotopy coherent adjunction. Our proof is essentially constructive, enumerating the choices necessary to make each stage of the extension. In §9.4, we give precise characterizations of the homotopical uniqueness of such extensions, proving that the appropriate spaces of extensions are contractible Kan complexes. Via the 2-categorical self-duality of $\mathcal{A}dj$ described in Remark 9.2.12, there is a dual proof that instead exploits the universal property of the counit, the main steps of which are alluded to in the exercises.

Our proof that any adjunction extends to a homotopy coherent adjunction inductively specifies the data in the image of a homotopy coherent adjunction by choosing fillers for horns corresponding to "fillable" arrows.

9.2.1. Definition. An arrow $\underline{b}$ of $\mathcal{A}dj$ is **fillable** if
- it is non-degenerate and atomic,
- its codomain $b_0 = +$, and
- $b_i \neq b_1$ for $i > 1$.



Write $\mathrm{Fill}_n \subset \mathrm{Atom}_n$ for the subset of fillable n-arrows of the subset of atomic and non-degenerate n-arrows.

9.2.2. Definition (distinguished faces of fillable arrows). On account of the graphical calculus, we refer to $h(\underline{b}) \coloneqq b_1 - 1$, an integer in $\{0, \ldots, n-1\}$ that labels the line immediately above the position of the left-most turn-around, as the **height** of the fillable arrow $\underline{b}$.[5] The fillability of $\underline{b}$ implies that no "tautening" is required in computing the distinguished codimension-one face $\underline{b} \cdot \delta^{h(\underline{b})}$, which then is non-degenerate and has the same width as $\underline{b}$. Further analysis of this face differs by case:

---

[4] For instance, there is a commutative diagram $\begin{array}{ccc} A & \xrightarrow{\text{bar}} & B^{\Delta_\top} \\ {\scriptstyle u}\downarrow & & \downdownarrows{\scriptstyle \text{res}} \\ B & \xrightarrow[(uf)^\bullet]{} & B^{\Delta_+} \end{array}$.

[5] The unique fillable 0-arrow $\underline{u}$ behaves somewhat differently, but nonetheless it is linguistically convenient to include it among the fillable arrows.

- **case $h(\underline{b}) > 0$:** The face $\underline{b} \cdot \delta^{h(\underline{b})}$ is non-degenerate and atomic, but it is not fillable, since non-degeneracy implies that there is some $i > 1$ with $b_i = h(\underline{b})$, whence the entries of $\underline{b} \cdot \delta^{h(\underline{b})}$ at 1 and i both equal $h(\underline{b})$.
- **case $h(\underline{b}) = 0$:** The face $\underline{b} \cdot \delta^{h(\underline{b})}$ decomposes as $\underline{u} \cdot \underline{a}$, where $\underline{a}$ is non-degenerate, atomic, has width one less than the width of $\underline{b}$, and has codomain $-$.

We write

$$\underline{b}^{\circ} := \begin{cases} \underline{b} \cdot \delta^{h(\underline{b})} & h(\underline{b}) > 0 \\ \underline{a} & h(\underline{b}) = 0. \end{cases}$$

for the non-degenerate atomic $h(\underline{b})$th face in the positive height case and for the non-degenerate atomic factor of the 0th face $\underline{ua}$ in the height 0 case and refer to the atomic $n-1$-arrow defined in either case as the **distinguished face** of the fillable arrow.

We now argue that any non-degenerate and atomic n-arrow of $\mathcal{A}dj$ that is *not* fillable arises as the distinguished face of a unique fillable $n+1$-arrow in the form of the first case just described when its codomain is $+$ and in the form of the second case just described when its codomain is $-$.

9.2.3. Lemma (identifying fillable parents).

*(i) If $\underline{b}$ is a non-degenerate and atomic n-arrow of $\mathcal{A}dj$ with codomain $+$ that is not fillable then it is the codimension-one face of exactly two fillable $(n+1)$-arrows with the same width, both of which has $\underline{b}$ as its b_1th face: one which has height b_1 that we refer to as its **fillable parent** and denote by $\underline{b}^{\dagger}$ and the other which has height $b_1 - 1$.*

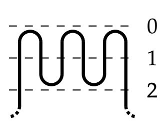
*(ii) If $\underline{a}$ is a non-degenerate and atomic n-arrow of $\mathcal{A}dj$ with codomain $-$, then the composite arrow $\underline{ua}$ is a codimension-one face of exactly one fillable $(n+1)$-arrow, which we call the **fillable parent** of $\underline{a}$ and denote by $\underline{a}^{\dagger}$. The fillable parent $\underline{a}^{\dagger}$ has width one greater than the width of $\underline{a}$, has height 0, and has $\underline{ua}$ as its 0th face.*

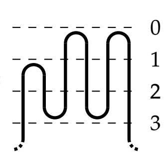
Together, these cases define a "fillable parent/distinguished face" bijection

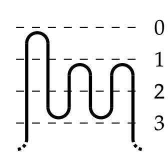
$$\mathrm{Atom}_n \backslash \mathrm{Fill}_n \underset{(-)^{\circ}}{\overset{(-)^{\dagger}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \mathrm{Atom}_{n+1}$$

between fillable $n+1$-arrows and non-degenerate atomic n-arrows which are not fillable.

Proof. For 9.2, if $\underline{b}$ is a non-degenerate atomic n-arrow with domain $+$ that is not fillable, then it must be the case that $b_i = b_1$ for some $i > 1$. This arrow is then a codimension-one face of the two atomic $n+1$-arrows that are formed by inserting an extra line into the gap labeled b_1 separate the entry b_1 from the other turn-arounds that occur in the same gap; the $n+1$-arrows obtained in this way will then clearly have $\underline{b}$ as their b_1th face. There are exactly two ways to do this, as illustrated below:

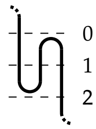
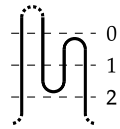
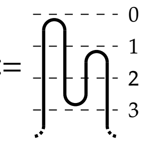
$\underline{b} :=$ 

$\rightsquigarrow$ $\underline{b}^{\dagger} :=$ 

or 

For (ii), given a non-degenerate atomic n-arrow $\underline{a}$ with codomain $-$, there is a unique way to make $\underline{ua}$ into the face of an atomic $n+1$-arrow with domain $+$ whose width is only one greater: by adding

an extra line in the upper-most space above the squiggle diagram.

$\underline{a} :=$  $\rightsquigarrow$ $\underline{ua} :=$  $\rightsquigarrow$ $\underline{a}^\dagger :=$  $\square$

9.2.4. Definition. A subcomputad $\mathcal{A}$ of $\mathcal{A}dj$ is **parental** if it contains at least one non-identity arrow and satisfies the condition:

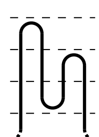
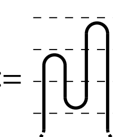
- if $\underline{c}$ is a non-degenerate atomic arrow in $\mathcal{A}$, then either it is fillable or its fillable parent $\underline{c}^\dagger$ is also in $\mathcal{A}$.

The condition implies that any parental subcomputad $\mathcal{A} \subset \mathcal{A}dj$ contains at least one fillable arrow. The last vertex of any fillable arrow has the form $\underline{ua}$ for some 0-arrow $\underline{a}$, so it follows that the 0-arrow $\underline{u}$ is contained in any parental subcomputad.

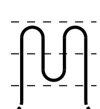
Recall from Definition 6.1.8, that any collection of arrows S in a computad generates a minimal subcomputad $\overline{S}$.

9.2.5. Example (notable parental subcomputads).

- The unique fillable 0-arrow $\underline{u}$ generates the **minimal parental subcomputad** $\overline{\{\underline{u}\}} \subset \mathcal{A}dj$ containing only $\underline{u}$ and its degenerate copies.
- The unit 1-arrow $\underline{\eta}$ is fillable and the subcomputad $\overline{\{\underline{\eta}\}} \subset \mathcal{A}dj$ it generates has $\underline{u}$, $\underline{f}$, and $\underline{\eta}$ as its only non-degenerate atomic arrows. Of these, $\underline{u}$ and $\underline{\eta}$ are parental and $\underline{\eta}$ is the fillable parent of $\underline{f}$, so this subcomputad is parental.
- The triangle identity witness $\underline{\beta}$ of Example 9.1.11 is fillable and the subcomputad $\overline{\{\underline{\beta}\}} \subset \mathcal{A}dj$ it generates has $\underline{u}$, $\underline{f}$, $\underline{\eta}$, $\underline{\epsilon}$, and $\underline{\beta}$ as its only non-degenerate atomic arrows. Since $\underline{\beta}$ is the fillable parent of $\underline{\epsilon}$, this subcomputad is parental.
- The pair of fillable 3-arrows

  $\underline{\omega} :=$  and $\underline{\tau} :=$ 

  generate a subcomputad $\overline{\{\underline{\omega}, \underline{\tau}\}} \subset \mathcal{A}dj$ that has $\underline{u}$, $\underline{f}$, $\underline{\eta}$, $\underline{\epsilon}$, both triangle identity witnesses $\underline{\beta}$ and $\underline{\alpha}$ of Example 9.1.11, and $\underline{\nu} := \underline{\omega} \cdot \delta^1 = \underline{\tau} \cdot \delta^1$ as its only non-degenerate atomic arrows. Since $\underline{\omega}$ is the fillable parent of $\underline{\alpha}$ and $\underline{\tau}$ is the fillable parent of

  $\underline{\nu} :=$ 

  this subcomputad is parental.
- $\mathcal{A}dj$ is trivially a parental subcomputad of itself.

9.2.6. Non-Example.

- The subcomputad $\overline{\{\underline{\epsilon}\}}$ has $\underline{u}$, $\underline{f}$, and $\underline{\epsilon}$ as its only non-degenerate atomic arrows. Since both $\underline{f}$ and $\underline{\epsilon}$ are missing their fillable parents, this subcomputad is not parental.

- The subcomputad $\overline{\{\underline{\eta},\underline{\epsilon}\}}$ that has $\underline{u}$, $\underline{f}$, $\underline{\eta}$, and $\underline{\epsilon}$ as its only non-degenerate atomic arrows still fails to be parental since the fillable parent of $\underline{\epsilon}$ is missing.
- The subcomputad $\overline{\{\underline{\beta},\underline{\alpha}\}}$ that has $\underline{u}$, $\underline{f}$, $\underline{\eta}$, $\underline{\epsilon}$, and both triangle identity witnesses $\underline{\beta}$ and $\underline{\alpha}$ as its atomic non-degenerate arrows is not parental since the fillable parent $\underline{\omega}$ of $\underline{\alpha}$ is missing.

These examples establish a chain of parental subcomputad inclusions

$$\overline{\{\underline{u}\}} \subset \overline{\{\underline{\eta}\}} \subset \overline{\{\underline{\beta}\}} \subset \overline{\{\underline{\omega},\underline{\tau}\}} \subset \mathcal{A}dj.$$

Our aim in the remainder of this section is to filter a general parental subcomputad inclusion $\mathcal{A} \subset \mathcal{A}'$ as a countable tower of parental subcomputad inclusions, with each sequential inclusion presented as the pushout of an explicit simplicial subcomputad inclusion. The subcomputad inclusions $\mathbb{2}[\Lambda^k[n]] \hookrightarrow \mathbb{2}[\Delta[n]]$ from Example 6.1.5 will feature, where $\Lambda^k[n]$ is an inner horn. A second family of inclusions will also be needed, which we now introduce.

9.2.7. Definition. For any simplicial set U, let $\mathbb{3}[U]$ denote the simplicial category whose three non-trivial homs are displayed in the following cartoon:

$$\begin{tikzcd} & - \arrow[dr, "\mathbb{1}"] & \\ \top \arrow[ur, "U"] \arrow[rr, "\mathbb{1}\star U"'] & & + \end{tikzcd}$$

That is $\mathbb{3}[U]$ has objects "$\top$", "$-$", and "$+$" and non-trivial hom-sets

$$\mathbb{3}[U](\top,-) \coloneqq U\,, \qquad \mathbb{3}[U](-,+) \coloneqq \mathbb{1}\,, \qquad \mathbb{3}[U](\top,+) \coloneqq \mathbb{1} \star U,$$

and whose only non-trivial composition operation is defined by the canonical inclusion

$$\begin{tikzcd} \mathbb{3}[U](-,+) \times \mathbb{3}[U](\top,-) \arrow[r, hook] \arrow[d, phantom, "\wr\Vert"] & \mathbb{3}[U](\top,+) \arrow[d, phantom, "\wr\Vert"] \\ U \arrow[r, hook] & \mathbb{1} \star U \end{tikzcd}$$

Here we define the endo hom-spaces to contain only the respective identities and the remaining hom-spaces to be empty.

9.2.8. Lemma. *A simplicial functor $\mathbb{3}[U] \to \mathcal{K}$ is uniquely determined by the data:*
- *a pair of 0-arrows $u\colon A \to B$ and $b\colon X \to B$*
- *a cone with summit U over the cospan*

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, dashed] & {}^{b/}\mathsf{Fun}(X,B) \arrow[d, two heads, "\pi"] \\ \mathsf{Fun}(X,A) \arrow[r, "u_*"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

or equivalently a map $U \to {}^{b/}u_*$ whose codomain is its pullback.

Proof. The objects X, A, and B are the images of $\top$, $-$, and $+$ respectively. The map $u\colon A \to B$ is the image of the unique 0-arrow from $-$ to $+$. Simplicial functoriality then demands the specification

of the vertical maps below making the square commute

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, dashed] & \mathbb{1} \star U \arrow[d, dashed] \\ \mathsf{Fun}(X, A) \arrow[r, "u_*"] & \mathsf{Fun}(X, B) \end{tikzcd}$$

By the join $\dashv$ slice adjunction of Proposition 4.2.5, the simplicial map $\mathbb{1} \star U \to \mathsf{Fun}(X, B)$ may be defined by specifying a 0-arrow $b\colon X \to B$, the image of the cone point of $\mathbb{1} \star U$, together with a map $U \to {}^{b/}\mathsf{Fun}(X, B)$. Now the above commutative square transposes to the one of the statement. $\square$

Similarly a simplicial functor $\mathbb{2}[U] \to \mathcal{K}$ is uniquely determined by the data $U \to \mathsf{Fun}(X, A)$ of a simplicial map valued in one of the functor spaces of $\mathcal{K}$. In particular, simplicial functors $\mathbb{2}[\Delta[n]] \to \mathcal{K}$ correspond to n-arrows in $\mathcal{K}$. $\square$

9.2.9. Notation.

- For each fillable n-arrow $\underline{b}$ of positive height, let $F_{\underline{b}}\colon \mathbb{2}[\Delta[n]] \to \mathcal{A}dj$ be the simplicial functor classified by the n-arrow $\underline{b}$ of $\mathcal{A}dj$.
- For each fillable n-arrow $\underline{b}$ of height 0, let $F_{\underline{b}}\colon \mathbb{3}[\Delta[n-1]] \to \mathcal{A}dj$ be the simplicial functor defined on objects by $- \mapsto -$, $+ \mapsto +$, and $\top \mapsto b_{w(\underline{b})}$, the domain of $\underline{b}$, and on the three non-trivial homs by

$$\mathbb{3}[\Delta[n-1]](\top, -) \cong \Delta[n-1] \xrightarrow{b^\circ} \mathcal{A}dj(b_{w(\underline{b})}, -) \qquad \mathbb{3}[\Delta[n-1]](-, +) \cong \Delta[0] \xrightarrow{u} \mathcal{A}dj(-, +)$$

$$\mathbb{3}[\Delta[n-1]](\top, +) \cong \Delta[n] \xrightarrow{b} \mathcal{A}dj(b_{w(\underline{b})}, +)$$

9.2.10. Lemma (extending parental subcomputads). *Suppose $\mathcal{A} \subset \mathcal{A}dj$ is a parental subcomputad and $\underline{b}$ is a fillable n-arrow of height k that is not a member of $\mathcal{A}$ but whose faces $\underline{b} \cdot \delta^i$ are in $\mathcal{A}$ for all $i \neq k$. Then the subcomputad $\mathcal{A}' := \overline{\{\mathcal{A}, \underline{b}\}}$ generated by $\mathcal{A}$ and $\underline{b}$ is defined by the pushout on the left below in the case $k > 0$ and on the right below in the case $k = 0$*

$$\begin{tikzcd} \mathbb{2}[\Lambda^k[n]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[d, hook] \\ \mathbb{2}[\Delta[n]] \arrow[r, "F_{\underline{b}}"'] & \mathcal{A}' \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{3}[\partial\Delta[n-1]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[d, hook] \\ \mathbb{3}[\Delta[n-1]] \arrow[r, "F_{\underline{b}}"'] & \mathcal{A}' \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

and in both cases $\mathcal{A}'$ is again a parental subcomputad.

Proof. Because $\underline{b}$ is the fillable parent of its distinguished face $\underline{b}^\circ$, this non-degenerate, atomic, non-fillable $(n-1)$-arrow cannot be a member of the parental subcomputad $\mathcal{A}$. Since the other faces of $\underline{b}$ are assumed to belong to $\mathcal{A}$, $\underline{b}$ and $\underline{b}^\circ$ are the only two atomic arrows that are in $\mathcal{A}'$ but not in $\mathcal{A}$. Since the first is fillable and the second has the first as its fillable parent, it is clear that the subcomputad $\mathcal{A}'$ is again parental.

To verify the claimed pushouts, we consider what is required to extend a simplicial functor $F\colon \mathcal{A} \to \mathcal{K}$ to a simplicial functor $F\colon \mathcal{A}' \to \mathcal{K}$. In the positive height case, all that is needed for such an extension is an n-arrow f in $\mathcal{K}$ with the property that $f \cdot \delta^i = F(\underline{b} \cdot \delta^i)$ for all $i \neq k$, which may be specified

by a simplicial functor $f\colon \mathbb{2}[\Delta[n]] \to \mathcal{K}$ that makes the following square commute:

$$\begin{tikzcd} \mathbb{2}[\Lambda^k[n]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[d, "F"] \\ \mathbb{2}[\Delta[n]] \arrow[r, dashed, "f"'] & \mathcal{K} \end{tikzcd}$$

If the height of $\underline{b}$ is zero, then both its 0th face $\underline{b}\cdot\delta^0 = \underline{u}\cdot\underline{a}$ and its atomic non-degenerate factor $\underline{a}$ are missing from $\mathcal{A}$. So to extend a simplicial functor $F\colon \mathcal{A} \to \mathcal{K}$ to a simplicial functor $F\colon \mathcal{A}' \to \mathcal{K}$ requires an $(n-1)$ arrow g and an n-arrow f in $\mathcal{K}$ so that $g\cdot\delta^i = F(\underline{a}\cdot\delta^i)$ for all $i \in [n-1]$ and $f\cdot\delta^i = F(\underline{b}\cdot\delta^i)$ for all $i \neq 0 \in [n]$, so that $f\cdot\delta^0 = F\underline{u}\cdot g$. By Lemma 9.2.8 this data may be specified by a simplicial functor $f\colon \mathbb{3}[\Delta[n-1]] \to \mathcal{K}$ that makes the following square commute:

$$\begin{tikzcd} \mathbb{3}[\partial\Delta[n-1]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[d, "F"] \\ \mathbb{3}[\Delta[n-1]] \arrow[r, dashed, "f"'] & \mathcal{K} \end{tikzcd}$$

□

9.2.11. Proposition. *Any inclusion $\mathcal{A} \hookrightarrow \mathcal{A}'$ of parental subcomputads of $\mathcal{A}dj$ may be filtered as a countable tower of parental subcomputad inclusions*

$$\mathcal{A} = \mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \cdots \hookrightarrow \bigcup_{i\geq 0} \mathcal{A}_i = \mathcal{A}'$$

in such a way that for each $i \geq 1$ there is a finite non-empty set S_i of fillable arrows that are not themselves contained in but which have all faces except the distinguished one contained in $\mathcal{A}_{i-1}$ so that the parental subcomputad $\mathcal{A}_i$ is generated by $\mathcal{A}_{i-1} \cup S_i$. Hence, the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}'$ may be expressed as a countable composite of inclusions which are constructed by pushouts of the form

$$\begin{tikzcd} \left(\coprod\limits_{\substack{\underline{b}\in S_i \\ h(\underline{b})>0}} \mathbb{2}[\Lambda^{h(\underline{b})}[\dim(\underline{b})]]\right) \sqcup \left(\coprod\limits_{\substack{\underline{b}\in S_i \\ h(\underline{b})=0}} \mathbb{3}[\partial\Delta[\dim(\underline{b})]]\right) \arrow[r, "\langle F_{\underline{b}}\rangle_{\underline{b}\in S_i}"] \arrow[d, hook] & \mathcal{A}_{i-1} \arrow[d, hook] \\ \left(\coprod\limits_{\substack{\underline{b}\in S_i \\ h(\underline{b})>0}} \mathbb{2}[\Delta[\dim(\underline{b})]]\right) \sqcup \left(\coprod\limits_{\substack{\underline{b}\in S_i \\ h(\underline{b})=0}} \mathbb{3}[\Delta[\dim(\underline{b})]]\right) \arrow[r, "\langle F_{\underline{b}}\rangle_{\underline{b}\in S_i}"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathcal{A}_i \end{tikzcd}$$

Proof. Let S denote the set of fillable arrows in $\mathcal{A}'$ which are not in $\mathcal{A}$ and let $S_{w,n,k}$ denote the subset of arrows with width w, dimension n, and height k. Note that any non-degenerate arrow of $\mathcal{A}dj$ must have dimension strictly less than its width, and there are only finitely many non-degenerate arrows of any given width. The height is strictly less than the dimension so each set $S_{w,n,k}$ is finite and if non-empty we must have $k < n < w$. Now we order the triples (w, n, k) that index non-empty subsets of fillable arrows lexicographically by increasing width, increasing dimension, and *decreasing* height and let $S_i \coloneqq S_{w_i,n_i,k_i}$ denote subset of fillable arrows in the ith triple in this ordering for $i \geq 1$.

Let $\mathcal{A}_i$ be the subcomputad of $\mathcal{A}dj$ generated by $\mathcal{A} \cup (\bigcup_{j=1}^{i} S_i)$. By construction this family filters the inclusion of $\mathcal{A}$ into $\mathcal{A}' = \mathcal{A} \cup (\bigcup_i S_i)$.

We complete the proof by induction on the index starting from the parental subcomputad $\mathcal{A}_0 = \mathcal{A}$. For the inductive step, we must verify that all but the distinguished face of each $\underline{b} \in S_i \coloneqq S_{w_i,n_i,k_i}$ lie in $\mathcal{A}_{i-1}$. By iterating Lemma 9.2.10, it will follow that $\mathcal{A}_i$ is again parental.

By construction $\mathcal{A}_{i-1}$ is the smallest subcomputad of $\mathcal{A}dj$ which contains $\mathcal{A}$ and all of the fillable arrows of $\mathcal{A}'$ which have:

- width less that w_i or
- width w_i and dimension less than n_i or
- width w_i, dimension n_i, and height greater than k_i.

Each fillable arrow $\underline{b} \in S_i$ has width w_i, dimension n_i, and height k_i. We must show that its faces $\underline{b} \cdot \delta^j$ lie in $\mathcal{A}_{i-1}$ for each $j \neq k_i$, which we verify by a tedious but straightforward case analysis.

case $j \neq k_i+1$**:** If $j \neq k_i+1$ then since $j \neq k_i$ the line numbered j is not one of the ones separating the gap b_1 from the other entries of $\underline{b}$, which means that this entry will not be eliminated when computing the jth face. Consider the unique factorization

$$\underline{b} \cdot \delta^j = (\underline{b}^1 \cdot \alpha_1) \circ \cdots \circ (\underline{b}^r \cdot \alpha_r)$$

of the face $\underline{b} \cdot \delta^j$ into non-degenerate and atomic arrows of $\mathcal{A}'$ acted upon by degeneracy operators. By the analysis just given, $\underline{b}^1$ is fillable with width at most w_i, height k_i or $k_i - 1$ depending on whether $j > k_i + 1$ or $j < k_i$, and dimension less than n_i. Thus $\underline{b}^1$ it is contained in $\mathcal{A}_{i-1}$ by the hypothesis that this subcomputad contains all fillable arrows of width at most w_i and dimension less than n_i. Since b_1 has width at least 2, the other atomic factors have width less than or equal to $w_i - 2$. Thus each of these is either a fillable arrow with width at most $w_i - 2$, which means that it is in $\mathcal{A}_{i-1}$, or its fillable parent in $\mathcal{A}'$ has width at most $w_i - 1$, which means that this fillable parent is also in $\mathcal{A}_{i-1}$. As $\underline{b} \cdot \delta^j$ is a composite of degenerate images of arrows in $\mathcal{A}_{i-1}$ it too lies in $\mathcal{A}_{i-1}$.

case $j = k_i + 1$**:** Since $\underline{b} \notin \mathcal{A}_{i-1}$ and all parental subcomputads contain the unique fillable 0-arrow $\underline{u}$, we know that $\underline{b} \neq \underline{u}$ and so has width at least 2 and dimension at least 1. The only fillable arrow of width 2 is $\underline{\eta}$ which has depth 0 and face $\underline{\eta} \cdot \delta^1 = \mathrm{id}_+$, which certainly lies in $\mathcal{A}_{i-1}$, so we can safely assume that the width of $\underline{b}$ is at least 3 which forces the height k_i to be at most $n_i - 2$; otherwise $\underline{b}$ would not be atomic. Thus j is at most $n_i - 1$, which tells us that $\underline{b} \cdot \delta^j$ is again atomic. Now we have two subcases:

- **case** $b_2 = b_1 + 1$**:** In this case, the line $j = k_1 + 1 = b_1$ separates the gaps b_1 and b_2, so the face $\underline{b} \cdot \delta^j$ will have width at least two less than the width of $\underline{b}$. This face may also be degenerate but in any case the fillable parent of the non-degenerate arrow that it represents has width less than w_i and so this face is present in $\mathcal{A}_{i-1}$.
- **case** $b_2 > b_1+1$**:** In this case, the line $j = k_1+1 = b_1$ separates the gap b_1 from the gap immediately below it, which contains neither b_0 nor b_2. So the face $\underline{b} \cdot \delta^j$ has width w_i and is non-degenerate. Because $\underline{b}$ is non-degenerate, there is some $s > 2$ so that $b_s = b_1 + 1$, so $\underline{b} \cdot \delta^j$ is not fillable. Now Lemma 9.2.3 tells us that its fillable parent has width w_i, dimension n_i, and height $k_i + 1$. Thus, this fillable parent lies in $\mathcal{A}_{i-1}$ by our hypothesis that that $\mathcal{A}_{i-1}$ contains all fillable arrows of $\mathcal{A}'$ of width w_i, dimension n_i, and height greater than k_i, so $\underline{b} \cdot \delta^j$ must also be in there, as desired.

$\square$

9.2.12. Remark (a dual form). In the 2-category $\mathcal{A}dj^{\mathrm{co}}$, the 0-arrow $\underline{u}$ is left adjoint to the 0-arrow $\underline{f}$, with unit $\underline{\epsilon}$ and counit $\underline{\eta}$. The 2-functor $\mathcal{A}dj \to \mathcal{A}dj^{\mathrm{co}}$ that classifies this adjunction is an isomorphism. Via this duality, we could have introduced a variant notion of "fillable n-arrow" (a subset of those n-arrows with codomain $-$) and "parental subcomputad" of $\mathcal{A}dj$ so that every parental subcomputad contained the 0-arrow $\underline{f}$. Dualizing the argument of Example 9.2.5, the subcomputads $\overline{\{\underline{f}\}} \subset \overline{\{\underline{\epsilon}\}} \subset \overline{\{\underline{\alpha}\}}$ would all be parental, but the subcomputads of Example 9.2.5 would no longer be so.

Exercises.

9.2.i. Exercise. Give a graphical description of the dual fillable n-arrows discussed in Remark 9.2.12.

9.3. Building homotopy coherent adjunctions

We now employ Proposition 9.2.11 to prove that every adjunction in an ∞-cosmos $\mathcal{K}$ extends to a homotopy coherent adjunction $\mathcal{A}dj \to \mathcal{K}$ which carries the canonical adjunction in $\mathcal{A}dj$ to the chosen adjunction in $\mathcal{K}$. The data used to present an adjunction in a quasi-categorically enriched category $\mathcal{K}$ in the sense of Definition 2.1.1 — a pair of ∞-categories A and B, a pair of 0-arrows $u \in \mathsf{Fun}(A,B)$ and $f \in \mathsf{Fun}(B,A)$, a pair of 1-arrows $\eta\colon \mathrm{id}_B \to uf \in \mathsf{Fun}(B,B)$ and $\epsilon\colon fu \to \mathrm{id}_A \in \mathsf{Fun}(A,A)$, and witnesses to the triangle equalities given by a pair of 2-arrows

$$\begin{tikzcd} & ufu \arrow[dr, "u\epsilon"] \arrow[d, phantom, "\beta"] & \\ u \arrow[ur, "\eta u"] \arrow[rr, equal] & {} & u \end{tikzcd} \in \mathsf{Fun}(A,B) \qquad \begin{tikzcd} & fuf \arrow[dr, "\epsilon f"] \arrow[d, phantom, "\alpha"] & \\ f \arrow[ur, "f\eta"] \arrow[rr, equal] & {} & f \end{tikzcd} \in \mathsf{Fun}(B,A)$$

— determines a simplicial functor $T\colon \overline{\{\underline{\beta},\underline{\alpha}\}} \to \mathcal{K}$ whose domain is the subcomputad of $\mathcal{A}dj$ generated by the triangle identity 2-arrows. This functor then is defined by mapping $-$ to A and $+$ to B and acting on the non-degenerate and atomic arrows in the way suggested by their syntax: $T(\underline{u}) \coloneqq u$, $T(\underline{f}) \coloneqq f$, $T(\underline{\eta}) \coloneqq \eta$, $T(\underline{\epsilon}) \coloneqq \epsilon$, $T(\underline{\beta}) \coloneqq \beta$, and $T(\underline{\alpha}) \coloneqq \alpha$. In Theorem 9.3.4, we prove that it is always possible to extend the data $(f \dashv u, \eta, \epsilon)$ to a homotopy coherent adjunction $\mathcal{A}dj \to \mathcal{K}$, but unless β and α satisfy a coherence condition that will be described, doing so might require making a different choice for one of these triangle equality witnesses.

By Proposition 9.2.11, extensions along parental subcomputad inclusions may be built inductively from two types of extension problem, one of which involves attaching fillable arrows with positive height and the other of which involves attaching fillable arrows of height zero. We prove that both simplicial subcomputad extension problems can be solved relatively against any simplicial functor $P\colon \mathcal{K} \to \mathcal{L}$ between quasi-categorically enriched categories that defines a "local isofibration," meaning that the action of P on functor spaces is by isofibrations. This relative lifting result will be used in the proof of the homotopical uniqueness of such extensions in §9.4.

9.3.1. Lemma. *Let $P\colon \mathcal{K} \to \mathcal{L}$ be a simplicial functor that defines a local isofibration of quasi-categories. Then any lifting problem against the directed suspension of an inner horn inclusion*

$$\begin{tikzcd} \mathbb{2}[\Lambda^k[n]] \arrow[r] \arrow[d, hook] & \mathcal{K} \arrow[d, "P"] \\ \mathbb{2}[\Delta[n]] \arrow[r] \arrow[ur, dashed] & \mathcal{L} \end{tikzcd}$$

has a solution.

PROOF. The lifting problem of the statement is equivalent to asking for a lift of the inner horn inclusion

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(A,B) \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(PA,PB) \end{tikzcd}$$

against the action of P on some functor spaces of $\mathcal{K}$ and $\mathcal{L}$. By hypothesis this action is an isofibration of quasi-categories and so the claimed lift exists. □

9.3.2. PROPOSITION (the relative universal property of the unit). *If $\mathcal{K}$ has an adjunction $(f \dashv u, \eta)$, $P\colon \mathcal{K} \to \mathcal{L}$ is a local isofibration of quasi-categorically enriched categories, and if $T\colon \mathfrak{Z}[\partial\Delta[n]] \to \mathcal{K}$ is defined to carry the unique 0-arrow $- \to +$ to $u\colon A \to B$, the cone point of $\mathbb{1} \star \partial\Delta[n] \cong \Lambda^0[n]$ to $b\colon X \to B$ and the 1-arrow from 0 to 1 in $\Lambda^0[n]$ to $\eta b\colon b \to ufb$ for some $n \geq 1$, then any lifting problem*

$$\begin{tikzcd} \mathfrak{Z}[\partial\Delta[n]] \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[d, "P"] \\ \mathfrak{Z}[\Delta[n]] \arrow[r] \arrow[ur, dashed] & \mathcal{L} \end{tikzcd}$$

has a solution.

PROOF. By Lemma 9.2.8, the lifting problem of the statement translates to a lifting problem of the following form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "t"] \arrow[d, hook] & {}^{b/}u_* \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & {}^{Pb/}Pu_* \end{tikzcd}$$

where the upper horizontal map carries the initial vertex $0 \in \partial\Delta[n]$ to the object $\eta b \in {}^{b/}u_*$, which is initial in there by Proposition 4.1.4 applied to the adjunction between functor spaces

$$\begin{tikzcd} \mathsf{Fun}(X,B) \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\perp"] & & \mathsf{Fun}(X,A) \arrow[ll, bend left, "u_*"] \end{tikzcd}$$

and Appendix C, which proves that Joyal's slices are fibered equivalent to comma quasi-categories. The right-hand vertical is an isofibration that preserves this initial element, since it carries it to a component of the unit for the adjunction $(Pf \dashv Pu, P\eta)$ in $\mathcal{L}$. Lemma F.1.2 proven in Appendix F now proves that the desired lift exists. □

Our theorems about extensions to homotopy coherent adjunctions will follow as special cases of the following relative extension and lifting result for parental subcomputads containing the unit of an adjunction.

9.3.3. THEOREM. *Suppose $\underline{\overline{\{\eta\}}} \subset \mathcal{A} \subset \mathcal{A}' \subset \mathcal{A}dj$ are parental subcomputads. Then if $P\colon \mathcal{K} \to \mathcal{L}$ is a local isofibration between quasi-categorically enriched categories and if $\mathcal{K}$ has an adjunction $(f \dashv u, \eta)$ then*

we may solve any lifting problem

$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[d, "P"] \\ \mathcal{A}' \arrow[r] \arrow[ur, dashed] & \mathcal{L} \end{tikzcd}$$

so long as $T(\underline{f}) = f$, $T(\underline{u}) = u$, *and* $T(\underline{\eta}) = \eta$.

Proof. By Proposition 9.2.11 the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}'$ can be fillered as a sequential composite of pushouts of coproducts of maps of two basic forms, so it suffices to demonstrate that we may solve lifting problems of the following two forms

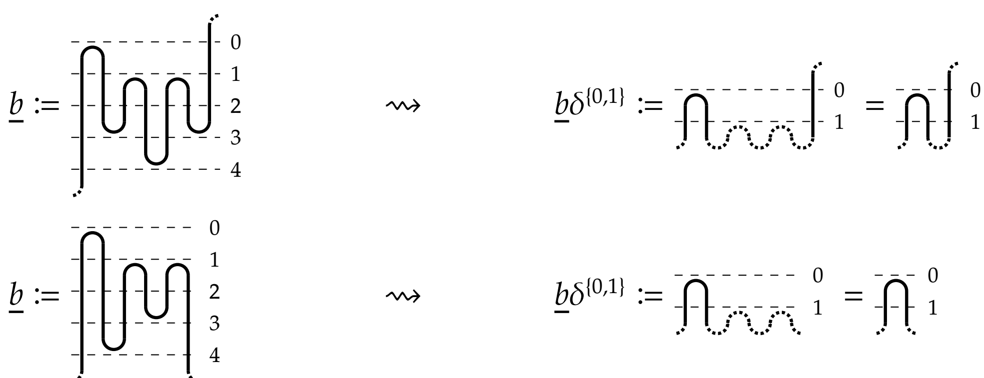
$$\begin{tikzcd} \mathbb{2}[\Lambda^k[n]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[d, "P"] \\ \mathbb{2}[\Delta[n]] \arrow[r, "F_{\underline{b}}"'] \arrow[urr, dashed] \arrow[ur, phantom, "\ulcorner"] & \mathcal{A}' \arrow[r] & \mathcal{L} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{3}[\partial\Delta[n-1]] \arrow[r, "F_{\underline{b}}"] \arrow[d, hook] & \mathcal{A} \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[d, "P"] \\ \mathbb{3}[\Delta[n-1]] \arrow[r, "F_{\underline{b}}"'] \arrow[urr, dashed] \arrow[ur, phantom, "\ulcorner"] & \mathcal{A}' \arrow[r] & \mathcal{L} \end{tikzcd}$$

for some fillable n-arrow $\underline{b} \in \mathcal{A}dj$, of positive height in the left-hand case and of height zero in the right-hand case. In the case of the left-hand lifting problem, such lifts exist immediately from Lemma 9.3.1.

In the case of the right-hand lifting problem, we must verify that the hypotheses of Proposition 9.3.2 are satisfied, which amounts to verifying that for any fillable n-arrow $\underline{b}$ of height zero, the functor $F_{\underline{b}} \colon \mathbb{3}[\Delta[n-1]] \to \mathcal{A}dj$ defined in Notation 9.2.9 in relavant part by the map

$$\mathbb{3}[\Delta[n-1]](\top, +) \cong \Delta[n] \xrightarrow{\underline{b}} \mathcal{A}dj(b_{w(\underline{b})}, +)$$

carries the 1-arrow from 0 to 1 in $\Delta[n]$ to a component of the unit in $\mathcal{A} \subset \mathcal{A}dj$. This is easily verified using the graphical calculus. The image of the 1-arrow from 0 to 1 under the map $\underline{b} \colon \Delta[n] \to \mathcal{A}dj(b_{w(\underline{b})}, +)$ is simply the initial edge of $\underline{b}$, which may be computed by removing all of the other lines from the strictly undulating squiggle diagram. By the specifications of a fillable arrow with height 0, there are only two possibilities, depending on whether the domain of $\underline{b}$ is $+$ or $-$, as illustrated by the following diagrams:



Thus we see that $\underline{b} \cdot \delta^{\{0,1\}} = \underline{\eta}\underline{u}$ or $\underline{b} \cdot \delta^{\{0,1\}} = \underline{\eta}$, both of which satisfy the conditions required for Proposition 9.3.2 to provide the desired lift. $\square$

9.3.4. Theorem (homotopy coherent adjunctions). *If $\mathcal{K}$ is a quasi-categorically enriched category containing an adjunction $(f \dashv u, \eta, \epsilon, \beta, \alpha)$:*

(i) There exists a simplicial functor $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ for which $\mathbb{A}(\underline{u}) = u$, $\mathbb{A}(\underline{f}) = f$, and $\mathbb{A}(\underline{\eta}) = \eta$.

(ii) There exists a simplicial functor $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ for which $\mathbb{A}(\underline{u}) = u$, $\mathbb{A}(\underline{f}) = f$, $\mathbb{A}(\underline{\eta}) = \eta$, $\mathbb{A}(\underline{\epsilon}) = \epsilon$, and $\mathbb{A}(\underline{\beta}) = \beta$.

(iii) If there exist a pair of 3-arrows ω and τ in $\mathcal{K}$

$$\omega \coloneqq \begin{tikzcd} & uf \arrow[d, "uf\eta"'] \arrow[dr, equal] & \\ \mathrm{id}_B \arrow[ur, "\eta"] \arrow[rr, "\eta"] \arrow[dr, "\eta\cdot\eta"'] & ufuf \arrow[r, "u\epsilon f"'] & uf \end{tikzcd} \qquad \tau \coloneqq \begin{tikzcd} & uf \arrow[d, "\eta uf"'] \arrow[dr, equal] & \\ \mathrm{id}_B \arrow[ur, "\eta"] \arrow[rr, "\eta"] \arrow[dr, "\eta\cdot\eta"'] & ufuf \arrow[r, "u\epsilon f"'] & uf \end{tikzcd}$$

witnessing the coherence relations $\omega\delta^0 = u\alpha$, $\tau\delta^0 = \beta f$, $\omega\delta^1 = \tau\delta^1$, $\omega\delta^2 = \tau\delta^2 = \eta\sigma^1$, with the 3rd faces of these simplices defined by the pair of 2-arrows given by the horizontal composite

$$\Delta[1] \times \Delta[1] \xrightarrow{\eta\times\eta} \mathrm{Fun}(B,B) \times \mathrm{Fun}(B,B) \xrightarrow{\circ} \mathrm{Fun}(B,B)$$

then there exists a simplicial functor $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ for which $\mathbb{A}(\underline{u}) = u$, $\mathbb{A}(\underline{f}) = f$, $\mathbb{A}(\underline{\eta}) = \eta$, $\mathbb{A}(\underline{\epsilon}) = \epsilon$, $\mathbb{A}(\underline{\beta}) = \beta$, $\mathbb{A}(\underline{\alpha}) = \alpha$, $\mathbb{A}(\underline{\omega}) = \omega$, and $\mathbb{A}(\underline{\tau}) = \tau$.

Coherence conditions of the form stated in ((iii)) appear in the definition of a biadjoint pair in a strongly bicategorically enriched category in [**103**, 1.3.8].

Exercises.

9.3.i. Exercise. Unpack Proposition 9.3.2 in the case $n = 1$ and $n = 2$.

9.3.ii. Exercise. Use Remark 9.2.12 to formulate the dual of Theorem 9.3.4, describing homotopy coherent adjunctions generated by a given counit.

9.4. Homotopical uniqueness of homotopy coherent adjunctions

The homotopy 2-category of an ∞-cosmos can be regarded as a simplicial category of the form described in Lemma 9.1.3, in which context it is equipped with a canonical quotient functor $Q \colon \mathcal{K} \to \mathfrak{h}\mathcal{K}$ of simplicial categories. Formally, Q is a component of the counit of the adjunction described in Digression 1.4.2. It follows easily from the characterization of 2-categories in Lemma 9.1.3 that Q is a local isofibration.

By Proposition 9.1.13, any adjunction in the homotopy 2-category of an ∞-cosmos is represented by a unique simplicial functor $T \colon \mathcal{A}dj \to \mathfrak{h}\mathcal{K}$. To say that a homotopy coherent adjunction $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ lifts the adjunction in the homotopy 2-category means that $\mathbb{A}$ is a lift of T along Q. Theorem 9.3.4 proves that a lift of any adjunction in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ can be constructed by specifying a lift $T \colon \overline{\{\underline{\beta}\}} \to \mathcal{K}$ of $T \colon \mathcal{A}dj \to \mathfrak{h}\mathcal{K}$ — this amounting to a choice of 1-arrows representing the unit

and counit and a 2-arrow witnessing one of the triangle equalities — and then extending along the parental subcomputad inclusion to define the homotopy coherent adjunction $\mathbb{A}$.

$$\begin{tikzcd} \underline{\overline{\{\beta\}}} \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[d, "Q"] \\ \mathcal{A}dj \arrow[r, "T"'] \arrow[ur, dashed, "\mathbb{A}"] & \mathfrak{h}\mathcal{K} \end{tikzcd}$$

Note that the commutativity of the top left triangle implies the commutativity of the bottom right one, so instead of thinking of $\mathbb{A}$ as a *lift* of $T\colon \mathcal{A}dj \to \mathfrak{h}\mathcal{K}$ to a homotopy coherent adjunction, we can equally think of $\mathbb{A}$ as an *extension* of $T\colon \underline{\overline{\{\beta\}}} \to \mathcal{K}$ to a homotopy coherent adjunction.

In this section, we will define the space of extensions of given adjunction data to a homotopy coherent adjunction. Our main theorem in this section is that if the base diagram to be extended is indexed by a parental subcomputad, then the space of lifts is a contractible Kan complex.

The first step is to construct a (possibly large) simplicial hom-space between two simplicial categories.

9.4.1. Definition. Define the cotensor $\mathcal{L}^U$ of a simplicial category $\mathcal{L}$ with a simplicial set U to be the simplicial category with $\mathsf{obj}\,\mathcal{L}^U \coloneqq \mathsf{obj}\,\mathcal{L}$ and hom-spaces

$$\mathcal{L}^U(X,Y) \coloneqq \mathcal{L}(X,Y)^U.$$

Any simplicial set U defines a comonoid $(U, !\colon U \to \mathbb{1}, \Delta\colon U \to U\times U)$ with respect to the cartesian product and diagonal maps, so the endofunctor $(-)^U\colon s\mathcal{S}et\text{-}\mathcal{C}at \to s\mathcal{S}et\text{-}\mathcal{C}at$ is a monad whose unit and multiplication are defined by restricting along these maps.

For simplicial categories $\mathcal{K}$ and $\mathcal{L}$, let $\mathsf{Icon}(\mathcal{K},\mathcal{L})$ denote the large simplicial sets defined by the natural isomorphism

$$\mathsf{Icon}(\mathcal{K},\mathcal{L})^U \cong \mathsf{Icon}(\mathcal{K},\mathcal{L}^U) \qquad \text{i.e.,} \qquad \mathsf{Icon}(\mathcal{K},\mathcal{L})_n \coloneqq \mathsf{Icon}(\mathcal{K},\mathcal{L}^{\Delta[n]}).$$

The composition map

$$\mathsf{Icon}(\mathcal{L},\mathcal{M}) \times \mathsf{Icon}(\mathcal{K},\mathcal{L}) \xrightarrow{\ \circ\ } \mathsf{Icon}(\mathcal{K},\mathcal{M})$$

is given by defining the composite of a pair of n-arrows $F\colon \mathcal{K} \to \mathcal{L}^{\Delta[n]}$ and $G\colon \mathcal{L} \to \mathcal{M}^{\Delta[n]}$ to be the Kleisli composite n-arrow

$$\mathcal{K} \xrightarrow{\ F\ } \mathcal{L}^{\Delta[n]} \xrightarrow{\ G^{\Delta[n]}\ } (\mathcal{M}^{\Delta[n]})^{\Delta[n]} \cong \mathcal{M}^{\Delta[n]\times\Delta[n]} \xrightarrow{\ \Delta^*\ } \mathcal{M}^{\Delta[n]}$$

9.4.2. Warning. The simplicial category $\mathcal{L}^U$ is not the category of "U-shaped diagrams in $\mathcal{L}$." In particular, the cotensor $\mathcal{L}^{\Delta[1]}$ is distinct from the category $\mathcal{L}^2$ of arrows (or isofibrations) considered elsewhere.

9.4.3. Remark. The vertices of $\mathsf{Icon}(\mathcal{K},\mathcal{L})$ are simplicial functors $\mathcal{K} \to \mathcal{L}$. The name "icon" is chosen because the 1-simplices are analogous to the "identity component oplax natural transformations" in 2-category theory as defined by Lack [59]. In particular, each 1-simplex $\mathcal{K} \to \mathcal{L}^{\Delta[1]}$ or n-simplex $\mathcal{K} \to \mathcal{L}^{\Delta[n]}$ spans simplicial functors $\mathcal{K} \to \mathcal{L}$ that agree on objects.

9.4.4. LEMMA. *If $P\colon \mathcal{K} \twoheadrightarrow \mathcal{L}$ is a local isofibration between quasi-categorically enriched categories and $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is a simplicial subcomputad inclusion, then if either P is bijective on objects or I is injective on objects then the Leibniz map*

$$\widehat{\mathsf{Icon}}(I,P)\colon \mathsf{Icon}(\mathcal{B},\mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\mathcal{A},\mathcal{K}) \underset{\mathsf{Icon}(\mathcal{A},\mathcal{L})}{\times} \mathsf{Icon}(\mathcal{B},\mathcal{L})$$

is an isofibration between quasi-categories.

PROOF. A lifting problem of simplicial sets as below-left transposes into a lifting problem of simplicial categories below-right:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "i"'] & \mathsf{Icon}(\mathcal{B},\mathcal{K}) \arrow[d, "\widehat{\mathsf{Icon}}(I,P)"] \\ V \arrow[r] \arrow[ur, dashed] & \mathsf{Icon}(\mathcal{A},\mathcal{K}) \underset{\mathsf{Icon}(\mathcal{A},\mathcal{L})}{\times} \mathsf{Icon}(\mathcal{B},\mathcal{L}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, hook, "I"'] & \mathcal{K}^V \arrow[d, "\widehat{\{i,P\}}"] \\ \mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathcal{K}^U \underset{\mathcal{L}^U}{\times} \mathcal{L}^V \end{tikzcd}$$

When P is surjective on objects, so is the simplicial functor $\widehat{\{i,P\}}$ and in general the action on homs is given by the Leibniz map

$$\widehat{\{i,P\}}\colon \mathsf{Fun}(X,Y)^V \twoheadrightarrow \mathsf{Fun}(X,Y)^U \underset{\mathsf{Fun}(PX,PY)^U}{\times} \mathsf{Fun}(PX,PY)^V,$$

which is a trivial fibration whenever $U \hookrightarrow V$ is an inner horn inclusion or is the inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$.

By Definition 6.1.11 to solve the right-hand lifting problem, it suffices to solve the two lifting problems

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & \mathcal{K}^V \arrow[d, "\widehat{\{i,P\}}"] \\ \mathbb{1} \arrow[r] \arrow[ur, dashed] & \mathcal{K}^U \underset{\mathcal{L}^U}{\times} \mathcal{L}^V \end{tikzcd} \qquad \begin{tikzcd} \mathbb{2}[\partial\Delta[n]] \arrow[r] \arrow[d, hook] & \mathcal{K}^V \arrow[d, "\widehat{\{i,P\}}"] \\ \mathbb{2}[\Delta[n]] \arrow[r] \arrow[ur, dashed] & \mathcal{K}^U \underset{\mathcal{L}^U}{\times} \mathcal{L}^V \end{tikzcd}$$

where the left-hand lifting problem is not needed if I is bijective on objects. The right-hand lifting problem can be solved because $\widehat{\{i,P\}}$ is a local trivial fibration and the left-hand lifting problem can be solved whenever P is surjective on objects. □

9.4.5. COROLLARY. *If $\mathcal{K}$ is a quasi-categorically enriched category, if $\mathcal{A}$ is a simplicial computad, and if $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is a simplicial subcomputad inclusion, then*

$$\mathsf{Icon}(I,\mathcal{K})\colon \mathsf{Icon}(\mathcal{B},\mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\mathcal{A},\mathcal{K})$$

is an isofibration between quasi-categories. □

9.4.6. LEMMA. *Suppose $P\colon \mathcal{K} \to \mathcal{L}$ is a simplicial functor between quasi-categories that is locally conservative in the sense that each $\mathsf{Fun}(A,B) \to \mathsf{Fun}(PA,PB)$ reflects isomorphisms. Then if $\mathcal{A}$ is a simplicial computad and $\mathcal{A} \hookrightarrow \mathcal{B}$ is a bijective-on-objects simplicial subcomputad inclusion, then the Leibniz map*

$$\widehat{\mathsf{Icon}}(I,P)\colon \mathsf{Icon}(\mathcal{B},\mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\mathcal{A},\mathcal{K}) \underset{\mathsf{Icon}(\mathcal{A},\mathcal{L})}{\times} \mathsf{Icon}(\mathcal{B},\mathcal{L})$$

is a conservative functor of quasi-categories.

PROOF. In the language of marked simplicial sets, we must show that any lifting problem below-left has a solution

$$\begin{tikzcd} \mathbb{2} \arrow[r] \arrow[d, hook] & \mathsf{Icon}(\mathcal{B},\mathcal{K}) \arrow[d, two heads, "\widehat{\mathsf{Icon}}(I,P)"] \\ \mathbb{2}^\sharp \arrow[r] \arrow[ur, dashed] & \mathsf{Icon}(\mathcal{A},\mathcal{K}) \underset{\mathsf{Icon}(\mathcal{A},\mathcal{L})}{\times} \mathsf{Icon}(\mathcal{B},\mathcal{L}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, hook] & \mathcal{K}^{\mathbb{2}^\sharp} \arrow[d, two heads] \\ \mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathcal{K}^{\mathbb{2}} \underset{\mathcal{L}^{\mathbb{2}}}{\times} \mathcal{L}^{\mathbb{2}^\sharp} \end{tikzcd}$$

where $\mathbb{2}^\sharp$ represents invertible 1-arrows. By adjunction, this transposes to the lifting problem above-right. As in the proof of Lemma 9.4.4, it suffices to consider the case where $\mathcal{A} \hookrightarrow \mathcal{B}$ is $\mathbb{2}[\partial[\Delta[n]]] \hookrightarrow \mathbb{2}[\Delta[n]]$ for some $n \geq 0$, in which case the lifting problem of simplicial categories reduces to one of simplicial sets

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(A,B)^{\mathbb{2}^\sharp} \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(A,B)^{\mathbb{2}} \underset{\mathsf{Fun}(PA,PB)^{\mathbb{2}}}{\times} \mathsf{Fun}(PA,PB)^{\mathbb{2}^\sharp} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} (\partial\Delta[n]\times\mathbb{2}^\sharp)\cup(\Delta[n]\times\mathbb{2}) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(A,B) \arrow[d] \\ \Delta[n]\times\mathbb{2}^\sharp \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(PA,PB) \end{tikzcd}$$

The marked simplicial sets on the left have the same underlying sets, differing only in their markings, so by the hypothesis that $\mathsf{Fun}(A,B) \to \mathsf{Fun}(PA,PB)$ is conservative, the result follows. ☐

9.4.7. DEFINITION. The **space of homotopy coherent adjunctions** in a quasi-categorically enriched category $\mathcal{K}$ is

$$\mathsf{cohadj}(\mathcal{K}) \coloneqq \mathsf{Icon}(\mathcal{A}dj,\mathcal{K}).$$

9.4.8. PROPOSITION. *The space of homotopy coherent adjunctions in $\mathcal{K}$ is a Kan complex, possibly a large one.*

PROOF. Since $\mathcal{A}dj$ is a simplicial computad, Corollary 9.4.5 implies that $\mathsf{cohadj}(\mathcal{K})$ is a quasi-category. By Corollary 1.1.15, we need only show that all of its arrows are isomorphism. Since a 1-arrow in a functor space of $\mathcal{K}$ is an isomorphism if and only if it represents an invertible 2-cell in $\mathfrak{h}\mathcal{K}$, the quotient simplicial functor $Q\colon \mathcal{K} \to \mathfrak{h}\mathcal{K}$ is locally conservative, and by Lemma 9.4.6, it suffices to show that $\mathsf{cohadj}(\mathfrak{h}\mathcal{K})$ is a Kan complex.

From Proposition 9.1.13 we can extract an explicit description of $\mathsf{cohadj}(\mathfrak{h}\mathcal{K})$ that reveals that it is actually isomorphic the nerve of a 1-category: its

- objects are adjunctions $(f \dashv u, \eta, \epsilon)$ in $\mathfrak{h}\mathcal{K}$
- arrows $(\phi,\psi)\colon (f \dashv u,\eta,\epsilon) \to (f' \dashv u',\eta',\epsilon')$ consist of a pair of 2-cells $\phi\colon f \Rightarrow f'$ and $\psi\colon u \Rightarrow u'$ so that $\epsilon' \cdot (\phi\psi) = \epsilon$ and $\eta' = (\psi\phi)\cdot\eta$, and
- identities and composition are given componentwise.

The isomorphisms in $\mathsf{cohadj}(\mathfrak{h}\mathcal{K})$ are those pairs (ϕ,ψ) whose components ϕ and ψ are both invertible. From the defining equations $\epsilon' \cdot (\phi\psi) = \epsilon$ and $\eta' = (\psi\phi)\cdot\eta$ it follows that the mate of ψ is an inverse to ϕ and the mate of ϕ is an inverse to ψ, so every arrow is in fact an isomorphism. ☐

9.4.9. PROPOSITION (homotopical uniqueness of parental subcomputad extensions). *Suppose $\overline{\{\eta\}} \subset \mathcal{A} \subset \mathcal{A}' \subset \mathcal{A}dj$ are parental subcomputads. Suppose $T\colon \mathcal{A} \to \mathcal{K}$ is a simplicial functor so that $T(\underline{f}) = f$, $T(\underline{u}) = u$, and $T(\underline{\eta}) = \eta$ define an adjunction in $\mathcal{K}$. Then the fiber of the isofibration*

$$\mathsf{Icon}(\mathcal{A}',\mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\mathcal{A},\mathcal{K})$$

over T is a contractible Kan complex.

Proof. The fibers over T are those simplicial functors $T\colon \mathcal{A}' \to \mathcal{K}$ extending $T\colon \mathcal{A} \to \mathcal{K}$. By Theorem 9.3.3, this fiber is non-empty. We must show for any inclusion $U \hookrightarrow V$ of simplicial sets, the lifting problem

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "i"'] & E_T \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Icon}(\mathcal{A}', \mathcal{K}) \arrow[d, two heads] \\ V \arrow[r] \arrow[ur, dashed] & \mathbb{1} \arrow[r, "T"] & \mathsf{Icon}(\mathcal{A}, \mathcal{K}) \end{tikzcd}$$

has a solution. Transposing, we obtain a lifting problem of simplicial categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[d, hook] & \mathcal{K} \arrow[r, "\mathcal{K}^!"] & \mathcal{K}^V \arrow[d, two heads] \\ \mathcal{A}' \arrow[rr] \arrow[urr, dashed] & & \mathcal{K}^U \end{tikzcd}$$

against the local isofibration $\mathcal{K}^V \twoheadrightarrow \mathcal{K}^U$. The simplicial functor $\mathcal{K}^!\colon \mathcal{K} \to \mathcal{K}^V$, like all simplicial functors, preserves the adjunction $(f \dashv u, \eta)$ in $\mathcal{K}$, so Theorem 9.3.3 applies to provide a solution. ☐

Taking $\mathcal{A}' = \mathcal{A}dj$, Proposition 9.4.9 tells us that the space of homotopy coherent adjunctions extending a simplicial functor $\mathcal{A} \to \mathcal{K}$ indexed by a parental subcomputad whose image specifies the unit of an adjunction is a contractible Kan complex. This proves that such extensions are "homotopically unique." We conclude this section with more refined presentations of this kind of result for two instances of basic adjunction data of interest.

9.4.10. Definition (the space of units). Simplicial functors $\underline{\overline{\{\eta\}}} \to \mathcal{K}$ correspond bijectively to the choice of a pair of 0-arrows $f \in \mathsf{Fun}(B, A)$, $u \in \mathsf{Fun}(A, B)$, together with a 1-arrow $\eta\colon \mathrm{id}_B \to uf \in \mathsf{Fun}(B, B)$. We refer to the simplicial subset $\mathsf{unit}(\mathcal{K}) \subset \mathsf{Icon}(\underline{\overline{\{\eta\}}}, \mathcal{K})$ of those triples (f, u, η) that specify the unit of an adjunction and those 1-simplices that define isomorphisms between them as the **space of units**, and denote its objects by $(f \dashv u, \eta)$. Since $\mathsf{Icon}(\underline{\overline{\{\eta\}}}, \mathcal{K})$ is a quasi-category, $\mathsf{unit}(\mathcal{K})$ is a Kan complex.

9.4.11. Lemma. *There is an isomorphism of quasi-categories*

$$\mathsf{Icon}(\underline{\overline{\{\eta\}}}, \mathcal{K}) \cong \coprod_{A,B \in \mathcal{K}} \mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, \circ) \qquad \begin{tikzcd} & \mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, \circ) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ \mathsf{Fun}(A,B) \times \mathsf{Fun}(B,A) \arrow[dr, "\circ"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathbb{1} \arrow[dl, "\circ"] \\ & \mathsf{Fun}(B,B) & \end{tikzcd}$$

Proof. Exercise 9.4.ii. ☐

9.4.12. Theorem (uniqueness of homotopy coherent extensions of a unit).

(i) The space E_η of homotopy coherent adjunctions extending the counit η is a contractible Kan complex.

(ii) *The forgetful functor* $p_U \colon \mathsf{cohadj}(\mathcal{K}) \xrightarrow{\sim\!\!\!\twoheadrightarrow} \mathsf{unit}(\mathcal{K})$ *is a trivial fibration of Kan complexes.*

PROOF. Both statements follow from specializing Proposition 9.4.9 to the parental subcomputad $\overline{\{\underline{\eta}\}} \subset \mathcal{A}dj$. If $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ is a homotopy coherent adjunction, its restriction to $\overline{\{\underline{\eta}\}} \to \mathcal{K}$ defines an object of $\mathsf{unit}(\mathcal{K}) \subset \mathsf{Icon}(\overline{\{\underline{\eta}\}}, \mathcal{K})$. The fiber of the isofibration $p_U \colon \mathsf{cohadj}(\mathcal{K}) \xrightarrow{\sim\!\!\!\twoheadrightarrow} \mathsf{unit}(\mathcal{K})$ over $(f \dashv u, \eta)$ coincides with the fiber considered in Proposition 9.4.9 and is thus a contractible Kan complex.

By Corollay 9.4.5, the map $p_U \colon \mathsf{cohadj}(\mathcal{K}) \xrightarrow{\sim\!\!\!\twoheadrightarrow} \mathsf{unit}(\mathcal{K})$ is an isofibration between Kan complexes, and is thus a Kan fibration. By Proposition D.5.4 any Kan fibration between Kan complexes with contractible fibers is a trivial fibration, proving (ii). □

9.4.13. DEFINITION (the space of right adjoints). Simplicial functors $\overline{\{\underline{u}\}} \to \mathcal{K}$ correspond bijectively to the choice of a 0-arrow u in $\mathcal{K}$. Indeed:

$$\mathsf{Icon}(\overline{\{\underline{u}\}}, \mathcal{K}) \cong \coprod_{A,B \in \mathcal{K}} \mathsf{Fun}(A, B).$$

We refer to the simplicial subset $\mathsf{rightadj}(\mathcal{K}) \subset \mathsf{Icon}(\overline{\{\underline{u}\}}, \mathcal{K})$ of those 0-arrows that possess a left adjoint and the isomorphisms between them as the **space of right adjoints**. As a quasi-category whose 1-arrows are all isomorphisms, $\mathsf{rightadj}(\mathcal{K})$ is a Kan complex.

The isofibration $\mathsf{Icon}(\overline{\{\underline{\eta}\}}, \mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\overline{\{\underline{u}\}}, \mathcal{K})$ restricts to define an isofibration $q_R \colon \mathsf{unit}(\mathcal{K}) \twoheadrightarrow \mathsf{rightadj}(\mathcal{K})$ of Kan complexes.

9.4.14. PROPOSITION. *The isofibration* $q_R \colon \mathsf{unit}(\mathcal{K}) \twoheadrightarrow \mathsf{rightadj}(\mathcal{K})$ *is a trivial fibration of Kan complexes.*

PROOF. Since both $\mathsf{unit}(\mathcal{K})$ and $\mathsf{rightadj}(\mathcal{K})$ are Kan complexes, automatically q_R is a Kan fibration. By Proposition D.5.4, we need only show that its fibers are contractible. The fiber of q_R over $u \colon A \to B$ is isomorphic to the sub-quasi-category of the fiber of the isofibration $\mathsf{Icon}(\overline{\{\underline{\eta}\}}, \mathcal{K}) \twoheadrightarrow \mathsf{Icon}(\overline{\{\underline{u}\}}, \mathcal{K})$ over u whose objects are pairs (f, η) which have the property that f is a left adjoint to the fixed 0-arrow u with unit represented by η and whose 1-simplices are all invertible. By Lemma 9.4.11, this isofibration is isomorphic to the coproduct of the family of projections

$$\coprod_{A,B \in \mathcal{K}} \mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, \circ) \xrightarrow{p_1} \mathsf{Fun}(A, B) \times \mathsf{Fun}(B, A) \xrightarrow{\pi} \mathsf{Fun}(A, B)$$

whose fiber over u is isomorphic to $\mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, u_*)$.

Applying Proposition 4.1.4 to the adjunction between functor spaces

$$\begin{tikzcd} \mathsf{Fun}(B,B) \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(B,A) \arrow[l, bend left, "u_*"] \end{tikzcd}$$

and the element $\mathrm{id}_B \in \mathsf{Fun}(B, B)$ reveals that (f, η) is a initial in $\mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, u_*)$. So the fiber of q_R over u is isomorphic to the sub-quasi-category of $\mathsf{Hom}_{\mathsf{Fun}(B,B)}(\mathrm{id}_B, u_*)$ spanned by its initial elements and as such is a contractible Kan complex.[6] □

[6]In any quasi-category in which every element is initial, all 1-arrows are isomorphisms, since the initial elements are also initial in its homotopy category, which must then be a groupoid. This proves that the quasi-category spanned by initial

9.4.15. Theorem (uniqueness of homotopy coherent extensions of a right adjoint).

(i) The space E_u of homotopy coherent adjunctions with right adjoint u is a contractible Kan complex.
(ii) The forgetful functor $p_R \colon \mathsf{cohadj}(\mathcal{K}) \twoheadrightarrow \mathsf{rightadj}(\mathcal{K})$ is a trivial fibration of Kan complexes.

Proof. The space E_u is defined as the fiber of the composite fibration

$$p_R \colon \mathsf{cohadj}(\mathcal{K}) \xrightarrow[\sim]{p_U} \mathsf{unit}(\mathcal{K}) \xrightarrow[\sim]{q_R} \mathsf{rightadj}(\mathcal{K})$$

By Theorem 9.4.12 and Proposition 9.4.14, both maps are trivial fibrations of Kan complexes, hence so is the composite and thus its fiber is a contractible Kan complex. □

Exercises.

9.4.i. Exercise. State and prove a relative version of Proposition 9.4.9, establishing the homotopical uniqueness of solutions to lifting problems of parental subcomputad inclusions against local isofibrations of quasi-categorically enriched categories.

9.4.ii. Exercise. Prove Lemma 9.4.11.

elements is in fact a Kan complex. Any adjunction between Kan complexes is an adjoint equivalence, so in particular the right adjoint $! \colon A \twoheadrightarrow \mathbb{1}$ defines an equivalence, and hence a trivial fibration .

CHAPTER 10

The formal theory of homotopy coherent monads

10.1. Homotopy coherent monads

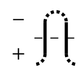
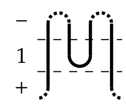
The free homotopy coherent monad is defined as a full subcategory $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ on the object $+$. Via this definition, it inherits a graphical calculus from the graphical calculus established for $\mathcal{A}dj$ in §9.1.

10.1.1. Definition (the free homotopy coherent monad). The **free homotopy coherent monad** $\mathcal{M}nd$ is the full subcategory of the free homotopy coherent adjunction $\mathcal{A}dj$ on the object $+$. Proposition 9.1.10 gives us two definitions of $\mathcal{M}nd$:

- It is the 2-category regarded as a simplicial category with one object, with the hom-category $\mathbf{\Delta}_+$, and with horizontal composition given by ordinal sum $\oplus\colon \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \to \mathbf{\Delta}_+$.
- It is the simplicial category with one object whose n-arrows are strictly undulating squiggles on $(n+1)$-lines that start and end in the gap labeled $+$.

10.1.2. Lemma. *The simplicial category $\mathcal{M}nd$ is a simplicial computad, though $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ is not a simplicial subcomputad inclusion.*

Proof. Horizontal composition in $\mathcal{M}nd$ is given by horizontal juxtaposition of squiggle diagrams that start and end at $+$. Thus, an n-arrow is atomic if and only if it has no instances of $+$ in its interior. This proves that $\mathcal{M}nd$ is a simplicial computad, but note that $\mathcal{M}nd$ includes atomic arrows such as

$\underline{t} \coloneqq \underline{uf} \coloneqq$  and $\underline{\mu} \coloneqq \underline{u\epsilon f} \coloneqq$  (10.1.3)

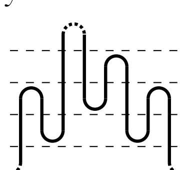
that do have $-$ in their interiors and thus fail to be atomic in $\mathcal{A}dj$. □

Employing the graphical calculus, we discover another characterization of the atomic n-arrows of $\mathcal{M}nd$ in reference to the atomic 0-arrow $\underline{t}$ defined in (10.1.3):

10.1.4. Lemma. *An n-arrow in $\mathcal{M}nd$ is atomic if and only if its final vertex is $\underline{t}$.*

In particular, $\underline{t}$ is the unique atomic 0-arrow.

Proof. The n-arrows are strictly undulating squiggles on $(n+1)$-lines that start and end at the space labeled $+$; these are atomic if and only if there are no instances of $+$ in their interiors.

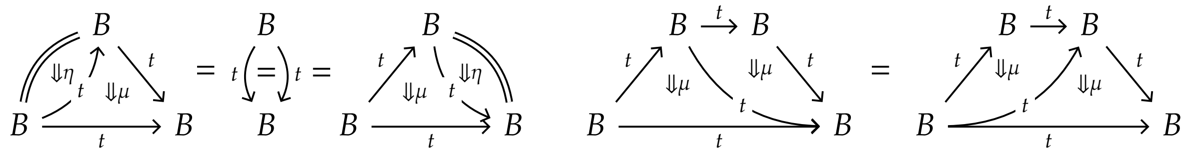


This condition implies that if all the lines are removed except the bottom one, a process that computes the final vertex of the n-simplex, the resulting squiggle looks like a single hump over one line, which is the graphical representation of the 0-arrow $\underline{t}$. □

10.1.5. Definition. A **monad** in a 2-category is given by:

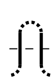
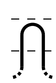
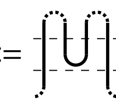
- an object B,
- an endofunctor $t\colon B \to B$,
- and a pair of 2-cells $\eta\colon \mathrm{id}_B \Rightarrow t$ and $\mu\colon t^2 \Rightarrow t$

so that the "unit" and "associativity" pasting equalities hold:



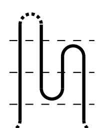
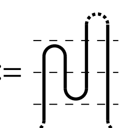
When these conditions are satisfied, we say that (t, η, μ) defines a monad on B.

10.1.6. Lemma. *The atomic arrows*

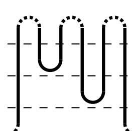
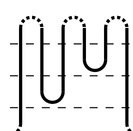
$\underline{t} :=$  , $\underline{\eta} :=$  , *and* $\underline{\mu} := \underline{u\epsilon f} :=$ 

define a monad $(\underline{t}, \underline{\eta}, \underline{\mu})$ *on* $+$ *in the 2-category* $\mathcal{M}nd$.

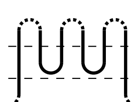
Proof. The unit pasting identities of Definition 10.1.5 are witnessed by the atomic 2-arrows

$\underline{u\alpha} :=$  and $\underline{\beta f} :=$  .

The pair of atomic 2-arrows

 and 

demonstrate that the left hand side and right-hand side of the associativity pasting equality have a common composite, namely the common first face



□

As a 2-category $\mathcal{M}nd$ has a familiar universal property. Lawvere's characterization of $(\mathbf{\Delta}_+, \oplus, [-1])$ as the free strict monoidal category containing a monoid $([0], !\colon [-1] \to [0], \sigma^0\colon [1] \to [0])$ tells us that $\mathcal{M}nd$ is the free 2-category containing a monad [**60**]:

10.1.7. Proposition. *For any 2-category* $\mathcal{C}$*, 2-functors* $\mathcal{M}nd \to \mathcal{C}$ *correspond to monads in* $\mathcal{C}$. □

These considerations motivate the following definition:

10.1.8. Definition (homotopy coherent monad). A **homotopy coherent monad** in a quasi-categorically enriched category $\mathcal{K}$ is a simplicial functor $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$. Explicitly, a homotopy coherent monad consists of

- an object $B \in \mathcal{K}$ and
- a homotopy coherent diagram $\mathbb{T} \colon \mathbf{\Delta}_+ \to \mathsf{Fun}(B, B)$ that we refer to as the **monad resolution** of $\mathbb{T}$

so that the diagram

$$\begin{tikzcd} \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \arrow[r, "\mathbb{T}\times\mathbb{T}"] \arrow[d, "\oplus"'] & \mathsf{Fun}(B,B) \times \mathsf{Fun}(B,B) \arrow[d, "\circ"] \\ \mathbf{\Delta}_+ \arrow[r, "\mathbb{T}"'] & \mathsf{Fun}(B,B) \end{tikzcd}$$

commutes. This simplicial functoriality condition implies that the generating 0- and 1-arrows of the monad resolution have the following form:

$$\begin{tikzcd} \mathrm{id}_B \arrow[r, "\eta" description] & t \arrow[r, shift left=2, "\eta t"] \arrow[r, leftarrow, "\mu" description] \arrow[r, shift right=2, "t\eta"'] & tt \arrow[r, shift left=4, "\eta tt"] \arrow[r, shift left=2, leftarrow, "\mu t" description] \arrow[r, "t\eta t" description] \arrow[r, shift right=2, leftarrow, "t\mu" description] \arrow[r, shift right=4, "tt\eta"'] & ttt & \cdots & \in \mathsf{Fun}(B,B) \end{tikzcd} \qquad (10.1.9)$$

where (t, η, μ) denotes the image of the monad $(\underline{t}, \underline{\eta}, \underline{\mu})$ of Lemma 10.1.6 under $\mathbb{T} \colon \mathcal{M}nd \to \mathcal{K}$. We refer to the 0-arrow $t \colon B \to B$ as the **functor part** of the homotopy coherent monad $\mathbb{T}$ and refer to the 1-arrows η and μ as the **unit** and **associativity** maps.

Note that for any generalized element $b \colon X \to B$, the monad resolution (10.1.9) restricts to define a monad resolution

$$\begin{tikzcd} b \arrow[r, "\eta b" description] & tb \arrow[r, shift left=2, "\eta tb"] \arrow[r, leftarrow, "\mu b" description] \arrow[r, shift right=2, "t\eta b"'] & ttb \arrow[r, shift left=4, "\eta ttb"] \arrow[r, shift left=2, leftarrow, "\mu tb" description] \arrow[r, "t\eta tb" description] \arrow[r, shift right=2, leftarrow, "t\mu b" description] \arrow[r, shift right=4, "tt\eta b"'] & tttb & \cdots & \in \mathsf{Fun}(X,B) \end{tikzcd} \qquad (10.1.10)$$

10.1.11. Example (free monoid monad). Let $\mathcal{M}$ be a Kan-complex (or topologically) enriched category equipped with an enriched monoidal structure $- \otimes - \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ that admits countable conical coproducts that are preserved by the monoidal product separately in each variable. Then there exists a simplicially enriched endofunctor $T \colon \mathcal{M} \to \mathcal{M}$ defined on objects by

$$T(X) := \coprod_{n \geq 0} X^{\otimes n}$$

equipped with simplicial natural transformations $\eta \colon \mathrm{id}_{\mathcal{M}} \Rightarrow T$ and $\mu \colon T^2 \Rightarrow T$ defined by including at the degree-one component and "distributing" the coproduct

$$T^2(X) \cong \coprod_{n \geq 0} \left(\coprod_{m \geq 0} X^{\otimes m} \right)^{\otimes n}.$$

The monad resolution of this simplicially enriched monad defines a simplicial functor $\mathbf{\Delta}_+ \times \mathcal{M} \to \mathcal{M}$, regarding $\mathbf{\Delta}_+$ as a simplicial category whose hom-spaces are sets. Applying the homotopy coherent nerve, $\mathfrak{N} \colon \mathcal{K}an\text{-}s\mathcal{S}et \to \mathcal{Q}\mathcal{C}at$, this simplicially enriched monad defines the left action $\mathbf{\Delta}_+ \times \mathfrak{N}\mathcal{M} \to \mathfrak{N}\mathcal{M}$ of a homotopy coherent monad in $\mathcal{Q}\mathcal{C}at$ on $\mathfrak{N}\mathcal{M}$.

More generally, any topologically enriched monad on a topologically enriched category defines a homotopy coherent monad on its homotopy coherent nerve.

Any homotopy coherent monad $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$ defines a monad in the homotopy 2-category, simply by composing with the canonical quotient functor discussed at the beginning of §9.4 and applying Proposition 10.1.7:

$$\mathcal{M}nd \xrightarrow{\mathbb{T}} \mathcal{K} \xrightarrow{Q} \mathfrak{h}\mathcal{K}$$

However "monads up to homotopy" — that is, monads in the homotopy 2-category of an ∞-cosmos — cannot necessarily be made homotopy coherent.

10.1.12. Non-Example (a monad in the homotopy 2-category that is not homotopy coherent). Stasheff identifies homotopy associative H-spaces that do not extend to A_∞-spaces; that is, monoids up to homotopy cannot necessarily be rectified into homotopy coherent monoids. Let

$$(M, \eta\colon * \to M, \eta\colon M \times M \to M)$$

denote such an up-to-homotopy monoid. This structure defines a monad up to homotopy on the (large) quasi-category of spaces by applying the homotopy coherent nerve to the endofunctor $M \times -\colon \mathcal{K}an \to \mathcal{K}an$ and natural transformations induced by η and ϵ. This monad in $\mathfrak{h}\mathcal{QC}at$ cannot be made homotopy coherent.

Exercises.

10.1.i. Exercise.

(i) Show that $\mathcal{M}nd$ contains a unique atomic 1-arrow $\underline{\mu}_n \colon \underline{t}^n \to \underline{t}$ for each $n \geq 0$, $\underline{\mu}_1 = \mathrm{id}_{\underline{t}}$ being degenerate, but each of the other $\underline{\mu}_n$ being non-degenerate.

(ii) Identify the images of these atomic 1-arrows in the monad resolution (10.1.9).

(iii) Given an interpretation for the 1-arrow $\underline{\mu}_n$ that acknowledges the role played by $\underline{\mu}_3$ in the proof of Lemma 10.1.6.

10.2. Homotopy coherent algebras and the monadic adjunction

Homotopy coherent monads can be defined in any quasi-categorically (or merely simplically) enriched category but we are particularly interested in homotopy coherent monads valued in ∞-cosmoi because the flexible weighted limits guaranteed by Corollary 7.3.3 permit us to construct the *monadic adjunction*, which relates the ∞-category on which the monad acts to the ∞-category of *algebras*.

The universal property of the monadic adjunction associated to a homotopy coherent monad $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$ is very easy to describe, though some more work will be required to demonstrate that any ∞-cosmos $\mathcal{K}$ admits such a construction. The monadic adjunction is the terminal adjunction extending the homotopy coherent monad, which means that it is given by the right Kan extension along the inclusion

$$\begin{tikzcd} & \mathcal{A}dj \arrow[dr, dashed, "\mathbb{A}^{\mathbb{T}}"] & \\ \mathcal{M}nd \arrow[ur, hook] \arrow[rr, "\mathbb{T}"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\cong"] & & \mathcal{K} \end{tikzcd}$$

Since $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ is fully faithful, the value of the right Kan extension at $+ \in \mathcal{M}nd$ is isomorphic to $B \coloneqq \mathbb{T}(+)$. By Example 7.1.19, the value of the right Kan extension at $- \in \mathcal{M}nd$ is computed as the limit of $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$ weighted by the restriction of the covariant representable of $\mathcal{A}dj$ at $-$ along $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$, which is how we will now define the weight W_- for the ∞-category of algebras.

10.2.1. Observation (weights on $\mathcal{M}nd$). A **weight on $\mathcal{M}nd$** is a simplicial functor $W\colon \mathcal{M}nd \to s\mathcal{S}et$. Explicitly, to specify a weight on $\mathcal{M}nd$ is equivalent to specifying

- a simplicial set $W := W(+)$
- equipped with a left action of the simplicial monoid[1] $(\mathbf{\Delta}_+, \oplus, [-1])$

$$\mathbf{\Delta}_+ \times W \xrightarrow{\cdot} W \qquad \text{so that} \qquad \begin{tikzcd} \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \times W \arrow[r, "\mathrm{id}\times\cdot"] \arrow[d, "\oplus\times\mathrm{id}"'] & \mathbf{\Delta}_+ \times W \arrow[d, "\cdot"] \\ \mathbf{\Delta}_+ \times W \arrow[r, "\cdot"'] & W \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "{[-1]\times\mathrm{id}}"] \arrow[dr, equal] & \mathbf{\Delta}_+ \times W \arrow[d, "\cdot"] \\ & W \end{tikzcd}$$

Frequently, the simplicial set W happens to be a quasi-category, in which case the weight W on $\mathcal{M}nd$ is precisely a homotopy coherent monad on the quasi-category W.

Relative to the encoding of weights on $\mathcal{M}nd$ as simplicial sets with a left $\mathbf{\Delta}_+$-action, a map $f\colon V \to W \in s\mathcal{S}et^{\mathcal{M}nd}$ is given by a map $f\colon V \to W$ of simplicial sets that is $\mathbf{\Delta}_+$-equivariant in the sense that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "{[-1]\times\mathrm{id}}"'] & W \arrow[d, "{[-1]\times\mathrm{id}}"] \\ \mathbf{\Delta}_+ \times V \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\cdot"'] & \mathbf{\Delta}_+ \times W \arrow[d, "\cdot"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes.

10.2.2. Lemma. *Let $W\colon \mathcal{M}nd \to s\mathcal{S}et$ be a weight on $\mathcal{M}nd$ and let $T\colon \mathcal{M}nd \to \mathcal{K}$ be a homotopy coherent monad on $B \in \mathcal{K}$. Then a W-shaped cone over $\mathbb{T}$ with summit X is specified by a simplicial map $\gamma\colon W \to \mathsf{Fun}(X, B)$ which makes the square*

$$\begin{tikzcd} \mathbf{\Delta}_+ \times W \arrow[r, "\mathbb{T}\times\gamma"] \arrow[d, "\cdot"'] & \mathsf{Fun}(B,B) \times \mathsf{Fun}(X,B) \arrow[d, "\circ"] \\ W \arrow[r, "\gamma"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

Proof. By Observation 10.2.1, a simplicial natural transformation $\gamma\colon W \to \mathsf{Fun}(X, \mathbb{T}-)$ is given by its unique component $\gamma\colon W \to \mathsf{Fun}(X, B)$ subject to the equivariance condition:

$$\begin{tikzcd} \mathbf{\Delta}_+ \times W \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "\cdot"'] & \mathbf{\Delta}_+ \times \mathsf{Fun}(X,B) \arrow[d, "\cdot"] \\ W \arrow[r, "\gamma"'] & \mathsf{Fun}(X,B) \end{tikzcd}$$

The right-hand action map is the transpose of the action map of the composite simplicial functor $\mathsf{Fun}(X, \mathbb{T}-)\colon \mathcal{M}nd \to \mathcal{QC}at$, this being

$$\mathbf{\Delta}_+ \xrightarrow{\mathbb{T}} \mathsf{Fun}(B,B) \xrightarrow{\mathsf{Fun}(X,-)} \mathsf{Fun}(\mathsf{Fun}(X,B), \mathsf{Fun}(X,B))$$

[1]The strict monoidal category $(\mathbf{\Delta}_+, \oplus, [-1])$ is a monoid in $(s\mathcal{S}et, \times, \mathbb{1})$. Applying the nerve functor, $(\mathbf{\Delta}_+, \oplus, [-1])$ also defines a monoid in $(s\mathcal{S}et, \times, \mathbb{1})$.

which transposes to

$$\mathbf{\Delta}_+ \times \mathsf{Fun}(X,B) \xrightarrow{\mathbb{T}\times\mathrm{id}} \mathsf{Fun}(B,B)\times\mathsf{Fun}(X,B) \xrightarrow{\circ} \mathsf{Fun}(X,B).$$

Now the equivariance square coincides with the commutative square of the statement. □

10.2.3. EXAMPLE (notable weights on $\mathcal{M}nd$). We fix notation for a few notable weights on $\mathcal{M}nd$.

(i) Write $W_+ : \mathcal{M}nd \to s\mathcal{S}et$ for the unique represented functor on $\mathcal{M}nd$, which is given by the quasi-category $\mathbf{\Delta}_+ = \mathcal{M}nd(+,+) = \mathcal{A}dj(+,+)$ acted upon the left by itself via the ordinal sum map

$$\mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \xrightarrow{\oplus} \mathbf{\Delta}_+$$

(ii) Write $W_- : \mathcal{M}nd \to s\mathcal{S}et$ for the restriction of the covariant representable functor $\mathcal{A}dj \to s\mathcal{S}et$ on $-$ along $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$. This weight is presented by the quasi-category $\mathbf{\Delta}_\top = \mathcal{A}dj(-,+)$ acted upon the left by $\mathbf{\Delta}_+$ by the ordinal sum map

$$\mathbf{\Delta}_+ \times \mathbf{\Delta}_\top \xrightarrow{\oplus} \mathbf{\Delta}_\top$$

which defines the horizontal composition in $\mathcal{A}dj$ in Definition 9.1.1.

There is a natural inclusion

$$\begin{tikzcd} \mathbf{\Delta}_+ \arrow[r, hook, "-\oplus[0]"] \arrow[d, phantom, "\wr\Vert"] & \mathbf{\Delta}_\top \arrow[d, phantom, "\wr\Vert"] \\ \mathcal{A}dj(+,+) \arrow[r, hook, "-\circ\underline{u}"'] & \mathcal{A}dj(-,+) \end{tikzcd}$$

that "freely adjoins a top element in each ordinal" or, in the graphical calculus of Definition 9.1.9, "precomposes a strictly undulating squiggle with $\underline{u} := (+,-)$." This commutes with the left $\mathbf{\Delta}_+$-actions and so defines an inclusion of weights $W_+ \hookrightarrow W_-$ by Observation 10.2.1.[2]

Since W_+ is the representable weight it is automatically flexible. By the first axiom of Definition 7.1.3 the W_+-weighted limit of a homotopy coherent monad $\mathbb{T}$ recovers the ∞-category $\lim_{W_+}\mathbb{T} \cong B$ on which $\mathbb{T}$ acts.

10.2.4. LEMMA. *The inclusion $W_+ \hookrightarrow W_-$ is a projective cell complex in $s\mathcal{S}et^{\mathcal{M}nd}$ built by attaching projective n-cells $\partial\Delta[n]\times W_+ \hookrightarrow \Delta[n]\times W_+$ in dimensions $n>0$. In particular, W_- is a flexible weight.*

PROOF. We apply Theorem 7.2.12 and prove that $W_+ \hookrightarrow W_-$ is a projective cell complex by verifying that $\mathrm{coll}(W_+) \hookrightarrow \mathrm{coll}(W_-)$ is a relative simplicial computad. Since W_+ is flexible, it follows then that W_- is too. The collage $\mathrm{coll}(W_-)$ can be identified with the non-full simplicial subcategory of $\mathcal{A}dj$ containing two objects $-$ (aka $\top$) and $+$ and the hom-spaces $\mathcal{A}dj(-,+)$ and $\mathcal{A}dj(+,+)$ but with the hom-space from $-$ to $-$ trivial and the hom-space from $+$ to $-$ empty. Via the graphical calculus of Definition 9.1.9, we see that $\mathrm{coll}(W_-)$ is a simplicial category whose n-arrows are strictly undulating squiggles from $-$ to $+$ or from $+$ to $+$ with composition defined by concatenation at $+$. Its *atomic* n-arrows are then those that have no instances of $+$ in their interiors.

Similarly, $\mathrm{coll}(W_+)$ is the simplicial category with two objects $\top$ and $+$ and with the hom-spaces from $\top$ to $+$ and from $+$ to $+$ both defined to be $\mathcal{A}dj(+,+)$, with the hom space from $\top$ to $\top$ trivial and the hom-space from $+$ to $\top$ empty. This is also a simplicial computad in which the only atomic arrow from $\top$ to $+$ is the identity 0-arrow corresponding to $[-1] \in \mathcal{A}dj(+,+) = \mathbf{\Delta}_+$; as before, the atomic arrows from $+$ to $+$ are the strictly undulating squiggles which have no instances of $+$ in

[2] A "right adjoint" to the inclusion $W_+ \hookrightarrow W_-$ will be described in the proof of Proposition 10.2.11.

their interiors. To provide intuition for this simplicial computad structure on $\mathrm{coll}(W_+)$, recall that since the representable W_+ defines a projective cell complex $\emptyset \hookrightarrow W_+$ built by attaching a single projective 0-cell at $+$, the proof of Theorem 7.2.12 tells us that the simplicial subcomputad inclusion $\mathbb{1} + \mathcal{M}nd \hookrightarrow \mathrm{coll}(W_+)$ is defined by adjoining a single atomic 0-arrow from $\top$ to $+$ to the simplicial computad $\mathbb{1} + \mathcal{M}nd$.

The inclusion $\mathrm{coll}(W_+) \hookrightarrow \mathrm{coll}(W_-)$ is bijective on the common subcategory $\mathbb{1}+\mathcal{M}nd$ and defined by sending each n-arrow from $\top$ to $+$ in $\mathrm{coll}(W_+)$, represented as a squiggle from $+$ to $+$, to the squiggle from $-$ to $+$ defined by precomposing with $\underline{u}$. This function carries the unique atomic 0-arrow from $\top$ to $+$ in $\mathrm{coll}(W_+)$ to $\underline{u}$, which is the unique atomic 0-arrow in $\mathrm{coll}(W_-)$ from $\top$ to $+$. Now Theorem 7.2.12 proves that $W_+ \hookrightarrow W_-$ is a projective cell complex. Furthermore, since $\mathrm{coll}(W_+) \hookrightarrow \mathrm{coll}(W_-)$ is surjective on atomic 0-arrows, only projective cells of positive dimension are needed to present $W_+ \hookrightarrow W_-$ as a sequential composite of pushouts of projective n-cells $\partial\Delta[n]\times W_+ \hookrightarrow \Delta[n]\times W_+$. □

Now let $\mathcal{K}$ be an ∞-cosmos.

10.2.5. Definition. The **∞-category of $\mathbb{T}$-algebras** for a homotopy coherent monad $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$ in an ∞-cosmos $\mathcal{K}$ is the flexible weighted limit $\lim_{W_-} \mathbb{T}$. When $\mathbb{T}$ acts on the ∞-category B via the monad resolution (10.1.9) with functor part $t\colon B \to B$, we write

$$\mathrm{Alg}_{\mathbb{T}}(B) \coloneqq \lim_{W_-} \mathbb{T}$$

for the ∞-category of algebras. By Proposition 7.3.1(ii), the projective cell complex $W_+ \hookrightarrow W_-$ induces an isofibration

$$\lim_{W_-} \mathbb{T} \twoheadrightarrow \lim_{W_+} \mathbb{T}$$

upon taking weighted limits defining a map that we denote by $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ and refer to as the **monadic forgetful functor**. This map is the leg of the W_--shaped limit cone indexed by the unique object $+ \in \mathcal{M}nd$.

By Corollary 7.3.3 and Lemma 10.2.4:

10.2.6. Proposition. *Any homotopy coherent monad in an ∞-cosmos admits an ∞-category of algebras.* □

We now introduce the generic bar resolution $\mathbf{\Delta}_\top \to \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)$ associated to the ∞-category of $\mathbb{T}$-algebras for a homotopy coherent monad acting on B.

10.2.7. Definition (generic bar resolution). The limit cone $\beta\colon W_- \Rightarrow \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), \mathbb{T}(-))$ defines the **generic bar resolution** of a homotopy coherent monad $\mathbb{T}$ acting on an ∞-category B. By Lemma 10.2.2 and Example 10.2.3, a W_--cone with summit $\mathrm{Alg}_{\mathbb{T}}(B) \coloneqq \lim_{W_-} \mathbb{T}$ over a homotopy coherent monad acting on $\mathbb{T}$ is given by a simplicial map $\beta\colon \mathbf{\Delta}_\top \to \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)$ so that the square

$$\begin{tikzcd}
\mathbf{\Delta}_+ \times \mathbf{\Delta}_\top \arrow[r, "\mathbb{T}\times\beta"] \arrow[d, "\circ"'] & \mathsf{Fun}(B,B) \times \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B) \arrow[d, "\circ"] \\
\mathbf{\Delta}_\top \arrow[r, "\beta"'] & \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)
\end{tikzcd}$$

commutes. Under the identification $\mathbf{\Delta}_\top \cong \mathcal{A}dj(-,+)$, we write u^t and $\beta^t\colon tu^t \to u^t$ for the 0- and 1-arrows of $\mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)$ defined to be the images of $\underline{u}$ and $\underline{u\epsilon}$ under $\beta\colon \mathbf{\Delta}_\top \to \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)$

respectively. This 0-arrow u^t is the monadic forgetful functor of Definition 10.2.5. Then in the notation of (10.1.9), the generic bar resolution has the form of a homotopy coherent diagram

$$\begin{tikzcd} u^t \arrow[r, shift left, "\eta u^t"] & tu^t \arrow[l, shift left, "\beta^t"] \arrow[r, shift left=3, "\eta t u^t"] \arrow[r, shift right, "t\eta u^t"'] & ttu^t \arrow[l, shift right=1, "\mu u^t"'] \arrow[l, shift left=3, "t\beta^t"] \arrow[r, shift left=5] \arrow[r] \arrow[r, shift right=5] & tttu^t \arrow[l, shift right=2.5] \arrow[l, shift left=2.5] \arrow[l, shift left=7.5] & \cdots & \in \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B) \end{tikzcd} \tag{10.2.8}$$

that restricts along the embedding $\mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}_\top$ that freely adjoins the top element in each ordinal to the monad resolution (10.1.10) applied to u^t.

For any generalized element $X \to \mathrm{Alg}_{\mathbb{T}}(B)$ of the ∞-category of $\mathbb{T}$-algebras associated to a homotopy coherent monad acting on B, **an X-family of $\mathbb{T}$-algebras in B**, the generic bar resolution (10.2.8) restricts to define a **bar resolution**

$$\begin{tikzcd} b \arrow[r, shift left, "\eta b"] & tb \arrow[l, shift left, "\beta"] \arrow[r, shift left=3, "\eta t b"] \arrow[r, shift right, "t\eta b"'] & ttb \arrow[l, shift right=1, "\mu b"'] \arrow[l, shift left=3, "t\beta"] \arrow[r, shift left=5] \arrow[r] \arrow[r, shift right=5] & tttb \arrow[l, shift right=2.5] \arrow[l, shift left=2.5] \arrow[l, shift left=7.5] & \cdots \end{tikzcd} \tag{10.2.9}$$

10.2.10. PROPOSITION. *The monadic forgetful functor $u^t \colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ is conservative: for any 2-cell γ with codomain $\mathrm{Alg}_{\mathbb{T}}(B)$ if $u^t\gamma$ is invertible, then so is γ.*

PROOF. Conservativity of the functor u^t asserts that for all X the isofibration of quasi-categories $u^t_* \colon \mathsf{Fun}(X, \mathrm{Alg}_{\mathbb{T}}(B)) \twoheadrightarrow \mathsf{Fun}(X, B)$ reflects invertible 1-cells. Working with marked simplicial sets, this is the case just when this map has the right lifting property with respect to the inclusion $\mathbb{2} \hookrightarrow \mathbb{2}^\sharp$ of the walking arrow into the walking marked arrow.

By Definition 10.2.5, the monadic forgetful functor is defined by applying $\lim_{-} \mathbb{T}$ to the projective cell complex $W_+ \hookrightarrow W_-$ of Lemma 10.2.4. By Proposition 7.3.1(i), the isofibration $u^t \colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ then factors as the inverse limit of a tower of isofibrations, each layer of which is constructed as the pullback of products of projective cells $B^{\Delta[n]} \twoheadrightarrow B^{\partial\Delta[n]}$ for $n \geq 1$. The cosmological functor $\mathsf{Fun}(X, -) \colon \mathcal{K} \to \mathcal{QC}at$ preserves this limit, so $u^t_* \colon \mathsf{Fun}(X, \mathrm{Alg}_{\mathbb{T}}(B)) \twoheadrightarrow \mathsf{Fun}(X, B)$ is similarly the inverse limit of pullbacks of products of maps $\mathsf{Fun}(X, B)^{\Delta[n]} \twoheadrightarrow \mathsf{Fun}(X, B)^{\partial\Delta[n]}$ for $n \geq 1$. Since conservativity of a functor between quasi-categories may be captured by a lifting property, it suffices to show that the maps $\mathsf{Fun}(X, B)^{\Delta[n]} \twoheadrightarrow \mathsf{Fun}(X, B)^{\partial\Delta[n]}$ reflective invertibility of 1-simplices. Since for $n \geq 1$, the inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ is bijective on vertices, this is immediate from Corollary D.4.17, which says that invertibility in exponentiated quasi-categories is detected pointwise. $\square$

We now show that any homotopy coherent monad $\mathbb{T} \colon \mathcal{M}nd \to \mathcal{K}$ on an ∞-category B in an ∞-cosmos extends to a homotopy coherent adjunction $\mathbb{A}^{\mathbb{T}} \colon \mathcal{A}dj \to \mathcal{K}$ whose right adjoint is u^t.

10.2.11. PROPOSITION (the monadic adjunction). *For any homotopy coherent monad $\mathbb{T} \colon \mathcal{M}nd \to \mathcal{K}$ on B, the monadic forgetful functor $u^t \colon \mathrm{Alg}_{\mathbb{T}}(B) \to B$ is the right adjoint of a homotopy coherent adjunction $\mathbb{A}^{\mathbb{T}} \colon \mathcal{A}dj \to \mathcal{K}$*

$$\begin{tikzcd} B \arrow[r, bend left, "f^t"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[l, bend left, "u^t"] \end{tikzcd} \qquad \eta^t \colon \mathrm{id}_B \Rightarrow u^t f^t \qquad \epsilon^t \colon f^t u^t \Rightarrow \mathrm{id}_{\mathrm{Alg}_{\mathbb{T}}(B)}.$$

whose underlying homotopy coherent monad is $\mathbb{T}$. This constructs the ***monadic adjunction*** *of the homotopy coherent monad.*

In particular, the triple $(u^t f^t, \eta^t, u^t\epsilon^t f^t)$ recovers the monad (t, η, μ) on B defined in 10.1.8.

Proof. Recall the weights W_+ and W_- are defined in Example 10.2.3 to be restrictions of the representable functors on $\mathcal{A}dj$; in the case of W_+ this restriction defines the representable functor for $\mathcal{M}nd$ since the inclusion $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ is full on $+$. The weight for the monadic homotopy coherent adjunction is defined to be the composite of the Yoneda embedding with the restriction functor

$$\begin{tikzcd} \mathcal{A}dj^{\mathrm{op}} \arrow[r, "\text{よ}"] & s\mathcal{S}et^{\mathcal{A}dj} \arrow[r, "\mathrm{res}"] & s\mathcal{S}et^{\mathcal{M}nd} \end{tikzcd}$$

$$\begin{tikzcd} + \arrow[d, bend right, "\underline{f}"'] \arrow[rr, mapsto] \arrow[d, phantom, "\dashv"] & & W_+ \arrow[d, bend right, "-\circ \underline{f}"'] \arrow[d, phantom, "\vdash"] \\ - \arrow[u, bend right, "\underline{u}"'] \arrow[rr, mapsto] \arrow[r, phantom, "\mapsto"] & & W_- \arrow[u, bend right, "-\circ \underline{u}"'] \end{tikzcd}$$

which can be interpreted as defining an adjunction of weights whose left and right adjoints, in the encoding of Observation 10.2.1, are given by the maps

$$\begin{tikzcd} \mathbf{\Delta}_+ \arrow[r, bend left, "-\circ \underline{u}"] \arrow[r, phantom, "\perp"] & \mathbf{\Delta}_\top \arrow[l, bend left, "-\circ \underline{f}"] \end{tikzcd}$$

that act on strictly undulating n-arrows by precomposing with $\underline{u} = (+,-)$ or $\underline{f} = (-,+)$ as appropriate; these maps commute with the left $\mathbf{\Delta}_+$-actions by postcomposition with a strictly undulating squiggle from $+$ to $+$.

Composing with the weighted limit functor $\lim_{-} \mathbb{T}$ defines a simplicial functor

$$\mathbb{A}^{\mathbb{T}} := \lim_{\mathrm{res}\,\text{よ}(-)} \mathbb{T} \colon \mathcal{A}dj \to \mathcal{K},$$

i.e., a homotopy coherent adjunction between $\lim_{W_+} \mathbb{T} \cong B$ and $\lim_{W_-} \mathbb{T} \cong \mathrm{Alg}_{\mathbb{T}}(B)$ whose right adjoint is given by the action of the 0-arrow $\underline{u}$, which is the monadic forgetful functor $u^t \colon \mathrm{Alg}_{\mathbb{T}}(B) \to B$ is defined in 10.2.5.

Finally, the underlying homotopy coherent monad of the homotopy coherent adjunction just constructed is defined to be the limit of $\mathbb{T} \colon \mathcal{M}nd \to \mathcal{K}$ weighted by

$$\begin{tikzcd} \mathcal{M}nd^{\mathrm{op}} \arrow[r, hook] & \mathcal{A}dj^{\mathrm{op}} \arrow[r, "\text{よ}"] & s\mathcal{S}et^{\mathcal{A}dj} \arrow[r, "\mathrm{res}"] & s\mathcal{S}et^{\mathcal{M}nd} \end{tikzcd}$$

which is just the Yoneda embedding for $\mathcal{M}nd$. By the first axiom of Definition 7.1.3, this functor is isomorphic to $\mathbb{T}$. □

In §10.4, we give a characterization of the monadic adjunction of a homotopy coherent monad. To build towards this result, we spend the next section establishing important special properties of the monadic forgetful functor $u^t \colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ and its left adjoint $f^t \colon B \to \mathrm{Alg}_{\mathbb{T}}(B)$, whose essential image identifies the **free $\mathbb{T}$-algebras**.

Exercises.

10.2.i. Exercise. Prove that the ∞-category of algebras associated to the homotopy coherent monad $W_+ \colon \mathcal{M}nd \to \mathcal{Q}Cat$ on $\mathbf{\Delta}_+$ is $\mathbf{\Delta}_\perp$. What is the monadic adjunction?

10.3. Limits and colimits in the ∞-category of algebras

The key technical insight enabling Beck's proof of the monadicity theorem [**5**] is the observation that any algebra is canonically a colimit of a particular diagram of free algebras. In the case of a monad (t, η, μ) acting on a 1-category B, the data of a t-algebra in B is given by a *u^t-split coequalizer diagram*

$$\begin{tikzcd} t^2b \arrow[r, shift left=4.5, "\mu b"] \arrow[r, shift right=4.5, "t\beta"'] & tb \arrow[l, dashed, shift right=9, "\eta tb"'] \arrow[l, shift left=0, "t\eta b"] \arrow[r, "\beta"] & b \arrow[l, dashed, shift right=4.5, "\eta b"'] \end{tikzcd} \tag{10.3.1}$$

Here the solid arrows are maps which respect the t-algebra structure where the dotted splittings do not. Split coequalizers are examples of *absolute colimits*, which are preserved by any functor, and in particular by $t: B \to B$, a fact we may exploit to show that the underlying fork of (10.3.1) defines a reflexive coequalizer diagram in the category of t-algebras.

In the ∞-categorical context, we require a higher-dimensional version of the diagram (10.3.1), namely the bar resolution constructed in (10.2.9) for any generalized element $X \to \mathrm{Alg}_{\mathbb{T}}(B)$ of the ∞-category of $\mathbb{T}$-algebras for a homotopy coherent monad acting on B. This replaces the u^t-split coequalizer by a canonically-defined *u^t-split augmented simplicial object*.

Before defining this special class of colimits, we establish a more general result:

10.3.2. Proposition. *Let $\mathbb{T}: \mathcal{M}nd \to \mathcal{K}$ be a homotopy coherent monad on an ∞-category B with functor part $t: B \to B$. Then if B admits and t preserves colimits of shape J, then the monadic forgetful functor $u^t: \mathrm{Alg}_{\mathbb{T}}(B) \to B$ creates colimits of shape J.*

Proof. The 0-arrows in the image of a homotopy coherent monad $\mathbb{T}: \mathcal{M}nd \to \mathcal{K}$ are given by the identity functor at B, the "functor part" $t: B \to B$ defined as the image of the unique atomic 0-arrow of $\mathcal{M}nd$, and finite composites $t^n: B \to B$ for each $n \geq 1$. If t preserves colimits of shape J in B, then so does t^n. Thus, in the case where B admits and t preserves J-shaped colimits, the homotopy coherent monad lifts to homotopy coherent monad $\mathbb{T}: \mathcal{M}nd \to \mathcal{K}_{\perp,J}$ in the ∞-cosmos of Proposition 8.2.11. Since the inclusion $\mathcal{K}_{\perp,J} \hookrightarrow \mathcal{K}$ creates flexible weighted limits, such as those weighted by W_-, it follows that the limit cone $u^t: \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ lifts to $\mathcal{K}_{\perp,B}$. This monadic forgetful functor is the unique 0-arrow component of the limit cone, so by Proposition 8.2.11 this tells us that $\mathrm{Alg}_{\mathbb{T}}(B)$ admits and $u^t: \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ creates J-shaped colimits. □

A dual argument, employing the ∞-cosmos $\mathcal{K}_{\top,J}$ of ∞-categories that admit and functors that preserve J-indexed limits, proves that if B admits and t preserves limits of shape J, then the monadic forgetful functor $u^t: \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ creates limits of shape J. We don't explicitly consider this dual version here, however, because we will prove a stronger result in Theorem 10.3.9 that drops the hypotheses on t.

10.3.3. Definition (u-split simplicial objects). The image of the embedding

$$\begin{tikzcd} \mathbf{\Delta}_+^{\mathrm{op}} \arrow[r, hook, "{[0]\oplus-}"] \arrow[d, phantom, "\wr\Vert"] & \mathbf{\Delta}_\top \arrow[d, phantom, "\wr\Vert"] \\ \mathcal{A}dj(-,-) \arrow[r, hook, "\underline{u}\circ-"'] & \mathcal{A}dj(-,+) \end{tikzcd}$$

is the subcategory of $\mathbf{\Delta}_\top$ generated by all of its elementary operators except for the face operators $\delta^0: [n-1] \to [n]$ for each $n \geq 1$. We refer to these extra face maps as **splitting operators**. By Definition 2.3.9, a simplicial object $X \to B^{\mathbf{\Delta}^{\mathrm{op}}}$ in B **admits an augmentation** if it lifts along the restriction

functor $B^{\mathbf{\Delta}_+^{\mathrm{op}}} \twoheadrightarrow B^{\mathbf{\Delta}^{\mathrm{op}}}$, and an augmented simplicial object $X \to B^{\mathbf{\Delta}_+^{\mathrm{op}}}$ in B **admits a splitting** if it lifts along the restriction functor $B^{\mathbf{\Delta}_\top} \twoheadrightarrow B^{\mathbf{\Delta}_+^{\mathrm{op}}}$. Thus, given any functor $u\colon A \to B$ of ∞-categories, the ∞-categories $S^{\mathbf{\Delta}^{\mathrm{op}}}(u)$ and $S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u)$ of **u-split simplicial objects** and **u-split augmented simplicial objects in A** are defined by the pullbacks

$$\begin{tikzcd} S^{\mathbf{\Delta}^{\mathrm{op}}}(u) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & B^{\mathbf{\Delta}_\top} \arrow[d, two heads, "\mathrm{res}"] \\ A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \qquad \begin{tikzcd} S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & B^{\mathbf{\Delta}_\top} \arrow[d, two heads, "\mathrm{res}"] \\ A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}_+^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \end{tikzcd}$$

and there exists a forgetful functor

$$\begin{tikzcd} A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[d, two heads, "\mathrm{res}"'] & S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u) \arrow[l, two heads] \arrow[d, dashed] \arrow[dr] & \\ A^{\mathbf{\Delta}^{\mathrm{op}}} & S^{\mathbf{\Delta}^{\mathrm{op}}}(u) \arrow[l, two heads] \arrow[r] & B^{\mathbf{\Delta}_\top} \end{tikzcd}$$

Our interest in these notions is explained by the following example: if $u\colon A \to B$ is a right adjoint functor between ∞-categories, any homotopy coherent adjunction extending u defines a canonical u-split augmented simplicial object.

10.3.4. Lemma. *Let $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ be a homotopy coherent adjunction with right adjoint $u\colon A \to B$. Then the comonad resolution and bar resolution*

$$\begin{tikzcd} A \arrow[r, "\mathrm{bar}"] \arrow[d, "k_\bullet"'] & B^{\mathbf{\Delta}_\top} \arrow[d, two heads, "\mathrm{res}"] \\ A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}_+^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \end{tikzcd}$$

jointly define a u-split augmented simplicial object $A \to S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u)$.

Proof. Functoriality of $\mathbb{A}$ supplies a commutative diagram below-left

$$\begin{tikzcd} \mathbf{\Delta}_+^{\mathrm{op}} \cong \mathcal{A}dj(-,-) \arrow[r, "\mathbb{A}"] \arrow[d, "\underline{u}\circ-"'] & \mathsf{Fun}(A,A) \arrow[d, "u\circ-"] \\ \mathbf{\Delta}_\top \cong \mathcal{A}dj(-,+) \arrow[r, "\mathbb{A}"'] & \mathsf{Fun}(A,B) \end{tikzcd}$$

which internalizes to the commutative diagram of the statement. By the definition of the ∞-category of u-split augmented simplicial objects in 10.3.3, this induces the claimed functor $A \to S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u)$. Thus the comonad resolution $k_\bullet\colon A \to A^{\mathbf{\Delta}_+^{\mathrm{op}}}$ defines an augmented simplicial object in A that is u-split by the bar resolution for $\mathbb{A}$. $\square$

10.3.5. Proposition. *The monadic forgetful functor $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B$ creates colimits of u^t-split simplicial objects. Moreover, for any u^t-split augmented simplicial object, the augmentation defines the colimit cone for the underlying simplicial object in $\mathrm{Alg}_{\mathbb{T}}(B)$.*

Proof. The ∞-category of u^t-split simplicial objects is defined by the pullback

$$\begin{tikzcd} S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathbf{\Delta}_\top} \arrow[d, two heads, "\mathrm{res}"] \\ \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, two heads, "(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

By Proposition 2.3.11, the canonical 2-cell (2.3.10) defined by the initial object in $\mathbf{\Delta}_+$ defines an absolute left lifting diagram

$$\begin{tikzcd} & & & B \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t) \arrow[r, two heads] & B^{\mathbf{\Delta}_\top} \arrow[r, two heads, "\mathrm{res}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow B^{\nu^{\mathrm{op}}}", shift right=3] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.3.6}$$

that is also an absolute colimit in B, preserved by all functors and in particular by $t\colon B \to B$.

Now Proposition 10.3.2 tells us that $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ creates this colimit, which means that there exists an absolute left lifting diagram as below-left

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t) \arrow[r, two heads] \arrow[ur, dashed, "\mathrm{colim}"] \arrow[ur, phantom, "\Uparrow\lambda", shift right=3] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "u^t"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t) \arrow[r, two heads] \arrow[ur, dashed, "\mathrm{colim}"] \arrow[ur, phantom, "\Uparrow\lambda", shift right=3] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, two heads, "(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & B \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t) \arrow[r, two heads] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow B^{\nu^{\mathrm{op}}}", shift right=3] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

that when postcomposed with $(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}\colon \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \twoheadrightarrow B^{\mathbf{\Delta}^{\mathrm{op}}}$ recovers the absolute left lifting diagram (10.3.6), in the sense expressed by the pasting equality above-right. Thus, the monadic forgetful functor creates colimits of u^t-split simplicial objects in $\mathrm{Alg}_{\mathbb{T}}(B)$.

Upon precomposing with the $S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u^t) \to S^{\mathbf{\Delta}^{\mathrm{op}}}(u^t)$ the fact that Proposition 10.3.2 tells us that $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ creates the colimit (10.3.6) also means that whenever there exists a pasting equality

$$\begin{tikzcd} & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "u^t"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u^t) \arrow[r, two heads] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow \mathrm{Alg}_{\mathbb{T}}(B)^{\nu^{\mathrm{op}}}", shift right=3] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, two heads, "(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & & B \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u^t) \arrow[r, two heads] & B^{\mathbf{\Delta}_\top} \arrow[r, two heads, "\mathrm{res}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow B^{\nu^{\mathrm{op}}}", shift right=3] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

such as arises here by 2-functoriality of the simplicial cotensor construction, the 2-cell

$$\begin{tikzcd} & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u^t) \arrow[r, two heads] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow \mathrm{Alg}_{\mathbb{T}}(B)^{\nu^{\mathrm{op}}}", shift right=3] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

is an absolute left lifting diagram. This proves that $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \twoheadrightarrow B$ creates colimits of u^t-split simplicial objects. $\square$

Now, as displayed by the bar resolution (10.2.8), any $\mathbb{T}$-algebra in B canonically gives rise to a u^t-split simplicial object to which Proposition 10.3.5 applies; the bar resolution $\mathbf{\Delta}_\top \to \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), B)$

internalizes to a diagram $\mathrm{bar}\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B^{\mathbf{\Delta}_\top}$. The colimit cone in $\mathrm{Alg}_{\mathbb{T}}(B)$ is given by the $\mathbf{\Delta}_+^{\mathrm{op}}$-shaped subdiagram of the bar resolution that omits the dashed maps

$$\begin{tikzcd}
u^t \arrow[r, dashed, shift left=1, "\eta u^t"] & tu^t \arrow[l, shift left=1, "\beta^t"] \arrow[r, dashed, shift left=3, "\eta t u^t"] \arrow[r, shift right=1, "t\eta u^t"'] & ttu^t \arrow[l, shift right=1, "\mu u^t"'] \arrow[l, shift left=3, "t\beta^t"] \arrow[r, dashed, shift left=5] \arrow[r, shift left=1] \arrow[r, shift right=3] & tttu^t \arrow[l, shift left=3] \arrow[l, shift right=1] \arrow[l, shift right=5] & \cdots
\end{tikzcd}$$

This subdiagram admits a concise description: it is the comonad resolution for the comonad induced by the monadic adjunction $f^t \dashv u^t$ on $\mathrm{Alg}_{\mathbb{T}}(B)$, this being a functor $\mathbf{\Delta}_+^{\mathrm{op}} \to \mathsf{Fun}(\mathrm{Alg}_{\mathbb{T}}(B), \mathrm{Alg}_{\mathbb{T}}(B))$ that internalizes to a functor $k^t_\bullet\colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}}$.

10.3.7. Theorem (canonical colimit representation of algebras). *For any homotopy coherent monad $\mathbb{T}$ on B, the induced comonad resolution $k^t_\bullet\colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}}$ on the ∞-category of $\mathbb{T}$-algebras in B encodes an absolute left lifting diagram*

$$\begin{tikzcd}
& \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, shift right=4, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
& & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "k^t_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=4, "\Uparrow \mathrm{Alg}_{\mathbb{T}}(B)^{\nu^{\mathrm{op}}}"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
\tag{10.3.8}$$

created from the u^t-split simplicial object in B.

$$\begin{tikzcd}
& & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "u^t"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "k^t_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=4, "\Uparrow \mathrm{Alg}_{\mathbb{T}}(B)^{\nu^{\mathrm{op}}}"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
=
\begin{tikzcd}
& & & B \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "\mathrm{bar}"'] & B^{\mathbf{\Delta}_\top} \arrow[r, "\mathrm{res}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=4, "\Uparrow B^{\nu^{\mathrm{op}}}"] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}$$

Thus (10.3.8) *exists the algebras for a homotopy coherent monad as colimits of canonical simplicial objects of free algebras.*

Proof. Applying Lemma 10.3.4 to the monadic adjunction of Proposition 10.2.11, we see that the comonad resolution $k^t_\bullet\colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}}$ on $\mathrm{Alg}_{\mathbb{T}}(B)$ and the bar resolution $\mathrm{bar}\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B^{\mathbf{\Delta}_\top}$ defined in (10.2.8) together define a canonical u^t-split augmented cosimplicial object:

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{bar}"] \arrow[ddr, bend right, "k^t_\bullet"'] \\
& S^{\mathbf{\Delta}_+^{\mathrm{op}}}(u^t) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & B^{\mathbf{\Delta}_\top} \arrow[d, two heads, "\mathrm{res}"] \\
& \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[r, two heads, "(u^t)^{\mathbf{\Delta}_+^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}_+^{\mathrm{op}}}
\end{tikzcd}$$

Now the claimed result follows immediately from Proposition 10.3.5. $\square$

Our final task for this section is to generalize the dual of Proposition 10.3.2, proving that the monadic forgetful functor creates all limits that B admits, whether or not t preserves them.

10.3.9. THEOREM. *Let $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$ be a homotopy coherent monad on an ∞-category B. Then the monadic forgetful functor $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B$ creates all limits that B admits.*

PROOF. See [**83**, §5] for now. □

Exercises.

10.3.i. EXERCISE ([**40**, 4.2.3]). Let $t\colon B \to B$ be an endofunctor of an ∞-category, and define the **∞-category of t-algebras in B** by the pullback

$$\begin{tikzcd} \mathrm{alg}_t(B) \arrow[r] \arrow[d, two heads, "u^t"'] \arrow[dr, phantom, "\lrcorner" very near start] & B^2 \arrow[d, two heads, "{(p_1,p_0)}"] \\ B \arrow[r, "{(t,\mathrm{id}_B)}"'] & B \times B \end{tikzcd}$$

Adapt the proof of Proposition 10.3.2 to show that if B admits and t preserves colimits of shape J, then the forgetful functor $u^t\colon \mathrm{alg}_t(B) \to B$ creates colimits of shape J.

10.4. The monadicity theorem

Consider an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \qquad \eta\colon \mathrm{id}_B \Rightarrow uf, \quad \epsilon\colon fu \Rightarrow \mathrm{id}_A$$

between ∞-categories, that is in the homotopy 2-category of an ∞-cosmos. Theorem 9.3.4 proves that this data lifts to a homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$, which then restricts to define a homotopy coherent monad $\mathbb{T}\colon \mathcal{A}dj \to \mathcal{K}$ on B. Proposition 10.2.11 then constructs a new homotopy coherent adjunction with $\mathbb{T}$ as its underlying homotopy coherent monad: namely the **monadic adjunction** $f^t \dashv u^t$ between B and the ∞-category of $\mathbb{T}$-algebras $\mathrm{Alg}_{\mathbb{T}}(B)$. Immediately from its construction as a right Kan extension — there is a simplicial natural transformation from the first homotopy coherent adjunction to the second whose component at $+$ is the identity and whose component at $-$ defines a functor that we call $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ commuting strictly with all of the data of each homotopy coherent adjunction

$$\begin{tikzcd} A \arrow[rr, dashed, bend left, "r"] \arrow[dr, bend right, "u"'] & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, bend right, "u^t"'] \\ & B \arrow[ul, bend right, "f"'] \arrow[ul, phantom, "\dashv"] \arrow[ur, bend right, "f^t"'] \arrow[ur, phantom, "\vdash"] & \end{tikzcd}$$

This **monadicity theorem**, originally proven for 1-categories by Beck [**5**] and first proven for quasi-categories by Lurie [**62**], supplies conditions under which this comparison functor r is an equivalence, so that the ∞-category A can be identified with the ∞-category of $\mathbb{T}$-algebras.

To construct this simplicial natural transformation, we re-express the ∞-category of algebras as a weighted limit of the full homotopy coherent adjunction diagram, not merely as a weighted limit of its underlying homotopy coherent monad.

10.4.1. Proposition. *The ∞-category of algebras associated to the homotopy coherent monad underlying a homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ is the limit weighted by the weight $\operatorname{lan} W_-$ defined by the left Kan extension*

$$\begin{tikzcd} & \mathcal{A}dj \arrow[dr, dashed, "\operatorname{lan} W_-"] \arrow[d, phantom, "\Uparrow\cong"] & \\ \mathcal{M}nd \arrow[ur, hook] \arrow[rr, "W_-"'] & {} & s\mathcal{S}et \end{tikzcd}$$

Explicitly, $\operatorname{lan} W_-\colon \mathcal{A}dj \to s\mathcal{S}et$ is the homotopy coherent adjunction displayed on the top below

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}} \arrow[d, hook] \arrow[r, bend right, "\underline{u}\circ-"'] \arrow[r, phantom, "\perp"] & \mathbf{\Delta}_\top \arrow[l, bend right, "\underline{f}\circ-"'] \arrow[d, equal] & \leftrightsquigarrow & \mathcal{A}dj \xrightarrow{\operatorname{lan} W_-} s\mathcal{S}et \\ \mathcal{A}dj(-,-) \arrow[r, bend right, "\underline{u}\circ-"'] \arrow[r, phantom, "\perp"] & \mathcal{A}dj(-,+) \arrow[l, bend right, "\underline{f}\circ-"'] & \leftrightsquigarrow & \mathcal{A}dj \xrightarrow{\mathcal{A}dj_-} s\mathcal{S}et \end{tikzcd}$$

defined by restricting the domain of the right adjoint and codomain of the left adjoint of the representable adjunction $\mathcal{A}dj_-$ along the canonical inclusion $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$

Proof. Recall Lemma 7.1.20, which says that the weighted limit of a restricted diagram can be computed as the limit of the original diagram weighted by the left Kan extension of the weight. Thus

$$\lim^{\mathcal{A}dj}_{\operatorname{lan} W_-} \mathbb{A} \cong \lim^{\mathcal{M}nd}_{W_-} \operatorname{res} \mathbb{A}$$

recovers the ∞-category of algebras for the homotopy coherent monad underlying $\mathbb{A}$.

All that remains is to compute the functor $\operatorname{lan} W_-\colon \mathcal{A}dj \to s\mathcal{S}et$ explicitly. Because the inclusion $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ is full on $+$, $\operatorname{lan} W_-(+) \cong W_-(+) \cong \mathcal{A}dj(-,+)$, since W_- was defined as the restriction of the covariant representable functor $\mathcal{A}dj_-\colon \mathcal{A}dj \to s\mathcal{S}et$ along $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$. By the standard formula for left Kan extensions reviewed in Example 7.1.19 presented in the form of (7.1.6), the value of $\operatorname{lan} W_-$ at the object $-$ is computed by

$$\begin{aligned} \operatorname{lan} W_-(-) &\cong \int^{\mathcal{M}nd} \mathcal{A}dj(+,-) \times \mathcal{A}dj(-,+) \\ &\cong \operatorname{coeq}\left(\mathcal{A}dj(+,-) \times \mathcal{A}dj(+,+) \times \mathcal{A}dj(-,+) \overset{\circ\times\mathrm{id}}{\underset{\mathrm{id}\times\circ}{\rightrightarrows}} \mathcal{A}dj(+,-) \times \mathcal{A}dj(-,+) \right). \end{aligned}$$

By associativity of composition in $\mathcal{A}dj$, the composition map

$$\mathcal{A}dj(+,-) \times \mathcal{A}dj(-,+) \xrightarrow{\ \circ\ } \mathcal{A}dj(-,-)$$

defines a cone under the coequalizer diagram. By the graphical calculus and Proposition 9.1.10, the image of this map in $\mathcal{A}dj(-,-) \cong \mathbf{\Delta}_+^{\mathrm{op}}$ is comprised of those strictly undulating squiggles from $-$ to $-$

that pass through $+$. This is the subcategory $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$. In fact, we claim that

$$\mathcal{A}dj(+,-) \times \mathcal{A}dj(+,+) \times \mathcal{A}dj(-,+) \underset{\mathrm{id}\times\circ}{\overset{\circ\times\mathrm{id}}{\rightrightarrows}} \mathcal{A}dj(+,-) \times \mathcal{A}dj(-,+) \xrightarrow{\circ} \mathbf{\Delta}^{\mathrm{op}} \tag{10.4.2}$$

is a coequalizer diagram. The map from the coequalizer to $\mathbf{\Delta}^{\mathrm{op}}$ is surjective for the reason just described: a strictly undulating squiggle from $-$ to $-$ that passes through $+$ can be decomposed as a horizontal composite of a squiggle in $\mathcal{A}dj(-,+)$ followed by a squiggle in $\mathcal{A}dj(+,-)$. To see that the map from the coequalizer to $\mathbf{\Delta}^{\mathrm{op}}$ is injective, consider two distinct subdivisions of a squiggle from $-$ to $-$ into a pair of squiggles from $-$ to $+$ and from $+$ to $-$. The subsquiggle between the two chosen $+$ symbols in this an element of $\mathcal{A}dj(+,+)$, and thus this pair of elements of $\mathcal{A}dj(+,-) \times \mathcal{A}dj(-,+)$ are identified in the coequalizer diagram. ☐

10.4.3. Lemma. *For any homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$, there exists a simplicial natural transformation from $\mathbb{A}$ to the monadic adjunction $\lim^{\mathcal{A}dj}_{\mathrm{lan}\, W_-} \mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ whose components at $+$ and $-$ defined on weights by the counit of the adjunction*

$$\begin{tikzcd} s\mathcal{S}et^{\mathcal{A}dj} \arrow[r, bend right, "\mathrm{res}"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{\mathcal{M}nd} \arrow[l, bend right, "\mathrm{lan}"'] \end{tikzcd} \qquad \begin{array}{c} \mathrm{lan}\, W_- = \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_- \xrightarrow{\epsilon_-} \mathcal{A}dj_- \\[1em] \mathcal{A}dj_+ \cong \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_+ \underset{\cong}{\xrightarrow{\epsilon_+}} \mathcal{A}dj_+ \end{array}$$

Proof. Consider the diagram of weights in $s\mathcal{S}et^{\mathcal{A}dj}$.

$$\begin{tikzcd} \mathcal{A}dj_- \arrow[dr, bend left, "-\circ\underline{f}"] \arrow[dr, phantom, "\dashv"] & & \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_- \arrow[ll, bend right, "\epsilon"'] \arrow[dl, bend left, "\mathrm{lan}\,\mathrm{res}(-\circ\underline{f})"] \arrow[dl, phantom, "\dashv"] \\ & \mathcal{A}dj_+ \cong \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_+ \arrow[ul, bend left, "-\circ\underline{u}"] \arrow[ur, bend left, "\mathrm{lan}\,\mathrm{res}(-\circ\underline{u})"] & \end{tikzcd}$$

Applying these weights to a homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ with underlying adjunction $f \dashv u\colon A \to B$ yields

$$\begin{tikzcd} A \arrow[rr, dashed, bend left, "r"] \arrow[dr, bend right, "u"'] \arrow[dr, phantom, "\dashv"] & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, bend right, "u^t"'] \arrow[dl, phantom, "\dashv"] \\ & B \arrow[ul, bend right, "f"'] \arrow[ur, bend right, "f^t"'] & \end{tikzcd}$$

with the component ϵ_- inducing the non-identity component of the canonical comparison functor with the monadic adjunction. ☐

10.4.4. Example. Returning to Example 10.1.11, there is a Kan-complex enriched category $\mathcal{M}on(\mathcal{M})$ of monoids in $\mathcal{M}$ equipped with a simplicially enriched adjunction

$$\begin{tikzcd} \mathcal{M}on(\mathcal{M}) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Applying the homotopy coherent nerve, this defines a homotopy coherent adjunction between the quasi-categories $\mathfrak{N}\mathcal{M}on(\mathcal{M})$ and $\mathfrak{N}\mathcal{M}$. By Lemma 10.4.3, there is a canonical comparison map to the monadic homotopy coherent adjunction

$$\begin{tikzcd}
\mathfrak{N}\mathcal{M}on(\mathcal{M}) \arrow[dr, bend right, "u"'] \arrow[rr, dashed, bend left, "r"] & & \mathrm{Alg}_{\mathbb{T}}(\mathfrak{N}\mathcal{M}) \arrow[dl, bend right, "u^t"'] \\
& \mathfrak{N}\mathcal{M} \arrow[ul, bend right, "f"'] \arrow[ul, phantom, "\perp"] \arrow[ur, bend right, "f^t"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

that is not an equivalence. Elements of $\mathfrak{N}\mathcal{M}on(\mathcal{M})$ are strict monoids in $\mathcal{M}$, while elements of $\mathrm{Alg}_{\mathbb{T}}(\mathfrak{N}\mathcal{M})$ are homotopy coherent monoids, objects $X \in \mathcal{M}$ equipped with n-ary multiplication maps $\mu_n \colon X^{\otimes n} \to X$ for all n that are coherently associative up to higher homotopy.

10.4.5. Lemma. *Let $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ be a homotopy coherent adjunction with right adjoint $u \colon A \to B$ and let $\mathbb{A}^{\mathbb{T}} \colon \mathcal{A}dj \to \mathcal{K}$ be the associated monadic adjunction. Then there is a canonical functor $L \colon \mathrm{Alg}_{\mathbb{T}}(B) \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ that*

(i) is u-split by the bar resolution $\mathrm{bar} \colon \mathrm{Alg}_{\mathbb{T}}(B) \to B^{\mathbf{\Delta}^{\mathrm{op}}}$,
(ii) is so that the composite $L \circ r \colon A \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ is the simplicial object underlying the comonad resolution $k_\bullet \colon A \to A^{\mathbf{\Delta}_+^{\mathrm{op}}}$, and
(iii) is so that the composite $r^{\mathbf{\Delta}^{\mathrm{op}}} \circ L \colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}}$ is the simplicial object underlying the comonad resolution $k^t_\bullet \colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}_+^{\mathrm{op}}}$.

Proof. The first two statements ask for a functor L that fits into a commutative diagram below-left

$$\begin{tikzcd}
A \arrow[r, "k_\bullet"] \arrow[d, "r"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[d, two heads, "\mathrm{res}"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, dashed, "L"] \arrow[d, "\mathrm{bar}"'] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d, "u^{\mathbf{\Delta}^{\mathrm{op}}}"] \\
B^{\mathbf{\Delta}_\top} \arrow[r, two heads, "\mathrm{res}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
\qquad \leftrightsquigarrow \qquad
\begin{tikzcd}
\mathcal{A}dj_ & \mathcal{A}dj_ \times \mathbf{\Delta}_+^{\mathrm{op}} \arrow[l, "\circ"'] \\
\mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_ \arrow[u, "\epsilon"] & \mathcal{A}dj_ \times \mathbf{\Delta}^{\mathrm{op}} \arrow[l, dashed, "\circ"'] \arrow[u] \\
\mathcal{A}dj_+ \times \mathbf{\Delta}_\top \arrow[u, "\circ"] & \mathcal{A}dj_+ \times \mathbf{\Delta}^{\mathrm{op}} \arrow[l, hook', "\mathrm{id}\times(\underline{u}\circ-)"] \arrow[u, "(-\circ\underline{u})\times\mathrm{id}"']
\end{tikzcd}$$

in which each of the objects and all but the map L have been described as maps induced by taking weighted limits of the homotopy coherent adjunction diagram, with the weights in $s\mathcal{S}et^{\mathcal{A}dj}$ displayed above-right. By the Yoneda lemma, each of the three maps of weights labeled "$\circ$" are defined by a single map of simplicial sets. In the case of $\circ \colon \mathcal{A}dj\_ \times \mathbf{\Delta}_+^{\mathrm{op}} \to \mathcal{A}dj\_$ the Yoneda lemma says it suffices to define a map $\mathbf{\Delta}_+^{\mathrm{op}} \to \mathcal{A}dj\_(-) = \mathbf{\Delta}_+^{\mathrm{op}}$; we take this map to be the identity, which implies that $\circ \colon \mathcal{A}dj\_ \times \mathbf{\Delta}_+^{\mathrm{op}} \to \mathcal{A}dj\_$ acts in both components by composing over $-$ in $\mathcal{A}dj$. In light of the explicit description of the adjunction $\mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj\_$ given in Proposition 10.4.1 the other two maps labelled "$\circ$" may be defined similarly by identity maps. Since the dashed map makes the right-hand diagram of weights commute, the induced functor on weighted limits has the desired properties (i) and (ii).

The final statements demands commutativity of the diagram below left, which again follows from the commutativity of the corresponding diagram of weights below-right

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "L"'] \arrow[dr, "k_\bullet^t"] & & & \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_{-} & \\ A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} & \leftrightsquigarrow & \mathcal{A}dj_{-} \times \mathbf{\Delta}^{\mathrm{op}} \arrow[u, "\circ"] & \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_{-} \times \mathbf{\Delta}^{\mathrm{op}} \arrow[l, "\epsilon\times\mathrm{id}"] \arrow[ul, "\circ"'] \end{tikzcd}$$

this just amounting to the simple observation that the counit component $\epsilon\colon \mathrm{lan}\,\mathrm{res}\,\mathcal{A}dj_{-} \hookrightarrow \mathcal{A}dj_{-}$ is just given by the natural inclusion $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$. □

10.4.6. Lemma. *Given any homotopy coherent adjunction with left adjoint $f\colon B \to A$, the diagram defined by restricting the canonical cone* (2.3.10) *built from the interalized comonad resolution $k_\bullet\colon A \to A^{\mathbf{\Delta}_+^{\mathrm{op}}}$ along $f\colon B \to A$*

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ B \arrow[ur, "f"] \arrow[r, "k_\bullet f"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\zeta"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \quad := \quad \begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ B \arrow[r, "f"'] & A \arrow[r, "k_\bullet"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

displays f as an absolute colimit of the family of diagrams $k_\bullet f\colon B \to A^{\mathbf{\Delta}^{\mathrm{op}}}$.

Proof. The homotopy coherent adjunction provides a commutative diagram below-left

$$\begin{tikzcd} \mathbf{\Delta}_+^{\mathrm{op}} \arrow[r, "\mathbb{A}"] \arrow[d, hook, "\underline{f}\circ-"'] & \mathsf{Fun}(A,A) \arrow[d, "f\circ-"] \\ \mathbf{\Delta}_\perp \arrow[r, "\mathbb{A}"] & \mathsf{Fun}(B,A) \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\mathrm{cobar}"'] & A \arrow[d, "k_\bullet"] \\ A^{\mathbf{\Delta}_\top} \arrow[r, two heads, "\mathrm{res}"] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \end{tikzcd}$$

which transposes to the commutative diagram above-right, which tells us that when the internalized comonad resolution $k_\bullet\colon A \to A^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is restricted along f, it extends to a split augmented simplicial object, with the splittings on the opposite side as usual; this is no matter since $\mathbf{\Delta}_+^{\mathrm{op}}$, considered as a full sub 2-category of $s\mathcal{S}et$ spanned by finite ordinals, is isomorphic to its co-dual via an isomorphism that commutes with $\mathbf{\Delta}_\top^{\mathrm{co}} \cong \mathbf{\Delta}_\perp$ This tells us that the colimit cone of the statement is the one of Proposition 2.3.11. □

There are many versions of the monadicity theorem. For expediency's sake, we prove just one for now. We break its statement into two parts, first constructing a left adjoint to the canonical comparison functor, which under additional hypotheses we prove defines an adjoint equivalence.

10.4.7. Theorem. *Let $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ be a homotopy coherent adjunction with right adjoint $u\colon A \to B$ with underlying homotopy coherent monad $\mathbb{T}\colon \mathcal{M}nd \to \mathcal{K}$. If A admits colimits of u-split simplicial objects then the canonical comparison functor admits a left adjoint:*

$$\begin{tikzcd} A \arrow[r, bend right, "r"'] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[l, bend right, dashed, "\ell"'] \end{tikzcd}$$

Proof. If A admits colimits of u-split simplicial objects, then there exists an absolute left lifting of the u-split simplicial object $L\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ defined in Lemma 10.4.5

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, dashed, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.4.8}$$

whose functor part we take to be the definition of the left adjoint $\ell\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$. By Lemma 10.4.5(ii), the diagram defined by restricting along r agrees with the cosimplicial object underlying the comonad resolution, which has a canonical cone (2.3.10) as displayed below-left:

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A \arrow[urr, bend left, equal] \arrow[r, "k_\bullet"'] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A \arrow[urr, bend left, equal] \arrow[r, "r"'] \arrow[ur, phantom, "\exists!\Uparrow\epsilon"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.4.9}$$

By the universal property of the absolute left lifting diagram $(\ell r, \lambda r)$, this induces a unique 2-cell $\epsilon\colon \ell r \Rightarrow \mathrm{id}_A$.

The unit is induced from the absolute left lifting diagram (10.3.8). By Lemma 10.4.5(iii), the comonad resolution $k^t_\bullet\colon \mathrm{Alg}_{\mathbb{T}}(B) \to \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}}$ factors as $r^{\mathbf{\Delta}^{\mathrm{op}}} \cdot L$, so the pasted composite below left factors through the absolute left lifting diagram as below right.

$$\begin{tikzcd} & A \arrow[r, "r"] \arrow[d, "\Delta"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} A \arrow[r, "r"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[u, "\ell"] \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, shift left=3, "\exists!\Uparrow\eta"] \arrow[ur, phantom, shift right=3, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.4.10}$$

To verify the triangle equalities, note that by construction

$$\begin{tikzcd} & A \arrow[r, "r"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ A \arrow[ur, equal] \arrow[r, "r"'] \arrow[ur, phantom, shift right=3, "\Uparrow\epsilon"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[u, "\ell"] \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, shift left=3, "\Uparrow\eta"] \arrow[ur, phantom, shift right=3, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & A \arrow[r, "r"] \arrow[d, "\Delta"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ A \arrow[urr, bend left, equal] \arrow[r, "r"'] \arrow[ur, phantom, "\Uparrow\epsilon"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

$$= \begin{tikzcd} & A \arrow[r, "r"] \arrow[d, "\Delta"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ A \arrow[ur, equal] \arrow[r, "k_\bullet"'] \arrow[ur, phantom, shift right=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ A \arrow[r, "r"'] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, shift right=3, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

the last equality following from simplicial naturality of r and the definition of β as $\mathrm{Alg}_{\mathbb{T}}(B)^{\nu^{\mathrm{op}}}$ in Theorem 10.3.7. Thus the triangle equality composite $r\epsilon \cdot \eta r = \mathrm{id}_r$.

It follows that the other triangle equality composite $\phi \coloneqq \epsilon\ell \cdot \ell\eta$ is an idempotent:

$$\phi \cdot \phi \coloneqq (\epsilon\ell \cdot \ell\eta) \cdot (\epsilon\ell \cdot \ell\eta) = \epsilon\ell \cdot \epsilon\ell r\ell \cdot \ell r\ell\eta \cdot \ell\eta = \epsilon\ell \cdot \ell r\epsilon\ell \cdot \ell\eta r\ell \cdot \ell\eta = \epsilon\ell \cdot \ell\eta =: \phi,$$

so to prove that $\epsilon\ell \cdot \ell\eta = \mathrm{id}_\ell$ it suffices to show that ϕ is an isomorphism. To demonstrate this, we will show:

(i) that ϕf^t is invertible, i.e., that ϕ is an isomorphism when restricted to free $\mathbb{T}$-algebras
(ii) and that the putative left adjoint ℓ preserves the canonical colimit (10.3.8) that expresses every $\mathbb{T}$-algebra as a colimit of free $\mathbb{T}$-algebras.

We then combine (i) and (ii) to argue that ϕ is invertible.

To this end, we first observe by Lemma 10.4.6 and the definition of ℓ above that we have a pair of absolute left lifting diagrams:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ B \arrow[ur, "f"] \arrow[r, "k_\bullet f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \quad\cong\quad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ B \arrow[r, "f^t"'] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

By simplicial naturality of the canonical comparison map $rf = f^t$, and by Lemma 10.4.5(ii) $Lf^t = Lrf = k_\bullet f$. Thus the absolute left lifting problems coincide and we obtain a canonical natural isomorphism $\gamma\colon \ell f^t \cong f$.

Now to prove the claim of (i) that ϕf^t is an isomorphism it suffices to prove that ηf^t is an isomorphism and that $\epsilon\ell f^t$ is an isomorphism — and by naturality of whiskering and the isomorphism $\gamma\colon \ell f^t \cong f$ just constructed, $\epsilon\ell f^t$ is an isomorphism if and only if ϵf is an isomorphism.

By (10.4.10), the construction of γ, and simplicial naturality of r, which implies that $r^{\mathbf{\Delta}^{\mathrm{op}}}k_\bullet = k^t_\bullet r$:

$$\begin{tikzcd} & A \arrow[r, "r"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\eta"] \\ B \arrow[ur, "f"] \arrow[r, "f^t"'] \arrow[ur, phantom, shift right=3, "\cong\Uparrow\gamma"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[u, "\ell"] \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[r, phantom, shift left=4, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & A \arrow[r, "r"] \arrow[d, "\Delta"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ B \arrow[urr, bend left, "f"] \arrow[r, "f^t"'] \arrow[r, phantom, shift left=5, "\cong\Uparrow\gamma"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

$$= \begin{tikzcd} & A \arrow[r, "r"] \arrow[d, "\Delta"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ B \arrow[ur, "f"] \arrow[r, "k_\bullet f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "r^{\mathbf{\Delta}^{\mathrm{op}}}"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[d, "\Delta"] \\ B \arrow[r, "f^t"'] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] \arrow[r, phantom, shift left=4, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

So ηf^t is the inverse of the isomorphism $r\gamma$, and is consequently invertible.

Similarly, by (10.4.9),

$$\begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ B \arrow[r, "f"'] & A \arrow[urr, bend left, equal] \arrow[r, "r"'] \arrow[urr, phantom, shift right=2, "\Uparrow\epsilon"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & A \arrow[d, "\Delta"] \\ B \arrow[ur, "f"] \arrow[r, "k_\bullet f"'] \arrow[ur, phantom, shift right=3, "\Uparrow\zeta"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ B \arrow[urr, bend left, "f"] \arrow[r, "f^t"'] \arrow[r, phantom, shift left=5, "\Uparrow\cong\gamma"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

so $\epsilon f = \gamma$ is also an isomorphism. Thus, we conclude that ϕf^t is invertible as claimed in (i).

To prove (ii), we must show that

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "\ell"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equals] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"'] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.4.11}$$

is an absolute left lifting diagram. Of course, we expect this to be true because left adjoints preserve colimits by Theorem 2.4.2, but as we have not yet shown that ℓ is a left adjoint, this requires a direct argument.

By the equational characterization of Theorem ??, the cosmological functor $(-)^{\mathbf{\Delta}^{\mathrm{op}}}\colon \mathcal{K} \to \mathcal{K}$ preserves the absolute left lifting diagram (10.4.8); thus

$$\begin{tikzcd} & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d, "\Delta^{\mathbf{\Delta}^{\mathrm{op}}}"] \\ \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[ur, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"] \arrow[r, "L^{\mathbf{\Delta}^{\mathrm{op}}}"'] \arrow[ur, phantom, "\Uparrow\lambda^{\mathbf{\Delta}^{\mathrm{op}}}"] & (A^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

is absolute left lifting. Since $\lambda^{\mathbf{\Delta}^{\mathrm{op}}} \cdot \Delta = \Delta^{\mathbf{\Delta}^{\mathrm{op}}} \cdot \lambda$, there are two equivalent ways to compute the horizontal composite of this 2-cell with β, displayed below-left and below-right

$$\begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[urr, "\ell"] \arrow[d, "\Delta"] & & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d, "\Delta^{\mathbf{\Delta}^{\mathrm{op}}}"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equals] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[urr, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"] \arrow[rr, "L^{\mathbf{\Delta}^{\mathrm{op}}}"'] \arrow[urr, phantom, "\Uparrow\lambda^{\mathbf{\Delta}^{\mathrm{op}}}"] & & (A^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[urr, "\ell"] \arrow[rr, "L"'] \arrow[urr, phantom, "\Uparrow\lambda"] \arrow[d, "\Delta"] & & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d, "\Delta^{\mathbf{\Delta}^{\mathrm{op}}}"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equals] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[rr, "L^{\mathbf{\Delta}^{\mathrm{op}}}"'] & & (A^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

By Lemma 2.4.1, to show that (10.4.11) is absolute left lifting, i.e., that ℓ preserves the absolute left lifting diagram β, it suffices to prove that L preserves the absolute left lifting diagram β. By Proposition 4.3.15, to show that this diagram is absolute left lifting it suffices to show that for each $[n] \in \mathbf{\Delta}$, that the diagram

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "L"] \arrow[d, "\Delta"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\mathrm{ev}_n"] \arrow[d, "\Delta^{\mathbf{\Delta}^{\mathrm{op}}}"] & A \arrow[d, "\Delta"] \\ \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equals] \arrow[r, "k^t_\bullet"'] \arrow[ur, phantom, "\Uparrow\beta"] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "L^{\mathbf{\Delta}^{\mathrm{op}}}"'] & (A^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\mathrm{ev}_n"'] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

is absolute left lifting.

By the construction of L in Lemma 10.4.5, $\mathrm{ev}_n L\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$ is the map induced by taking the weighted limits of the homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$ by the map of weights $- \circ (\underline{f}\underline{u})^{n+1}\colon \mathcal{A}dj_{\_} \to \operatorname{lan}\operatorname{res}\mathcal{A}dj_{\_}$. Thus we see that $\mathrm{ev}_n L$ is the map $ft^n u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$, and in particular factors through $u^t\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B$. Since the canonical colimit of Theorem 10.3.7 is u^t-split, $(u^t, u^t\beta)$ is an absolute left lifting diagram preserved under postcomposition by all functors, and in particular by $f(uf)^n$. Thus, the above diagram is absolute left lifting as claimed, which tells us that L and thus ℓ preserves the colimit (10.3.8).

It remains only to combine (i) and (ii) to argue that ϕ is invertible. For this, we consider the pasting equality

$$
\begin{tikzcd}
& \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "\ell"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\beta"] & A \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"] \arrow[r, bend right, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"'] \arrow[r, phantom, "\Uparrow\phi^{\mathbf{\Delta}^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
=
\begin{tikzcd}
& \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, bend left, "\ell"] \arrow[r, bend right, "\ell"'] \arrow[r, phantom, "\Uparrow\phi"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\beta"] & A \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\ell^{\mathbf{\Delta}^{\mathrm{op}}}"'] & A^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
$$

By the definition of $k^t_\bullet$, the components of the whiskered natural transformation $\phi^{\mathbf{\Delta}^{\mathrm{op}}} \cdot k^t_\bullet$ at $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ is $\phi(f^t u^t)^{n+1}$, which is an isomorphism by (i). By Corollary D.4.17, this proves that $\phi^{\mathbf{\Delta}^{\mathrm{op}}} \cdot k^t_\bullet$ is invertible. Thus, by (ii) the left hand diagram is isomorphic to an absolute left lifting diagram and thus is absolute left lifting. The pasting equality describes a factorization of the left hand absolute left lifting diagram through the absolute left lifting diagram of (ii) via ϕ, so by the uniqueness in the universal property of absolute left lifting diagrams we conclude that ϕ is invertible as desired. This proves that $(\ell \dashv r, \eta, \epsilon)$ defines an adjunction as claimed. ☐

We now describe conditions under which the adjunction just constructed defines an adjoint equivalence. As the proof will reveal, condition (ii) implies that the unit is an isomorphism, while conditions (ii) and (iii) together imply that the counit is an isomorphism.

10.4.12. Theorem (monadicity). *Let $\mathbb{A}\colon \mathcal{Adj} \to \mathcal{K}$ be a homotopy coherent adjunction with right adjoint $u\colon A \to B$ with underlying homotopy coherent monad $\mathbb{T}\colon \mathcal{Mnd} \to \mathcal{K}$. If*

(i) A admits colimits of u-split simplicial objects,
(ii) $u\colon A \to B$ preserves them, and
(iii) $u\colon A \to B$ is conservative

then the canonical comparison functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ admits a left adjoint $\ell\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$ defining an adjoint equivalence.

Note that Theorem 10.3.7 and Proposition 10.2.10 establish these properties for the monadic adjunction.

Proof. Theorem 10.4.7 constructs an adjunction $(\ell \dashv r, \eta, \epsilon)$ under the hypothesis (i), with the left adjoint $\ell\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$ defined as the colimit of the u-split family of diagrams $L\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ with colimit cone $\lambda\colon \Delta\ell \Rightarrow L$. It remains to show that this defines an adjoint equivalence.

By hypothesis (ii), u preserves the u-split colimit diagram that defines ℓ. By Lemma 10.4.5(i), $u^{\mathbf{\Delta}^{\mathrm{op}}}L\colon \mathrm{Alg}_{\mathbb{T}}(B) \to B^{\mathbf{\Delta}^{\mathrm{op}}}$ is the monadic bar resolution, so the absolute left lifting diagrams below-left and below-center are isomorphic:

$$
\begin{tikzcd}
& A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, "\ell"] \arrow[r, "L"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
\cong
\begin{tikzcd}
& & B \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[urr, bend left, "u^t"] \arrow[r, "\mathrm{bar}"'] & B^{\mathbf{\Delta}_\top} \arrow[ur, "\mathrm{ev}_0"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, shift right=3, "\Uparrow B^{\nu^{\mathrm{op}}}"] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
=
\begin{tikzcd}
& \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "u^t"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Uparrow\beta"] & B \arrow[d, "\Delta"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[ur, equal] \arrow[r, "k^t_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "(u^t)^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}
$$

By the construction of the bar resolution in Lemma 10.3.4, the absolute left lifting diagram above-left coincides with the one above-right. Now consulting (10.4.10), we see that the left-hand diagram

above factors through the right-hand diagram above via $u^t\eta$. Thus $u^t\eta$ is invertible, and by Proposition 10.2.10, $\eta\colon \mathrm{id}_{\mathrm{Alg}_{\mathbb{T}}(B)} \cong r\ell$ is also an isomorphism.

By the same line of reasoning, the diagrams $Lr\colon A \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ and $k_\bullet\colon A \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ are both u-split by the bar resolution by Lemmas 10.4.5 and 10.3.4 respectively. Again by the hypothesis that $u\colon A \to B$ preserves u-split colimits, the absolute left lifting diagram below-left must be isomorphic to the one below-center, which equals the absolute left lifting diagram below-right:

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ A \arrow[ur, "\ell r"] \arrow[r, "Lr"'] \arrow[ur, phantom, "\Uparrow\lambda r"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \cong \begin{tikzcd} & & B \arrow[d, "\Delta"] \\ A \arrow[urr, bend left, "u"] \arrow[r, "\mathrm{bar}"'] & B^{\mathbf{\Delta}_\top} \arrow[ur, "\mathrm{ev}_0"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Uparrow B^{\nu^{\mathrm{op}}}"] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} = \begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ A \arrow[ur, equal] \arrow[r, "k_\bullet"'] \arrow[ur, phantom, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "u^{\mathbf{\Delta}^{\mathrm{op}}}"'] & B^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

By the uniqueness of factorization through absolute lifting diagrams it follows that $u\epsilon$ must be an isomorphism, and since u is conservative, this means that $\epsilon\colon \ell r \cong \mathrm{id}_A$ is also invertible.

This completes the proof, but in fact we can sidestep the most difficult part of the proof of Theorem 10.4.7 — the proof of the triangle equality $\epsilon\ell \cdot \ell\eta = \mathrm{id}_\ell$ — under the additional hypotheses (ii) and (iii) that imply invertibility of the unit and counit constructed there. Since η and ϵ are both invertible, the triangle equality composite $\epsilon\ell \cdot \ell\eta$ is an isomorphism. Since the other triangle equality composite is an identity, as in the proof of Proposition 2.1.11 this composite is also an idempotent and hence, by cancelation, this idempotent isomorphism is also an identity. □

The dual to Theorem 10.4.12 characterizes those left adjoints that define *comonadic* functors, whose domain is equivalent to the ∞-category of coalgebras for the associated homotopy coherent comonad; see Exercise 10.4.i. When a functor $u\colon A \to B$ admits both left and right adjoints, then it turns out that these conditions coincide: if the adjunction $\ell \dashv u$ is monadic then $u \dashv r$ is comonadic and conversely, as the following result shows.

10.4.13. PROPOSITION. *Suppose $u\colon A \to B$ admits both left and right adjoints*

$$\begin{tikzcd} A \arrow[r, "u" description] & B \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

Then if u is monadic then u is also comonadic, and if u is comonadic, then u is also monadic.

PROOF. We'll show that if u is monadic, then u is also comonadic. The converse implication is dual. If u is monadic, then u is conservative, by Proposition 10.2.10. By Proposition 2.3.11, B admits limits of u-split cosimplicial objects, and by Theorem 10.3.9, $u\colon A \to B$ creates them, so A admits and u preserves such colimits as well. Since we have assumed that u admits a right adjoint, Theorem 9.3.4 can be used to construct a homotopy coherent adjunction with left adjoint u. Now, by the dual of Theorem 10.4.12, we conclude that u is comonadic as claimed. □

Exercises.

10.4.i. EXERCISE. Dualize the work of this section to define and characterize the *comonadic* adjunction between A and the ∞-category of coalgebras for the homotopy coherent comonad acting on A.

10.5. Monadic descent

Consider once more the canonical comparison functor

$$\begin{tikzcd} A \arrow[rr, dashed, bend left, "r"] \arrow[dr, bend right, "u"'] & & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, bend right, "u^t"'] \\ & B \arrow[ul, bend right, "f"'] \arrow[ur, bend right, "f^t"'] & \end{tikzcd}$$

defining the component at $- \in \mathcal{A}dj$ of a simplicial natural transformation $\mathbb{A} \to \mathbb{A}^{\mathbb{T}}$ from a homotopy coherent adjunction to the monadic homotopy coherent adjunction built from its underlying homotopy coherent monad. The monadicity theorem of the previous section characterizes when the comparison functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ is an equivalence. Our aim in this section is to present a theorem proven by Sulyma [**98**], which characterizes when the functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ is fully faithful, which is the case just when the canonical map

$$\ulcorner \mathrm{id}_r \urcorner \colon A^2 \to \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(r,r)$$

is an equivalence over $A \times A$.

To analyze this situation, we consider the homotopy coherent comonad $\mathbb{K}\colon \mathcal{C}md \to \mathcal{K}$ underlying the homotopy coherent adjunction $\mathbb{A}\colon \mathcal{A}dj \to \mathcal{K}$. We write $k := fu$ for the functor part of the homotopy coherent comonad, an endomorphism $k\colon A \to A$ defined as the image of the unique atomic 0-arrow in $\mathcal{C}md$. Our first partial result, true for formal reasons under no additional hypotheses, observes that the canonical comparison functor r is always fully faithful on maps out of the comonad $k := fu\colon A \to A$ of the homotopy coherent adjunction.

10.5.1. Lemma ([**98**, 3.5]). *The canonical comparison map pulls back along the substitution of the comonad $k\colon A \to A$ into its domain variable to define a fibered equivalence:*

$$\begin{tikzcd} \mathsf{Hom}_A(k,A) \arrow[rr] \arrow[dr, dashed, "\sim"] \arrow[ddr] & & A^2 \arrow[dr, "\ulcorner \mathrm{id}_r \urcorner"] \arrow[ddr] & \\ & \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(rk,r) \arrow[rr] \arrow[d] & & \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(r,r) \arrow[d] \\ & A \times A \arrow[rr, "\mathrm{id} \times k"'] & & A \times A \end{tikzcd}$$

Proof. By Proposition 4.1.1 applied to the adjunctions $f \dashv u$ and $f^t \dashv u^t$ and the relations $k = fu$, $u = u^t r$, and $rf = f^t$:

$$\begin{aligned} \mathsf{Hom}_A(k,A) &:= \mathsf{Hom}_A(fu,A) \simeq_{A\times A} \mathsf{Hom}_A(u,u) = \mathsf{Hom}_A(u^t r, u^t r) \simeq_{A \times A} \mathsf{Hom}_A(f^t u^t r, r) \\ &= \mathsf{Hom}_A(rfu, r) =: \mathsf{Hom}_A(rk, r), \end{aligned}$$

and this equivalence is the one induced by id_r: the first equivalence is induced by pasting with $\eta = \eta^t$, while the second equivalence is induced by pasting with $\epsilon^t r = r\epsilon$. By the triangle equality $\epsilon f \cdot f\epsilon = \mathrm{id}_f$, the composite map is the one induced by pasting with id_r. $\square$

The statement of the main theorem requires the following definition, recall that the homotopy coherent comonad $\mathbb{K}$ internalizes to define a functor $k_\bullet \colon A \to A^{\mathbf{\Delta}_+^{\mathrm{op}}}$ that we call the **comonad resolution**, which restricts to a canonical cosimplicial object together with a cone under it.

10.5.2. DEFINITION. A generalized element $a\colon X \to A$ is $\mathbb{K}$**-cocomplete** if the restriction of the canonical diagram

$$\begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ X \arrow[r, "a"'] & A \arrow[r, "k_\bullet"'] \arrow[urr, equal, bend left] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, phantom, shift left=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

is an absolute left lifting diagram.

The terminology is adapted from Hess [**43**], who refers to such elements as "strongly $\mathbb{K}$-cocomplete."

10.5.3. EXAMPLE (algebras are cocomplete). Theorem 10.3.7 proves that in a monadic homotopy coherent adjunction, the identity functor at the ∞-category of algebras is $\mathbb{K}^{\mathbb{T}}$-cocomplete. By restricting the absolute left lifting diagram, all generalized elements in the ∞-category of $\mathbb{T}$-algebras are $\mathbb{K}^{\mathbb{T}}$-cocomplete.

We now argue that the canonical comparison functor associated to a homotopy coherent adjunction is full and faithful if and only if all generalized elements of A are $\mathbb{K}$-cocomplete.

10.5.4. THEOREM ([**98**, 3.14]). *The canonical comparison functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ for a homotopy coherent adjunction $\mathbb{A}$ with underlying homotopy coherent comonad $\mathbb{K}$ is full and faithful if and only if the identity $\mathrm{id}_A\colon A \to A$ is $\mathbb{K}$-cocomplete, i.e.,*

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A \arrow[r, "k_\bullet"'] \arrow[ur, equal] \arrow[r, phantom, shift left=3, "\Uparrow\alpha"] & A^{\Delta^{\mathrm{op}}} \end{tikzcd} \quad := \quad \begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A \arrow[r, "k_\bullet"'] \arrow[urr, equal, bend left] & A^{\mathbf{\Delta}_+^{\mathrm{op}}} \arrow[r, two heads, "\mathrm{res}"'] \arrow[ur, "\mathrm{ev}_{-1}"] \arrow[r, phantom, shift left=3, "\Uparrow A^{\nu^{\mathrm{op}}}"] & A^{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd} \tag{10.5.5}$$

is an absolute left lifting diagram.

In the special case where A admits colimits of u-split simplicial objects, the diagram (10.4.9) in the proof of Theorem 10.4.12 establishes the result: the composite $\ell r\colon A \to A$ defines the colimit of the comonad resolution $k_\bullet\colon A \to A^{\mathbf{\Delta}^{\mathrm{op}}}$, and this functor is related to the identity via the counit $\epsilon\colon \ell r \Rightarrow \mathrm{id}_A$. To say that id_A is $\mathbb{K}$-cocomplete, is to say that $\mathrm{id}_A\colon A \to A$ is the colimit of this diagram, which is the case if and only if ϵ is fully faithful. Since a right adjoint r is fully faithful just when the counit is an isomorphism (see Proposition 13.4.5 or Exercise 10.5.i), the result follows.

In the absence of this hypothesis, we cannot construct the left adjoint $\ell\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A$. In its place, we make use of the functor $L\colon \mathrm{Alg}_{\mathbb{T}}(B) \to A^{\mathbf{\Delta}^{\mathrm{op}}}$ of Lemma 10.4.5. In the terminology of §3.5, the following lemma proves that the ∞-category of cones under this diagram is right-represented by the canonical comparison functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$.

10.5.6. LEMMA. *There is a fibered equivalence* $\mathsf{Hom}_{A^{\mathbf{\Delta}^{\mathrm{op}}}}(L, \Delta) \simeq \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(\mathrm{Alg}_{\mathbb{T}}(B), r)$ *over* $A \times \mathrm{Alg}_{\mathbb{T}}(B)$.

Proof. The inverse equivalences are defined by 1-cell induction:

$$
\begin{tikzcd}
& \mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(L,\Delta) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[d, "r"'] \arrow[dr, "\Delta"'] & {} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, "L"] \arrow[ddl, bend left, "k^t_\bullet"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[dr, "\Delta"'] & A^{\Delta^{\mathrm{op}}} \arrow[d, "r^{\Delta^{\mathrm{op}}}"] & \\
& \mathrm{Alg}_{\mathbb{T}}(B)^{\Delta^{\mathrm{op}}} &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(L,\Delta) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\exists!\zeta}{\Leftarrow}"] & \\
A \arrow[d, "r"'] & {} \arrow[d, phantom, "\overset{\beta}{\Leftarrow}"] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dll, equal] \arrow[dl, bend left, "k^t_\bullet"] \\
\mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "\Delta"'] & \mathrm{Alg}_{\mathbb{T}}(B)^{\Delta^{\mathrm{op}}} &
\end{tikzcd}
$$

and then

$$
\begin{tikzcd}
& \mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(L,\Delta) \arrow[dl, two heads, bend right, "p_1"'] \arrow[dr, two heads, bend left, "p_0"] \arrow[d, phantom, "\overset{\zeta}{\Leftarrow}"] & \\
A \arrow[rr, "r"'] & {} & \mathrm{Alg}_{\mathbb{T}}(B)
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(L,\Delta) \arrow[d, dashed, "\ulcorner\zeta\urcorner"] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(\mathrm{Alg}_{\mathbb{T}}(B), r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "r"'] & {} & \mathrm{Alg}_{\mathbb{T}}(B)
\end{tikzcd}
$$

And conversely,

$$
\begin{tikzcd}
& \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(\mathrm{Alg}_{\mathbb{T}}(B), r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, "r"] \arrow[dr, bend left, "k_\bullet"] \arrow[dr, bend right, "\Delta"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & {} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, "L"] \\
& A^{\Delta^{\mathrm{op}}} &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(\mathrm{Alg}_{\mathbb{T}}(B), r) \arrow[d, dashed] \arrow[ddl, two heads, bend right, "p_1"'] \arrow[ddr, two heads, bend left, "p_0"] & \\
& \mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(L,\Delta) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
A \arrow[dr, "\Delta"'] & \overset{\phi}{\Leftarrow} & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[dl, "L"] \\
& A^{\Delta^{\mathrm{op}}} &
\end{tikzcd}
$$

We leave the verification that these maps define inverse images to the reader. □

Proof of Theorem 10.5.4. By Lemma 10.4.5(ii), the fibered equivalence of Lemma 10.5.6 pulls back along $A \times r \colon A \times A \to A \times \mathrm{Alg}_{\mathbb{T}}(B)$ to a fibered equivalence

$$\mathsf{Hom}_{A^{\Delta^{\mathrm{op}}}}(k_\bullet, \Delta) \simeq \mathsf{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(r, r)$$

over $A \times A$. Moreover this equivalence commutes with the canonical functors up to natural isomorphism

$$\begin{tikzcd} & A^2 \arrow[dl, "\ulcorner\alpha\urcorner"'] \arrow[dr, "\ulcorner\mathrm{id}_r\urcorner"] \arrow[d, phantom, "\cong"] & \\ \mathrm{Hom}_{A^{\Delta^{\mathrm{op}}}}(k_\bullet, \Delta) & \simeq & \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{T}}(B)}(r, r) \end{tikzcd}$$

Under the universal property of Proposition 3.4.7, the functor $\ulcorner\mathrm{id}_r\urcorner$ is classified by the 2-cell

$$\begin{tikzcd} A^2 \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_0"'] \arrow[r, phantom, "\Uparrow\kappa"] & A \arrow[r, "r"] & \mathrm{Alg}_{\mathbb{T}}(B) \end{tikzcd}$$

defined as the whiskered composite of the generic arrow over A. The equivalence converts this into a 2-cell under A^2 and over the cospan $A \xrightarrow{\Delta} A^{\Delta^{\mathrm{op}}} \xleftarrow{k_\bullet} A$ by forming the horizontal composite of κ with α:

$$\begin{tikzcd} & & & A \arrow[d, "\Delta"] \\ A^2 \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_0"'] \arrow[r, phantom, "\Uparrow\kappa"] & A \arrow[urr, equal, bend left] \arrow[r, "r"] \arrow[rr, bend right, "k_\bullet"'] & \mathrm{Alg}_{\mathbb{T}}(B) \arrow[r, "L"] \arrow[ur, phantom, "\Uparrow\alpha"] & A^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

and this is the 2-cell that classifies the map labelled $\ulcorner\alpha\urcorner$.

By Theorem 3.5.3, the left-diagonal map is an equivalence if and only if (10.5.5) is an absolute left lifting diagram. By Corollary 3.5.6, the right-diagonal map is an equivalence if and only if r is fully faithful. By the 2-of-3 property, either map is an equivalence if and only if both are, which is what we wished to show. □

10.5.7. Remark. Sulyma observes that the same argument shows that the canonical comparison functor $r\colon A \to \mathrm{Alg}_{\mathbb{T}}(B)$ is fully faithful on maps out of $a\colon X \to A$ if and only if $a\colon X \to A$ is $\mathbb{K}$-cocomplete [**98**, 3.14].

The motivation for the result of Theorem 10.5.4 arises from the theory of monadic descent. To explain, we first require some definitions.

10.5.8. Definition. The **∞-category of descent data** $\mathrm{Dsc}_{\mathbb{T}}(B)$ of a homotopy coherent monad $\mathbb{T}$ acting on an ∞-category B is the ∞-category of $\mathbb{K}^{\mathbb{T}}$-coalgebras in the ∞-category of $\mathbb{T}$-algebras:

$$\mathrm{Dsc}_{\mathbb{T}}(B) \coloneqq \mathrm{Coalg}_{\mathbb{K}^{\mathbb{T}}}(\mathrm{Alg}_{\mathbb{T}}(B)).$$

Elements are called **descent data**.

Unpacking Definition 10.5.8, we can clarify the meaning of "descent data."

10.5.9. Proposition. *Let $\mathbb{T}$ be a homotopy coherent monad acting on an ∞-category $B \in \mathcal{K}$. The ∞-category of descent data is the limit of $\mathbb{T}\colon \mathcal{Mnd} \to \mathcal{K}$ weighted by the weight $W_{\mathrm{dsc}}\colon \mathcal{Mnd} \to s\mathcal{Set}$ given by the category $\mathbf{\Delta}$ with the left $\mathbf{\Delta}_+$-action by ordinal sum.*

Proof. Recall from Proposition 10.2.11 that the monadic adjunction associated to a homotopy coherent monad is defined as the limit of the homotopy coherent monad diagram $\mathbb{T}$ weighted by

the restriction of the Yoneda embedding $よ \in sSet^{\mathcal{A}dj^{\mathrm{op}}\times\mathcal{A}dj}$ along the inclusion $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$ in the codomain variable, a weight we denote by $\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ$. From the monadic homotopy coherent adjunction

$$\mathbb{A}^{\mathbb{T}} := \lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T} \colon \mathcal{A}dj \to \mathcal{K},$$

the induced homotopy coherent comonad on $\mathrm{Alg}_{\mathbb{T}}(B)$ is defined by restricting along the inclusion $\mathcal{C}md \hookrightarrow \mathcal{A}dj$, producing a diagram

$$\mathbb{T}^{\mathbb{K}} := \mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}(\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}) \colon \mathcal{C}md \to \mathcal{K}.$$

Now dualizing Definition 10.2.5, the descent object is defined to be the ∞-category of coalgebras for this homotopy coherent comonad, which is define to be the lifted weighted by the restriction of the representable functor $\mathcal{A}dj_{+} \colon \mathcal{A}dj \to sSet$ along the inclusion $\mathcal{C}md \hookrightarrow \mathcal{A}dj$ in its codomain variable:

$$\mathrm{Dsc}_{\mathbb{T}}(B) := \lim^{\mathcal{C}md}_{\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}} \mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}(\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}).$$

Our aim is now to express this iterated weighted limit as a single weighted limit of the diagram $\mathbb{T}$. By Lemma 7.1.20, the weighted limit of the restricted diagram $\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}(\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}) \colon \mathcal{C}md \to \mathcal{K}$ is isomorphic to the limit of the diagram $\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}) \colon \mathcal{A}dj \to \mathcal{K}$ weighted by the left Kan extension of the weight along $\mathcal{C}md \hookrightarrow \mathcal{A}dj$. Thus:

$$\begin{aligned}\mathrm{Dsc}_{\mathbb{T}}(B) &:= \lim^{\mathcal{C}md}_{\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}} \mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}(\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T})\\ &\cong \lim^{\mathcal{A}dj}_{\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}}(\lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}).\end{aligned}$$

By Definition 7.1.3(ii), the weighted limit of a weighted limit of a diagram is the limit of that diagram weighted by the weighted colimit of the weights:

$$\cong \lim^{\mathcal{M}nd}_{\mathrm{colim}^{\mathcal{A}dj}_{\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}}\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathbb{T}.$$

By the symmetry of Definition 7.1.4, this weighted colimit of the weights is isomorphic to the weighted colimit of weights with the weight and diagram swapped, yielding the first isomorphism below. Now by Definition 7.1.3(i), this reduces to the restricting of the diagram along $\mathcal{M}nd \hookrightarrow \mathcal{A}dj$, yielding the second isomorphism below:

$$\mathrm{colim}^{\mathcal{A}dj}_{\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}}\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ \cong \mathrm{colim}^{\mathcal{A}dj}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\,よ}\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+} \cong \mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}.$$

Upon substituting into our formula for $\mathrm{Dsc}_{\mathbb{T}}(B)$ we conclude that

$$\cong \lim^{\mathcal{M}nd}_{\mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+}}\mathbb{T}$$

as desired.

It remains only to simplify the description of the weight

$$W_{\mathrm{dsc}} := \mathrm{res}^{\mathcal{M}nd}_{\mathcal{A}dj}\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\mathcal{A}dj_{+} \in sSet^{\mathcal{M}nd}.$$

By Observation 10.2.1, this diagram defines a category $W_{\mathrm{dsc}}(+)$ — which we will shortly identify, defined by evaluating at $+ \in \mathcal{M}nd$ — and a homotopy coherent monad on it. This is the dual of

the calculation of Proposition 10.4.1. By the formula for pointwise left Kan extensions, the value of $\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\,\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\,\mathcal{A}dj_{+}$ at the object $+$ is computed by

$$\begin{aligned}
W_{\mathrm{dsc}}(+) &:= (\mathrm{lan}^{\mathcal{A}dj}_{\mathcal{C}md}\,\mathrm{res}^{\mathcal{C}md}_{\mathcal{A}dj}\,\mathcal{A}dj_{+})(+) \\
&\cong \int^{\mathcal{C}md} \mathcal{A}dj(-,+)\times\mathcal{A}dj(+,-) \\
&\cong \mathrm{coeq}\left(\mathcal{A}dj(-,+)\times\mathcal{A}dj(-,-)\times\mathcal{A}dj(+,-) \underset{\mathrm{id}\times\circ}{\overset{\circ\times\mathrm{id}}{\rightrightarrows}} \mathcal{A}dj(-,+)\times\mathcal{A}dj(+,-) \right) \\
&\cong \mathrm{coeq}\left(\mathbf{\Delta}_{\top}\times\mathbf{\Delta}_{+}^{\mathrm{op}}\times\mathbf{\Delta}_{\perp} \underset{\mathrm{id}\times\circ}{\overset{\circ\times\mathrm{id}}{\rightrightarrows}} \mathbf{\Delta}_{\top}\times\mathbf{\Delta}_{\perp} \right) \\
&\cong \mathbf{\Delta}.
\end{aligned}$$

The monad structure is given by a "left action of $\mathbf{\Delta}_{+}$," in this case by the functor

$$\oplus\colon \mathbf{\Delta}_{+}\times\mathbf{\Delta}\to\mathbf{\Delta}$$

obtained by restricting the ordinal sum composition in $\mathcal{A}dj$. □

10.5.10. Remark. The W_{dsc}-weighted limit cone on $\mathrm{Dsc}_{\mathbb{T}}(B)$ takes the form of a $\mathbf{\Delta}_{+}$-invariant diagram $\mathbf{\Delta}\to\mathsf{Fun}(\mathrm{Dsc}_{\mathbb{T}}(B),B)$ that we refer to as the **generic descent datum**. Writing $b\colon \mathrm{Dsc}_{\mathbb{T}}(B)\to B$ for the 0-arrow in the image of the terminal element of $\mathbf{\Delta}$, this diagram has the form

$$\begin{array}{ccccccc}
 & & & & \xrightarrow{\ \eta\ } & & \\
 & \xrightarrow{\ \eta\ } & & & \xleftarrow{\ \mu\ } & & \\
b & \xleftarrow{\ \beta\ } & tb & & \xrightarrow{\ t\eta\ } & t^2b & \cdots \\
 & \xrightarrow{\ \theta\ } & & & \xleftarrow{\ t\beta\ } & & \\
 & & & & \xrightarrow{\ t\theta\ } & &
\end{array}$$

The generic descent datum, internalizes to a functor $\mathrm{Dsc}_{\mathbb{T}}(B)\to B^{\mathbf{\Delta}}$.

10.5.11. Observation. The weights $W_{-}, W_{\mathrm{dsc}}, W_{+}\in s\mathcal{S}et^{\mathcal{M}nd}$ are given by the categories $\mathbf{\Delta}$, $\mathbf{\Delta}_{\top}$, and $\mathbf{\Delta}_{+}$, respectively, with the left $\mathbf{\Delta}_{+}$-action in each case given by ordinal sum. Thus the inclusions

$$\mathbf{\Delta}\hookrightarrow\mathbf{\Delta}_{\top}\hookrightarrow\mathbf{\Delta}_{+} \qquad \leftrightsquigarrow \qquad W_{-}\to W_{\mathrm{dsc}}\to W_{+}\in s\mathcal{S}et^{\mathcal{M}nd}$$

are $\mathbf{\Delta}_{+}$-equivariant, defining maps of weights. On weighted limits, we thus get canonical maps that fit into a commutative diagram

$$\begin{tikzcd}
\lim_{W_{+}}\mathbb{T} \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \lim_{W_{\mathrm{dsc}}}\mathbb{T} \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \lim_{W_{-}}\mathbb{T} \arrow[d, phantom, "\wr\Vert"] \\
B \arrow[r, "c"] \arrow[rr, bend right, "f^t"'] & \mathrm{Dsc}_{\mathbb{T}}(B) \arrow[r] & \mathrm{Alg}_{\mathbb{T}}(B)
\end{tikzcd}$$

The canonical functor $c\colon B\to\mathrm{Dsc}_{\mathbb{T}}(B)$ can also be described as the non-identity component of the simplicial natural transformation from the monadic homotopy coherent adjunction $\mathbb{A}^{\mathbb{T}}$ to the comonadic homotopy coherent adjunction associated to its homotopy coherent comonad $\mathbb{K}^{\mathbb{T}}$.

$$\begin{tikzcd}
B \arrow[rr, dashed, bend left, "c"] \arrow[dr, bend right, "f^t"'] \arrow[dr, phantom, "\perp"] & & \mathrm{Dsc}_{\mathbb{T}}(B) \arrow[dl, bend right, "u^{kt}"'] \arrow[dl, phantom, "\perp"] \\
& \mathrm{Alg}_{\mathbb{T}}(B) \arrow[ul, bend right, "u^t"'] \arrow[ur, bend right, "c^{kt}"'] &
\end{tikzcd}$$

10.5.12. Definition. The homotopy coherent monad $\mathbb{T}$ satisfies **descent** if $c\colon B \to \mathrm{Dsc}_{\mathbb{T}}(B)$ is fully faithful and satisfies **effective descent** if $c\colon B \to \mathrm{Dsc}_{\mathbb{T}}(B)$ is an equivalence. The homotopy coherent monad $\mathbb{T}$ satisfies effective descent just when the monadic homotopy coherent adjunction is also *comonadic*.

Specializing the dual of Theorem 10.5.4, we have:

10.5.13. Corollary. *A homotopy coherent monad $\mathbb{T}$ acting on B satisfies descent if and only if $\mathrm{id}_B\colon B \to B$ is $\mathbb{T}$-complete.* *finish*

Dually, for a homotopy coherent comonad $\mathbb{K}$ acting on an ∞-category A, the **∞-category of codescent data** is
$$\mathrm{Codesc}_{\mathbb{K}}(A) \coloneqq \mathrm{Alg}_{\mathbb{T}^{\mathbb{K}}}(\mathrm{Coalg}_{\mathbb{K}}(A)).$$
The homotopy coherent comonad $\mathbb{K}$ satisfies **codescent** if the canonical comparison map $c\colon A \to \mathrm{Codesc}_{\mathbb{K}}(A)$ is fully faithful and satisfies **effective codescent** if $c\colon A \to \mathrm{Codesc}_{\mathbb{K}}(A)$ is an equivalence, which is the case just when the comonadic homotopy coherent adjunction is also *monadic*. The former holds just when $\mathrm{id}_A\colon A \to A$ is $\mathbb{K}$-cocomplete.

Exercises.

10.5.i. Exercise. Use Corollary 3.5.10 and Proposition 4.1.1 to anticipate the proof of Proposition 13.4.5: show that a right adjoint is fully faithful if and only if the counit of the adjunction is an isomorphism.

10.5.ii. Exercise. Prove the claim made in Remark 10.5.7.

10.6. Homotopy coherent monad maps

Two adjunctions are *equivalent* just when there exists a pair of equivalences as displayed horizontally below that commute up to isomorphism with the right adjoints

$$\begin{tikzcd}
A \arrow[r, "a", "\sim"'] \arrow[d, "u"] \arrow[d, phantom, "\dashv" left, xshift=-1ex] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "u'"'] \arrow[d, phantom, "\vdash" right, xshift=1ex] \\
B \arrow[u, bend left, "f"] \arrow[r, "\sim", "b"'] & B' \arrow[u, bend right, "f'"']
\end{tikzcd}$$

and so that the mate of the isomorphism $bu \cong u'a$ defines an isomorphism $f'b \cong af$ in the square formed with the left adjoints. The pair $f \dashv u$ extends to a homotopy coherent adjunction, which defines a homotopy coherent monad on B, and similarly $f' \dashv u'$ defines a homotopy coherent monad on B'. But are the monadic adjunctions induced by these homotopy coherent monads similarly equivalent?

A simpler question also requires some argument. Consider just a single adjunction $f \dashv u$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of an ∞-cosmos and two extensions to homotopy coherent adjunctions $\mathbb{A}, \mathbb{A}' \colon \mathcal{A}dj \to \mathcal{K}$. By Proposition 9.4.9, $\mathbb{A}$ and $\mathbb{A}'$ are isomorphic as vertices of the Kan complex $\mathsf{cohadj}(\mathcal{K})$. But what does this actually mean?

Unpacking Definition 9.4.1, we are given a homotopy coherent adjunction $\hat{\mathbb{A}} \colon \mathcal{A}dj \to \mathcal{K}^{\mathbb{I}}$ whose target is the quasi-categorically enriched category whose objects are the same as in $\mathcal{K}$ and whose functor-spaces are defined as the cotensor of the functor spaces of $\mathcal{K}$ with $\mathbb{I}$ so that when $\hat{\mathbb{A}}$ is evaluated at the endpoints of $\mathbb{I}$, this returns the homotopy coherent adjunctions $\mathbb{A}$ and $\mathbb{A}'$. Note this data does *not* define a simplicial natural transformation $\mathbb{A} \to \mathbb{A}'$, in particular, there would be no obvious choices for its components other than the identity functors at A and B, so we cannot directly apply Proposition 7.3.1 to construct an equivalence between the ∞-categories of algebras. The following result explains how to construct an equivalence between two homotopy coherent adjunctions that are isomorphic as vertices of $\mathsf{cohadj}(\mathcal{K})$ in slightly greater generality than described here.

10.6.1. PROPOSITION. *Suppose $u, u' \colon A \to B$ are naturally isomorphic right adjoints in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of an ∞-cosmos $\mathcal{K}$. Then any homotopy coherent adjunctions $\mathbb{A} \colon \mathcal{A}dj \to \mathcal{K}$ and $\mathbb{A}' \colon \mathcal{A}dj \to \mathcal{K}$ extending u and u' are connected by a zig zag of simplicial natural equivalences, and hence the monadic homotopy coherent adjunctions for $\mathbb{A}$ and $\mathbb{A}'$ are naturally equivalent.*

PROOF. By Theorem 9.4.15, the forgetful functor $p_R \colon \mathsf{cohadj}(\mathcal{K}) \overset{\sim}{\twoheadrightarrow} \mathsf{rightadj}(\mathcal{K})$ defines a trivial fibration of Kan complexes, where $\mathsf{rightadj}(\mathcal{K})$ is defined as a sub-quasi-category of $\coprod_{A,B\in\mathcal{K}} \mathsf{Fun}(A, B)$ containing the right adjoint functors and isomorphisms between them. The postulated natural isomorphism $\alpha \colon u \cong u'$ defines a lifting problem

$$\begin{tikzcd} \partial\mathbb{I} \arrow[r, "\mathbb{A}+\mathbb{A}'"] \arrow[d, hook] & \mathsf{cohadj}(\mathcal{K}) \arrow[d, two heads, "p_R", "\wr"'] \\ \mathbb{I} \arrow[r, "\alpha"'] \arrow[ur, dashed, "\hat{\mathbb{A}}"] & \mathsf{rightadj}(\mathcal{K}) \end{tikzcd}$$

whose solution defines a simplicial functor $\hat{\mathbb{A}} \colon \mathcal{A}dj \to \mathcal{K}^{\mathbb{I}}$.

Consulting the definition of the cotensor of a quasi-categorically enriched category with a simplicial set U in Definition 9.4.1, we see that when $\mathcal{K}$ is an ∞-cosmos (and in particular admits simplicial cotensors of objects inside $\mathcal{K}$), $\mathcal{K}^U$ is concisely defined as the Kleisli category for the monad $(-)^U \colon \mathcal{K} \to \mathcal{K}$, with the unit defined by the constant map $\Delta \colon A \to A^U$ and the multiplication defined by the fold map $\nabla \colon (A^U)^U \cong A^{U\times U} \to A^U$, both arising from the comonoid $(U, ! \colon U \to \mathbb{1}, \Delta \colon U \to U \times U)$ in $s\mathcal{S}et$. As such, the quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{K}^U$ are connected by the Kleisli adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{K}^U \arrow[l, bend left, "K"] \end{tikzcd}$$

whose left adjoint is identity on objects and acts on homs by post-composing with the constant map

$$\mathsf{Fun}(X, A) \xrightarrow{\Delta_*} \mathsf{Fun}(X, A^U) \cong \mathsf{Fun}(X, A)^U,$$

while the right adjoint sends A to A^U and acts on homs by

$$\mathsf{Fun}(X, A)^U \cong \mathsf{Fun}(X, A^U) \xrightarrow{(-)^U} \mathsf{Fun}(X^U, (A^U)^U)) \xrightarrow{\nabla_*} \mathsf{Fun}(X^U, A^U).$$

Now for each vertex $u\colon \mathbb{1} \to U$, there is a simplicial natural transformation $K \Rightarrow \mathrm{ev}_u$ whose components are given by $\mathrm{ev}_u\colon A^U \twoheadrightarrow A$.

Specializing to $U = \mathbb{I}$, we obtain a zig-zag of simplicial natural transformations

$$\begin{tikzcd} \mathcal{K}^{\mathbb{I}} \arrow[r, bend left=50, "\mathrm{ev}_0"] \arrow[r, "K" description] \arrow[r, bend right=50, "\mathrm{ev}_1"'] \arrow[r, phantom, shift left=2.5ex, "\Uparrow\wr"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\wr"] & \mathcal{K} \end{tikzcd}$$

whose components are the trivial fibrations $\mathrm{ev}_0, \mathrm{ev}_1\colon A^{\mathbb{I}} \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow A$, and in particular, define equivalences. Precomposing with $\hat{\mathbb{A}}'$, this defines a zig-zag of simplicial natural equivalences between the homotopy coherent adjunction $\mathbb{A}$ and the homotopy coherent adjunction $\mathbb{A}'$. The homotopy coherent monadic adjunction is computed as a flexible weighted limit of these diagrams, so Proposition 7.3.1 implies that the homotopy coherent monadic adjunctions are equivalent, as claimed. □

Proposition 10.6.1 can be understood as presenting an affirmative answer to the question posed at the start of this section. The equivalence a and b can be promoted to adjoint equivalences by Proposition 2.1.11, then composed with the adjunctions $f \dashv u$ and $f' \dashv u'$ to produce a pair of naturally isomorphic adjunctions between the same ∞-categories, to which Proposition 10.6.1 applies. The details are left as Exercise 10.6.i. But even with this question resolved, such considerations inspire a more general avenue of inquiry that is worth pursuing, which we do following Zaganidis work in his PhD thesis [**106**].

To state the question, we first review a bit of classical 2-category theory.

10.6.2. Definition (monad morphism in a 2-category). Let (t, η, μ) be a monad on B and let (s, ι, ν) be a monad on A in a 2-category. A (**lax**) **monad morphism** $(f, \chi)\colon (t, \eta, \mu) \to (s, \iota, \nu)$ is given by:

- a functor $f\colon B \to A$
- a 2-cell $\chi\colon sf \Rightarrow ft$

so that the following pasting equalities hold

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "s"] \arrow[d, phantom, "\Downarrow\nu"] & \\ A \arrow[ur, "s"] \arrow[rr, "s"] & {} & A \\ B \arrow[u, "f"] \arrow[rr, "t"'] \arrow[urr, phantom, "\Downarrow\chi"] & & B \arrow[u, "f"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dr, "s"] & \\ A \arrow[ur, "s"] \arrow[r, phantom, "\Downarrow\chi"] & B \arrow[u, "f"] \arrow[r, phantom, "\Downarrow\chi"] \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & A \\ B \arrow[u, "f"] \arrow[ur, "t"] \arrow[rr, "t"'] & {} & B \arrow[u, "f"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "s"] \arrow[r, bend left=60, equal] \arrow[r, phantom, shift left=2ex, "\Downarrow\iota"] & A \\ B \arrow[u, "f"] \arrow[r, "t"'] \arrow[ur, phantom, "\Downarrow\chi"] & B \arrow[u, "f"'] \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal] & A \\ B \arrow[u, "f"] \arrow[r, equal] \arrow[r, bend right=60, "t"'] \arrow[r, phantom, shift right=2ex, "\Downarrow\eta"] & B \arrow[u, "f"'] \end{tikzcd}$$

If $(g, \psi)\colon (t, \eta, \mu) \to (s, \iota, \nu)$ is a second monad morphism, a **monad transformation** $\alpha\colon (f, \chi) \Rightarrow (g, \psi)$ is given by a 2-cell $\alpha\colon f \Rightarrow g$ so that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\Downarrow\chi"] & A \arrow[d, "s"] \\ B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} = \begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "t"'] \arrow[dr, phantom, "\Downarrow\psi"] & A \arrow[d, "s"] \\ B \arrow[r, "g"'] & A \end{tikzcd}$$

This defines the 2-category of monads and monad morphisms in a fixed 2-category $\mathcal{C}$ [**92**]. If $\mathcal{C}$ admits *Eilenberg-Moore objects*, which are to represent the "category of algebras functor,"[3] then this 2-category defines a reflective subcategory of a particular 2-category of adjunctions that we now introduce.

10.6.3. Definition. A **right adjunction morphism** is a commutative square between the right adjoints

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"] & A' \arrow[d, "u'"'] \\ B \arrow[u, bend left, dashed, "f"] \arrow[r, "b"'] & B' \arrow[u, bend right, dashed, "f'"'] \end{tikzcd}$$

and a **right adjunction transformation** is a pair of natural transformations

$$\begin{tikzcd} A \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "u"] & A' \arrow[d, "u'"'] \\ B \arrow[u, bend left, dashed, "f"] \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B' \arrow[u, bend right, dashed, "f'"'] \end{tikzcd}$$

so that $u'\alpha = \beta u$.

10.6.4. Proposition. *If $\mathcal{C}$ is a 2-category admitting the construction of Eilenberg-Moore objects, the forgetful 2-functor from the 2-category of adjunctions, right adjunction morphisms, and right adjunction transformations to the 2-category of monads admits a fully faithful right 2-adjoint.*

Proof. Exercise 10.6.ii or see [**24**]. □

10.6.5. Remark. A lax morphism of monads $(f : B \to A, \chi : sf \Rightarrow ft)$ as in Definition 10.6.2 corresponds to a lift of f along the right adjoints of the monadic adjunctions for t and s, but this lifted functor does not commute with the left adjoints. In general, the mate of the identity functor is non-invertible.

The question is what is a homotopy coherent monad morphism?

10.6.6. Digression. Consider homotopy coherent monads $\mathbb{T} : \mathcal{M}nd \to \mathcal{K}$ and $\mathbb{S} : \mathcal{M}nd \to \mathcal{K}$ on B and A. A simplicial natural transformation $f : \mathbb{T} \Rightarrow \mathbb{S}$ is given by its unique component $f : B \to A$ satisfying a strict naturality condition relative to the bar resolutions

$$\begin{tikzcd} \mathbf{\Delta}_+ \arrow[r, "\mathbb{T}"] \arrow[d, "\mathbb{S}"'] & \mathsf{Fun}(B, B) \arrow[d, "f_*"] \\ \mathsf{Fun}(A, A) \arrow[r, "f^*"'] & \mathsf{Fun}(B, A) \end{tikzcd}$$

Such data defines a monad morphism in the homotopy 2-category whose component $\chi : sf \Rightarrow ft$ is the identity 2-cell. So in general, this definition is too strict.

By Remark 9.4.3, if the vertices $\mathbb{T}$ and $\mathbb{S}$ are in the same connected component of $\mathsf{Icon}(\mathcal{M}nd, \mathcal{K})$, then necessarily $B = A$, which is also too rigid a notion of monad morphism to consider in general.

[3] A 2-category $\mathcal{C}$ admits Eilenberg-Moore objects whenever it admits the PIE limits alluded to in Digression 7.2.6 [**55**].

We present the data of a homotopy coherent monad morphism by means of a simplicial computad $\mathcal{M}nd_2$ introduced by Zaganidis. Generalizing the relationship between the simplicial computad $\mathcal{M}nd$ and the simplicial computad $\mathcal{A}dj$, $\mathcal{M}nd_2$ defines a simplicial subcategory of a simplicial category $\mathcal{A}dj_{\mathbb{2}}$ that we introduce first via a graphical calculus developed in Zaganidis [**106**, ?]. This graphical calculus extends to a sequence of composable adjunction morphisms so we might as well introduce $\mathcal{A}dj_{\mathbb{n}}$ in its full generality.

The simplicial category $\mathcal{A}dj_{\mathbb{n}}$ is a 2-category whose hom-wise nerve can be presented via a graphical calculus, exactly as was the case for $\mathcal{A}dj$. A 2-categorical description of $\mathcal{A}dj_{\mathbb{n}}$ is given in [**106**, §3.1.1]. The graphical calculus that presents $\mathcal{A}dj_{\mathbb{n}}$ as a simplicial category is described in more detail in [**106**, §3.1.2].

10.6.7. Definition. The objects of $\mathcal{A}dj_{\mathbb{n}}$ are pairs whose first component is either $+$ or $-$ and whose second component $i \in \{0, 1, \ldots, n-1\}$ is best thought of as a *color* drawn from a linearly ordered set where the color 0 is the "lightest" and the color $n-1$ is the "darkest." Note the objects of $\mathcal{A}dj_{\mathbb{n}}$ are the objects of the product simplicial category $\mathcal{A}dj \times \mathbb{n}$; we write $+_i$ or $-_i$ for the two objects with color i or either $\pm_i$ or $\mp_i$ to denote a generic object of $\mathcal{A}dj \times \mathbb{n}$.

The n-arrows in $\mathcal{A}dj_{\mathbb{n}}$ are **strictly undulating colored squiggles on n-lines**. In more detail, an n-arrow $\pm_k \to \mp_j$ is permitted only if $k \geq j$; that is the color of the domain, appearing on the right, must be "darker" than the color of the codomain, appearing on the left. The data of a morphism $\pm_k \to \mp_j$ is given by a strictly undulating squiggle of $\mathcal{A}dj$ from $\pm$ to $\mp$ as appropriate, with all four choices of $+$ or $-$ possible, together with a **coloring**, with colors drawn from the interval $[j, k]$, of the connected components of each of the strips between the lines i and $i-1$ for some $i \in [n]$ of the shaded region under the squiggle diagram. This coloring must satisfy the axioms:

- In the strip between the ith and $i-1$th lines for $i \in [n]$, the coloring function is monotone, becoming darker as you move from left to right.
- If a connected component of a strip above the line i shares a boundary with a connected component of a strip below the line i, then color of the top strip is lighter than the color of the bottom strip, i.e., the coloring function is monotone, becoming darker as you move from top to bottom in a single vertical section of the squiggle diagram.
- Finally, in the case of a morphism $-_k \to \pm_j$, each of the strips that touches the right-hand boundary must be colored the maximal color k.

For a fully formal definition together with a description of the composition, face, and degeneracy actions are described in [**106**, §3.1.2].

The monochromatic strictly undulating squiggles in $\mathcal{A}dj_{\mathbb{n}}$ define the data found in just one of its n homotopy coherent adjunctions.

10.6.8. Example. The atomic 0-arrows of $\mathcal{A}dj_{\mathbb{n}}$ define n adjunctions that we denote by $\underline{f}_i \dashv \underline{u}_i$ and $n-1$ adjunction morphisms as depicted below:

$$\begin{tikzcd}
-_0 \arrow[d, bend left, "\underline{u}_0"] \arrow[d, phantom, "\dashv"] & -_1 \arrow[l, "\underline{a}_1"'] \arrow[d, bend left, "\underline{u}_1"] \arrow[d, phantom, "\dashv"] & \cdots \arrow[l] & -_{n-2} \arrow[l] \arrow[d, bend left, "\underline{u}_{n-2}"] \arrow[d, phantom, "\dashv"] & -_{n-1} \arrow[l, "\underline{a}_{n-1}"'] \arrow[d, bend left, "\underline{u}_{n-1}"] \arrow[d, phantom, "\dashv"] \\
+_0 \arrow[u, bend left, "\underline{f}_0"] & +_1 \arrow[l, "\underline{b}_1"] \arrow[u, bend left, "\underline{f}_1"] & \cdots \arrow[l] & +_{n-2} \arrow[l] \arrow[u, bend left, "\underline{f}_{n-2}"] & +_{n-1} \arrow[l, "\underline{b}_{n-1}"] \arrow[u, bend left, "\underline{f}_{n-1}"]
\end{tikzcd}$$

We now state a few of the key results from Zaganidis' thesis whose proofs are too long to reproduce here. Note that $\mathcal{A}dj_{\mathbb{n}}$ is *not* a simplicial computad, since the diagram of right adjoints and adjunction morphisms in Example 10.6.8 commutes strictly at the level of 0-arrows. However, this is the only obstruction, in a sense made precise by the following result

10.6.9. Proposition ([**106**, 3.2.5]). *Consider the inclusion $\mathbb{2} \times \mathbb{n} \hookrightarrow \mathcal{A}dj_{\mathbb{n}}$ whose image is the subcategory comprised of the right adjoints $\underline{u}_i$ and the 0-arrow components $\underline{a}_i$ and $\underline{b}_i$ of the adjunction morphisms. Then $\mathbb{2} \times \mathbb{n} \hookrightarrow \mathcal{A}dj_{\mathbb{n}}$ is a relative simplicial computad.*

10.6.10. Remark. Diagrams of morphisms between adjunctions of more general 2-categorical pasting diagram shapes are also of interest. Zaganidis constructs a generalization $\mathcal{A}dj^S_{\mathbb{n}}$ of $\mathcal{A}dj_{\mathbb{n}}$ for any "shape" $S \subset \mathfrak{C}\Delta[n-1]$ with the universal property that for a 2-category C, 2-functors $\mathcal{A}dj^S_{\mathbb{n}} \to C$ correspond to 2-functors from S into the 2-category of adjunctions, right adjunction morphisms, and right adjunction transformations of Definition 10.6.3. Zaganidis' construction requires S to be a simplicial computad, although $S \hookrightarrow \mathfrak{C}\Delta[n-1]$ need not be a simplicial subcomputad inclusion. Instead, any factorization in $\mathfrak{C}\Delta[n-1]$ of an atomic arrow of S must be through an object that is not contained in the subcategory S.

An argument along the lines of that given in §9.2 and §9.3 proves that homotopy coherent adjunction morphisms are generated by what we term a **diagram of n-right adjoints**: this being given by a commutative diagram in $\mathcal{K}$

$$\begin{tikzcd}
A_0 \arrow[d, "u_0"] & A_1 \arrow[l, "a_1"'] \arrow[d, "u_1"] & \cdots \arrow[l] & A_{n-2} \arrow[l] \arrow[d, "u_{n-2}"] & A_{n-1} \arrow[l, "a_{n-1}"] \arrow[d, "u_{n-1}"] \\
B_0 & B_1 \arrow[l, "b_1"] & \cdots \arrow[l] & B_{n-2} \arrow[l] & B_{n-1} \arrow[l, "b_{n-1}"]
\end{tikzcd}$$

together with a choice of left adjoint and unit for each u_i. We state just one of the many extension theorems proven as Theorems A and 5.12-13 in [**106**]:

10.6.11. Theorem. *Any diagram of n-right adjoints in a quasi-categorically enriched category extends to a simplicial functor $\mathcal{A}dj_{\mathbb{n}} \to \mathcal{K}$. And, moreover the space of extensions over a given diagram of right adjoints is a trivial Kan complex.*

10.6.12. Definition. Let $\mathcal{M}nd_{\mathbb{n}}$ be the full subcategory of $\mathcal{A}dj_{\mathbb{n}}$ on the objects $+_0, \ldots, +_{n-1}$. By [**106**, 5.19] it is a simplicial computad.

10.6.13. Theorem ([**106**, 5.20,5.24-5]). *For any homotopy coherent diagram of monads $\mathbb{T} \colon \mathcal{M}nd_{\mathbb{n}} \to \mathcal{K}$, the simplicially enriched right Kan extension $\mathbb{A}^{\mathbb{T}} \colon \mathcal{A}dj_{\mathbb{n}} \to \mathcal{K}$ exists. For each $i \in \{0, \ldots, n-1\}$, the object $\mathbb{A}^{\mathbb{T}}(-_i)$ is equivalent to the ∞-category of algebras for the ith underlying homotopy coherent monad.*

Note that its not the case that the value of the right Kan extension along $\mathcal{M}nd_{\mathbb{n}} \hookrightarrow \mathcal{A}dj_{\mathbb{n}}$ at $-_i$ recovers the ∞-category of algebras for the ith homotopy coherent monad on the nose. However, by applying Theorem 10.4.12 this ∞-category can be shown to be equivalent to the ∞-category of algebras. See §5.3 of [**106**] for more details.

Exercises.

10.6.i. Exercise. Use Proposition 10.6.1 to prove that the ∞-categories of algebras associated to any homotopy coherent adjunctions extending equivalent adjunctions, in the sense described at the start of this section, are equivalent.

10.6.ii. Exercise. Prove Proposition 10.6.4, perhaps just in the 2-category of categories.

Part III

The calculus of modules

CHAPTER 11

Two-sided fibrations

Recall from Proposition 8.2.12 that for any ∞-category B in an ∞-cosmos $\mathcal{K}$, the quasi-categorically enriched categories $\mathcal{coCart}(\mathcal{K})_{/B}$ and $\mathcal{Cart}(\mathcal{K})_{/B}$ define sub ∞-cosmoi of $\mathcal{K}_{/B}$. In this section, we introduce another sub ∞-cosmos ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B} \subset \mathcal{K}_{/A\times B}$ whose objects are *two-sided fibrations* from A to B. Several equivalent definitions of this notion are given in §11.1. Iterating Proposion 8.2.12 reveals that ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B}$ is again an ∞-cosmos, which we study in §11.2. Importantly, two-sided fibration from B to $\mathbb{1}$ is simply a cocartesian fibration over B, while a two-sided fibration from $\mathbb{1}$ to B is a cartesian fibration over B, so results about two-sided fibrations simultaneously generalize these one-sided notions. For instance, in §11.3, we introduce two-sided representables and prove the Yoneda lemma, generalizing Theorem 5.5.4 for representable (co)cartesian fibrations.

Another reason for our interest in two-sided fibrations is the fact that the discrete objects in ${}_{A}\mathcal{Fib}_{B}$ are precisely the *modules* from A to B, which we define and study in §11.4. The calculus of modules, developed in Chapter 12, is the main site of the formal category theory of ∞-categories, which is the subject of Chapter 13.

11.1. Four equivalent definitions of two-sided fibrations

By factoring, any span in $\mathcal{K}$ from A to B may be replaced up to equivalence by a **two-sided isofibration**, a span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ for which the functor $(q,p)\colon E \twoheadrightarrow A \times B$ is an isofibration. Two-sided isofibrations from A to B are the objects of the ∞-cosmos $\mathcal{K}_{/A\times B}$. In this section, we describe what it means for a two-sided isofibration to be *cocartesian on the left* or *cartesian on the right*, and then introduce *two-sided fibrations*, which integrate these notions.

11.1.1. Lemma (cocartesian on the left). *For a two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ in an ∞-cosmos $\mathcal{K}$, the following are equivalent:*

(i) The functor

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

is a cocartesian fibration in the slice ∞-cosmos $\mathcal{K}_{/B}$.

(ii) The functor

$$\begin{tikzcd} E \arrow[rr, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

in $\mathcal{K}_{/A}$ lies in the sub ∞-cosmos $\mathcal{coCart}(\mathcal{K})_{/A}$.

(iii) The functor induced by id_q

$$\begin{tikzcd} E \arrow[rr, "i"'] \arrow[rr, phantom, shift left=1.5, "\perp"] \arrow[dr, "{(q,p)}"'] & & \mathsf{Hom}_A(q,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, "{(p_1,pp_0)}"] \\ & A \times B & \end{tikzcd}$$

admits a left adjoint in $\mathcal{K}_{/A\times B}$.

(iv) The isofibration $q\colon E \twoheadrightarrow A$ *is a cocartesian fibration in* $\mathcal{K}$ *and for every* q*-cocartesian transformation*

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}, \text{ the composite } \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\chi"] & E \arrow[r, two heads, "p"] & B \end{tikzcd} \text{ is an isomorphism.}$$

If any of these equivalent conditions are satisfied, we say that $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is **cocartesian on the left**.

Proof. We first prove that the equivalence (i)⇔(iii) is exactly the interpretation of the equivalence Theorem 5.1.11(i)⇔(ii) applied to the isofibration $(q,p)\colon E \twoheadrightarrow A\times B$ in the ∞-cosmos $\mathcal{K}_{/B}$. This latter result asserts that the isofibration $(q,p)\colon E \twoheadrightarrow A \times B$ is a cocartesian fibration in $\mathcal{K}_{/B}$ if and only if a certain canonical functor from E to the left representation of the functor $(q,p)\colon E \twoheadrightarrow A\times B$ admits a left adjoint over the codomain $\pi\colon A\times B \twoheadrightarrow B$; since $(\mathcal{K}_{/B})_{/\pi\colon A\times B\twoheadrightarrow B} \cong \mathcal{K}_{/A\times B}$ this is the same as asserting this adjunction over $A\times B$.

The only subtlety in interpreting Theorem 5.1.11 in $\mathcal{K}_{/B}$ has to do with the correct interpretation of the left representable comma ∞-category in $\mathcal{K}_{/B}$ for the functor $(q,p)\colon E \twoheadrightarrow A\times B$. This comma ∞-category is constructed as a subobject of the $\mathbb{2}$-cotensor of the object $\pi\colon A\times B \to B$ in $\mathcal{K}_{/B}$, which Proposition 1.2.19 tells us is formed as the left-hand vertical of the pullback diagram

$$\begin{tikzcd} A^{\mathbb{2}} \times B \arrow[r] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & (A\times B)^{\mathbb{2}} \arrow[d, two heads, "\pi^{\mathbb{2}}"] \\ B \arrow[r, "\Delta"] & B^{\mathbb{2}} \end{tikzcd}$$

By (3.4.2) the comma ∞-category is constructed by the pullback in $\mathcal{K}_{/B}$

$$\begin{tikzcd} \mathsf{Hom}_A(q,A) \arrow[rr] \arrow[d, two heads, "p_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddr, dotted, "pp_0"] & & A^{\mathbb{2}}\times B \arrow[r, two heads, "p_1\times\mathrm{id}"] \arrow[d, two heads, "p_0\times\mathrm{id}"] \arrow[ddl, "\pi"'] & A\times B \arrow[ddll, bend left, "\pi"] \\ E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A\times B \arrow[dl, two heads, "\pi"] & \\ & B & & \end{tikzcd} \tag{11.1.2}$$

which is created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$, and its codomain-projection functor is the top composite $(p_1, pp_0)\colon \mathsf{Hom}_A(q,A) \twoheadrightarrow A\times B$. Now we see that the interpretation of Theorem 5.1.11(i)⇔(ii) is exactly the equivalence (i)⇔(iii).

It remains to demonstrate the equivalence with (ii) and 11.4. Assuming (iii) and composing with $\pi\colon A\times B \twoheadrightarrow A$, we are left with a fibered adjunction that demonstrates that q is a cocartesian fibration.

The unit of both fibered adjunctions is the same, and by Theorem 5.1.11(v) the composite

$$\begin{tikzcd} \mathsf{Hom}_A(q,A) \arrow[rr, equal, bend left] \arrow[dr, "\ell"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{Hom}_A(q,A) \arrow[r, two heads, "p_0"] & E \\ & E \arrow[ur, "i"'] & & \end{tikzcd}$$

is the generic q-cocartesian cell. Since η is fibered over $A \times B$, when we postcompose with p we get an identity, which tells us that $p\colon E \twoheadrightarrow B$ carries q-cocartesian cells to isomorphisms. This proves 11.4.

In fact, 11.4 can easily be seen to be equivalent to (ii). By Example 5.2.4, the cocartesian cells for the cocartesian fibration $\pi\colon A \times B \twoheadrightarrow A$ are precisely those 2-cells whose component with codomain B is an isomorphism, so 11.4 says exactly that $(q,p)\colon E \twoheadrightarrow A \times B$ defines a cartesian functor from q to π. Thus 11.4 implies (ii).

For the converse, assume (ii) and consider a 2-cell in $\mathcal{K}_{/B}$

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, bend right, "{(a,b)}"'] & E \arrow[d, two heads, "{(q,p)}"] \arrow[dl, phantom, "\Downarrow(\alpha,\mathrm{id})"] \\ & A \times B \end{tikzcd}$$

Because q is a cocartesian fibration, $\alpha\colon qe \Rightarrow a$ has a q-cocartesian lift $\chi\colon e \Rightarrow e'\colon X \to E$, and since (q,p) is a cartesian functor, the whiskered 2-cell $p\chi\colon b \Rightarrow pe'$ is an isomorphism. Because $(q,p)\colon E \twoheadrightarrow A \times B$ is an isofibration, we may lift the 2-cell $(\mathrm{id}, p\chi^{-1})\colon e' \Rightarrow (a,b)$ to an invertible 2-cell $\gamma\colon e' \Rightarrow e''$ with $q\gamma = \mathrm{id}_a$. The composite $\gamma \cdot \chi\colon e \Rightarrow e''$ is a lift of (α, id) along (q,p) over B, which is easily verified to define a (q,p)-cocartesian lift of (α,id) in $\mathcal{K}_{/B}$. □

If $(q,p)\colon E \twoheadrightarrow A \times B$ is a *discrete* cocartesian fibration in $\mathcal{K}_{/B}$ then the converse to the last statement of 11.4 holds: any 2-cell χ for which $p\chi$ is an isomorphism is q-cocartesian. See Exercise 11.1.i.

By Lemma 11.1.1 and its dual:

11.1.3. Corollary. *A two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is cocartesian on the left and cartesian on the right if the following equivalent conditions are satisfied:*

(i) The functor

$$\begin{tikzcd} E \arrow[rr, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

lies in $\mathcal{C}o\mathcal{C}art(\mathcal{K})_{/A}$ and defines a cartesian fibration in $\mathcal{K}_{/A}$.

(ii) The functor

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

lies in $\mathcal{C}art(\mathcal{K})_{/B}$ and defines a cocartesian fibration in $\mathcal{K}_{/B}$. □

A two-sided fibration is a span that is cocartesian on the left, cartesian on the right, and satisfies a further compatibility condition that can be stated in a number of equivalent ways, which boil down to the assertion that the processes of taking q-cocartesian and p-cartesian lifts commute:

11.1.4. Theorem. *For a two-sided isofibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ in $\mathcal{K}$, the following are equivalent:*

(i) The functor

$$\begin{tikzcd} E \arrow[rr, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

defines a cartesian fibration in $\mathcal{coCart}(\mathcal{K})_{/A}$.

(ii) The functor

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

defines a cocartesian fibration in $\mathcal{Cart}(\mathcal{K})_{/B}$.

(iii) The canonical functors admit the displayed adjoints in $\mathcal{K}_{/A\times B}$

$$\begin{tikzcd} E \arrow[rr, "i"', "\top"] \arrow[d, "i"] & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend right, "r"'] \arrow[d, "{(ip_1,\mathrm{id})}"'] \\ \mathsf{Hom}_A(q,A) \arrow[u, dashed, bend left, "\ell", "\dashv"'] \arrow[rr, "{(\mathrm{id},ip_0)}", "\perp"'] & & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[u, dashed, bend right, "\ell"', "\vdash"] \end{tikzcd}$$

and the mate of the identity 2-cell in this displayed commutative square defines an isomorphism $\ell r \cong r\ell$.

(iv) $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ *is cocartesian on the left, cartesian on the right, and for any* $a\colon X \to A$, $b\colon X \to B$, *and* $e\colon X \to E$ *and pair of 2-cells* $\alpha\colon qe \Rightarrow a$ *and* $\beta\colon b \Rightarrow pe$, *the map* $\beta^*\alpha_*(e)\colon X \to E$, *obtained as the domain of a* p*-cartesian lift of the composite of* β *with a* q*-cocartesian lift* $\chi_\alpha\colon e \Rightarrow \alpha_*(e)$ *of* α *over* B, *is isomorphic over* $A \times B$ *to the map* $\alpha_*\beta^*(e)\colon X \to E$ *obtained as the codomain of a* q*-cocartesian lift of the composite of* α *with a* p*-cartesian lift* $\chi_\beta\colon \beta^*(e) \Rightarrow e$ *of* β *over* A.

A span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a **two-sided fibration from** A **to** B if any of these equivalent conditions are satisfied:

Proof. Our strategy will be to show that condition (i) is equivalent to (iii), an equationally witnessed condition in the slice ∞-cosmos $\mathcal{K}_{/A\times B}$. A dual argument will show that condition (ii) is equivalent to (iii). We then unpack this condition to prove its equivalence with (iv).

We use Theorem 5.1.11(ii), which provides a condition that characterizes the cartesian fibrations in any ∞-cosmos via the presence of a fibered adjunction, to re-express (i) in $\mathcal{K}_{/A\times B}$. To apply this characterization to the map displayed in (i), we must first compute the right representable comma object in $\mathcal{coCart}(\mathcal{K})_{/A} \subset \mathcal{K}_{/A}$ associated to the functor $(q,p)\colon E \to A \times B$ over A. By Proposition 8.2.12, it suffices to compute this comma object in $\mathcal{K}_{/A}$. By the dual of the calculation (11.1.2) we gave in the proof of Lemma 11.1.1, the comma object covariantly representing the functor $(q,p)\colon E \to A\times B$ in $\mathcal{coCart}(\mathcal{K})_{/A} \subset \mathcal{K}_{/A}$ is the cocartesian fibration $qp_1\colon \mathsf{Hom}_B(B,p) \twoheadrightarrow A$.

Now Theorem 5.1.11(ii) applied in $\mathcal{coCart}(\mathcal{K})_{/A}$ tells us that condition (i) holds if and only if the canonical functor i admits a right r over $A \times B$

$$\begin{tikzcd} E \arrow[rr, "i", ""' name=I] \arrow[dr, two heads, "(q,p)"'] & & \mathrm{Hom}_B(B,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "(qp_1,p_0)"] \\ & A \times B & \end{tikzcd} \tag{11.1.5}$$

where the three displayed objects are all cocartesian fibrations over A and each of the four displayed maps are cartesian functors between these cocartesian fibrations.

By Lemma 11.1.1, to say that (q,p) defines a cartesian functor between cocartesian fibrations is to say that the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is cocartesian on the left, which is the case if and only if the canonical functor i admits a left adjoint

$$\begin{tikzcd} E \arrow[rr, "\perp", "i"'] \arrow[dr, "(q,p)"'] & & \mathrm{Hom}_A(q,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "(p_1,pp_0)"] \\ & A \times B & \end{tikzcd}$$

in $\mathcal{K}_{/A\times B}$. Similarly, to say that (qp_1,p_0) defines a cartesian functor between cocartesian fibrations is to say that the span $A \overset{qp_1}{\twoheadleftarrow} \mathrm{Hom}_B(B,p) \overset{p_0}{\twoheadrightarrow} B$ is cocartesian on the left. By Lemma 11.1.1 again, this is equivalent to the hypothesis that the canonical functor from $\mathrm{Hom}_B(B,p)$ to the comma object contravariantly representing qp_1 admits a left adjoint

$$\begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[rr, "\perp", "(ip_1,\mathrm{id})"'] \arrow[dr, "(qp_1,p_0)"'] & & \mathrm{Hom}_A(q,A) \underset{E}{\times} \mathrm{Hom}_B(B,p) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "(p_1,p_0)"] \\ & A \times B & \end{tikzcd}$$

in $\mathcal{K}_{/A\times B}$.

Finally, by Theorem 5.1.19, to say that the functor i of (11.1.5) is cartesian is to say that in the commutative solid-arrow diagram in $\mathcal{K}_{/A\times B}$,

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"] & \mathrm{Hom}_B(B,p) \arrow[d, "(ip_1,\mathrm{id})"'] \\ \mathrm{Hom}_A(q,A) \arrow[u, dashed, bend left, "\ell"] \arrow[u, phantom, "\dashv"] \arrow[r, "(\mathrm{id},ip_0)"'] & \mathrm{Hom}_A(q,A) \underset{E}{\times} \mathrm{Hom}_B(B,p) \arrow[u, dashed, bend right, "\ell"'] \arrow[u, phantom, "\vdash"] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\top", "i"'] \arrow[d, "i"'] & \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend right, "r"'] \arrow[d, "(ip_1,\mathrm{id})"] \\ \mathrm{Hom}_A(q,A) \arrow[r, "(\mathrm{id},ip_0)", "\perp"'] & \mathrm{Hom}_A(q,A) \underset{E}{\times} \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend left, "r"] \end{tikzcd} \tag{11.1.6}$$

the mate of this identity 2-cell involving the left adjoints ℓ is an isomorphism. This "Beck-Chevalley" condition is equivalent to saying that the other mate, displayed above right, associated to the functor r of (11.1.5) is an isomorphism. Finally, to say that r is also a cartesian functor between cocartesian

fibrations is to say that the further mate

$$\begin{tikzcd} E & & \mathsf{Hom}_B(B,p) \arrow[ll, dashed, "r"'] \\ \mathsf{Hom}_A(q,A) \arrow[u, dashed, "\ell"] & & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[ll, dashed, "r"] \arrow[u, dashed, "\ell"'] \arrow[ul, phantom, "\Uparrow\cong"] \end{tikzcd} \tag{11.1.7}$$

is an isomorphism.

Thus, we have shown that condition (i) is equivalent to (iii) positing the existence of adjunctions (11.1.6) in $\mathcal{K}_{/A\times B}$ so that all of the mates of the solid-arrow diagram are isomorphisms. Dualizing and reversing this argument, we see that this is equivalent to condition (ii).

Finally, (iii) and (iv) are equivalent since the existence of the left adjoints in (11.1.6) is equivalent to the span being cocartesian on the left, the existence of the right adjoints is equivalent to being cartesian on the right, and the compatibility condition for the cartesian and cocartesian lifts is the meaning of the isomorphism (11.1.7). ☐

11.1.8. Corollary. *Any two-sided isofibration $(a,b)\colon X \twoheadrightarrow A \times B$ that is equivalent over $A \times B$ to a two-sided fibration $(q,p)\colon E \twoheadrightarrow A\times B$ is a two-sided fibration.*

Proof. The assertion of Theorem 11.1.4(iii) is invariant under fibered equivalence. ☐

Theorem 11.1.4 will help us establish an important family of examples.

11.1.9. Proposition. *For any ∞-category A and any $n \geq 2$, the two-sided isofibration $A \xleftarrow{p_{n-1}} A^{\mathbb{n}} \xrightarrow{p_0} A$ defines a two-sided fibration.*

This result is a generalization of Proposition 5.1.23 and its dual and the proof uses similar ideas.

Proof. We use Theorem 11.1.4(iii). The right representable comma ∞-category associated to $p_0\colon A^{\mathbb{n}} \twoheadrightarrow A$ is constructed by the pullback

$$\begin{tikzcd} A^{\mathbb{2}\vee\mathbb{n}} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "p_1"] \\ A^{\mathbb{n}} \arrow[r, two heads, "p_0"'] & A \end{tikzcd}$$

which is equivalent to $(p_n, p_0)\colon A^{\mathbb{n}+\mathbb{1}} \twoheadrightarrow A \times A$ over the endpoint evaluation maps. The canonical map

$$\begin{tikzcd} A^{\mathbb{n}} \arrow[rr, "A^{\sigma^0}"', "\top"] \arrow[dr, two heads, "(p_{n-1},p_0)"'] & & A^{\mathbb{n}+\mathbb{1}} \arrow[ll, dashed, bend right, "A^{\delta^1}"'] \arrow[dl, two heads, "(p_n,p_0)"] \\ & A \times A & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{n} \arrow[rr, dashed, bend left, "\delta^1"] & & \mathbb{n}+\mathbb{1} \arrow[ll, "\sigma_0", "\perp"'] \\ & \mathbb{1}+\mathbb{1} \arrow[ul, hook, "(n-1{,}0)"] \arrow[ur, hook, "(n{,}0)"'] & \end{tikzcd}$$

that tests whether $(p_{n-1}, p_0)\colon A^{\mathbb{n}} \twoheadrightarrow A \times A$ is cartesian on the right is given by restriction along the epimorphism $\sigma^0\colon \mathbb{n}+\mathbb{1} \to \mathbb{n}$ that sends the objects $0,1 \in \mathbb{n}+\mathbb{1}$ to $0 \in \mathbb{n}$. This functor admits a left adjoint under the endpoint inclusions displayed above-right, which provides the desired fibered right adjoint displayed above left.

A dual argument shows that $(p_{n-1}, p_0)\colon A^{\mathbb{n}} \twoheadrightarrow A\times A$ is cocartesian on the left. The final condition asks that the mate of the commutative square defined by the degeneracy maps

$$\begin{tikzcd}
\mathbb{n} \arrow[d, bend right, dashed, "\delta^n"'] \arrow[r, bend left, dashed, "\delta^1"] \arrow[r, phantom, "\perp"] & \mathbb{n}+\mathbb{1} \arrow[l, "\sigma^0"] \arrow[d, bend left, dashed, "\delta^{n+1}"] \\
\mathbb{n}+\mathbb{1} \arrow[u, "\sigma^n"'] \arrow[u, phantom, "\vdash"] \arrow[r, bend right, dashed, "\delta^1"'] \arrow[r, phantom, "\top"] & \mathbb{n}+\mathbb{2} \arrow[l, "\sigma^0"'] \arrow[u, "\sigma^{n+1}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

is an isomorphism, as is easily verified. The square in Theorem 11.1.4(iii) is obtained by applying $A^{(-)}$. $\square$

Theorem 11.1.4 has a relative analogue, whose proof is left to the reader, which characterizes what we call a **cartesian functor** between two-sided fibrations from A to B.

11.1.10. LEMMA. *A map of spans between a pair of two-sided fibrations from A to B*

$$\begin{tikzcd}
& E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\
A & & B \\
& F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] &
\end{tikzcd} \tag{11.1.11}$$

defines a cartesian functor between the cartesian fibrations in $\mathcal{coCart}(\mathcal{K})_{/A}$ if and only if it defines a cartesian functor between the cocartesian fibrations in $\mathcal{Cart}(\mathcal{K})_{/B}$.

PROOF. A similar argument to that given in Theorem 11.1.4 shows that the map $g\colon E \to F$ of (11.1.11) is a cartesian functor between cartesian fibrations in $\mathcal{coCart}(\mathcal{K})_{/A}$ if and only if in two commutative diagrams over $A\times B$, the mates are isomorphisms. This condition also characterizes cartesian functors between cocartesian fibrations in $\mathcal{Cart}(\mathcal{K})_{/B}$. The details are left as Exercise 11.1.ii. $\square$

A cartesian functor is just a map of spans (11.1.11) that defines a cartesian functor between the cocartesian fibrations q and s and also a cartesian functor between the cartesian fibrations p and r. It follows from the internal characterization (iii) of Theorem 11.1.4 and a similar internal characterization of cartesian functors derived from Theorem 5.1.19 that:

11.1.12. COROLLARY. *Any cosmological functor preserves two-sided fibrations and cartesian functors between them.* $\square$

Exercises.

11.1.i. EXERCISE. Suppose $(q,p)\colon E \twoheadrightarrow A\times B$ is a *discrete* cocartesian fibration in $\mathcal{K}_{/B}$. Prove, for any 2-cell χ with codomain E, that if $p\chi$ is an isomorphism, then χ is q-cocartesian.

11.1.ii. EXERCISE. Prove Lemma 11.1.10.

11.1.iii. EXERCISE. Prove that

(i) A two-sided isofibration $\mathbb{1} \xleftarrow{!} E \overset{p}{\twoheadrightarrow} B$ defines a two-sided fibration from $\mathbb{1}$ to B if and only if $p\colon E \twoheadrightarrow B$ is a cartesian fibration.

(ii) A map of spans

$$\begin{tikzcd} & E \arrow[dl, "!"'] \arrow[dr, "p"] \arrow[dd, "g"] & \\ 1 & & B \\ & F \arrow[ul, "!"] \arrow[ur, "q"'] & \end{tikzcd}$$

defines a cartesian functor of two-sided fibrations if and only if $g: E \to F$ defines a cartesian functor from p to q.

11.2. The ∞-cosmos of two-sided fibrations

The equivalent conditions (i) and (ii) of Theorem 11.1.4 provide two equivalent ways to define the ∞-cosmos of two-sided fibrations.

11.2.1. Definition (the ∞-cosmos of two-sided fibrations). By Theorem 11.1.4 and Lemma 11.1.10, the pair of quasi-categorically enriched subcategories

$$\mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A\times B \twoheadrightarrow A} \qquad \text{and} \qquad co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/\pi\colon A\times B \twoheadrightarrow B}$$

of $\mathcal{K}_{/A\times B}$ coincide. By Proposition 8.2.12, applied twice, this subcategory is an ∞-cosmos, which we call the *∞-cosmos of two-sided fibrations* from A to B and denote by

$${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \subset \mathcal{K}_{/A\times B}.$$

By definition

$$\mathsf{Fun}_{{}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}}(E,F) \coloneqq \mathsf{Fun}^{\mathrm{c}}_{A\times B}(E,F) \subset \mathsf{Fun}_{A\times B}(E,F)$$

is the quasi-category of maps of spans that define cartesian functors from E to F in the sense of Lemma 11.1.10.

11.2.2. Proposition. *The simplicial subcategory ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$ creates an ∞-cosmos structure on the ∞-cosmos of two-sided fibrations from the ∞-cosmos of two-sided isofibrations.*

Proof. This inclusion factors as

$${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \cong co\mathcal{C}art(\mathcal{C}art(\mathcal{K})_{/B})_{/A\times B\twoheadrightarrow B} \hookrightarrow (\mathcal{C}art(\mathcal{K})_{/B})_{/A\times B\twoheadrightarrow B} \hookrightarrow (\mathcal{K}_{/B})_{/A\times B\twoheadrightarrow B} \cong \mathcal{K}_{/A\times B}$$

Applying Proposition 8.2.12, we see that both inclusions create ∞-cosmos structures. ☐

11.2.3. Observation (two-sided fibrations generalize (co)cartesian fibrations). By Exercise 11.1.iii, a two-sided fibration from B to 1 is a cocartesian fibration over B, while a two-sided fibration from 1 to B is a cartesian fibration over B. Indeed,

$$co\mathcal{C}art(\mathcal{K})_{/B} \cong {}_{B\backslash}\mathcal{F}ib(\mathcal{K})_{/1} \qquad \text{and} \qquad \mathcal{C}art(\mathcal{K})_{/B} \cong {}_{1\backslash}\mathcal{F}ib(\mathcal{K})_{/B}.$$

In this sense, statements about two-sided fibrations simultaneously generalize statements about cartesian and cocartesian fibrations.

11.2.4. Proposition. *For any pair of functors $a: A' \to A$ and $b: B' \to B$, the cosmological pullback functor*

$$(a,b)^*: \mathcal{K}_{/A\times B} \longrightarrow \mathcal{K}_{/A'\times B'}$$

restricts to define a cosmological functor

$$(a,b)^*: {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \longrightarrow {}_{A'\backslash}\mathcal{F}ib(\mathcal{K})_{/B'}.$$

In particular, the pullback of a two-sided fibration is again a two-sided fibration.

PROOF. By factoring (a, b) as $A' \times B' \xrightarrow{\mathrm{id} \times b} A' \times B \xrightarrow{a \times \mathrm{id}} A \times B$ we see that it suffices to consider pullback along one side at a time. Proposition 5.1.20 proves that pullback along $b\colon B' \to B$ preserves cartesian fibrations and cartesian functors, defining a restricted functor

$$\begin{tikzcd} \mathcal{C}art(\mathcal{K})_{/B} \arrow[r, hook] \arrow[d, dashed, "b^*"'] & \mathcal{K}_{/B} \arrow[d, "b^*"] \\ \mathcal{C}art(\mathcal{K})_{/B'} \arrow[r, hook] & \mathcal{K}_{/B'} \end{tikzcd}$$

Since limits and isofibrations in $\mathcal{C}art(\mathcal{K})_{/B}$ are created in $\mathcal{K}_{/B}$, this restricted functor is a cosmological functor. Applying this result to the map

$$\begin{tikzcd} A \times B' \arrow[rr, "\mathrm{id} \times b"] \arrow[dr, two heads, "\pi"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

in the ∞-cosmos $co\mathcal{C}art(\mathcal{K})_{/A}$, we conclude that pullback restricts to define a cosmological functor

$$\begin{tikzcd} {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A \times B \twoheadrightarrow A} \arrow[r, hook] \arrow[d, dashed, "(\mathrm{id} \times b)^*"'] & (co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A \times B \twoheadrightarrow A} \arrow[r, hook] \arrow[d, "(\mathrm{id} \times b)^*"'] & \mathcal{K}_{/A \times B} \arrow[d, "(\mathrm{id} \times b)^*"] \\ {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B'} \cong \mathcal{C}art(co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A \times B' \twoheadrightarrow A} \arrow[r, hook] & (co\mathcal{C}art(\mathcal{K})_{/A})_{/\pi\colon A \times B' \twoheadrightarrow A} \arrow[r, hook] & \mathcal{K}_{/A \times B'} \end{tikzcd}$$

□

11.2.5. LEMMA. *If $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is a two-sided fibration from A to B, $v\colon A \twoheadrightarrow C$ is a cocartesian fibration and $u\colon B \twoheadrightarrow D$ is a cartesian fibration, then the composite span*

$$C \overset{v}{\twoheadleftarrow} A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B \overset{u}{\twoheadrightarrow} D$$

defines a two-sided fibration from C to D. Moreover, a cartesian functor between two-sided fibrations from A to B induces a cartesian functor

$$\begin{tikzcd} & & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & & & \\ C & A \arrow[l, two heads, "v"'] & & B \arrow[r, two heads, "u"] & D \\ & & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & & \end{tikzcd}$$

between two-sided fibrations from C to D.

Note composition in the sense being described here does not define a cosmological functor from ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$ to ${}_{C\backslash}\mathcal{F}ib(\mathcal{K})_{/D}$ because it does not preserve flexible limits.

PROOF. By Theorem 11.1.4, it suffices to consider composition on one side at a time, say with a cocartesian fibration $v\colon A \twoheadrightarrow C$. Working in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/B}$, we are given cocartesian fibrations

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[rr, two heads, "v \times \mathrm{id}"] \arrow[dr, two heads, "\pi"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

These compose vertically to define a cocartesian fibration

$$\begin{tikzcd} E \arrow[rr, two heads, "(vq,p)"] \arrow[dr, two heads, "p"'] & & C \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd}$$

and hence a two-sided fibration from C to B, as desired.

Now the vq-cocartesian cells are the q-cocartesian lifts of the v-cocartesian cells. If g is a cartesian functor from q to s, then these are clearly preserved, proving that g also defines a cartesian functor from vq to vs. $\square$

Proposition 11.2.4 and Lemma 11.2.5 combine to prove that two-sided fibrations can be composed "horizontally."

11.2.6. Proposition. *The pullback of a two-sided fibration from A to B along a two-sided fibration from B to C*

$$\begin{tikzcd} & & E \underset{B}{\times} F \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\ A & & B & & C \end{tikzcd}$$

defines a two-sided fibration from A to C as displayed.

Proof. The composite two-sided fibration is constructed in two stages, first by pulling back

$$\begin{tikzcd} E \underset{B}{\times} F \arrow[r] \arrow[d, two heads, "(\pi_1, r\pi_0)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, two heads, "(s,r)"] \\ E \times C \arrow[r, two heads, "p \times C"'] & B \times C \end{tikzcd}$$

and then by composing the left leg with the cocartesian fibration $q\colon E \twoheadrightarrow A$. By Proposition 11.2.4 and Lemma 11.2.5, this results is a two-sided fibration from A to C. Alternatively, the composite can be constructed by pulling back along $A \times s$ and composing with the cartesian fibration $r\colon F \twoheadrightarrow C$, resulting in another two-sided fibration from A to C that is canonically isomorphic to the first. $\square$

11.2.7. Example. If $p\colon E \twoheadrightarrow B$ is a cartesian fibration and $q\colon F \twoheadrightarrow A$ is a cocartesian fibration, then the span

$$A \overset{q}{\twoheadleftarrow} F \overset{\pi}{\twoheadleftarrow} F \times E \overset{\pi}{\twoheadrightarrow} E \overset{p}{\twoheadrightarrow} B$$

defines a two-sided fibration from A to B.

11.2.8. Example. By Proposition 11.1.9 and Proposition 11.2.4, a general comma span

$$C \xleftarrow{p_1} \mathsf{Hom}_A(f,g) \xrightarrow{p_0} B$$

is a two-sided fibration, as a pullback of $A \overset{p_1}{\twoheadleftarrow} A^{\mathbb{2}} \overset{p_0}{\twoheadrightarrow} A$. By Proposition 11.2.6, "horizontal composites" of comma spans are also two-sided fibrations. In certain cases, a horizontal composite again

defines a span that is equivalent to a comma span, as is the case for:

$$\begin{tikzcd}[column sep=small]
& & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[dl, twoheadrightarrow, "\pi_1"'] \arrow[dr, twoheadrightarrow, "\pi_0"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(A,g) \arrow[dl, twoheadrightarrow, "p_1"'] \arrow[dr, twoheadrightarrow, "p_0"] & & \mathsf{Hom}_A(f,A) \arrow[dl, twoheadrightarrow, "p_1"'] \arrow[dr, twoheadrightarrow, "p_0"] & \\
C & & A & & B
\end{tikzcd}$$

which is equivalent to $(p_1, p_0)\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C\times B$ over $C\times B$. Certain other horizontal composites are not equivalent to comma spans but nonetheless define two-sided fibrations, as is the case for:

$$\begin{tikzcd}[column sep=small]
& & \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \arrow[dl, twoheadrightarrow, "\pi_1"'] \arrow[dr, twoheadrightarrow, "\pi_0"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(a,A) \arrow[dl, twoheadrightarrow, "p_1"'] \arrow[dr, twoheadrightarrow, "p_0"] & & \mathsf{Hom}_B(B,b) \arrow[dl, twoheadrightarrow, "p_1"'] \arrow[dr, twoheadrightarrow, "p_0"] & \\
A & & X & & B
\end{tikzcd}$$

Exercises.

11.2.i. Exercise. Consider a diagram

$$\begin{tikzcd}
& E \arrow[dl, twoheadrightarrow, "q"'] \arrow[dr, twoheadrightarrow, "p"] \arrow[dd, "g"] & & F \arrow[dl, twoheadrightarrow, "s"'] \arrow[dr, twoheadrightarrow, "r"] \arrow[dd, "h"] & \\
A & & B & & C \\
& E' \arrow[ul, twoheadrightarrow, "q'"] \arrow[ur, twoheadrightarrow, "p'"'] & & B' \arrow[ul, twoheadrightarrow, "s'"] \arrow[ur, twoheadrightarrow, "r'"'] &
\end{tikzcd}$$

in which g and h define cartesian functors between two-sided fibrations, from A to B and from B to C, respectively. Prove that g and h pullback to define a cartesian functor $(g,h)\colon E \underset{B}{\times} F \to E' \underset{B}{\times} F'$ between two-sided fibrations from A to C.

11.2.ii. Exercise. Prove the assertion made in Example 11.2.7.

11.3. Representable two-sided fibrations and the Yoneda lemma

We now introduce representable two-sided fibrations and prove a two-sided version of the external Yoneda lemma.

11.3.1. Definition. Specializing Example 11.2.7, for any pair of elements $a\colon 1 \to A$ and $b\colon 1 \to B$, the span

$$A \overset{p_1\pi}{\twoheadleftarrow} \mathsf{Hom}_A(a,A) \times \mathsf{Hom}_B(B,b) \overset{p_0\pi}{\twoheadrightarrow} B$$

defines a two-sided fibration from A to B that we refer to as the **two-sided fibration represented by** a and b. Note there is a canonical element $(\mathrm{id}_a, \mathrm{id}_b)\colon 1 \to \mathsf{Hom}_A(a,A) \times \mathsf{Hom}_B(B,b)$ in the fiber over $(a,b)\colon 1 \to A\times B$.

The terminology of Definition 11.3.1 is justified by the Yoneda lemma for two-sided fibrations.

11.3.2. THEOREM (Yoneda lemma). *For any elements $a\colon 1 \to A$ and $b\colon 1 \to B$ and any two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, restriction along $(\ulcorner \mathrm{id}_a \urcorner, \ulcorner \mathrm{id}_b \urcorner)\colon 1 \to \mathsf{Hom}_A(a,A) \times \mathsf{Hom}_B(B,b)$ defines an equivalence of quasi-categories*

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b)\\ \downarrow\!\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}1\\ \downarrow{\scriptstyle (a,b)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right).$$

Interpreting this result in a slice ∞-cosmos will enable us to replace the elements a and b with a pair of generalized elements $a\colon X \to A$ and $b\colon X \to B$; see Corollary 11.3.5.

PROOF. The two-sided fibration represented by a and b is defined by a pullback

$$\begin{tikzcd} \mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b) \arrow[r, two heads, "\mathrm{id}\times p_0"] \arrow[d, two heads, "p_1\times \mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_A(a,A)\times B \arrow[d, two heads, "p_1\times\mathrm{id}"] \\ A\times \mathsf{Hom}_B(B,b) \arrow[r, two heads, "\mathrm{id}\times p_0"'] & A\times B \end{tikzcd}$$

We will argue by applying the Yoneda lemma of Theorem 5.5.4 twice: first for cocartesian fibrations over the object $\pi\colon A \times \mathsf{Hom}_B(B,b) \twoheadrightarrow \mathsf{Hom}_B(B,b)$ the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/\mathsf{Hom}_B(B,b)}$, and then for cartesian fibrations over the object B in the $\mathcal{K}$.

To set up the first use of the Yoneda lemma, we begin by pulling back the two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ along $p_0\colon \mathsf{Hom}_B(B,b) \to B$ to obtain a two-sided fibration

$$\begin{tikzcd} & & \mathsf{Hom}_B(p,b) \arrow[dl, two heads, "p_0"'] \arrow[dr, two heads, "p'"] \arrow[dd, phantom, "\vee", very near start] & \\ & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & \mathsf{Hom}_B(B,b) \arrow[dl, two heads, "p_0"] \\ A & & B & \end{tikzcd}$$

from A to $\mathsf{Hom}_B(B,b)$. By the adjoint correspondence between cartesian functors established by Lemma 11.3.3(iii) below, there is an isomorphism

$$\begin{aligned} &\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b)\\ \downarrow\!\!\!\downarrow{\scriptstyle p_1\times p_0} \\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right) \\ &\cong \mathsf{Fun}^{\mathrm{c}}_{A\times \mathsf{Hom}_B(B,b)}\left(\begin{array}{c}\mathsf{Hom}_A(a,A)\times \mathsf{Hom}_B(B,b)\\ \downarrow\!\!\!\downarrow{\scriptstyle p_1\times \mathrm{id}} \\ A\times \mathsf{Hom}_B(B,b)\end{array}, \begin{array}{c}\mathsf{Hom}_B(p,b)\\ \downarrow\!\!\!\downarrow{\scriptstyle (qp_0,p')}\\ A\times \mathsf{Hom}_B(B,b)\end{array}\right). \end{aligned}$$

Consider the object $\pi\colon A \times \mathrm{Hom}_B(B,b) \twoheadrightarrow \mathrm{Hom}_B(B,b)$ in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/\mathrm{Hom}_B(B,b)}$. The element

$$\begin{tikzcd} \mathrm{Hom}_B(B,b) \arrow[rr, "(a!,\mathrm{id})"] \arrow[dr, equal] & & A \times \mathrm{Hom}_B(B,b) \arrow[dl, two heads, "\pi"] \\ & \mathrm{Hom}_B(B,b) & \end{tikzcd}$$

has $p_1 \times \mathrm{id}\colon \mathrm{Hom}_A(a,A) \times \mathrm{Hom}_B(B,b) \to A \times \mathrm{Hom}_B(B,b)$ as its representing cocartesian fibration in $\mathcal{C}art(\mathcal{K})_{/\mathrm{Hom}_B(B,b)}$. Applying the Yoneda lemma to this representable cocartesian fibration in $\mathcal{C}art(\mathcal{K})_{/\mathrm{Hom}_B(B,b)}$, we see that restricting along this map defines an equivalence whose codomain

$$\mathrm{Fun}^{\mathrm{c}}_{A\times\mathrm{Hom}_B(B,b)}(\mathrm{Hom}_B(B,b) \to A \times \mathrm{Hom}_B(B,b), \mathrm{Hom}_B(p,b) \twoheadrightarrow A \times \mathrm{Hom}_B(B,b))$$

is the mapping quasi-category defined by the pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{c}}_{\mathrm{Hom}_B(B,b)}(\mathrm{id}_{\mathrm{Hom}_B(B,b)}, \mathrm{Hom}_B(p,b) \twoheadrightarrow \mathrm{Hom}_B(B,b)) \arrow[d, two heads, "(qp_0,p')\circ-"] \\ 1 \arrow[r, "(a!,\mathrm{id})"] & \mathrm{Fun}^{\mathrm{c}}_{\mathrm{Hom}_B(B,b)}(\mathrm{id}_{\mathrm{Hom}_B(B,b)}, A \times \mathrm{Hom}_B(B,b) \overset{\pi}{\twoheadrightarrow} \mathrm{Hom}_B(B,b)) \end{tikzcd}$$

By Lemma 11.3.3(iii) applied to the adjunction $(p_0 \circ -) \dashv p_0^*$ the right-hand vertical map is isomorphic to the left-hand vertical map displayed below

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{c}}_B(\mathrm{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, E \overset{p}{\twoheadrightarrow} B) \arrow[r, "\sim"] \arrow[d, two heads, "(q,p)\circ-"'] & \mathrm{Fun}_B(1 \xrightarrow{b} B, E \overset{p}{\twoheadrightarrow} B) \arrow[d, two heads, "(q,p)\circ-"] \\ \mathrm{Fun}^{\mathrm{c}}_B(\mathrm{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} \mathrm{Hom}_B(B,b), A \times B \overset{\pi}{\twoheadrightarrow} B) \arrow[r, "\sim"] & \mathrm{Fun}_B(1 \xrightarrow{b} B, A \times B \overset{\pi}{\twoheadrightarrow} B) \end{tikzcd}$$

which is equivalent to the vertical map on the right by the Yoneda lemma applied to the cartesian fibration $p_0\colon \mathrm{Hom}_B(B,b) \twoheadrightarrow B$ corresponding to the element $b\colon 1 \to B$. Combining these observations, we obtain an equivalence to the pullback

$$\begin{tikzcd} \mathrm{Fun}_{A\times B}(1 \xrightarrow{(a,b)} A \times B, E \overset{(q,p)}{\twoheadrightarrow} A \times B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}_B(1 \xrightarrow{b} B, E \overset{p}{\twoheadrightarrow} B) \arrow[d, two heads, "(q,p)\circ-"] \\ 1 \arrow[r, "(a,b)"] & \mathrm{Fun}_B(1 \xrightarrow{b} B, A \times B \overset{\pi}{\twoheadrightarrow} B) \end{tikzcd}$$

This composite equivalence is the map defined by precomposition with the canonical element

$$(\ulcorner \mathrm{id}_a \urcorner, \ulcorner \mathrm{id}_b \urcorner)\colon 1 \to \mathrm{Hom}_A(a,A) \times \mathrm{Hom}_B(B,b),$$

completing the proof. □

11.3.3. Lemma.

(i) Suppose $s\colon B \to A$ is a discrete cartesian fibration and $p\colon E \to B$ is an isofibration. Then $p\colon E \to B$ is a cocartesian fibration if and only if $sp\colon E \twoheadrightarrow A$ is a cocartesian fibration.

(ii) Suppose that s is a discrete cocartesian fibration and p and q are cocartesian fibrations. Then (g, s) defines a cartesian functor from p to q if and only if g defines a cartesian functor from sp to r.

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, two heads, "p"'] & F \arrow[d, two heads, "q"] \\ B \arrow[r, two heads, "s"'] & A \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "sp"'] & & F \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd} \tag{11.3.4}$$

(iii) If $s\colon B \twoheadrightarrow A$ is a discrete cocartesian fibration, then the adjunction $s \circ - \dashv s^*$

$$\mathsf{Fun}_A(E \overset{p}{\twoheadrightarrow} B \overset{s}{\twoheadrightarrow} A, F \overset{q}{\twoheadrightarrow} A) \cong \mathsf{Fun}_B(E \overset{p}{\twoheadrightarrow} B, s^*F \overset{s^*q}{\twoheadrightarrow} B)$$

between composition with and pullback along s restricts to cartesian functors:

$$\mathsf{Fun}^{\mathrm{c}}_A(E \overset{p}{\twoheadrightarrow} B \overset{s}{\twoheadrightarrow} A, F \overset{q}{\twoheadrightarrow} A) \cong \mathsf{Fun}^{\mathrm{c}}_B(E \overset{p}{\twoheadrightarrow} B, s^*F \overset{s^*q}{\twoheadrightarrow} B).$$

PROOF. For (i) recall that Lemma 5.1.7 proves that cocartesian fibrations compose, with sp-cocartesian cells being the p-cocartesian lifts of s-cocartesian cells. This proves that sp is cocartesian if p is, and the converse follows as well by the proof of that result: if s is a discrete cocartesian fibration, then any 2-cell with codomain B is s-cocartesian, so we may take the p-cocartesian cells to be precisely the sp-cocartesian cells.

For (ii), note that in proving (i), we have just argued that sp-cocartesian cells are precisely the same as p-cocartesian cells when p is a cocartesian fibration and s is a discrete cocartesian fibration. This proves that the two notions of cartesian functor coincide.[1]

Finally (iii) follows immediately from (ii) and Proposition 5.1.20, which tells us that a map to the pullback of a cocartesian fibration along $s\colon B \to A$ defines a cartesian functor over B if and only if the square as displayed on the left of (11.3.4) defines a cartesian functor. □

11.3.5. COROLLARY. *The inclusion*

$${}_{A\backslash}\mathcal{F}ib_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$$

admits a left biadjoint defined by sending a two-sided isofibration $A \overset{a}{\leftarrow} X \overset{b}{\rightarrow} B$ to the composite two-sided fibration

$$\begin{tikzcd} & & \mathsf{Hom}_A(a, A) \times_X \mathsf{Hom}_B(B, b) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & \mathsf{Hom}_A(a, A) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & & \mathsf{Hom}_B(B, b) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ A & & X & & B \end{tikzcd}$$

and equipped with a natural equivalence

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_A(a, A) \times_X \mathsf{Hom}_B(B, b) \\ \downarrow \\ A \times B \end{array}, \begin{array}{c} E \\ \downarrow \\ A \times B \end{array} \right) \simeq \mathsf{Fun}_{A\times B}\left(\begin{array}{c} X \\ \downarrow {\scriptstyle (a,b)} \\ A \times B \end{array}, \begin{array}{c} E \\ \downarrow \\ A \times B \end{array} \right).$$

[1]If s is cocartesian but not discrete, then if the left-hand diagram defines a cartesian functor, then so does the right one, but the converse no longer holds.

of functor spaces.

Proof. Let $(q,p)\colon E \twoheadrightarrow A \times B$ be a two-sided fibration in $\mathcal{K}$. Then since $- \times X\colon \mathcal{K} \to \mathcal{K}_{/X}$ is a cosmological functor, the span

$$A \times X \xtwoheadleftarrow{q\times\mathrm{id}} E \times X \xtwoheadrightarrow{p\times\mathrm{id}} B \times X$$

is a two-sided fibration in $\mathcal{K}_{/X}$. The maps $a\colon X \to A$ and $b\colon X \to B$ induce a pair of elements $(a,\mathrm{id})\colon X \to A \times X$ and $(b,\mathrm{id})\colon X \to B \times X$ in $\mathcal{K}_{/X}$ which are respectively covariantly and contravariantly represented by

$$(p_1,p_0)\colon \mathsf{Hom}_A(a,A) \twoheadrightarrow A \times X \qquad \text{and} \qquad (p_1,p_0)\colon \mathsf{Hom}_B(B,b) \twoheadrightarrow B \times X.$$

Applying Theorem 11.3.2 in $\mathcal{K}_{/X}$ to this two-sided isofibration and pair of elements, we find that restriction along the canonical map

$$i\colon X \to \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b)$$

defines an equivalence of quasi-categories

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B\times X}\left(\begin{array}{c}\mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b)\\ \downarrow\!\!\downarrow \\ A\times B\times X\end{array}, \begin{array}{c} E\times X \\ \downarrow\!\!\downarrow \\ A\times B\times X\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B\times X}\left(\begin{array}{c} X \\ \downarrow{\scriptstyle(a,b,\mathrm{id})} \\ A\times B\times X\end{array}, \begin{array}{c} E\times X \\ \downarrow\!\!\downarrow \\ A\times B\times X\end{array}\right).$$

Transposing the domain and codomain across the adjunction

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend right, "-\times X"'] \arrow[r, phantom, "\perp"] & \mathcal{K}_{/X} \arrow[l, bend right, "\Sigma_X"'] \end{tikzcd}$$

yields the equivalence of functor spaces

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b)\\ \downarrow\!\!\downarrow \\ A\times B\end{array}, \begin{array}{c} E \\ \downarrow\!\!\downarrow \\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} X \\ \downarrow{\scriptstyle(a,b)} \\ A\times B\end{array}, \begin{array}{c} E \\ \downarrow\!\!\downarrow \\ A\times B\end{array}\right)$$

of the statement. ☐

The argument used to establish the equivalence of functor spaces in Corollary 11.3.5 works for any pair of generalized elements $a\colon X \to A$ and $b\colon X \to B$, whether or not this span defines a two-sided isofibration. In the case of spans represented by a single non-identity functor a "one-sided" version of Corollary 11.3.5, which is much more simply established, may be preferred:

11.3.6. PROPOSITION (one-sided Yoneda for two-sided fibrations). *For any two-sided isofibration* $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ *and functor* $f\colon A \to B$, *restriction along the canonical functor*

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, "\ulcorner \mathrm{id}_f \urcorner"] & \\ A & \mathsf{Hom}_B(B,f) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B \end{tikzcd}$$

induce equivalences of functor spaces

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_B(B,f)\\ \downarrow\!\!\!\downarrow{\scriptstyle (p_1,p_0)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}A\\ \downarrow{\scriptstyle (\mathrm{id}_A,f)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right).$$

PROOF. This follows immediately from the external Yoneda lemma of Theorem 5.5.4 applied to the element $(\mathrm{id}_A, f)\colon A \to A \times B$ and the cartesian fibration $(q,p)\colon E \twoheadrightarrow A\times B$ in the ∞-cosmos $\mathcal{coCart}(\mathcal{K})_{/A}$. □

Exercises.

11.3.i. EXERCISE. State and prove the other one-sided Yoneda lemma for two-sided fibrations, establishing an equivalence of functor spaces below-left induced by restricting along the functor below-right

$$\mathsf{Fun}^{\mathrm{c}}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(f,A)\\ \downarrow\!\!\!\downarrow{\scriptstyle (p_1,p_0)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}B\\ \downarrow{\scriptstyle (f,\mathrm{id}_B)}\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow{\scriptstyle (q,p)}\\ A\times B\end{array}\right) \qquad \begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, equal] \arrow[d, "i"] & \\ A & \mathsf{Hom}_A(f,A) \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_0"'] & B \end{tikzcd}$$

11.4. Modules as discrete two-sided fibrations

We are not so much interested in two-sided fibrations but the special case of those two-sided fibrations that are discrete as objects in $\mathcal{K}_{/A\times B}$.

11.4.1. DEFINITION. A **module from** A **to** B is a two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ that is a discrete object in ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B}$.

Note that an object in the replete subcosmos ${}_{A\backslash}\mathcal{Fib}(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/A\times B}$ is discrete in there if and only if it is discrete as an object of $\mathcal{K}_{/A\times B}$; see Exercise 8.2.iv. Our work in this chapter enables us to give a direct characterization of modules:

11.4.2. LEMMA. *A two-sided isofibration* $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ *defines a module from* A *to* B *if and only if it is*

(i) *cocartesian on the left,*
(ii) *cartesian on the right,*
(iii) *and discrete as an object of* $\mathcal{K}_{/A\times B}$.

Proof. This follows immediately from the characterization of two-sided fibrations given as Theorem 11.1.4(iv). $\square$

Unpacking the definition, we easily establish the following properties of modules.

11.4.3. Lemma. *If $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a module from A to B, then*

(i) The functors

$$\begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "p"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, two heads, "{(q,p)}"] \arrow[dr, two heads, "q"'] & & A \times B \arrow[dl, two heads, "\pi"] \\ & A & \end{tikzcd}$$

define a discrete cocartesian fibration and a discrete cartesian fibration, respectively, in the slice ∞-cosmoi $\mathcal{K}_{/B}$ and $\mathcal{K}_{/A}$.

(ii) The functors $q\colon E \twoheadrightarrow A$ and $p\colon E \twoheadrightarrow B$ respectively define a cocartesian fibration and a cartesian fibration in $\mathcal{K}$.

(iii) For any 2-cell χ with codomain E, χ is p-cartesian if and only if $q\chi$ is invertible, and χ is q-cocartesian if and only if $p\chi$ is invertible.

(iv) In particular, any 2-cell that is fibered over $A \times B$ is both p- and q-cocartesian and any map of spans from a two-sided fibration

$$\begin{tikzcd} & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "q"] \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

to a module defines a cartesian functor of two-sided fibrations in the sense of Lemma 11.1.10, and also a cartesian functor from s to q and from r to p.

Proof. By Lemma 11.1.1, conditions (i) and (iii) together assert exactly that $(q,p)\colon E \twoheadrightarrow A \times B$ defines a discrete cocartesian fibration in $\mathcal{K}_{/B}$, proving (i). Statement (ii) holds for any two-sided fibration; the point of reasserting it here is that it is not necessarily the case that the legs of a module are themselves discrete fibrations (see Exercise 11.4.i).

One direction of statement (iii) is proven as Lemma 11.1.1 while the converse is proven in Exercise 11.1.i. Statement (iv) follows. $\square$

An important property of modules is that they are stable under pullback:

11.4.4. Proposition. *If $A \twoheadleftarrow E \twoheadrightarrow B$ is a module from A to B and $a\colon A' \to A$ and $b\colon B' \to B$ are any pair of functors, then the pullback defines a module $A' \twoheadleftarrow E(b,a) \twoheadrightarrow B'$ from A' to B'.*

Proof. The cosmological functor

$$(a,b)^*\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \longrightarrow {}_{A'\backslash}\mathcal{F}ib(\mathcal{K})_{/B'}$$

of Proposition 11.2.4 preserves discrete objects in these ∞-cosmoi. $\square$

Applying Proposition 11.4.4 to a pair of elements $a\colon 1 \to A$ and $b\colon 1 \to B$, we see that a module from A to B is a two-sided fibration whose fibers $E(b,a)$ are discrete objects. The converse does not generally hold: being discrete as an object of the sliced ∞-cosmos $\mathcal{K}_{/A\times B}$ is a stronger condition than merely having discrete fibers. However, in the ∞-cosmos $\mathcal{QC}at$ the point 1 is a generator in a suitable sense and we have:

11.4.5. LEMMA. *If $E \twoheadrightarrow A \times B$ is a two-sided fibration of quasi-categories whose fibers are Kan complexes, then E is a module.*

PROOF. By definition, $E \twoheadrightarrow A \times B$ defines a discrete object in $\mathcal{QC}at_{/A\times B}$ if any 1-simplex in E^X that lies over a degenerate 1-simplex in $(A \times B)^X$ is an isomorphism. As isomorphisms in functor quasi-categories are determined pointwise (see Lemma 16.2.1), it suffices to consider the case $X = 1$ and now this reduces to the hypothesis that the fibers $E(b,a)$ over any pair of points $(a,b) \in A \times B$ are Kan complexes. $\square$

The prototypical example of a module is given by the arrow ∞-category construction.

11.4.6. PROPOSITION. *For any ∞-category A, the arrow ∞-category A^2 defines a module from A to A.*

The fact that $A^2 \twoheadrightarrow A\times A$ is discrete is related to but stronger than the fact that each fiber over a pair of elements in A, the **internal hom-space** between those elements of A, is a discrete ∞-category, proven in Proposition 3.4.10.

PROOF. Proposition 11.1.9 proves that $(p_1, p_0)\colon A^2 \twoheadrightarrow A\times A$ is a two-sided fibration, so it remains only to verify the discreteness condition, which we can do in $\mathcal{K}_{/A\times A}$ Discreteness of $A^2 \twoheadrightarrow A \times A$ as an object of $\mathcal{K}_{/A\times A}$ is an immediate consequence of 2-cell conservativity of Proposition 3.2.5: if γ is any 2-cell with codomain A^2 so that $p_0\gamma$ and $p_1\gamma$ are invertible, then γ is itself invertible. $\square$

By the construction of (3.4.2) and Proposition 11.4.4:

11.4.7. COROLLARY. *The comma ∞-category $\mathsf{Hom}_A(f,g) \twoheadrightarrow C\times B$ associated to a pair of functors $f\colon B \to A$ and $g\colon C \to A$ defines a module from C to B.* $\square$

Two special cases of these comma modules, those studied in §3.5, deserve a special name:

11.4.8. DEFINITION. To any functor $f\colon A \to B$ between ∞-categories

(i) the module $\mathsf{Hom}_B(B,f)$ from A to B is **right** or **covariantly represented** by f, while
(ii) the module $\mathsf{Hom}_B(f,B)$ from B to A is **left** or **contravariantly represented** by f.

More generally, a module $A \overset{E}{\twoheadrightarrow\!\!\!\!\!\!+} B$ is **covariantly** or **contravariantly represented** by f if E is equivalent to the left or right represented modules over $A \times B$.

As we saw in §5.5, the Yoneda lemma for two-sided fibrations simplifies when mapping into a module on account of the observation in Lemma 11.4.3(iv) that any map of spans from a two-sided fibration to a module defines a cartesian functor.

11.4.9. THEOREM (Yoneda lemma for modules). *For any elements $a\colon X \to A$ and $b\colon X \to B$ and any module $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, restriction along $(\ulcorner \mathrm{id}_a \urcorner, \ulcorner \mathrm{id}_b \urcorner)\colon X \to \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b)$ defines an equivalence of Kan complexes*

$$\mathsf{Fun}_{A\times B}\left(\begin{array}{c} \mathsf{Hom}_A(a,A) \underset{X}{\times} \mathsf{Hom}_B(B,b) \\ \big\downarrow\!\!\downarrow \\ A\times B \end{array} , \begin{array}{c} E \\ \big\downarrow\!\!\downarrow \\ A\times B \end{array} \right) \xrightarrow{\sim} \mathsf{Fun}_{A\times B}\left(\begin{array}{c} X \\ {\scriptstyle (a,b)}\big\downarrow \\ A\times B \end{array} , \begin{array}{c} E \\ \big\downarrow\!\!\downarrow \\ A\times B \end{array} \right).$$

Similarly for any functors $f\colon A \to B$ or $g\colon B \to A$, restriction along $\ulcorner \mathrm{id}_f \urcorner\colon A \to \mathsf{Hom}_B(B,f)$ or $\ulcorner \mathrm{id}_g \urcorner\colon B \to \mathsf{Hom}_A(g,A)$ define equivalences of Kan complexes

$$\mathsf{Fun}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_B(B,f)\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right) \stackrel{\sim}{\twoheadrightarrow} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}A\\ {\scriptstyle(\mathrm{id},f)}\downarrow\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right)$$

$$\mathsf{Fun}_{A\times B}\left(\begin{array}{c}\mathsf{Hom}_A(g,A)\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right) \stackrel{\sim}{\twoheadrightarrow} \mathsf{Fun}_{A\times B}\left(\begin{array}{c}B\\ {\scriptstyle(g,\mathrm{id})}\downarrow\\ A\times B\end{array}, \begin{array}{c}E\\ \downarrow\!\!\!\downarrow\\ A\times B\end{array}\right). \qquad \square$$

Exercises.

11.4.i. Exercise. Demonstrate by means of an example that if $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ defines a module from A to B then it is not necessarily the case that $p\colon E \twoheadrightarrow B$ is a discrete cartesian fibration or $q\colon E \twoheadrightarrow A$ is a discrete cocartesian fibration.

11.4.ii. Exercise.

(i) Explain why the two-sided fibration $(p_{n-1}, p_n)\colon A^{\mathbb{n}} \twoheadrightarrow A \times A$ of Proposition 11.1.9 does not define a module for $n > 2$.

(ii) Conclude that the horizontal composite of modules, as defined in Proposition 11.2.6, is not necessarily a module.

CHAPTER 12

The calculus of modules

The calculus of modules between ∞-categories bears a strong resemblance to the calculus of (bi)modules between unital rings. Here ∞-categories take the place of rings, with functors between ∞-categories playing the role of ring homomorphisms, which we display vertically on the table below. A module E from A to B, like the two-sided fibrations considered in Chapter 11, is an ∞-category on which A "acts on the left" and B "acts on the right" and these actions commute; this is analogous to the situation for bimodules in ring theory and explains our choice of terminology.[1] Modules will now be depicted as $A \overset{E}{\nrightarrow} B$ whenever explicit names for the legs of the constituent span are not needed.

unital rings	A	∞-categories
ring homomorphisms	$\begin{array}{c} A' \\ {\scriptstyle a}\downarrow \\ A \end{array}$	∞-functors
bimodules between rings	$A \overset{E}{\nrightarrow} B$	modules between ∞-categories
module maps	$\begin{array}{ccc} A' & \overset{E'}{\nrightarrow} & B' \\ {\scriptstyle a}\downarrow & \Downarrow & \downarrow{\scriptstyle b} \\ A & \underset{E}{\nrightarrow} & B \end{array}$	module maps

Finally, there is a notion of module map, that we shall introduce below, whose boundary in the most general case is a square comprised of two modules and two functors as above. In ring theory, a module map with this boundary is given by an A'–B' module homomorphism $E' \to E(b, a)$, whose codomain is the A'–B' bimodule defined by restricting the scalar multiplication in the A–B module E along the ring homomorphisms a and b.

The analogy extends deeper than this: unital rings, ring homomorphisms, bimodules, and module maps define a *proarrow equipment*, in the sense of Wood [**105**].[2] Our main result in this chapter is Theorem 12.2.6, which asserts that ∞-categories, functors, modules, and module maps in any ∞-cosmos define a *virtual equipment*, in the sense of Cruttwell and Shulman [**28**].

As a first step, in §12.1 we introduce the double category of two-sided fibrations, which restricts to define a *virtual double category* of modules. A *double category* is a sort of 2-dimensional category with objects; two varieties of 1-morphisms, the "horizontal" and the "vertical"; and 2-dimensional cells fitting

[1] In the 1- and ∞-categorical literature, the names "profunctor," "correspondence," and "distributor" are all used as synonyms for "module."

[2] This can be seen as a special case of the prototypical equipment comprised of $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-modules, and $\mathcal{V}$-natural transformations between then, for any closed symmetric monoidal category $\mathcal{V}$. The equipment for rings is obtained from the case where $\mathcal{V}$ is the category of abelian groups by restricting to abelian group enriched categories with a single object.

into "squares" whose boundaries consist of horizontal and vertical 1-morphisms with compatible domains and codomains. A motivating example from abstract algebra is the double category of modules: objects are rings, vertical morphisms are ring homomorphisms, horizontal morphisms are bimodules, and whose squares are bimodule homomorphisms. In the literature, this sort of structure is sometimes called a *pseudo double category* — morphisms and squares compose strictly in the "vertical" direction but only up to isomorphism in the "horizontal" direction — but we'll refer to this simply as a "double category" as it is the only variety that we will consider.

Our aim in §12.1 is to describe a similar structure whose objects and vertical morphisms are the ∞-categories and functors in any fixed ∞-cosmos, whose horizontal morphisms are modules, and whose squares are module maps, as will be defined in 12.1.6. If the horizontal morphisms are replaced by the larger class of two-sided fibrations, this does define a double category with the horizontal composition operation defined by Proposition 11.2.6. However, on account of Exercise 11.4.ii, the horizontal composition of two-sided fibrations does not preserve the class of modules: the arrow ∞-category $A^{\mathbb{2}}$ defines a module from A to A whose horizontal composite with itself is equivalent to the two-sided fibration $(p_2, p_0)\colon A^{\mathbb{3}} \twoheadrightarrow A \times A$ of Proposition 11.1.9, which is not discrete in $\mathcal{K}_{/A\times A}$. To define a genuine "tensor product for modules" operation would require a two-stage construction: first forming the pullback that defines a composite two-sided fibration as in Proposition 11.2.6, and then reflecting this into a two-sided *discrete* fibration by means of some sort of "homotopy coinverter" construction. As colimits that are not within the purview of the axioms of an ∞-cosmos, this presents somewhat of an obstacle.

Rather than leave the comfort of our axiomatic framework in pursuit of a double category of modules, we instead describe the structure that naturally arises within the axiomatization: it turns out to be familiar to category theorists and robust enough for our desired applications. We first demonstrate that ∞-categories, functors, modules, and module maps assemble into a *virtual double category*, a weaker structure than a double category in which cells are permitted to have a multi horizontal source, as a "virtual" replacement for horizontal composition of modules.

Once the definition of a virtual equipment is given in §12.2, these axioms are very easily checked. The final two sections are devoted to exploring the consequences of this structure, which will be put to full use in Chapter 13, which develops the formal category theory of ∞-categories by introducing Kan extensions in the virtual equipment of modules. In §12.3, we explain how certain horizontal composites of modules can be recognized in the virtual equipment, even if the general construction of the tensor product of an arbitrary composable pair of modules is not known. The final §12.4 collects together many special properties of the modules $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ represented by a functor $f\colon A \to B$ of ∞-categories, revisiting some of the properties first established in §3.5.

12.1. The double category of two-sided fibrations

Our first task is to define the 2-dimensional morphisms in the double categories that we will introduce.

12.1.1. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be two-sided isofibrations. A **map of spans** from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ is a fibered isomorphism class of strictly commuting functors

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

where two such functors g and g' are considered equivalent if there exists a natural isomorphism $\alpha \colon g \cong g'$ so that $r\alpha = \mathrm{id}_q$ and $s\alpha = \mathrm{id}_p$.

12.1.2. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be two-sided fibrations. A **map of two-sided fibrations** from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ is a map of spans in which any and hence every representing map

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

defines a cartesian functor of two-sided fibrations defined in Lemma 11.1.10.

12.1.3. Definition. Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$ be modules. A **map of modules** from E to F is just a map of spans from $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ to $A \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} B$, that is a fibered isomorphism class of strictly commuting functors

$$\begin{tikzcd} & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] \arrow[dd, "g"] & \\ A & & B \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

12.1.4. Observation (the 1-categories of spans and maps). The 1-category of two-sided isofibrations from A to B and maps of spans may be obtained as a quotient of the quasi-categorically enriched category $\mathcal{K}_{/A\times B}$, or of its homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/A\times B})$, or of the slice homotopy 2-category $\mathfrak{h}\mathcal{K}_{/A\times B}$: it is the 1-category with the same set of objects and in which the morphisms are isomorphism classes of 0-arrows.

Similarly, the 1-category of two-sided fibrations from A to B and maps of two-sided fibrations may be obtained as a quotient of the quasi-categorically enriched category ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$, or of its homotopy 2-category $\mathfrak{h}({}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B})$: it is the 1-category with the same set of objects and in which the morphisms are isomorphism classes of 0-arrows.

By Lemma 11.4.3, the 1-category of modules from A to B and modules maps is a full subcategory of either of the two 1-categories just considered.

The 1-categories of Observation 12.1.4 are of interest because they precisely capture the correct notion of equivalence between two-sided isofibrations, two-sided fibrations, or modules first introduced in Definition 3.2.7.

12.1.5. LEMMA.

(i) A pair of two-sided isofibrations are equivalent in $\mathcal{K}_{/A\times B}$ if and only if they are isomorphic in the 1-category of spans from A to B.

(ii) A pair of two-sided fibrations are equivalent in ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$ if and only if they are isomorphic in the 1-category of two-sided fibrations from A to B.

(iii) A pair of modules are equivalent over A times B if and only if they are isomorphic in the 1-category of modules from A to B. □

Each of the Definitions 12.1.3 admits a common generalization, which defines the 2-dimensional maps inhabiting squares.

12.1.6. DEFINITION (maps in squares). A **map of modules** or **map of two-sided fibrations** or **map of two-sided isofibration** from $A' \overset{q'}{\twoheadleftarrow} E' \overset{p'}{\twoheadrightarrow} B'$ to $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ over $a\colon A' \to A$ and $b\colon B' \to B$ is

$$\begin{tikzcd} A' \arrow[r, "E'", "|"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "|"{marking}] & B \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "g"] \arrow[d, two heads, "{(q',p')}"'] & E \arrow[d, two heads, "{(q,p)}"] \\ A' \times B' \arrow[r, "a\times b"] & A \times B \end{tikzcd}$$

is a fibered isomorphism class of strictly commuting functors g as displayed above-right, which in the case of two-sided fibrations must preserve the cartesian and cocartesian transformations, where two such functors g and g' are considered equivalent if there exists a natural isomorphism $\alpha\colon g \cong g'$ so that $q\alpha = \mathrm{id}_{aq'}$ and $p\alpha = \mathrm{id}_{bp'}$.

In the case of modules or two-sided isofibrations, the functor-space $\mathsf{Fun}_{a\times b}(E', E)$ of maps from E' to E over $a \times b$ is defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_{a\times b}(E',E) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E',E) \arrow[d, two heads, "{(q,p)}"] \\ \mathbb{1} \arrow[r, "{(aq',bp')}"'] & \mathsf{Fun}(E', A \times B) \end{tikzcd}$$

In the case of two-sided fibrations, the functor space is taken to be the full sub quasi-category

$$\mathsf{Fun}^{c}_{a\times b}(E',E) \subset \mathsf{Fun}_{a\times b}(E',E)$$

of all n-simplices whose vertices define cartesian functors.

We occasionally extend the notion of map to allow the domain to be an identity span $C \xleftarrow{\mathrm{id}_C} C \xrightarrow{\mathrm{id}_C} C$, but unless the domain is the identity span, we always require the codomain to be at least a two-sided isofibration.

We now introduce the double categories of isofibrations and of two-sided fibrations. These structures can be viewed either as a collection of data present in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ of an ∞-cosmos or as a quotient of quasi-categorically enriched structures, presented by a non-unital internal category defined up to natural isomorphism in the category of ∞-cosmoi and cosmological functors; see Exercise 12.1.i. For economy of language, we adopt the former approach.

12.1.7. PROPOSITION (the double category of two-sided (iso)fibrations). *The homotopy 2-category of an ∞-cosmos supports a* ***double category***[3] ***of two-sided isofibrations*** *whose:*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows $A \overset{E}{\twoheadrightarrow} B$ are two-sided isofibrations $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, and*
- *2-cells with boundary*

$$\begin{tikzcd} A' \arrow[r, "E'", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "\shortmid"{marking}] & B \end{tikzcd}$$

are maps of two-sided isofibrations as defined in 12.1.6, or equivalently, are isomorphism classes of objects in the quasi-category $\mathsf{Fun}_{a\times b}(E', E)$.

Vertical composition of arrows and 2-cells is by composition in $\mathcal{K}$, while horizontal composition of arrows and 2-cells is by pullback, which is well-defined and associative up to isomorphism. The ***double category of two-sided fibrations*** *is the sub double category that has the same objects and vertical arrows but whose:*

- *horizontal arrows $A \overset{E}{\twoheadrightarrow} B$ are two-sided fibrations $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ and*
- *2-cells with boundary*

$$\begin{tikzcd} A' \arrow[r, "E'", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "\shortmid"{marking}] & B \end{tikzcd}$$

are maps of two-sided fibrations as defined in 12.1.6, or equivalently, are isomorphism classes of objects in $\mathsf{Fun}^{c}_{a\times b}(E', E)$.

PROOF. The composition of horizontal arrows is defined in Proposition 11.2.6, while the horizontal composition of 2-cells is defined in Exercise 11.2.i. By simplicial functoriality of pullback and composition in $\mathcal{K}$, both constructions are associative up to canonical natural isomorphism. □

12.1.8. REMARK (why we left out the horizontal unit). We could have formally added the identity span $\Delta\colon A \to A \times A$ to serve as a horizontal unit in the double categories of Proposition 12.1.7 but we find it less confusing to leave them out because when we restrict to the structure of greatest interest, the virtual equipment category of modules, we will see that the arrow ∞-category $A \overset{\mathsf{Hom}_A}{\twoheadrightarrow} A$ plays the role of the horizontal unit for composition in a sense to be described in Proposition 12.2.4.

12.1.9. DEFINITION (virtual double category). A **virtual double category** consists of

- a category of **objects** and **vertical arrows**
- for any pair of objects A, B, a class of **horizontal arrows** $A \twoheadrightarrow B$

[3]As discussed previously, our double categories support vertical composition laws that are strictly unital and associative but horizontal composition laws that are only associative and unital up to isomorphism. For reasons to be explained in Remarks 12.1.8 and 12.1.13, we choose not to require a horizontal unit arrow or 2-cell.

- **cells**, with boundary depicted as follows

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1"] \arrow[d, "f"'] & A_1 \arrow[r, "E_2"] & \cdots \arrow[r, "E_n"] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "F"'] & & \arrow[u, phantom, "\Downarrow"] & B_n
\end{tikzcd}
\tag{12.1.10}
$$

  including those whose horizontal source has length zero, in the case $A_0 = A_n$.
- a **composite cell** as below-right, for any configuration as below-left

$$
\begin{tikzcd}
A_0 \arrow[r, dotted, "{E_{11},\ldots,E_{1k_1}}"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow"] & A_1 \arrow[r, dotted, "{E_{21},\ldots,E_{2k_2}}"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, dotted, "{E_{n1},\ldots,E_{nk_n}}"] \arrow[d, "\cdots"] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1"] \arrow[d, "g"'] & B_1 \arrow[r, "F_2"] & \cdots \arrow[r, "F_n"] & B_n \arrow[d, "h"] \\
C_0 \arrow[rrr, "G"'] & \arrow[u, phantom, "\Downarrow"] & & C_n
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}
A_0 \arrow[r, "E_{11}"] \arrow[d, "gf_0"'] & \cdot \arrow[r, dotted, "{E_{12},\ldots,E_{nk_n-1}}"] & \cdot \arrow[r, "E_{nk_n}"] & A_n \arrow[d, "hf_n"] \\
C_0 \arrow[rrr, "G"'] & \arrow[u, phantom, "\Downarrow"] & & C_n
\end{tikzcd}
$$

- an **identity cell** for every horizontal arrow

$$
\begin{tikzcd}
A \arrow[r, "E"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_E"] & B \arrow[d, equal] \\
A \arrow[r, "E"'] & B
\end{tikzcd}
$$

so that composition of cells is associative and unital in the usual multi-categorical sense.

12.1.11. Lemma. *The double category of two-sided isofibrations and the double category of two-sided fibrations extend to a virtual double category in which the*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows are two-sided isofibrations or two-sided fibrations as appropriate, and*
- *n-ary cells* (12.1.10) *are 2-cells*

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "g"] \\
B_0 \arrow[r, "F"'] & B_n
\end{tikzcd}
\tag{12.1.12}
$$

  whose single vertical source is the $(n-1)$-fold pullback of the sequence of spans comprising the vertical source in (12.1.10).

Proof. The only thing perhaps worth commenting on is the nullary cells which have an empty sequence as their vertical domain

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "g"] \\
B \arrow[r, "F"'] & C
\end{tikzcd}
$$

which we interpret as a 0-fold pullback, this being the identity span from A to A. So the nullary cells displayed above are fibered isomorphism classes of maps

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\ B & & C \\ & F \arrow[ul, two heads, "s"] \arrow[ur, two heads, "r"'] & \end{tikzcd}$$

where h and h' lie in the same equivalence class if there exists a natural isomorphism $\alpha\colon h \cong h'$ so that $s\alpha = \mathrm{id}_f$ and $r\alpha = \mathrm{id}_g$. □

12.1.13. Remark. For instance, the map

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta"] & \\ A & & A \\ & A^{\mathbb{I}} \arrow[ul, two heads, "q_1"] \arrow[ur, two heads, "q_0"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\ A \arrow[r, "|" marking, "A^{\mathbb{I}}"'] & A \end{tikzcd}$$

defines a nullary morphism with codomain $A \overset{A^{\mathbb{I}}}{\nrightarrow} A$ in the virtual double category of two-sided isofibrations. Note, however, that despite the fact that $\Delta\colon A \xrightarrow{\sim} A^{\mathbb{I}}$ defines an equivalence in $\mathcal{K}$, this cell does not define an isomorphism in the virtual double category of any kind. It is for this sort of reason that we left out the identity horizontal arrows in Proposition 12.1.7.

Our main example of interest is a full sub virtual double category defined by restricting the class of horizontal arrows and taking all cells between them. Since the only operations given in the structure of a virtual double category are vertical sources and targets, vertical identities, and vertical composition, it is clear that this substructure is closed under all of these operations, and thus inherits the structure of a virtual double category:

12.1.14. Proposition. *For any ∞-cosmos $\mathcal{K}$, there is a virtual double category of modules $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ defined as a full subcategory of either the virtual double categories of isofibrations or the virtual double category of two-sided fibrations whose*

- *objects are ∞-categories,*
- *vertical arrows are functors,*
- *horizontal arrows $A \overset{E}{\nrightarrow} B$ are modules E from A to B,*
- *n-ary cells are fibered isomorphism classes of maps of spans*

$$\begin{tikzcd} A_0 \arrow[r, "|" marking, "E_1"] \arrow[d, "f"'] & A_1 \arrow[r, "|" marking, "E_2"] & \cdots \arrow[r, "|" marking, "E_n"] & A_n \arrow[d, "g"] \\ B_0 \arrow[rrr, "|" marking, "F"'] \arrow[rrru, phantom, "\Downarrow"] & & & B_n \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \arrow[dl, two heads] \arrow[dr, two heads] \arrow[ddd, dashed] & \\ A_0 \arrow[d, "f"'] & & A_n \arrow[d, "g"] \\ B_0 & & B_n \\ & F \arrow[ul, two heads] \arrow[ur, two heads] & \end{tikzcd} \qquad (12.1.15)$$

These maps, introduced in Definition 12.1.6, can be thought of as special cases of the n-ary cells of Lemma 12.1.11 where $E_1, \ldots, E_n, F$ are all required to be modules: the single horizontal source in the diagram

(12.1.12) *is the two-sided fibration defined by the* $(n-1)$*-fold pullback of the sequence of modules comprising the horizontal source in the left-hand diagram.* ☐

We refer to (12.1.15) as an n-**ary module map**. Note a 1-ary module map is just a module map as in Definition 12.1.6. We refer to the finite sequence of modules occurring as the horizontal domain of an n-ary module map as a **composable sequence of modules**, which just means that their horizontal sources and targets are compatible in the evident way.

A hint at the relevance of this notion of n-ary module map is given by the following special case.

12.1.16. LEMMA. *There is a bijection between* n*-ary module maps whose codomain module* $B \overset{\mathrm{Hom}_D(h,k)}{\nrightarrow} C$ *is a comma as displayed below-left and 2-cells in the homotopy 2-category whose boundary is displayed above-right.*

$$\begin{tikzcd}
A_0 \arrow[r, "E_1", "\mid" marking] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\mid" marking] & \cdots \arrow[r, "E_n", "\mid" marking] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "\mathrm{Hom}_D(h,k)"', "\mid" marking] & \arrow[r, phantom, "\Downarrow"] & & C
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
& E_1 \times_{A_1} \cdots \times_{A_{n-1}} E_n \arrow[dl] \arrow[dr] & \\
A_0 \arrow[d, "f"'] & \Leftarrow & A_n \arrow[d, "g"] \\
B \arrow[dr, "k"'] & & C \arrow[dl, "h"] \\
& D &
\end{tikzcd}$$

PROOF. Combine Definition 12.1.6 with Proposition 3.4.7. ☐

For any pair of objects A and B in the virtual double category of modules, there is a vertical 1-category of modules from A to B and module maps over a pair of identity functors, which coincides with the 1-category of modules from A to B introduced in Observation 12.1.4.

12.1.17. LEMMA. *A parallel pair of modules* $A \overset{E}{\nrightarrow} B$ *and* $A \overset{F}{\nrightarrow} B$ *are isomorphic as objects of vertical 1-category of modules in the virtual double category of modules if and only if the modules* E *and* F *are equivalent as spans from* A *to* B*.*

PROOF. This is a restatement of Lemma 12.1.5(iii). ☐

For consistency with the rest of the text, we write $E \simeq F$ or $E \simeq_{A\times B} F$ whenever the modules $A \overset{E}{\nrightarrow} B$ and $A \overset{F}{\nrightarrow} B$ are isomorphic as objects of the vertical 1-category of modules from A to B. For instance, Proposition 4.1.1 proves that a functor $f\colon B \to A$ is left adjoint to a functor $u\colon A \to B$ if and only if $\mathrm{Hom}_A(f, A) \simeq \mathrm{Hom}_B(B, u)$ over $A \times B$, that is, if and only if the modules $B \overset{\mathrm{Hom}_A(f,A)}{\nrightarrow} A$ and $A \overset{\mathrm{Hom}_B(B,u)}{\nrightarrow} B$ are isomorphic as objects of the vertical 1-category of modules from A to B.

Exercises.

12.1.i. EXERCISE. Generalize the proof of Proposition 8.1.1 to prove that for any ∞-cosmos $\mathcal{K}$, there is an ∞-cosmos $\mathcal{K}^{\lrcorner\llcorner}$ whose objects are two-sided isofibrations between an arbitrary pair of ∞-categories. Prove that the domain and codomain define cosmological functors $\mathcal{K}^{\lrcorner\llcorner} \rightrightarrows \mathcal{K}$. Use this to give a second description of the double category of two-sided isofibrations as as quotient of a structure defined at the level of quasi-categorically enriched categories.

12.1.ii. EXERCISE. Prove that any double category defines a virtual double category.[4]

[4]If the double category lacks horizontal identity morphisms, the corresponding virtual double category may lack nullary morphisms — unless these can be defined in some other way as we did in the proof of Lemma 12.1.11. Note that if

12.2. The virtual equipment of modules

The virtual double category of modules $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ in an ∞-cosmos $\mathcal{K}$ has two special properties that characterize what Cruttwell and Shulman term a *virtual equipment*. Before stating the definition, we explore each of these in turn.

12.2.1. Proposition (restriction). *Any diagram in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ as below-left can be completed to a **cartesian cell** as below-right*

$$\begin{tikzcd} A' \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "|" marking] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A' \arrow[r, "E(b,a)", "|" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & B' \arrow[d, "b"] \\ A \arrow[r, "E"', "|" marking] & B \end{tikzcd}$$

characterized by the universal property that any cell as displayed below-left factors uniquely through ρ as below-right:

$$\begin{tikzcd} X_0 \arrow[r, "E_1", "|" marking] \arrow[d, "af"'] & X_1 \arrow[r, "E_2", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] \arrow[d, phantom, "\Downarrow"] & X_n \arrow[d, "bg"] \\ A \arrow[rrr, "E"', "|" marking] & & & B \end{tikzcd} \quad = \quad \begin{tikzcd} X_0 \arrow[r, "E_1", "|" marking] \arrow[d, "f"'] & X_1 \arrow[r, "E_2", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] \arrow[d, phantom, "\exists!\Downarrow"] & X_n \arrow[d, "g"] \\ A' \arrow[rrr, "E(b,a)", "|" marking] \arrow[d, "a"'] & & \arrow[d, phantom, "\Downarrow\rho"] & B' \arrow[d, "b"] \\ A \arrow[rrr, "E"', "|" marking] & & & B \end{tikzcd}$$

Proof. The horizontal source of the cartesian cell is defined by restricting the module $A \overset{E}{\nrightarrow} B$ along the functors a and b:

$$\begin{tikzcd} E(b,a) \arrow[r, "\rho"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads] \\ A' \times B' \arrow[r, "(a,b)"'] & A \times B \end{tikzcd} \tag{12.2.2}$$

By Proposition 11.4.4, this left-hand isofibration defines a module from A' to B', while by Definition 12.1.6 the top horizontal functor represents a module map inhabiting the desired square. As in Definition 12.1.6, the simplicial pullback in $\mathcal{K}$ induces an equivalence[5] of functor spaces:

$$\mathsf{Fun}_{f\times g}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, E(b,a)) \xrightarrow[\sim]{\rho\circ -} \mathsf{Fun}_{af\times bg}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, E)$$

which descends to a bijection on isomorphism classes of objects. This defines the unique factorization of cells as displayed above left through the cartesian restriction cell ρ. ☐

---

we added identity spans to our double category of two-sided isofibrations, then the corresponding virtual double category would be the correct one, which contains the virtual double category of modules. See Remark 12.1.13 however.

[5]If the pullbacks are defined strictly, then in fact pullback induces an *isomorphism* of functor spaces, but even if $E(b,a)$ is replaced by an equivalent module, the functor spaces are still equivalent, which enough to induce a bijection on isomorphism classes of objects.

We refer to the module $A' \overset{E(b,a)}{\nrightarrow} B'$ as the **restriction** of $A \overset{E}{\nrightarrow} B$ along the functors a and b, because the pullback (12.2.2) is analogous to the restriction of scalars of a bimodule along a pair of ring homomorphisms.

12.2.3. Example. The module $C \overset{\mathrm{Hom}_A(f,g)}{\nrightarrow} B$ is the restriction of the module $A \overset{A^2}{\nrightarrow} A$ along $g\colon C \to A$ and $f\colon B \to A$. To make this restriction relationship more transparent, we typically write $A \overset{\mathrm{Hom}_A}{\nrightarrow} A$ when regarding the arrow ∞-category as a module. Since the common notation for "homs" places the contravariant variable on the left and the covariant variable on the right, we've adopted a similar notation convention for restrictions in Proposition 12.2.1.

12.2.4. Proposition (units). *Any object A in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ is equipped with a canonical nullary* ***cocartesian cell*** *as displayed below*

$$\begin{tikzcd}
A \arrow[r, equals, "\shortmid" marking] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equals] \\
A \arrow[r, "\shortmid" marking, "\mathrm{Hom}_A"'] & A
\end{tikzcd}$$

characterized by the universal property that any cell in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ whose horizontal source includes the object A factors uniquely through ι as below-right:

$$\begin{tikzcd}
X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking, "E_n"] & A \arrow[r, "\shortmid" marking, "F_1"] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] & Y \arrow[d, "g"] \\
B \arrow[rrrr, "\shortmid" marking, "G"'] & & & & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_1}"] & \cdots \arrow[r, "\shortmid" marking, "E_n"] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_n}"] & A \arrow[r, equals, "\shortmid" marking] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "\shortmid" marking, "F_1"] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_1}"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_m}"] & Y \arrow[d, equals] \\
X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking, "E_n"] & A \arrow[r, "\shortmid" marking, "\mathrm{Hom}_A"] & A \arrow[r, "\shortmid" marking, "F_1"] \arrow[d, phantom, "\Downarrow\exists!"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] & Y \arrow[d, "g"] \\
B \arrow[rrrrr, "\shortmid" marking, "G"'] & & & & & C
\end{tikzcd}$$

Proof. The canonical nullary cell is represented by the map of spans

$$\begin{tikzcd}
& A \arrow[dl, equals] \arrow[dr, equals] \arrow[dd, "\iota"] & \\
A & & A \\
& \mathrm{Hom}_A \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] &
\end{tikzcd}$$

induced by the generic arrow with codomain A (3.2.3); recall from Example 12.2.3 that we write $A \overset{\mathrm{Hom}_A}{\nrightarrow} A$ for the module encoded by the arrow ∞-category construction.

In the case where both of the sequences E_i and F_j are empty, the one-sided version of the Yoneda lemma for modules given by Theorem 11.4.9 tells us that restriction along the this map induces an equivalence of functor spaces

$$\mathrm{Fun}_{A\times A}(\mathrm{Hom}_A, G(g,f)) \xrightarrow{\sim} \mathrm{Fun}_{A\times A}(A, G(g,f)).$$

Taking isomorphism classes of objects gives the bijection of the statement.

In the case where one or both of the sequences are non-empty, we may form their horizontal composite two-sided fibrations and then form either the horizontal composite $E \underset{A}{\times} \mathrm{Hom}_A$ or the horizontal

composite $\mathsf{Hom}_A \underset{A}{\times} F$ of the composable triple below:

$$\begin{tikzcd}[column sep=small]
& & & & E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & & \\
& & & \mathsf{Hom}_A(A,p) \arrow[dll, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & {} & \mathsf{Hom}_A(s,A) \arrow[dl, two heads] \arrow[drr, two heads] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\
& E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & {} & \mathsf{Hom}_A \arrow[dll, two heads, "p_1"'] \arrow[drr, two heads, "p_0"] & {} & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\
X & & A & & & & A & & Y
\end{tikzcd}$$

In the former case, this constructs the two-sided fibration $(qp_1, p_0)\colon \mathsf{Hom}_A(A,p) \twoheadrightarrow X \times A$, and in the latter case this constructs the two-sided fibration $(p_1, rp_0)\colon \mathsf{Hom}_A(s,A) \twoheadrightarrow A \times Y$. In both cases, by Lemma 11.1.1, the map ι pulls back to define a fibered adjoint functor

$$\begin{tikzcd}
E \arrow[rr, "E \underset{A}{\times} \iota", "\perp"'] \arrow[dr, two heads, "{(q,p)}"'] & & \mathsf{Hom}_A(A,p) \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads, "{(qp_1,p_0)}"] \\
& X \times A &
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[rr, "\perp", "\iota \underset{A}{\times} F"'] \arrow[dr, two heads, "{(s,r)}"'] & & \mathsf{Hom}_A(s,A) \arrow[ll, dashed, bend right, "\ell"'] \arrow[dl, two heads, "{(p_1,rp_0)}"] \\
& A \times Y &
\end{tikzcd}$$

We only require one of these adjunctions, so without loss of generality we use the former. This fibered adjunction pulls back along s and pushes forward along r to define a fibered adjunction

$$\begin{tikzcd}
E \underset{A}{\times} F \arrow[rr, "E \underset{A}{\times} \iota \underset{A}{\times} F", "\perp"'] \arrow[dr, two heads] & & E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F \arrow[ll, dashed, bend left, "r"] \arrow[dl, two heads] \\
& X \times Y &
\end{tikzcd}$$

between two-sided fibrations. Upon mapping into the discrete object $G(g,f) \twoheadrightarrow X \times Y$, this adjunction becomes an adjoint equivalence. In particular, restriction along ι induces an equivalence of Kan complexes

$$\mathsf{Fun}_{X \times Y}(E \underset{A}{\times} \mathsf{Hom}_A \underset{A}{\times} F, G(g,f)) \xrightarrow[\sim]{-\circ(E \underset{A}{\times} \iota \underset{A}{\times} F)} \mathsf{Fun}_{X\times Y}(E \underset{A}{\times} F, G(g,f)),$$

and once again taking isomorphism classes of objects gives the bijection of the statement. □

Propositions 12.2.1 and 12.2.4 imply that the virtual double category of modules is a *virtual equipment* in the sense introduced by Cruttwell and Shulman [**28**, §7].

12.2.5. Definition. A **virtual equipment** is a virtual double category so that

(i) For any horizontal arrow $A \overset{E}{\nrightarrow} B$ and pair of vertical arrows $a\colon A' \to A$ and $b\colon B' \to B$, there exists a horizontal arrow $B' \overset{E(b,a)}{\nrightarrow} A'$ and unary cartesian cell ρ satisfying the universal property of Proposition 12.2.1.

(ii) Every object A admits a **unit** horizontal arrow $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ equipped with a nullary cocartesian cell ι satisfying the universal property of Proposition 12.2.4.

Thus, Propositions 12.2.1 and 12.2.4 combine to prove:

12.2.6. THEOREM. *The virtual double category $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ of modules in an ∞-cosmos $\mathcal{K}$ is a virtual equipment.* □

By abstract nonsense, the relatively simple axioms (i) and (ii) established in Theorem 12.2.6 establish a robust "calculus of modules." In an effort to familiarize the reader with this little-known categorical structure and expedite the proofs of the formal category theory of ∞-categories in Chapter 13, we devote the remainder of this chapter to proving a plethora of results that actually follow formally from this axiomatization: namely, Lemmas 12.3.5 and 12.3.11, Proposition 12.4.1, Theorem 12.4.4, Corollary 12.4.6, Proposition 12.4.7, Corollary 12.4.8, and the bijection of Proposition 12.4.10 between unary cells in the virtual equipment of modules. One additional result is left as Exercise 12.4.iii for the reader.

12.2.7. NOTATION. We adopt the following notational conventions to streamline certain virtual equipment diagrams.

- We adopt the convention that an unlabeled unary cell whose vertical boundaries are identities and whose horizontal sources and targets is an identity cell.

$$\begin{tikzcd} A \arrow[r, "E", "\mid" marking] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "E"', "\mid" marking] & B \end{tikzcd} \quad := \quad \begin{tikzcd} A \arrow[r, "E", "\mid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_E"] & B \arrow[d, equal] \\ A \arrow[r, "E"', "\mid" marking] & B \end{tikzcd}$$

- Cells whose vertical boundary functors are identities and therefore whose source and target spans lie between the same pair of ∞-categories

$$\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E \qquad := \qquad \begin{tikzcd} A_0 \arrow[r, "E_1", "\mid" marking] \arrow[d, equal] & A_1 \arrow[r, "E_2", "\mid" marking] & \cdots \arrow[r, "E_n", "\mid" marking] & A_n \arrow[d, equal] \\ A_0 \arrow[rrr, "E"', "\mid" marking] \arrow[rrr, phantom, shift left=4, "\Downarrow\mu"] & & & A_n \end{tikzcd}$$

  may be displayed in line using the notation $\mu\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$, an expression which implicitly asserts that the modules appearing in the domain define a composable sequence, with the symbol "$\dot\times$" meant to suggest the pullback appearing as the horizontal domain of (12.1.10) rather than a product.

Exercises.

12.2.i. EXERCISE. Prove that unital rings, ring homomorphisms, bimodules, and bimodule maps also define a virtual equipment.

12.3. Composition of modules

In a virtual equipment one cannot define the composite of a generic pair $A \overset{E}{\twoheadrightarrow} B$ and $B \overset{F}{\twoheadrightarrow} C$ of horizontal morphisms but there is a mechanism by which a particular horizontal composite $A \overset{G}{\twoheadrightarrow} C$ that happens to exist can be recognized, in which case we write $E \otimes F \simeq G$ to reinforce the intuition provided by the analogy with bimodules. In the virtual equipment of modules, a composition relation $E \otimes F \simeq G$ does not mean that the module G is equivalent to the horizontal composite two-sided isofibration $E \mathbin{\dot\times} F$. Rather, in the notation of 12.2.7 a composition relation is witnessed by a module

map $E \mathbin{\underset{\cdot}{\times}} F \Rightarrow G$ that defines a cocartesian cell in a sense analogous to the universal property stated in Proposition 12.2.4.[6]

12.3.1. Definition. A composable sequence of modules

$$A_0 \overset{E_1}{\nrightarrow} A_1, A_1 \overset{E_2}{\nrightarrow} A_2, \ldots, A_{n-1} \overset{E_n}{\nrightarrow} A_n$$

admits a composite if there exists a module $A_0 \overset{E}{\nrightarrow} A_n$ and a cocartesian cell

$$\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$$

characterized by the universal property that any cell of the form

$$\begin{tikzcd}
X \arrow[r, "F_1", "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "F_m", "|" marking] & A_0 \arrow[r, "E_1", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] \arrow[d, phantom, "\Downarrow"] & A_n \arrow[r, "G_1", "|" marking] & \cdots \arrow[r, "G_k", "|" marking] & Y \arrow[d, "g"] \\
B \arrow[rrrrrr, "H"', "|" marking] & & & & & & C
\end{tikzcd}$$

factors uniquely through μ as follows:

$$\begin{tikzcd}
X \arrow[r, "F_1", "|" marking] \arrow[d, equals] & \cdots \arrow[r, "F_m", "|" marking] \arrow[d, phantom, "\cdots"] & A_0 \arrow[r, "E_1", "|" marking] \arrow[d, equals] & \cdots \arrow[r, "E_n", "|" marking] \arrow[d, phantom, "\Downarrow\mu"] & A_n \arrow[r, "G_1", "|" marking] \arrow[d, equals] & \cdots \arrow[r, "G_k", "|" marking] \arrow[d, phantom, "\cdots"] & Y \arrow[d, equals] \\
X \arrow[r, "F_1", "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "F_m", "|" marking] & A_0 \arrow[rr, "E", "|" marking] & \arrow[d, phantom, "\Downarrow\exists!"] & A_n \arrow[r, "G_1", "|" marking] & \cdots \arrow[r, "G_k", "|" marking] & Y \arrow[d, "g"] \\
B \arrow[rrrrrr, "H"', "|" marking] & & & & & & C
\end{tikzcd}$$

A composite $\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$ can be used to reduce the domain of a cell by replacing any occurrence of the composable sequence $E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n$ from A_0 to A_n by a single module E. Particularly in the case of binary composites, we write $E_1 \otimes E_2$ to denote the **composite** of the modules E_1 and E_2, appearing as the codomain of the cocartesian cell $E_1 \mathbin{\underset{\cdot}{\times}} E_2 \Rightarrow E_1 \otimes E_2$. Lemma 12.1.17 easily implies that composites are unique up to vertical isomorphism, i.e., by Lemma 12.1.17 up to equivalence of modules. Moreover:

12.3.2. Lemma. *Suppose the cells $\mu_i \colon E_{i1} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{ik_i} \Rightarrow E_i$ exhibit composites for $i = 1, \ldots, n$.*

(i) *If $\mu \colon E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \Rightarrow E$ exhibits a composite then the composite cell*

$$E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n} \xRightarrow{\mu_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} \mu_n} E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \xRightarrow{\ \mu\ } E$$

exhibits E as a composite of the sequence $E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n}$.

(ii) *If the composite cell*

$$E_{11} \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_{nk_n} \xRightarrow{\mu_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} \mu_n} E_1 \mathbin{\underset{\cdot}{\times}} \cdots \mathbin{\underset{\cdot}{\times}} E_n \xRightarrow{\ \mu\ } E$$

[6] In nearly all cases where $E \otimes F \simeq G$, precomposition with the cocartesian map $\mu \colon E \times_B F \Rightarrow G$, which by Definition 12.1.1 corresponds to a fibered isomorphism class of maps of spans over $A \times C$, induces an equivalence of Kan complexes $- \circ \mu \colon \mathsf{Fun}_{A\times C}(G, H) \xrightarrow{\sim} \mathsf{Fun}_{A\times C}(E \times_B F, H)$ for all modules H, so the module G may be understood as the "reflection" of the two-sided isofibration $E \times_B F$ into the subcategory of modules. Corollary 11.3.5, reappearing as Proposition 12.3.7 below, is one instance of this.

exhibits E as a composite of the sequence $E_{11} \underset{\cdot}{\times} \cdots \underset{\cdot}{\times} E_{nk_n}$, then $\mu \colon E_1 \underset{\cdot}{\times} \cdots \underset{\cdot}{\times} E_n \Rightarrow E$ exhibits E as a composite of the sequence $E_1 \underset{\cdot}{\times} \cdots \underset{\cdot}{\times} E_n$.

Proof. For (i), the required bijection factors as a composite of $n+1$ bijections induced by the maps $\mu_1, \ldots, \mu_n$ and μ. For (ii), the required bijection induced by μ composes with the bijections supplied by the maps $\mu_1, \ldots, \mu_n$ to a bijection and is thus itself a bijection. ☐

The trivial instances of composites are easily verified:

12.3.3. Lemma.

(i) The units $\iota \colon \varnothing \Rightarrow \mathsf{Hom}_A$ of Proposition 12.2.4 define nullary composites.

(ii) A unary cell $\mu \colon E \Rightarrow F$ is a composite if and only if it is an isomorphism in the vertical category of modules from A to B and module maps over identity functors, that is, if and only if the modules E and F are equivalent as spans.

Proof. Exercise 12.3.ii. ☐

12.3.4. Remark. On account of the universal property of restrictions established in Proposition 12.2.1, to prove that a cell $\mu \colon E_1 \underset{\cdot}{\times} \cdots \underset{\cdot}{\times} E_n \Rightarrow E$ exhibits a composite, it suffices to prove the factorization property of Definition 12.3.1 in the case where the vertical functors are identities.

As one might hope, the unit modules $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ are units for the composition of Definition 12.3.1: for any module $A \overset{E}{\nrightarrow} B$, $\mathsf{Hom}_A \otimes E \otimes \mathsf{Hom}_B \simeq E$.

12.3.5. Lemma (composites with units). *For any module $A \overset{E}{\nrightarrow} B$ the unique cell $\circ \colon \mathsf{Hom}_A \times E \times \mathsf{Hom}_B \Rightarrow E$ defined using the universal property of the unit cell*

$$\begin{tikzcd}
A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, equal] \arrow[ur, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "\shortmid" marking] & B \arrow[r, "\mathsf{Hom}_B", "\shortmid" marking] \arrow[ur, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[rrr, "E"', "\shortmid" marking] \arrow[urrr, phantom, "\Downarrow\circ"] & & & B
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "E"', "\shortmid" marking] & B
\end{tikzcd}$$

displays E as the composite $\mathsf{Hom}_A \otimes E \otimes \mathsf{Hom}_B$.

Proof. The result is immediate from Lemma 12.3.2(ii) and Lemma 12.3.3. ☐

The argument used in the proof of Proposition 12.2.4 to demonstrate that units define nullary composites in the virtual equipment of modules applies also to more general composites.

12.3.6. Lemma. *Consider an n-ary module morphism $\mu \colon E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \Rightarrow E$ in the virtual equipment of modules whose codomain is a module from A to B. If any representing map of spans*

$$\begin{tikzcd}
E_1 \underset{\cdot}{\times} \cdots \underset{\cdot}{\times} E_n \arrow[rr, "\ulcorner\mu\urcorner"] \arrow[dr, two heads] & & E \arrow[dl, two heads] \\
& A \times B &
\end{tikzcd}$$

admits a fibered adjoint over $A \times B$, then μ exihibits E as the composite $E_1 \otimes \cdots \otimes E_n$.

PROOF. To verify the universal property of Definition 12.3.1, consider a composable sequence of modules $F_1, \dots, F_m$ from X to A and a composable sequence of modules $G_1, \dots, G_k$ from B to Y and form the horizontal composite two-sided fibrations

$$\begin{tikzcd} & F \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & \\ X & & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & G \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\ B & & Y \end{tikzcd}$$

The fibered adjunction of the statement pulls back along $p \times s \colon F \times G \twoheadrightarrow A \times B$ and pushes forward along $q \times r \colon F \times G \to X \times Y$ to a fibered adjoint to

$$\begin{tikzcd} F \underset{A}{\times} E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \underset{B}{\times} G \arrow[rr, "F \underset{A}{\times} \ulcorner\mu\urcorner \underset{B}{\times} G"] \arrow[dr, two heads] & & F \underset{A}{\times} E \underset{B}{\times} G \arrow[dl, two heads] \\ & X \times Y & \end{tikzcd}$$

Via Remark 12.3.4, it suffices to verify the universal property of the composite for modules $X \overset{H}{\nrightarrow} Y$. Since modules are discrete, applying $\mathsf{Fun}_{X\times Y}(-, H)$ transforms this fibered adjunction into an adjoint equivalence

$$\mathsf{Fun}_{X\times Y}(F \underset{A}{\times} E \underset{B}{\times} G, H) \xrightarrow[\sim]{-\circ(F \underset{A}{\times} \ulcorner\mu\urcorner \underset{B}{\times} G)} \mathsf{Fun}_{X\times Y}(F \underset{A}{\times} E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n \underset{B}{\times} G, H).$$

Passing to isomorphism classes of objects this now gives the universal property of Definition 12.3.1. □

For instance:

12.3.7. PROPOSITION. *Let $A \overset{E}{\nrightarrow} B$ be a module encoded by the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$. Then the binary module map*

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_A(q,A)", "\shortmid" marking] \arrow[d, equal] & E \arrow[r, "\mathsf{Hom}_B(B,p)", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mu"] & B \arrow[d, equal] \\ A \arrow[rr, "E"', "\shortmid" marking] & & B \end{tikzcd}$$

represented by composite left and right adjoints of Theorem 11.1.4(iii) exhibits $E \simeq \mathsf{Hom}_A(q, A) \otimes \mathsf{Hom}_B(B, p)$ as the composite of the modules representing its legs.

PROOF. Immediate from Theorem 11.1.4(iii) and Lemma 12.3.6 applied twice.[7] □

12.3.8. REMARK. Nothing in the proof of Proposition 12.3.7 requires that the span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is actually a module except for the interpretation that μ is a binary cell in the virtual equipment of modules. For any two-sided fibration $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, it is still the case that restriction along μ defines a bijection between n-ary maps whose source includes the span E and whose codomain is a module stand in bijection with $(n + 1)$-ary maps whose source includes $\mathsf{Hom}_A(q, A) \underset{E}{\times} \mathsf{Hom}_B(B, p)$. By Theorem 11.4.9, this result can be extended further to mere spans E, not necessarily two-sided fibrations.

[7]Of course the composite of a left and a right adjoint is not an adjoint but here we're effectively composing adjoint equivalences in which case the direction does not matter.

12.3.9. Remark. We provide another description of composition map $\mu\colon \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \Rightarrow E$. The comma cones

$$\begin{tikzcd}
& & \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& \mathsf{Hom}_A(q,A) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathsf{Hom}_B(B,p) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A & & E \arrow[ll, two heads, "q"] \arrow[rr, two heads, "p"'] & & B
\end{tikzcd}$$

Define a pair of 2-cells to which the premises of Theorem 11.1.4(iv) apply. The conclusion asserts that there is a well-defined fibered isomorphism class of functors $\ulcorner\mu\urcorner\colon \mathsf{Hom}_A(q,A) \underset{E}{\times} \mathsf{Hom}_B(B,p) \to E$ defined by taking the q-cocartesian lift of the left comma cone, composing with the right comma cone, and then taking the codomain of a p-cartesian lift of this composite cell — or by first taking the p-cartesian lift, composing, and then taking the domain of a q-cartesian lift of this composite — this being the functor $\ell r \simeq r\ell$ in the notation of Theorem 11.1.4(iii). In the case where $A \overset{E}{\twoheadrightarrow} B$ is itself a comma module, the resulting fibered isomorphism class of functors $\ulcorner\mu\urcorner$ is the one that classifies the pasted composite above with the comma cone for E.

Any virtual double category has an identity cell for each horizontal arrow $A \overset{E}{\twoheadrightarrow} B$ whose vertical boundary arrows are identities. In a virtual equipment, we also have a **unit cell** for each vertical arrow $f\colon A \to B$ whose horizontal boundary is given by the unit modules $A \overset{\mathsf{Hom}_A}{\twoheadrightarrow} A$ and $B \overset{\mathsf{Hom}_B}{\twoheadrightarrow} B$ of Proposition 12.2.4.

12.3.10. Definition. Using the unit modules in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$, for any functor $f\colon A \to B$ we may define a unary **unit cell** as displayed below-left by appealing to the universal property of the nullary unit cell $\iota\colon \varnothing \Rightarrow \mathsf{Hom}_A$ for A in the equation below-right:

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, equals, "\shortmid" marking] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equals] \\
A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B"'] & B
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[r, equals, "\shortmid" marking] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, equals, "\shortmid" marking] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equals] \\
B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B"'] & B
\end{tikzcd}$$

In the characterization of Lemma 12.1.16, both sides of the pasting equality defining the unary unit cell correspond to the identity 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & B \end{tikzcd}$.

As one might hope, the unit cells are units for the vertical composition of cells in the virtual equipment of modules.

12.3.11. LEMMA (composition with unit cells). *Any cell α as below-right equals the pasted composite below-left:*

$$
\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A_0 \arrow[r, "E_1"] \arrow[d, equal] & A_1 \arrow[r, "E_2"] \arrow[d, equal] & \cdots \arrow[r, "E_n"] \arrow[d, phantom, "\cdots"] & A_n \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A_n \arrow[d, equal] \\
A_0 \arrow[r, "\mathrm{Hom}_{A_0}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_f"] & A_0 \arrow[r, "E_1"] \arrow[d, "f"] & A_1 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n"] & A_n \arrow[r, "\mathrm{Hom}_{A_n}"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_g"] & A_n \arrow[d, "g"] \\
B \arrow[r, "\mathrm{Hom}_B"'] \arrow[d, equal] & B \arrow[rrr, "E"'] \arrow[d, phantom, "\Downarrow\circ"] & & & C \arrow[r, "\mathrm{Hom}_C"'] & C \arrow[d, equal] \\
B \arrow[rrrrr, "E"'] & & & & & C
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "E_1"] \arrow[d, "f"'] & A_1 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n"] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "E"'] & & & C
\end{tikzcd}
$$

PROOF. By Definition 12.3.10 and the laws for composition with identity cells in a virtual double category stated in Definition 12.1.9, the left-hand composite of the statement equals the left-hand composite cell displayed below and the right-hand side of the statement equals the right-hand composite cell displayed below:

$$
\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, "f"'] & A_0 \arrow[r, "E_1"] \arrow[d, "f"] & A_1 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n"] & A_n \arrow[r, equal] \arrow[d, "g"'] & A_n \arrow[d, "g"] \\
B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[rrr, "E"'] \arrow[d, equal] & & & C \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B"'] \arrow[d, equal] & B \arrow[rrr, "E"'] \arrow[d, phantom, "\Downarrow\circ"] & & & C \arrow[r, "\mathrm{Hom}_C"'] & C \arrow[d, equal] \\
B \arrow[rrrrr, "E"'] & & & & & C
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "E_1"] \arrow[d, "f"'] & A_1 \arrow[r, "E_2"] \arrow[d, phantom, "\Downarrow\alpha"] & \cdots \arrow[r, "E_n"] & A_n \arrow[d, "g"] \\
B \arrow[rrr, "E"'] \arrow[d, equal] & & & C \arrow[d, equal] \\
B \arrow[rrr, "E"'] & & & C
\end{tikzcd}
$$

By Lemma 12.3.5, the left-hand side equals the right-hand side. ☐

12.3.12. DEFINITION (horizontal composition of cells). If given a horizontally composable sequence of unary cells

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_1 \arrow[r, "E_2"] \arrow[d, dashed, "f_1"] \arrow[dr, phantom, "\Downarrow\alpha_2"] & \cdots \arrow[r, "E_n"] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\Downarrow\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1"'] & B_1 \arrow[r, "F_2"'] & \cdots \arrow[r, "F_n"'] & B_n
\end{tikzcd}
$$

for which the composable sequences $E_1 \nrightarrow \cdots \nrightarrow E_n$ and $F_1 \nrightarrow \cdots \nrightarrow F_n$ both admit composites, then there exists a **horizontal composite unary cell** $\alpha_1 * \cdots * \alpha_n$ that is uniquely determined up to the

specification of the composites $\circ\colon E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow E$ and $F_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} F_n \Rightarrow F$ by the pasting identity:

$$
\begin{tikzcd}
A_0 \arrow[r, "E_1", "|"{marking}] \arrow[d, equal] & A_1 \arrow[r, "E_2", "|"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & \cdots \arrow[r, "E_n", "|"{marking}] & A_n \arrow[d, equal] \\
A_0 \arrow[rrr, "E", "|"{marking}] \arrow[d, "f_0"'] & & & A_n \arrow[d, "f_n"] \\
B_0 \arrow[rrr, "F"', "|"{marking}] \arrow[rrru, phantom, "\Downarrow\alpha_1*\cdots*\alpha_n"] & & & B_n
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "E_1", "|"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_1 \arrow[r, "E_2", "|"{marking}] \arrow[d, dashed, "f_1"] \arrow[dr, phantom, "\Downarrow\alpha_2"] & \cdots \arrow[r, "E_n", "|"{marking}] \arrow[dr, phantom, "\Downarrow\alpha_n"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1"', "|"{marking}] \arrow[d, equal] & B_1 \arrow[r, "F_2"', "|"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & \cdots \arrow[r, "F_n"', "|"{marking}] & B_n \arrow[d, equal] \\
B_0 \arrow[rrr, "F"', "|"{marking}] & & & B_n
\end{tikzcd}
$$

By an argument very similar to the proof of Lemma 12.3.11 using the composites of Lemma 12.3.5, the horizontal composite $\mathsf{Hom}_f * \alpha * \mathsf{Hom}_g$ of a unary cell α with the unit cells Hom_f and Hom_g at its vertical boundary functors recovers α. The upshot of Definition 12.3.12 is that $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can be understood to contain various "vertical" and "horizontal" bicategories.

12.3.13. Proposition (the vertical 2-category in the virtual equipment). *Any virtual equipment contains a **vertical 2-category** whose objects are the objects of the virtual equipment, whose arrows are the vertical arrows, and whose 2-cells are those unary cells*

$$
\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_A", "|"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathsf{Hom}_B"', "|"{marking}] & B
\end{tikzcd}
$$

whose horizontal boundary arrows are given by the unit modules.

Proof. To prove that these structures define a 2-category we must define "horizontal" composition of 2-cells (composing along a boundary 0-cell) and "vertical" composition of 2-cells (composing along a boundary 1-cell). The "horizontal" composition in the 2-category is defined via the vertical composition in the virtual double category described in Definition 12.1.9. The "vertical" composition in the 2-category is defined by Definition 12.3.12. To see that this yields a 2-category and not a bicategory note that any bicategory in which the composition of 1-cells is strictly associative and unital is a 2-category; in this case, the 1-cells are the vertical arrows of the virtual double category, which do indeed compose strictly. □

Proposition 12.4.10 will prove that the vertical 2-category in the virtual equipment $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ is isomorphic to the homotopy 2-category $\mathfrak{h}\mathcal{K}$.

12.3.14. Remark (horizontal bicategories in the virtual equipment). Via Definition 12.3.12, a virtual equipment can also be understood to contain various "horizontal" bicategories, defined by taking the 1-cells to be composable modules and the 2-cells to be unary module maps whose vertical boundary functors are identities. Particular horizontal bicategories of interest are described in Definition 12.4.11.

Exercises.

12.3.i. Exercise. Extending Exercise 12.1.ii, prove that in a virtual double category arising from an actual double category that every composable sequence of horizontal morphisms admits a composite in the sense of Definition 12.3.1.

12.3.ii. Exercise. Prove Lemma 12.3.3.

12.4. Representable modules

Any vertical arrow $f\colon A \to B$ in a virtual equipment has a pair of associated horizontal arrows $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ and $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$ — defined as restrictions of the horizontal unit arrows — that have universal properties similar to companions and conjoints in an ordinary double category. In the virtual equipment of modules, these are sensible referred to as the *left* and *right representations* of a functor as a module and coincide exactly with the left and right representable first introduced in §3.5 and reappearing in Definition 11.4.8.

12.4.1. Proposition (companion and conjoint relations for representables). *To any functor $f\colon A \to B$ in the virtual equipment of modules, there exist canonical restriction cells displayed below-left and application cells displayed below-right*

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, "f"] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

defining unary module maps between the unit modules $A \overset{\mathrm{Hom}_A}{\nrightarrow} A$ and $B \overset{\mathrm{Hom}_B}{\nrightarrow} B$ and the left and right representable modules $B \overset{\mathrm{Hom}_B(f,B)}{\nrightarrow} A$ and $A \overset{\mathrm{Hom}_B(B,f)}{\nrightarrow} B$. These satisfy the identities:

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{Hom}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, "f"] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\tag{12.4.2}
$$

and

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[d, equal] \\
B \arrow[r, "\mathrm{Hom}_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] & A \arrow[d, equal] \\
B \arrow[rr, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "\mathrm{Hom}_B(f,B)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] & A \arrow[d, equal] \\
B \arrow[rr, "\mathrm{Hom}_B(f,B)"', "\shortmid"{marking}] & & A
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathrm{Hom}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[d, equal] \\
A \arrow[r, "\mathrm{Hom}_B(B,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathrm{Hom}_B", "\shortmid"{marking}] & A \arrow[d, equal] \\
A \arrow[rr, "\mathrm{Hom}_B(B,f)"', "\shortmid"{marking}] & & B
\end{tikzcd}
\tag{12.4.3}
$$

Proof. The unary module maps κ are defined by the equations (12.4.2) by appealing to the universal property of the restriction cells in Proposition 12.2.1. The relations (12.4.3) could also be verified directly from the axioms of Theorem 12.2.6 via Propositions 12.2.4 and Lemmas 12.3.5 and 12.3.5. Instead, we appeal to Lemma 12.1.16 to characterize each of the cells in the virtual equipment as 2-cells in the homotopy 2-category.

We prove this for the right representables; the co-dual then proves this for the left representables. The binary module morphism on the left-hand side of the equality of (12.4.3) represents the 2-cell

below-left, while the right-hand composite is below-right:

$$
\begin{tikzcd}
& & A^2 \underset{A}{\times} \mathsf{Hom}_B(B,f) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"{rotate=-45}] & & \\
& A^2 \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & {} & \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
A \arrow[rr, equals] & {} & A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& & & A^2 \underset{A}{\times} \mathsf{Hom}_B(B,f) \arrow[dl] \arrow[drr] \arrow[dr, phantom, "\lrcorner"{rotate=-45}] & & & \\
& & A^2 \arrow[dl, "k"] \arrow[dr, "p_0"] \arrow[dll, bend right, "p_1"'] & & & \mathsf{Hom}_B(B,f) \arrow[dll, "p_1"'] \arrow[d, "\rho"] \arrow[dr, bend left, "p_0"] & \\
& \mathsf{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[drr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & & A \arrow[d, "f"] & & B^2 \arrow[dll, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & \\
A \arrow[rrr, "f"'] & {} & & B \arrow[rrr, equals] & & {} & B
\end{tikzcd}
$$

By the definition of the induced functor $k\colon A^2 \to \mathsf{Hom}_B(B,f)$, $\phi k = f\kappa$ and by the definition of the comma cone ϕ for $\mathsf{Hom}_B(B,f)$, $\phi = \kappa\rho$. Thus, the left-hand side equals the right-hand side. $\square$

12.4.4. Theorem. *In the virtual equipment of modules there are bijections between cells of the following four forms*

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equals] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[rr, "F"', "\shortmid"{marking}] & & D
\end{tikzcd}
\leftrightsquigarrow
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[r, "F"', "\shortmid"{marking}] & D
\end{tikzcd}
\leftrightsquigarrow
\begin{tikzcd}
A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & D \arrow[d, equals] \\
C \arrow[rr, "F"', "\shortmid"{marking}] & & D
\end{tikzcd}
$$

$$\leftrightsquigarrow \qquad\qquad \leftrightsquigarrow$$

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equals] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] & D \arrow[d, equals] \\
C \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}
$$

implemented by composing with the canonical cells κ and ρ of Proposition 12.4.1 and with the composition and nullary cells associated to the units.

Proof. The composite bijection carries cells α and β to the cells displayed below-left and below-right respectively:

$$
\hat{\alpha} :=
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equals] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid"{marking}] \arrow[d, equals] & C \arrow[r, "F", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid"{marking}] & D \arrow[d, equals] \\
C \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}
\quad \text{and} \quad
\check{\beta} :=
\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, equals] & B \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equals] \\
A \arrow[r, "\mathsf{Hom}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, equals] & B \arrow[r, "\mathsf{Hom}_B", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathsf{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equals] & A \arrow[r, "E", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\shortmid"{marking}] & D \arrow[d, equals] \\
C \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}
$$

By Proposition 12.4.1 and Lemma 12.3.11

$$
\check{\alpha} \coloneqq
\begin{tikzcd}
A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "F", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rrr, "F"', "\shortmid" marking] & & & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_f"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "\mathsf{Hom}_B", "\shortmid" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\mathsf{Hom}_g"] & B \arrow[d, "g"] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "F", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rrr, "F"', "\shortmid" marking] & & & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "E", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
C \arrow[r, "F"', "\shortmid" marking] & D
\end{tikzcd}
$$

The other composite is:

$$
\hat{\beta} \coloneqq
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathsf{Hom}_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[rrr, "F"', "\shortmid" marking] & \arrow[d, phantom, "\Downarrow\circ"] & & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"', "\shortmid" marking] & & & & & D
\end{tikzcd}
\tag{12.4.5}
$$

By Lemma 12.3.5, we have

$$
\begin{tikzcd}
C \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, equal] & D \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] & C \arrow[rrr, "F"', "\shortmid" marking] & \arrow[d, phantom, "\Downarrow\circ"] & & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"', "\shortmid" marking] & & & & & D
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, equal] & D \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & D \arrow[d, equal] \\
C \arrow[r, "\mathsf{Hom}_C", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\circ"] & C \arrow[r, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathsf{Hom}_D", "\shortmid" marking] & D \arrow[d, equal] \\
C \arrow[rr, "\mathsf{Hom}_C(f{,}C)", "\shortmid" marking] \arrow[d, equal] & & A \arrow[r, "E", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\beta"] & B \arrow[rr, "\mathsf{Hom}_D(D{,}g)", "\shortmid" marking] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"', "\shortmid" marking] & & & & & D
\end{tikzcd}
$$

since both composites equal β. By the universal property of the unit cells in Proposition 12.2.4, the bottom two rows of these diagrams are equal, so we may substitute the bottom two rows of the right-hand diagram for the bottom two rows of (12.4.5) to obtain:

$$\hat{\hat{\beta}} = \begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "\mathrm{Hom}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\kappa"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & C \arrow[r, "\mathrm{Hom}_C(f,C)"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & D \arrow[r, "\mathrm{Hom}_D"] & D \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_C(f,C)"'] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\beta"] & & A \arrow[r, "E"] & B \arrow[rr, "\mathrm{Hom}_D(D,g)"'] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd}$$

By Proposition 12.4.1 and Lemma 12.3.5 this reduces to β.

$$= \begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] \arrow[d, equal] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & A \arrow[r, "\mathrm{Hom}_A"] & A \arrow[r, "E"] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_C(f,C)"'] \arrow[d, equal] \arrow[drrrrr, phantom, "\Downarrow\beta"] & & A \arrow[r, "E"] & B \arrow[rr, "\mathrm{Hom}_D(D,g)"'] & & D \arrow[d, equal] \\
C \arrow[rrrrr, "F"'] & & & & & D
\end{tikzcd} = \begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\beta"] & A \arrow[r, "E"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd}$$

By vertically bisecting this construction, one obtains the one-sided bijections of the statement. ☐

We frequently apply this result in the following form:

12.4.6. Corollary. *Given modules* $A \overset{E}{\nrightarrow} B$ *and* $C \overset{F}{\nrightarrow} D$ *and functor* $f\colon A \to C$ *and* $g\colon B \to D$, *there is a bijection between ternary module maps* $\mathrm{Hom}_C(f,C) \times E \times \mathrm{Hom}_D(D,g) \Rightarrow F$ *and unary module maps* $E \Rightarrow F(g,f)$, *i.e., between cells*

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)"] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow"] & A \arrow[r, "E"] & B \arrow[r, "\mathrm{Hom}_D(D,g)"] & D \arrow[d, equal] \\
C \arrow[rrr, "F"'] & & & D
\end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}
A \arrow[r, "E"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\
A \arrow[r, "F(g,f)"'] & B
\end{tikzcd}$$

Proof. Combine Theorem 12.4.4 with Proposition 12.2.1. ☐

12.4.7. Proposition. *For any module* $A \overset{E}{\nrightarrow} B$ *and pair of functors* $a\colon X \to A$ *and* $b\colon Y \to B$, *the composite* $\mathrm{Hom}_A(A,a) \otimes E \otimes \mathrm{Hom}_B(b,B)$ *exists and is given by the restriction* $E(b,a)$, *with the ternary composite map*

$\mu\colon \mathrm{Hom}_A(A,a) \mathbin{\underset{\cdot}{\times}} E \mathbin{\underset{\cdot}{\times}} \mathrm{Hom}_B(b,B) \Rightarrow E(b,a)$ *defined by the universal property of the restriction by the pasting diagram:*

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{Hom}_A(A,a)", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\mu"] & B \arrow[r, "\mathrm{Hom}_B(b,B)", "\shortmid" marking] & Y \arrow[d, equal] \\
X \arrow[rrr, "E(b,a)"', "\shortmid" marking] \arrow[d, "a"'] & & & Y \arrow[d, "b"] \\
A \arrow[rrr, "E"', "\shortmid" marking] & & & B
\end{tikzcd}
\quad \coloneqq \quad
\begin{tikzcd}
X \arrow[r, "\mathrm{Hom}_A(A,a)", "\shortmid" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(b,B)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & Y \arrow[d, "b"] \\
A \arrow[r, "\mathrm{Hom}_A"', "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "E", "\shortmid" marking] \arrow[d, phantom, "\Downarrow\circ"] & B \arrow[r, "\mathrm{Hom}_B"', "\shortmid" marking] & B \arrow[d, equal] \\
A \arrow[rrr, "E"', "\shortmid" marking] & & & B
\end{tikzcd}
$$

PROOF. The horizontal composite two-sided fibration of the composable sequence is

$$
\begin{tikzcd}[column sep=small]
& & & \mathrm{Hom}_A(A,a) \underset{A}{\times} E \underset{B}{\times} \mathrm{Hom}_B(b,B) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\vee", very near start] & & & \\
& & \mathrm{Hom}_A(q,a) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\vee", very near start] & & \mathrm{Hom}_B(b,p) \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\vee", very near start] & & \\
& \mathrm{Hom}_A(A,a) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & \mathrm{Hom}_B(b,B) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\
X & & A & & B & & Y
\end{tikzcd}
$$

from which we see that the binary composite cell

$$
\begin{tikzcd}
\mathrm{Hom}_A(q,A) \underset{E}{\times} \mathrm{Hom}_B(B,p) \arrow[rr] \arrow[dr, two heads] & & E \arrow[dl, two heads, "{(q,p)}"] \\
& A \times B &
\end{tikzcd}
$$

of Proposition 12.3.7 pulls back along $a \times b\colon X \times Y \to A \times B$ to define the map μ. By Lemma 12.3.6, we conclude that $\mu\colon \mathrm{Hom}_A(A,a) \mathbin{\underset{\cdot}{\times}} E \mathbin{\underset{\cdot}{\times}} \mathrm{Hom}_B(b,B) \Rightarrow E(b,a)$ is a composite. $\square$

As a special case, right representable modules can always be composed with each other and dually left representable modules can always be composed with each other:

12.4.8. COROLLARY. *Any composable pair of functors $A \xrightarrow{f} B \xrightarrow{g} C$ defines a composable pair of right-represented modules and a composable pair of left-represented modules*

$$
A \overset{\mathrm{Hom}_B(B,f)}{\longrightarrow\!\!\!\!\!\!\!\shortmid\;\;\,} B \overset{\mathrm{Hom}_C(C,g)}{\longrightarrow\!\!\!\!\!\!\!\shortmid\;\;\,} C \qquad\qquad C \overset{\mathrm{Hom}_C(g,C)}{\longrightarrow\!\!\!\!\!\!\!\shortmid\;\;\,} B \overset{\mathrm{Hom}_B(f,B)}{\longrightarrow\!\!\!\!\!\!\!\shortmid\;\;\,} A
$$

and moreover:

$$
\mathrm{Hom}_B(B,f) \otimes \mathrm{Hom}_C(C,g) \simeq \mathrm{Hom}_C(C,gf) \quad \textit{and} \quad \mathrm{Hom}_C(g,C) \otimes \mathrm{Hom}_B(f,B) \simeq \mathrm{Hom}_C(gf,C). \quad \square
$$

Combining this result with Proposition 12.3.7 provides a generalization of Theorem 3.5.11, which allows us to detect representable modules.

12.4.9. PROPOSITION. *Let $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ encode a module.*

(i) The module $A \overset{E}{\nrightarrow} B$ is right representable if and only if its left leg $q\colon E \twoheadrightarrow A$ has a right adjoint $r\colon A \to E$ in which case $E \simeq \mathrm{Hom}_B(B, pr)$.

(ii) *The module $A \overset{E}{\twoheadrightarrow} B$ is left representable if and only if its right leg $p\colon E \twoheadrightarrow B$ has a left adjoint $\ell\colon B \to E$ in which case $E \simeq \mathsf{Hom}_A(q\ell, A)$.*

Proof. Lemma 3.5.8 proves that the claimed adjoints to the legs $p_1\colon \mathsf{Hom}_B(B,f) \twoheadrightarrow A$ and $p_0\colon \mathsf{Hom}_B(f,B) \twoheadrightarrow A$ of left or right representable modules exist, so it remains only to prove the converse. By Proposition 12.3.7, any module $A \overset{E}{\twoheadrightarrow} B$ can be expressed as a composite $E \simeq \mathsf{Hom}_A(q,A) \otimes \mathsf{Hom}_B(B,p)$ of the left representation of its left leg followed by the right right representation of its right leg. If $q \dashv r$, then by Proposition 4.1.1 $\mathsf{Hom}_A(q,A) \simeq \mathsf{Hom}_E(E,r)$, so $E \simeq \mathsf{Hom}_E(E,r) \otimes \mathsf{Hom}_B(B,p)$. By Corollary 12.4.8, $\mathsf{Hom}_E(E,r) \otimes \mathsf{Hom}_B(B,p) \simeq \mathsf{Hom}_B(B,pr)$ so it follows that $E \simeq \mathsf{Hom}_B(B,pr)$. The case of left representability is dual. □

Finally, we revisit the "cheap" version of the Yoneda lemma presented in Corollary 3.5.10, which encodes natural transformations in the homotopy 2-category as maps of represented modules.

12.4.10. Proposition. *For any parallel pair of functors there are natural bijections between 2-cells in the homotopy 2-category*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

and cells in the virtual equipment of modules:

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,f)", "|"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ A \arrow[r, "\mathsf{Hom}_B(B,g)"', "|"] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\mathsf{Hom}_A", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathsf{Hom}_B"', "|"] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "\mathsf{Hom}_B(g,B)", "|"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\ B \arrow[r, "\mathsf{Hom}_B(f,B)"', "|"] & A \end{tikzcd}$$

Proof. The bijection between 2-cells in the homotopy 2-category and unary module maps between left or right representables is simply a restatement of Corollary 3.5.10. Alternatively, the bijection between 2-cells in the homotopy 2-category and the cells displayed in the center follows from Lemma 12.1.16 and Proposition 12.2.4. The bijections in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can then be derived from Theorem 12.4.4 and Corollary 12.4.6. □

12.4.11. Definition (the covariant and contravariant embeddings). Proposition 12.4.10 defines the action on 2-cells of two identity-on-objects locally fully faithful homomorphisms

$$\begin{array}{ccc} \mathfrak{h}\mathcal{K} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K}) & & \mathfrak{h}\mathcal{K}^{\mathrm{coop}} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K}) \\ A \xrightarrow{f} B \longmapsto A \overset{\mathsf{Hom}_B(B,f)}{\twoheadrightarrow} B & & A \xrightarrow{f} B \longmapsto B \overset{\mathsf{Hom}_B(f,B)}{\twoheadrightarrow} A \end{array}$$

that embed the homotopy 2-category fully faithfully into the sub "bicategory" of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ containing only those unary cells whose vertical boundaries are identities.

This substructure of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ isn't quite a bicategory because not all horizontally composable modules can be composed, but if we restrict only to the right representable modules or only to the left representable modules, then by Corollary 12.4.8 the composites do exist and moreover the embeddings are horizontally as well as vertically pseudofunctorial: given $A \xrightarrow{f} B \xrightarrow{g} C$ we have $\mathsf{Hom}_B(B,f) \otimes \mathsf{Hom}_C(C,g) \simeq \mathsf{Hom}_C(C,gf)$ and $\mathsf{Hom}_C(g,C) \otimes \mathsf{Hom}_B(f,B) \simeq \mathsf{Hom}_C(gf,C)$. We refer to these as the **covariant** and **contravariant** embeddings respectively.

As described in Remark 12.3.14, $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ can be understood to contain genuine bicategories whose 2-cells are unary cells between right-represented modules (or between left-represented modules) whose vertical boundaries are identities. In this way, the covariant and contravariant embeddings can be understood to define genuine bicategorical homomorphisms.

There is a third locally fully faithful embedding of the homotopy 2-category $\mathfrak{h}\mathcal{K}$ into $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ that is identity on objects, sends $f\colon A \to B$ to the corresponding vertical 1-cell, and uses the third bijection of Proposition 12.4.10 to define the action on 2-cells. Since $\mathsf{Hom}_A \otimes \mathsf{Hom}_A \simeq \mathsf{Hom}_A$, the unary cells in this image of this embedding can be composed horizontally as well as vertically, and this embedding is functorial in both directions: vertical composites of natural transformations in $\mathfrak{h}\mathcal{K}$ coincide with horizontal composites of unary cells and horizontal composites of natural transformations in $\mathfrak{h}\mathcal{K}$ coincide with vertical composites of unary cells. The image is precisely the vertical 2-category of Proposition 12.3.13. We will make much greater use of the covariant and contravariant embeddings of Definition 12.4.11 however.

Exercises.

12.4.i. Exercise. Prove Proposition 12.4.1 in any virtual equipment, without appealing to Lemma 12.1.16.

12.4.ii. Exercise. Prove Proposition 12.4.7 in any virtual equipment, without appealing to Lemma 12.3.6.

12.4.iii. Exercise. For any functor $f\colon A \to B$, define a unary cell $\eta\colon \mathsf{Hom}_A \Rightarrow \mathsf{Hom}_B(f,f) \cong \mathsf{Hom}_B(B,f) \otimes \mathsf{Hom}_B(f,B)$ and a binary cell $\epsilon\colon \mathsf{Hom}_B(f,B) \times \mathsf{Hom}_B(B,f) \Rightarrow \mathsf{Hom}_B$. Use this data to demonstrate that the modules $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ are "adjoint" in a suitable sense.

CHAPTER 13

Formal category theory in a virtual equipment

Mac Lane famously asserted that "all concepts are Kan extensions" [**63**, §X.7] — at least in category theory. Kelly later amended this to assert that the *pointwise Kan extensions*, which he calls simply "Kan extensions" are the important ones, writing "Our present choice of nomenclature is based on our failure to find a single instance where a [non-pointwise] Kan extension plays any mathematical role whatsoever" [**56**, §4]. Using the calculus of modules we can now add the theory of pointwise Kan extensions of functors between ∞-categories to the basic ∞-category theory developed in Part I.

Right and left extensions of a functor $f: A \to C$ along a functor $k: A \to B$ can be defined internally to any 2-category — at this level of generality the eponym "Kan" is typically dropped. We review this notion in Definition 13.1.1. However, in the homotopy 2-category of an ∞-cosmos, the universal property defining left and right extensions is not strong enough, and indeed the correct universal property is associated to the stronger notion of a *pointwise extension*, for which the values of a right or left extension at an element of B can be computed as limits or colimits indexed by the appropriate comma ∞-category; see Corollary 13.5.3 for a precise statement.

Our aim in this chapter is to define and study pointwise extensions for functors between ∞-categories. In fact, we give multiple definitions of pointwise extension. One is fundamentally 2-categorical: a pointwise extension is an ordinary 2-categorical extension in the homotopy 2-category that is stable under pasting with comma squares. Another definition is that a 2-cell defines a pointwise right extension if and only if its image under the covariant embedding into the virtual equipment of modules defines a right extension there. Theorem 13.3.3 proves that these two notions coincide.

In §13.1, we introduce right liftings and right extensions in the virtual equipment of modules and being to familiarize ourselves with these notions. Before turning our attention to pointwise extensions, we first introduce exact squares in §13.2, a class of squares in the homotopy 2-category that include comma squares and which will be used to characterize the pointwise extensions internally to the homotopy 2-category. Pointwise extensions are introduced in a variety of equivalent ways in §13.3 and applied in §13.4 to develop a few aspects of the formal theory of ∞-categories. In §13.5, we conclude with a discussion of pointwise extensions in a cartesian closed ∞-cosmos, in which context these relate to the absolute lifting diagrams and limits and colimits studied in §2.3-2.4.

13.1. Liftings and extensions of modules

In this section we introduce and study liftings and extensions in the virtual equipment of modules. To motivate Definition 13.1.2, we briefly recall the standard 2-categorical definition:

13.1.1. Definition. A **right extension** of a 1-cell $f\colon A \to C$ along a 1-cell $k\colon A \to B$ is given by a pair $(r\colon B \to C, \nu\colon rk \Rightarrow f)$ as below-left

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, dashed, "r"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\gamma"] & C \\ B \arrow[ur, "g"'] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, "r"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Uparrow\exists!"] & \end{tikzcd}$$

so that any similar pair as above-center factors uniquely through ν as above right. The co-dual defines a **left extension** of a 1-cell $f\colon A \to C$ along a 1-cell $k\colon A \to B$.

The op-dual of Definition 13.1.1 defines a notion of **right lifting** in any 2-category. Analogous notions of right extension and right lifting can be defined for horizontal morphisms in a virtual double equipment, where in the presence of restrictions of modules it suffices to consider cells whose vertical functors are identities. We specialize our language to the virtual equipment of modules, as this will be the one case of interest:

13.1.2. Definition. A **right extension** of a module $A \overset{F}{\nrightarrow} C$ along a module $A \overset{K}{\nrightarrow} B$ consists of a pair given by a module $B \overset{R}{\nrightarrow} C$ together with a binary cell

$$\begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & C \arrow[d, equal] \\ A \arrow[rr, "F"', "\shortmid"{marking}] & & C \end{tikzcd}$$

with the property that every $n+1$-ary cell of the form displayed below-left factors uniquely through $\nu\colon K \times R \Rightarrow F$ as below-right:

$$\begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] & \cdots \arrow[d, phantom, "\Downarrow"] \arrow[r, "E_n", "\shortmid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] & & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[d, phantom, "\Downarrow\exists!"] \arrow[r, "E_n", "\shortmid"{marking}] & C \arrow[d, equal] \\ A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\shortmid"{marking}] & & & C \end{tikzcd}$$

Dually, a **right lifting** of $A \overset{F}{\nrightarrow} C$ through $B \overset{H}{\nrightarrow} C$ consists of a pair given by a module $A \overset{L}{\nrightarrow} B$ together with a binary cell

$$\begin{tikzcd} A \arrow[r, "L", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "H", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda"] & C \arrow[d, equal] \\ A \arrow[rr, "F"', "\shortmid"{marking}] & & C \end{tikzcd}$$

with the property that every $n+1$-ary cell of the form displayed below-left factors uniquely through $\lambda \colon L \times H \Rightarrow F$ as below-right:

$$\begin{tikzcd} A \arrow[r, "E_1", "\mid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\mid"{marking}] & B \arrow[r, "H", "\mid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\mid"{marking}] \arrow[urrr, phantom, "\Downarrow"] & & & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "E_1", "\mid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\mid"{marking}] & B \arrow[r, "H", "\mid"{marking}] \arrow[d, equal] & C \arrow[d, equal] \\ A \arrow[rr, "L", "\mid"{marking}] \arrow[d, equal] \arrow[urr, phantom, "\Downarrow\exists!"] & & B \arrow[r, "H", "\mid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "F"', "\mid"{marking}] \arrow[urrr, phantom, "\Downarrow\lambda"] & & & C \end{tikzcd}$$

Because of the asymmetry in Definition 12.1.9, there is no corresponding notion of *left* extension or *left* lifting. It follows easily from these definitions that right extensions or right liftings are unique up to vertical isomorphism in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$; see Exercise 13.1.i.

13.1.3. Lemma. *For any functor $f \colon A \to B$, the binary cell*

$$\begin{tikzcd} B \arrow[r, "\mathsf{Hom}_B(f,B)", "\mid"{marking}] \arrow[d, equal] & A \arrow[r, "\mathsf{Hom}_B(B,f)", "\mid"{marking}] & B \arrow[d, equal] \\ B \arrow[rr, "\mathsf{Hom}_B"', "\mid"{marking}] \arrow[urr, phantom, "\Downarrow\epsilon"] & & B \end{tikzcd}$$

*defined in Exercise 12.4.*iii *defines a right extension of $B \overset{\mathsf{Hom}_B}{\twoheadrightarrow} B$ through $B \overset{\mathsf{Hom}_B(f,B)}{\twoheadrightarrow} A$ and a right lifting of $B \overset{\mathsf{Hom}_B}{\twoheadrightarrow} B$ through $A \overset{\mathsf{Hom}_B(B,f)}{\twoheadrightarrow} B$.*

Proof. Exercise 13.3.i. $\square$

The result of Lemma 13.1.3 is a special case of a more general result.

13.1.4. Lemma. *For any module $C \overset{F}{\twoheadrightarrow} D$ and any pair of functors $f \colon A \to C$ and $g \colon B \to D$ the canonical cells*

$$\begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(f,C)", "\mid"{marking}] \arrow[d, equal] & A \arrow[r, "F(1,f)", "\mid"{marking}] & D \arrow[d, equal] \\ C \arrow[rr, "F"', "\mid"{marking}] \arrow[urr, phantom, "\Downarrow\rho"] & & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "F(g,1)", "\mid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_D(D,g)", "\mid"{marking}] & D \arrow[d, equal] \\ C \arrow[rr, "F"', "\mid"{marking}] \arrow[urr, phantom, "\Downarrow\rho"] & & D \end{tikzcd}$$

exhibit $F(1,f)$ as the right extension of F through $\mathsf{Hom}_C(f,C)$ and exhibit $F(g,1)$ as the right lifting of F through $\mathsf{Hom}_D(D,g)$.

Proof. The canonical cells of the statement arise from applying the bijection of Theorem 12.4.4 to the restriction cells of Proposition 12.2.1, which is to say that in the case of the right extension

diagram above-left this cell is obtained as the composite

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(f,C)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\rho"] & A \arrow[r, "F(1,f)", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\rho"] & D \arrow[d, equal] \\
C \arrow[r, "\mathrm{Hom}_C"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\circ"] & C \arrow[r, "F"', "\shortmid"{marking}] & D \arrow[d, equal] \\
C \arrow[rr, "F"', "\shortmid"{marking}] & & D
\end{tikzcd}$$

Now the bijections of Proposition 12.2.1 and Theorem 12.4.4, mildly generalized to allow the module E to be replaced by a composable sequence of modules, combine to exhibit the universal property of the right extension. □

Right extensions and right liftings can be understood as right adjoints to horizontal composition with a module on the left or on the right, respectively. This leads to the following "associativity" result, which we formulate for right extensions, leaving the dual result for right liftings to the reader.

13.1.5. Proposition. *Suppose $A \overset{K}{\nrightarrow} B$, $B \overset{H}{\nrightarrow} C$, and $A \overset{F}{\nrightarrow} D$ are modules so that the composite $K \otimes H$ and right extension $B \overset{R_K F}{\nrightarrow} D$ modules exist. Then the right extension of $R_K F$ along H exists if and only if the right extension of F along $K \otimes H$ exists, in which case $R_H(R_K F) \simeq_{C \times D} R_{K \otimes H} F$.*

Proof. The universal property of the binary composite cell $\mu\colon K \mathbin{\dot\times} H \Rightarrow K \otimes H$ and right extension cell $\nu\colon K \mathbin{\dot\times} R_K F \Rightarrow F$ provide a bijection between cells involving an arbitrary module $C \overset{R}{\nrightarrow} D$

$$\begin{tikzcd}
A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\mu"] & B \arrow[r, "H", "\shortmid"{marking}] & C \arrow[r, "R", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[d, equal] \\
A \arrow[rr, "K\otimes H"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\exists!"] & & C \arrow[r, "R"', "\shortmid"{marking}] & D \arrow[d, equal] \\
A \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "K", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "H", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\exists!"] & C \arrow[r, "R", "\shortmid"{marking}] & D \arrow[d, equal] \\
A \arrow[r, "K"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow\nu"] & B \arrow[rr, "R_K F"', "\shortmid"{marking}] & & D \arrow[d, equal] \\
A \arrow[rrr, "F"', "\shortmid"{marking}] & & & D
\end{tikzcd}$$

If the right extension $R_{K\otimes H}F$ exists we take this for R and use the binary cell on the lower-left to induce the binary cell on the upper-right, which we'll show exhibits $R_{K\otimes H}F$ as the right extension $R_H(R_K F)$ of $R_K F$ along H. Conversely, if the right extension $R_H(R_K F)$ exists we use the binary cell on the upper-right to induce the binary cell on the lower-left, which we'll show exhibits $R_H(R_K F)$ as the right extension $R_{K\otimes H}F$ of F along $K \otimes H$. This transference of universal properties is straightforward, following again from the bijection just exhibited, but with a composable sequence of modules in place of R.

Finally, if separately the right extensions $R_H(R_K F)$ and $R_{K\otimes H}F$ are known to exist, then the argument given above combined with the uniqueness of right extensions combines to show that $R_{K\otimes H}F \simeq_{C\times D} R_H(R_K F)$; see Exercise 13.1.i. □

We now explain how Definition 13.1.2 relates to Definition 13.1.1 via the covariant and contravariant embeddings of Definition 12.4.11.

13.1.6. Lemma. *If*

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_B(B,k)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_C(C,r)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & C \arrow[d, equal] \\ A \arrow[rr, "\mathrm{Hom}_C(C,f)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

defines a right extension in the virtual equipment of modules, then $\nu\colon rk \Rightarrow f$ defines a right extension in the homotopy 2-category. Dually if

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A(\ell,A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(h,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\lambda"] & C \arrow[d, equal] \\ A \arrow[rr, "\mathrm{Hom}_A(g,A)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

defines a right lifting in the virtual equipment of modules, then $\lambda\colon g \Rightarrow \ell h$ is a left extension in the homotopy 2-category.

Proof. By Corollary 12.4.8 binary cells $\nu\colon \mathrm{Hom}_B(B,k) \times \mathrm{Hom}_C(C,r) \Rightarrow \mathrm{Hom}_C(C,f)$ correspond to unary cells $\nu\colon \mathrm{Hom}_C(C,rk) \Rightarrow \mathrm{Hom}_C(C,f)$, which by Proposition 12.4.10 correspond to natural transformations $\nu\colon rk \Rightarrow f$ in the homotopy 2-category. Via this correspondence the universal property of Definition 13.1.2 clearly subsumes that of Definition 13.1.1. The left extension case is similar, via the contravariant embedding of Definition 12.4.11. ☐

A sharper characterization of the right extension diagrams of modules in the image of the covariant embedding will have to wait for Theorem 13.3.3, but we can characterize the right lifting diagrams of modules in the image of the covariant embedding now. Duals of these results apply to the right lifting and right extension diagrams of modules in the image of the contravariant embedding.

Recall the notion of absolute right lifting diagram introduced in Definition 2.3.4.

13.1.7. Proposition. *A 2-cell in the homotopy 2-category of an ∞-cosmos as below-left defines an absolute right lifting diagram if and only if the corresponding binary cell displayed below-right defines a right lifting in the virtual equipment of modules:*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C \arrow[r, "\mathrm{Hom}_B(B,r)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_A(A,f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\rho}"] & A \arrow[d, equal] \\ C \arrow[rr, "\mathrm{Hom}_A(A,g)"', "\shortmid"{marking}] & & A \end{tikzcd}$$

Dually, a 2-cell in the homotopy 2-category of an ∞-cosmos as below-left defines an absolute left lifting diagram if and only if the corresponding binary cell displayed below-right defines a right extension in the virtual equipment of modules:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\mathrm{Hom}_A(f,A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathrm{Hom}_B(\ell,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\lambda}"] & A \arrow[d, equal] \\ A \arrow[rr, "\mathrm{Hom}_A(g,A)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

Proof. By Proposition 12.4.10, natural transformations in the homotopy 2-category of an ∞-cosmos correspond bijectively to unary squares in the virtual equipment of modules of various forms. By this result, Corollary 12.4.6, and Corollary 12.4.8, there are canonical bijections:

$$
\begin{tikzcd}[column sep=small]
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\chi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
X \arrow[r, "\mathsf{Hom}_A(A{,}fb)", "\mid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\
X \arrow[r, "\mathsf{Hom}_A(A{,}gc)"', "\mid" marking] & A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(c{,}C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathsf{Hom}_B(B{,}b)", "\mid" marking] \arrow[d, phantom, "\Downarrow\hat{\chi}"] & B \arrow[r, "\mathsf{Hom}_A(A{,}f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathsf{Hom}_A(A{,}g)"', "\mid" marking] & & & A
\end{tikzcd}
\tag{13.1.8}
$$

If the binary cell $\hat{\rho}\colon \mathsf{Hom}_B(B,r) \times \mathsf{Hom}_A(A,f) \Rightarrow \mathsf{Hom}_A(A,g)$ defines a right lifting diagram in the virtual equipment of modules, then there is a unique factorization

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(c{,}C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathsf{Hom}_B(B{,}b)", "\mid" marking] \arrow[d, phantom, "\Downarrow\hat{\chi}"] & B \arrow[r, "\mathsf{Hom}_A(A{,}f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathsf{Hom}_A(A{,}g)"', "\mid" marking] & & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(c{,}C)", "\mid" marking] \arrow[d, equal] & X \arrow[r, "\mathsf{Hom}_B(B{,}b)", "\mid" marking] \arrow[d, phantom, "\exists!\Downarrow\hat{\zeta}"] & B \arrow[r, "\mathsf{Hom}_A(A{,}f)", "\mid" marking] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathsf{Hom}_B(B{,}r)"', "\mid" marking] \arrow[d, equal] & & B \arrow[r, "\mathsf{Hom}_A(A{,}f)"', "\mid" marking] \arrow[d, phantom, "\Downarrow\hat{\rho}"] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathsf{Hom}_A(A{,}g)"', "\mid" marking] & & & A
\end{tikzcd}
$$

Reversing the canonical bijection (13.1.8), this defines the desired unique factorization

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\Downarrow\zeta", near start] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] \arrow[ur, "r" description] \arrow[ur, phantom, "\Downarrow\rho", shift right=3ex] & A
\end{tikzcd}
$$

in the homotopy 2-category. Thus if $\hat{\rho}\colon \mathsf{Hom}_B(B,r) \times \mathsf{Hom}_A(A,f) \Rightarrow \mathsf{Hom}_A(A,g)$ is a right lifting, then $\rho\colon fr \Rightarrow g$ is an absolute right lifting.

Conversely, suppose $\rho\colon fr \Rightarrow g$ is an absolute right lifting and consider a cell in the virtual equipment of modules of the following form:

$$
\begin{tikzcd}
C \arrow[r, "E_1", "\mid" marking] \arrow[d, equal] & \cdots \arrow[r, "E_n", "\mid" marking] \arrow[d, phantom, "\Downarrow\bar{\psi}"] & B \arrow[r, "\mathsf{Hom}_A(A{,}f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathsf{Hom}_A(A{,}g)"', "\mid" marking] & & & A
\end{tikzcd}
$$

Let $C \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ denote the composite two-sided fibration $E_1 \times \cdots \times E_n$. By Remark 12.3.8 applied to $C \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, module maps $\bar{\psi}\colon E_1 \times \cdots \times E_n \times \mathsf{Hom}_A(A,f) \Rightarrow \mathsf{Hom}_A(A,g)$ stand in bijection with module maps $\hat{\psi}\colon \mathsf{Hom}_C(q,C) \times \mathsf{Hom}_B(B,p) \times \mathsf{Hom}_A(A,f) \Rightarrow \mathsf{Hom}_A(A,g)$, as displayed below-left. As argued in (13.1.8), these stand in canonical bijection with natural transformations as below-center:

$$
\begin{tikzcd}
C \arrow[r, "\mathsf{Hom}_C(q{,}C)", "\mid" marking] \arrow[d, equal] & E \arrow[r, "\mathsf{Hom}_B(B{,}p)", "\mid" marking] \arrow[d, phantom, "\Downarrow\hat{\psi}"] & B \arrow[r, "\mathsf{Hom}_A(A{,}f)", "\mid" marking] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathsf{Hom}_A(A{,}g)"', "\mid" marking] & & & A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}[column sep=small]
& E \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, phantom, "\overset{\psi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}[column sep=small]
& E \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[d, phantom, "\overset{\exists!\xi}{\Leftarrow}"] & \\
C \arrow[rr, "r" description] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

Since $\rho\colon fr \Rightarrow g$ is assumed to be an absolute right lifting, ψ factors uniquely through ρ to define a corresponding 2-cell $\xi\colon p \Rightarrow rq$ as above-right. Applying (13.1.8) again, this constructs a unique factorization in the virtual equipment of modules

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(q,C)"] \arrow[d, equal] & E \arrow[r, "\mathrm{Hom}_B(B,p)"] \arrow[d, phantom, "\Downarrow\hat{\psi}"] & B \arrow[r, "\mathrm{Hom}_A(A,f)"] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"'] & & & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}_C(q,C)"] \arrow[d, equal] & E \arrow[r, "\mathrm{Hom}_B(B,p)"] \arrow[d, phantom, "\exists!\Downarrow\hat{\xi}"] & B \arrow[r, "\mathrm{Hom}_A(A,f)"] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_B(B,r)"'] \arrow[d, equal] & & B \arrow[r, "\mathrm{Hom}_A(A,f)"'] \arrow[d, phantom, "\Downarrow\hat{\rho}"] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"'] & & & A
\end{tikzcd}
$$

By Remark 12.3.8, this defines a bijection

$$
\begin{tikzcd}
C \arrow[r, "E_1"] \arrow[d, equal] & \cdots \arrow[r, "E_n"] \arrow[d, phantom, "\Downarrow\bar{\psi}"] & B \arrow[r, "\mathrm{Hom}_A(A,f)"] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"'] & & & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "E_1"] \arrow[d, equal] & \cdots \arrow[r, "E_n"] \arrow[d, phantom, "\exists!\Downarrow\bar{\xi}"] & B \arrow[r, "\mathrm{Hom}_A(A,f)"] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[rr, "\mathrm{Hom}_B(B,r)"'] \arrow[d, equal] & & B \arrow[r, "\mathrm{Hom}_A(A,f)"'] \arrow[d, phantom, "\Downarrow\hat{\rho}"] & A \arrow[d, equal] \\
C \arrow[rrr, "\mathrm{Hom}_A(A,g)"'] & & & A
\end{tikzcd}
$$

Thus if $\rho\colon fr \Rightarrow g$ is an absolute right lifting, then $\hat{\rho}\colon \mathrm{Hom}_B(B,r) \times \mathrm{Hom}_A(A,f) \Rightarrow \mathrm{Hom}_A(A,g)$ is a right lifting. ☐

In Theorem 13.3.3 we will discover that right extensions of modules in the image of the covariant embedding are precisely characterized by the sought-for *pointwise* right extensions in the homotopy 2-category; dually pointwise left extensions correspond to right liftings of modules in the image of the contravariant embedding. In the next section, we build towards the 2-categorical definition of this notion.

Exercises.

13.1.i. Exercise. Suppose $B \overset{R}{\nrightarrow} C$ and $B \overset{S}{\nrightarrow} C$ both define right extensions of a module $A \overset{F}{\nrightarrow} C$ along a module $A \overset{K}{\nrightarrow} B$ in the sense of Definition 13.1.2. Prove that $R \simeq_{B\times C} S$.

13.1.ii. Exercise. Prove Lemma 13.1.3.

13.1.iii. Exercise. Dualize Proposition 13.1.5 to characterize right liftings through a composite of two modules.

13.1.iv. Exercise. Verify the dual statement of Proposition 13.1.7, that absolute left lifting diagrams $\lambda\colon g \Rightarrow f\ell$ correspond to right extension diagrams $\hat{\lambda}\colon \mathrm{Hom}_A(f,A) \times \mathrm{Hom}_B(\ell,B) \Rightarrow \mathrm{Hom}_A(g,A)$.

13.2. Exact squares

To motivate the main definition of this section, let us try to guess the 2-categorical universal property of a pointwise right extension by considering a special case that we already understand. If the ambient ∞-cosmos is cartesian closed, then the pointwise right extension of a diagram $f\colon A \to C$ along a functor $k\colon A \to B$ is intended to define the value of a right adjoint, which may or may not

exist in toto, to the restriction functor $\mathrm{res}_k\colon C^B \to C^A$ at the element $f\colon 1 \to C^A$. In the case of extensions along a functor $!\colon A \to 1$, the restriction functor is the constant diagram functor $\Delta\colon C \to C^A$ considered in Definition 2.3.2, and so via Definition 2.3.7 we can understand the pointwise right extension as computing the limit of f. The following lemma describes the transposed form of this universal property.

13.2.1. Lemma. *In a cartesian closed ∞-cosmos, the triangle below-left is an absolute right lifting diagram — defining the limit element and limit cone of f — if and only if the transposed triangle below-center has the property that for any X, the composite diagram below-right is a right extension diagram.*

$$
\begin{tikzcd}
 & C \arrow[d, "\Delta"] \\
1 \arrow[ur, dashed, "\lim f"] \arrow[r, "f"'] \arrow[ur, phantom, bend right, "\Downarrow\nu"] & C^A
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\
1 \arrow[ur, dashed, "\lim f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\
X \arrow[r, "!"] & 1 \arrow[ur, dashed, "\lim f"'] &
\end{tikzcd}
$$

Proof. A factorization of a cone with summit X through the absolute right lifting of f along the constant diagram functor

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\chi"] & C \arrow[d, "\Delta"] \\
1 \arrow[r, "f"'] & C^A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "!"'] \arrow[r, phantom, bend right, "\exists!\Downarrow\zeta"] & C \arrow[d, "\Delta"] \\
1 \arrow[ur, dashed, "\lim f"] \arrow[r, "f"'] \arrow[ur, phantom, bend right, "\Downarrow\nu"] & C^A
\end{tikzcd}
$$

transposes to a factorization as below:

$$
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] \arrow[drr, phantom, "\Uparrow\chi"] & A \arrow[r, "f"] & C \\
X \arrow[urr, bend right, "c"'] & &
\end{tikzcd}
=
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\
X \arrow[r, "!"] \arrow[urr, bend right=60, "c"'] & 1 \arrow[ur, dashed, "\lim f"'] \arrow[r, phantom, "\exists!\Uparrow\zeta"] &
\end{tikzcd}
\qquad \square
$$

Lemma 13.2.1 reveals that to define the limit of $f\colon A \to C$ in an ∞-cosmos that is not necessarily cartesian closed, it is not enough to form the right extension of $!\colon A \to 1$. In terminology we will introduce in Definition 13.3.1 below, we must ask in addition that the right extension diagram is *stable under pasting* with squares of the form:

$$
\begin{tikzcd}
X \times A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \arrow[d, "!"] \\
X \arrow[r, "!"'] & 1
\end{tikzcd}
$$

How might we characterize such squares? Firstly, they are pullbacks each of whose legs is a bifibration. Secondly, they are comma squares, where the comma cone is an identity 2-cell best regarded as pointing in a direction compatible with ν. By Lemmas 13.2.5 and 13.2.6, we shall see that both of these are instances of *exact squares*, which we now introduce.

By Proposition 12.4.10, natural transformations in the homotopy 2-category of an ∞-cosmos correspond bijectively to unary squares in the virtual equipment of modules of various forms, and in

particular, this result, Theorem 12.4.4, Proposition 12.2.4, and Proposition 12.2.1 defines a canonical bijection:

$$\begin{tikzcd} & D \arrow[dl, "k"'] \arrow[dr, "h"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} D \arrow[r, "\mathsf{Hom}_D", "\shortmid"{marking}] \arrow[d, "gk"'] \arrow[dr, phantom, "\Downarrow"] & D \arrow[d, "fh"] \\ A \arrow[r, "\mathsf{Hom}_A"', "\shortmid"{marking}] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k{,}C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_B(B{,}h)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[d, equal] \\ C \arrow[rr, "\mathsf{Hom}_A(f{,}g)"', "\shortmid"{marking}] & & B \end{tikzcd}$$

13.2.2. Definition (exact squares). A square in the homotopy 2-category of an ∞-cosmos

$$\begin{tikzcd} & D \arrow[dl, "k"'] \arrow[dr, "h"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

is **exact** if and only if the corresponding cell below-left, which under the bijection of Lemma 12.1.16 encodes the below-right pasted composite

$$\begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k{,}C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_B(B{,}h)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[d, equal] \\ C \arrow[rr, "\mathsf{Hom}_A(f{,}g)"', "\shortmid"{marking}] & & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd}[column sep=small] & & \mathsf{Hom}_C(k,C) \underset{D}{\times} \mathsf{Hom}_B(B,h) \arrow[dl, two heads] \arrow[dr, two heads] & & \\ & \mathsf{Hom}_C(k,C) \arrow[dr, "p_0", two heads] \arrow[ddl, bend right, "p_1"', two heads] & \overset{\phi}{\Leftarrow} \quad \quad \overset{\phi}{\Leftarrow} & \mathsf{Hom}_B(B,h) \arrow[dl, "p_1"', two heads] \arrow[ddr, bend left, "p_0", two heads] & \\ & & D \arrow[dl, "k"'] \arrow[dr, "h"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] & & \\ C \arrow[drr, "g"'] & & {} & & B \arrow[dll, "f"] \\ & & A & & \end{tikzcd} \tag{13.2.3}$$

displays $\mathsf{Hom}_A(f,g)$ as the composite $\mathsf{Hom}_C(k,C) \otimes \mathsf{Hom}_B(B,h)$ as defined in Definition 12.3.1.

When the boundary square is clear from context, for economy of language we may write that "$\alpha\colon fh \Rightarrow gk$ is an exact square" but note that the definition of exactness requires the specification of the four boundary components of the square inhabited by the 2-cell α.[1]

13.2.4. Remark (exactness as a Beck-Chevalley condition). By Proposition 12.4.7, the canonical cell $\mu \mathsf{Hom}_A(A,g) \times \mathsf{Hom}_A(f,A) \Rightarrow \mathsf{Hom}_A(f,g)$ encoded by the map of spans defined by 1-cell induction

$$\begin{tikzcd}[column sep=small] & & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[dl, two heads] \arrow[dr, two heads] & & \\ & \mathsf{Hom}_A(A,g) \arrow[dl, "p_1"', two heads] \arrow[dr, "p_0", two heads] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & & \mathsf{Hom}_A(f,A) \arrow[dl, "p_1"', two heads] \arrow[dr, "p_0", two heads] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[rr, "g"'] & {} & A & {} & B \arrow[ll, "f"] \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathsf{Hom}_A(A,g) \underset{A}{\times} \mathsf{Hom}_A(f,A) \arrow[d, dashed, "\mu"] \arrow[ddl, bend right, two heads] \arrow[ddr, bend left, two heads] & \\ & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"', two heads] \arrow[dr, "p_0", two heads] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

[1] For instance, compare the statements of Exercise 13.2.ii and Lemma 13.4.3.

is a composite. Exactness says that α induces an isomorphism

$$\hat{\alpha}\colon \mathsf{Hom}_C(k, C) \otimes \mathsf{Hom}_B(B, h) \simeq \mathsf{Hom}_A(A, g) \otimes \mathsf{Hom}_A(f, A)$$

of modules from C to B.

The remainder of this section is devoted to examples of exact squares.

13.2.5. Lemma (comma squares are exact). *For any cospan $C \xrightarrow{g} A \xleftarrow{f} B$, the comma cone defines an exact square:*

$$\begin{tikzcd} & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

Proof. By Proposition 12.3.7, the module $C \overset{\mathsf{Hom}_A(f,g)}{\twoheadrightarrow} B$ is the composite of the left representation of its left leg followed by the right representation of its right leg: $\mathsf{Hom}_C(p_1, C) \otimes \mathsf{Hom}_B(B, p_0) \simeq \mathsf{Hom}_A(f, g)$. By Remark 12.3.9, the composition map $\mathsf{Hom}_C(p_1, C) \times \mathsf{Hom}_B(B, p_0) \Rightarrow \mathsf{Hom}_A(f, g)$ classifies the pasted composite

$$\begin{tikzcd} & \mathsf{Hom}_C(p_1,C) \times \mathsf{Hom}_B(B,p_0) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & \\ \mathsf{Hom}_C(p_1,C) \arrow[dr, "p_0"] \arrow[ddr, phantom, "\overset{\phi}{\Leftarrow}"] \arrow[dd, bend right, "p_1"'] & {} & \mathsf{Hom}_B(B,p_0) \arrow[dl, "p_1"'] \arrow[dd, bend left, "p_0"] \\ & \mathsf{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

which recovers the cell $\hat{\phi}$ defined by (13.2.3) that tests the exactness of comma square $\phi\colon fp_0 \Rightarrow gp_1$. $\square$

13.2.6. Lemma. *If $g\colon C \to A$ is a cartesian fibration or $f\colon B \to A$ is a cocartesian fibration, then the pullback square*

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] \arrow[d, phantom, "\lrcorner"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

is exact.

Proof. The two statements are dual though the positions of the cocartesian and cartesian fibrations cannot be interchanged, as the proof will reveal. If $f\colon B \twoheadrightarrow A$ is a cocartesian fibration, observe

that the functor $i \colon P \to \mathrm{Hom}_A(f, g)$ induced by the identity 2-cell $f\pi_0 = g\pi_1$ is a pullback of the functor $i \colon B \to \mathrm{Hom}_A(f, A)$ induced by the identity 2-cell id_f.

$$
\begin{tikzcd}
P \arrow[rr, "\pi_0"] \arrow[dr, bend right, "i"'] \arrow[dd, "\pi_1"'] & & B \arrow[dr, bend right, "i"'] \arrow[dd, "f"] & \\
& \mathrm{Hom}_A(f,g) \arrow[ul, dashed, bend right, "\ell"'] \arrow[ul, phantom, "\perp"] \arrow[rr] \arrow[dl, "p_1"] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{Hom}_A(f,A) \arrow[ul, dashed, bend right, "\ell"'] \arrow[ul, phantom, "\perp"] \arrow[dl, "p_1"] \\
C \arrow[rr, "g"'] & & A &
\end{tikzcd}
$$

Since f is a cocartesian fibration, Theorem 5.1.11(ii) tells us that $i \colon B \to \mathrm{Hom}_A(f, A)$ has a fibered left adjoint over A. This fibered adjunction pulls back via the cosmological functor $g^* \colon \mathcal{K}_{/A} \to \mathcal{K}_{/C}$ to define a fibered left adjoint to $i \colon P \to \mathrm{Hom}_A(f, g)$.

Since $\pi_0 = p_0 i$, Corollary 12.4.8 implies that the canonical cell

$$\mathrm{Hom}_{\mathrm{Hom}_A(f,g)}(\mathrm{Hom}_A(f,g), i) \mathbin{\dot\times} \mathrm{Hom}_B(B, p_0) \Rightarrow \mathrm{Hom}_B(B, \pi_0)$$

is a composite. Since $\ell \dashv i$, $\mathrm{Hom}_P(\ell, P) \simeq \mathrm{Hom}_{\mathrm{Hom}_A(f,g)}(\mathrm{Hom}_A(f,g), i)$ by Proposition 4.1.1. And since $p_1 = \pi_1 \ell$, Corollary 12.4.8 again implies that the canonical cell

$$\mathrm{Hom}_C(\pi_1, C) \mathbin{\dot\times} \mathrm{Hom}_P(\ell, P) \Rightarrow \mathrm{Hom}_C(p_1, C)$$

is a composite. Composing these bijections, we see that cells with domain $\mathrm{Hom}_C(\pi_1, C) \mathbin{\dot\times} \mathrm{Hom}_B(B, \pi_0)$ correspond bijectively to cells with domain $\mathrm{Hom}_C(p_1, C) \mathbin{\dot\times} \mathrm{Hom}_B(B, p_0)$.

The equation $\mathrm{id} = \phi i \colon f\pi_0 = fp_0 i \Rightarrow gp_1 i = g\pi_1$ asserts that the identity is the transpose along $\ell \dashv i$ of the cell $\phi \colon fp_0 \Rightarrow gp_1 = g\pi_1\ell$, which tells us that the cells

$$\hat{\mathrm{id}} \colon \mathrm{Hom}_C(\pi_1, C) \mathbin{\dot\times} \mathrm{Hom}_B(B, \pi_0) \Rightarrow \mathrm{Hom}_A(f, g) \qquad \text{and}$$
$$\hat{\phi} \colon \mathrm{Hom}_C(p_1, C) \mathbin{\dot\times} \mathrm{Hom}_B(B, p_0) \Rightarrow \mathrm{Hom}_A(f, g)$$

correspond under the bijection just described. Since Lemma 13.2.5 proves that $\hat{\phi}$ is a composite, by Lemma 12.3.2 so is $\hat{\mathrm{id}}$. □

13.2.7. Lemma. *For any pair of functors $f \colon A \to B$ and $g \colon C \to D$ the square*

$$
\begin{tikzcd}
A \times C \arrow[r, "f \times C"] \arrow[d, "A \times g"'] & B \times C \arrow[d, "B \times g"] \\
A \times D \arrow[r, "f \times D"'] & B \times D
\end{tikzcd}
$$

is exact.

Proof. Exercise 13.2.iii. □

13.2.8. LEMMA. *The product of a comma square $\phi\colon fp_0 \Rightarrow gp_1$ with any ∞-category K defines an exact square*

$$\begin{tikzcd}[column sep=small] & \mathsf{Hom}_A(f,g)\times K \arrow[dl, two heads, "p_1\times K"'] \arrow[dr, "p_0\times K"] & \\ C\times K \arrow[dr, "g\times K"'] & \overset{\phi\times K}{\Leftarrow} & B\times K \arrow[dl, "f\times K"] \\ & A\times K & \end{tikzcd}$$

PROOF. Exercise 13.2.iv. ☐

Finally, the plethora of examples of exact squares just established can be composed to yield further comma squares:

13.2.9. LEMMA (composites of exact squares). *Given a diagram of squares in the homotopy 2-category*

$$\begin{tikzcd}[column sep=small] & H \arrow[dl, "s"'] \arrow[dr, "r"] & & G \arrow[dl, "q"'] \arrow[dr, "p"] & \\ F \arrow[dr, "c"'] & \overset{\gamma}{\Leftarrow} & D \arrow[dl, "k"'] \arrow[dr, "h"] & \overset{\beta}{\Leftarrow} & E \arrow[dl, "b"] \\ & C \arrow[dr, "g"'] & \overset{\alpha}{\Leftarrow} & B \arrow[dl, "f"] & \\ & & A & & \end{tikzcd}$$

if $\alpha\colon fh \Rightarrow hk$, $\beta\colon bp \Rightarrow hq$, and $\gamma\colon kr \Rightarrow cs$ are all exact squares then so are the composite rectangles $\alpha q\cdot f\beta$ and $g\gamma\cdot\alpha r$. Consequently, arbitrary "double categorical" composites of exact squares define exact squares.

PROOF. The two cases are co-duals, so it suffices to prove that the rectangle $\alpha q\cdot f\beta\colon g(kq) \Rightarrow (fb)p$ is exact. The corresponding cell $\widehat{\alpha q\cdot f\beta}$ displayed below-left factors as below-right:

$$\begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k,C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_D(q,D)", "\shortmid"{marking}] & G \arrow[r, "\mathsf{Hom}_E(E,p)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\widehat{\alpha q\cdot f\beta}"] & E \arrow[d, equal] \\ C \arrow[rrr, "\mathsf{Hom}_A(fb,g)"', "\shortmid"{marking}] & & & E \end{tikzcd} = \begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k,C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_D(q,D)", "\shortmid"{marking}] \arrow[d, equal] & G \arrow[r, "\mathsf{Hom}_E(E,p)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\beta}"] & E \arrow[d, equal] \\ C \arrow[r, "\mathsf{Hom}_C(k,C)"', "\shortmid"{marking}] \arrow[d, equal] & D \arrow[rr, "\mathsf{Hom}_B(b,h)"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\bar{\alpha}"] & & E \arrow[d, equal] \\ C \arrow[rrr, "\mathsf{Hom}_A(fb,g)"', "\shortmid"{marking}] & & & E \end{tikzcd}$$

through a cell $\bar{\alpha}$ defined by the universal property of the composite $\mathsf{Hom}_B(B,h)\otimes\mathsf{Hom}_B(b,B) \simeq \mathsf{Hom}_B(b,h)$ of Proposition 12.4.7 by the pasting equality:

$$\begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k,C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_B(B,h)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_B(b,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & E \arrow[d, equal] \\ C \arrow[r, "\mathsf{Hom}_C(k,C)"', "\shortmid"{marking}] \arrow[d, equal] & D \arrow[rr, "\mathsf{Hom}_B(b,h)"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\bar{\alpha}"] & & E \arrow[d, equal] \\ C \arrow[rrr, "\mathsf{Hom}_A(fb,g)"', "\shortmid"{marking}] & & & E \end{tikzcd} := \begin{tikzcd} C \arrow[r, "\mathsf{Hom}_C(k,C)", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "\mathsf{Hom}_B(B,h)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\hat{\alpha}"] & B \arrow[r, "\mathsf{Hom}_B(b,B)", "\shortmid"{marking}] \arrow[d, equal] & E \arrow[d, equal] \\ C \arrow[rr, "\mathsf{Hom}_A(f,g)"', "\shortmid"{marking}] \arrow[d, equal] & & B \arrow[r, "\mathsf{Hom}_B(b,B)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\circ"] & E \arrow[d, equal] \\ C \arrow[rrr, "\mathsf{Hom}_A(fb,g)"', "\shortmid"{marking}] & & & E \end{tikzcd}$$

By exactness of α, $\hat{\alpha}$ and both cells named $\circ$ are composites, so by Lemma 12.3.2 $\bar{\alpha}$ is a composite as well. By exactness of β and Lemma 12.3.2 again it now follows that $\widehat{\alpha q \cdot f\beta}$ is also a composite, proving exactness of the rectangle $\alpha q \cdot f\beta \colon g(kq) \Rightarrow (fb)p$. ☐

Exercises.

13.2.i. Exercise. Characterize the exact squares in the ∞-cosmos $\mathcal{C}at$ of strict 1-categories.

13.2.ii. Exercise. Prove that the identity cells

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "k"] & \\ A \arrow[dr, "k"'] & & B \arrow[dl, equal] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "k"'] \arrow[dr, equal] & \\ B \arrow[dr, equal] & & A \arrow[dl, "k"] \\ & B & \end{tikzcd}$$

define exact squares.

13.2.iii. Exercise. Prove Lemma 13.2.7.

13.2.iv. Exercise. Prove Lemma 13.2.8.

13.3. Pointwise right and left extensions

In this section we give four definitions of pointwise extension and prove they are equivalent. Our proof will reveal that the general 2-categorical notion of Definition 13.1.1 is on its own too weak.

13.3.1. Definition (stability of extensions under pasting). A right extension diagram $\nu \colon rk \Rightarrow f$ in a 2-category is said to be **stable under pasting with a square** α

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ B \arrow[ur, "r"'] & \end{tikzcd} \qquad \begin{tikzcd} & D \arrow[dl, "g"'] \arrow[dr, "h"] & \\ A \arrow[dr, "k"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & E \arrow[dl, "b"] \\ & B & \end{tikzcd}$$

if the pasted composite

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, phantom, "\Nearrow\alpha"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ E \arrow[r, "b"'] & B \arrow[ur, "r"'] & \end{tikzcd} \tag{13.3.2}$$

defines a right extension $rb \colon E \to C$ of $fg \colon D \to C$ along $h \colon D \to E$. The co-dual of this statement defines what it means for a left extension diagram to be stable under pasting with a square.

13.3.3. Theorem (pointwise right extensions). *For a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

in the homotopy 2-category of an ∞-cosmos $\mathcal{K}$ the following are equivalent:

(i) $\nu\colon rk \Rightarrow f$ *defines a right extension in* $\mathfrak{h}\mathcal{K}$ *that is stable under pasting with exact squares.*
(ii) $\nu\colon rk \Rightarrow f$ *defines a right extension in* $\mathfrak{h}\mathcal{K}$ *that is stable under pasting with comma squares.*
(iii) $\nu\colon \mathsf{Hom}_B(B,k) \mathbin{\dot\times} \mathsf{Hom}_C(C,r) \Rightarrow \mathsf{Hom}_C(C,f)$ *defines a right extension in* $\mathcal{M}od(\mathfrak{h}\mathcal{K})$.
(iv) For any exact square $\alpha\colon bh \Rightarrow kg$, $\nu g \cdot r\alpha\colon \mathsf{Hom}_E(E,h) \mathbin{\dot\times} \mathsf{Hom}_C(C,rb) \Rightarrow \mathsf{Hom}_C(C,fg)$ *defines a right extension in* $\mathcal{M}od(\mathfrak{h}\mathcal{K})$.

When these conditions hold, we say $r\colon B \to C$ *defines a* ***pointwise right extension*** *of* $f\colon A \to C$ *along* $k\colon A \to B$.

PROOF. Lemma 13.2.5 proves (i)⇒(ii).

To show (ii)⇒(iii), suppose $\nu\colon rk \Rightarrow f$ defines a right extension in $\mathfrak{h}\mathcal{K}$ that is stable under pasting with comma squares and consider a cell in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$:

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,k)", "|"'] \arrow[d, equal] & B \arrow[r, "E_1", "|"'] & \cdots \arrow[r, "E_n", "|"'] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "|"] & & & C \end{tikzcd}$$

Let $B \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} C$ denote the composite two-sided fibration $E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n$. By Remark 12.3.8 applied to $B \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} C$, module maps $\mathsf{Hom}_B(B,k) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,f)$ stand in bijection with module maps $\mathsf{Hom}_B(B,k) \mathbin{\dot\times} \mathsf{Hom}_B(q,B) \mathbin{\dot\times} \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,f)$. By Corollary 12.4.6 and Proposition 12.4.7, such module maps stand in bijection with module maps $\mathsf{Hom}_B(q,k) \Rightarrow \mathsf{Hom}_C(p,f)$. By Lemma 12.1.16, these module maps correspond bijectively to 2-cells

$$\begin{tikzcd} & \mathsf{Hom}_B(q,k) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] & \\ A \arrow[dr, "f"'] & \overset{\alpha}{\Leftarrow} & E \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

in the homotopy 2-category. By the hypothesis (ii),

$$\begin{tikzcd} \mathsf{Hom}_B(q,k) \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_0"'] \arrow[dr, phantom, "\Nearrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ E \arrow[r, two heads, "q"'] & B \arrow[ur, "r"'] & \end{tikzcd}$$

defines a right extension in $\mathfrak{h}\mathcal{K}$, so α factors uniquely through this pasted composite via a map $\gamma\colon p \Rightarrow rq$. By Proposition 12.4.10, this defines a cell $\gamma\colon \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,rq)$, which by Corollary 12.4.6 gives rise to a canonical cell $\hat\gamma\colon \mathsf{Hom}_B(q,B) \mathbin{\dot\times} \mathsf{Hom}_C(C,p) \Rightarrow \mathsf{Hom}_C(C,r)$. By Remark 12.3.8

again, this produces the desired unique factorization

$$
\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_B(B,k)", "|"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "|"{marking}] & \cdots \arrow[r, "E_n", "|"{marking}] & C \arrow[d, equal] \\
A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "|"{marking}] \arrow[urrr, phantom, "\Downarrow"] & & & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_B(B,k)", "|"{marking}] \arrow[d, equal] & B \arrow[r, "E_1", "|"{marking}] \arrow[d, equal] & \cdots \arrow[r, "E_n", "|"{marking}] \arrow[d, phantom, "\Downarrow\exists!"] & C \arrow[d, equal] \\
A \arrow[r, "\mathsf{Hom}_B(B,k)", "|"{marking}] \arrow[d, equal] & B \arrow[rr, "\mathsf{Hom}_C(C,r)", "|"{marking}] \arrow[d, phantom, "\Downarrow\nu"] & & C \arrow[d, equal] \\
A \arrow[rrr, "\mathsf{Hom}_C(C,f)"', "|"{marking}] & & & C
\end{tikzcd}
$$

To show (iii)⇒(iv), consider a diagram (13.3.2) in which $\alpha\colon bh \Rightarrow kg$ is exact and $\nu\colon \mathsf{Hom}_B(B,k) \mathbin{\dot\times} \mathsf{Hom}_C(C,r) \Rightarrow \mathsf{Hom}_C(C,f)$ defines a right extension diagram in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. Now by Corollary 12.4.6, a cell

$$\mathsf{Hom}_E(E,h) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,fg)$$

corresponds to a cell

$$\mathsf{Hom}_A(g,A) \mathbin{\dot\times} \mathsf{Hom}_E(E,h) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,f).$$

By exactness of α, this corresponds to a cell

$$\mathsf{Hom}_B(b,k) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,f),$$

or equivalently, upon restricting along the composite map $\mathsf{Hom}_B(B,k) \mathbin{\dot\times} \mathsf{Hom}_B(b,B) \Rightarrow \mathsf{Hom}_B(b,k)$ to a cell

$$\mathsf{Hom}_B(B,k) \mathbin{\dot\times} \mathsf{Hom}_B(b,B) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,f).$$

Since $B \overset{\mathsf{Hom}_C(C,r)}{\nrightarrow} C$ is the right extension of $A \overset{\mathsf{Hom}_C(C,f)}{\nrightarrow} C$ along $A \overset{\mathsf{Hom}_B(B,k)}{\nrightarrow} B$, this corresponds to a cell

$$\mathsf{Hom}_B(b,B) \mathbin{\dot\times} E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,r),$$

which transposes via Corollary 12.4.6 to a cell

$$E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,rb),$$

which gives the factorization required to prove that $E \overset{\mathsf{Hom}_C(C,rb)}{\nrightarrow} C$ is the right extension of $D \overset{\mathsf{Hom}_C(C,fg)}{\nrightarrow} C$ along $D \overset{\mathsf{Hom}_E(E,h)}{\nrightarrow} E$. A slightly more delicate argument is required to see that this bijection is implemented by composing with the map of modules $\mathsf{Hom}_E(E,h) \mathbin{\dot\times} \mathsf{Hom}_C(C,rb) \Rightarrow \mathsf{Hom}_C(C,fg)$ corresponding to the pasted composite (13.3.2), but for this it suffices by the Yoneda lemma to start with the identity cell $\mathrm{id}_{\mathsf{Hom}_C(C,rb)}$ and trace back up through the bijection just described. By Lemma 12.1.16 this is straightforward.

Finally Lemma 13.1.6 and the trivial example of Exercise 13.2.ii prove that (iv)⇒(i). □

13.3.4. Corollary. *The pasted composite* (13.3.2) *of a pointwise right extension with an exact square is a pointwise right extension.*

Proof. Lemma 13.2.9, the pasted composite of two exact squares remains an exact square, so by Theorem 13.3.3(i), the pasted composite of a pointwise right extension remains stable under pasting with exact squares. □

Exercises.

13.3.i. Exercise. Prove Lemma 13.1.3.

13.3.ii. Exercise. Using Lemma 13.1.6 as a hint, state and prove a dual version of Theorem 13.3.3 defining pointwise left extensions in the homotopy 2-category of an ∞-cosmos.

13.4. Formal category theory in a virtual equipment

13.4.1. Proposition. *For any pair of functors $u\colon A \to B$ the following are equivalent:*

(i) The natural transformation $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is the counit of an adjunction $f \dashv u$.

(ii) There is a pointwise right extension diagram in $\mathfrak{h}\mathcal{K}$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\Uparrow\epsilon" near start] & A \\ B \arrow[ur, "f"'] & \end{tikzcd} \tag{13.4.2}$$

*that is **absolute**, preserved by any functor $h\colon A \to C$.*

(iii) The square

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "u"] & \\ A \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A \arrow[ul, equal] & \end{tikzcd}$$

is exact.

Proof. Without the adjective "pointwise" the equivalence (i)$\Leftrightarrow$(ii) is the op-dual of Lemma 2.3.6, whose proof works in any 2-category and in particular in $\mathfrak{h}\mathcal{K}^{\mathrm{op}}$. It remains only to demonstrate that the right extension produced by (i)$\Rightarrow$(ii) is pointwise. To see this, recall that for any $f\colon B \to A$, Lemma 13.1.3 provides a right lifting in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ as below left

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_A(f,A)", "|" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_A(A,f)", "|" marking] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] \\ A \arrow[rr, "\mathsf{Hom}_A"', "|" marking] & & A \end{tikzcd} \quad \cong \quad \begin{tikzcd} A \arrow[r, "\mathsf{Hom}_B(B,u)", "|" marking] \arrow[d, equal] & B \arrow[r, "\mathsf{Hom}_A(A,f)", "|" marking] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] \\ A \arrow[rr, "\mathsf{Hom}_A"', "|" marking] & & A \end{tikzcd}$$

If $f \dashv u$, then by Proposition 4.1.1, the counit induces an equivalence $\mathsf{Hom}_B(B,u) \simeq \mathsf{Hom}_A(f,A)$, so we have an isomorphic right lifting diagram above right. By Proposition 12.4.10 and Theorem 13.3.3 this supplies a natural transformation $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ that defines a pointwise right extension (13.4.2).

Proposition 4.1.1, which demonstrates that $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is a counit if and only if it induces an equivalence $\mathsf{Hom}_B(B,u) \simeq \mathsf{Hom}_A(f,A)$ also proves (i)$\Leftrightarrow$(iii), via Lemma 12.3.5 which tells us that $\mathsf{Hom}_A \otimes \mathsf{Hom}_B(B,u) \simeq \mathsf{Hom}_B(B,u)$. $\square$

13.4.3. Lemma. *A functor $k\colon A \to B$ is **fully faithful** when any of the following equivalent conditions hold:*

(i) The square

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] & \\ A \arrow[dr, "k"'] & & A \arrow[dl, "k"] \\ & B & \end{tikzcd}$$

is exact.

(ii) The unary cell $\hat{\mathrm{id}}_k\colon \mathsf{Hom}_A \Rightarrow \mathsf{Hom}_B(k,k)$ *is a composite.*

(iii) The module map $\hat{\mathrm{id}}_k$ *defines an equivalence of modules* $\mathsf{Hom}_A \simeq \mathsf{Hom}_B(k,k)$ *from* A *to* A.

PROOF. The equivalence follows from Definition 13.2.2 and Lemma 12.3.3. □

13.4.4. PROPOSITION (extensions along fully faithful functors). *If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

is a pointwise right extension and k *is fully faithful, then* ν *is an isomorphism.*

PROOF. Pasting the pointwise right extension with the exact square of Lemma 13.4.3 yields a pointwise right extension diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ A \arrow[ur, "rk"'] & \end{tikzcd}$$

By Lemma 12.3.5 and Theorem 13.3.3(iii), $f\colon A \to C$ also defines a pointwise right extension of itself along the identity. The universal property of these extension diagrams in $\mathfrak{h}\mathcal{K}$ described in Definition 13.1.1 now suffices to construct an inverse isomorphism to ν. □

The dual result, that the 2-cell in a pointwise left extension along a fully faithful functor is invertible, is left to Exercise 13.4.i.

13.4.5. PROPOSITION. *A right adjoint* $u\colon A \to B$ *is fully faithful if and only if the counit* $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ *is an isomorphism.*

PROOF. If $f \dashv u$ with counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, then Proposition 4.1.1 reveals that pasting with ϵ induces an equivalence of modules $\mathsf{Hom}_B(B,u) \simeq \mathsf{Hom}_A(f,A)$. Equivalently, by Proposition 13.4.1, the counit $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is exact. If u is fully faithful, then by Lemma 13.2.9 the composite rectangle is also exact.

$$\begin{tikzcd} & & A \arrow[dl, equal] \arrow[dr, equal] & \\ & A \arrow[dl, equal] \arrow[dr, "u"] & & A \arrow[dl, "u"] \\ A \arrow[dr, equal] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] & \\ & A & & \end{tikzcd}$$

which by Lemma 12.3.3 is to say that ϵ induces an equivalence of modules $\mathsf{Hom}_A(A,A) \simeq \mathsf{Hom}_A(fu,A)$ under the contravariant embedding of Definition 12.4.11. By Proposition 12.4.10, which says this embedding is fully faithful, it follows that $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ is an isomorphism.

Conversely, assume $f \dashv u$ with invertible counit $\epsilon\colon fu \cong \mathrm{id}_A$. By Proposition 12.4.7 we have a composite $\mathsf{Hom}_B(B,u) \otimes \mathsf{Hom}_B(u,B) \simeq \mathsf{Hom}_B(u,u)$. Substituting the equivalence $\mathsf{Hom}_A(f,A) \simeq \mathsf{Hom}_B(B,u)$ gives another composite $\mathsf{Hom}_A(f,A) \otimes \mathsf{Hom}_B(u,B) \simeq \mathsf{Hom}_B(u,u)$, but now Corollary

12.4.8 gives a third composite $\mathsf{Hom}_A(f,A)\otimes\mathsf{Hom}_B(u,B)\simeq\mathsf{Hom}_A(fu,A)$. Factoring one composite through the other we obtain an equivalence $\mathsf{Hom}_A(fu,A)\simeq\mathsf{Hom}_B(u,u)$ which composes with the equivalence $\mathsf{Hom}_A\simeq\mathsf{Hom}_A(fu,A)$ induced by the invertible counit cell to define a composite $\mathsf{Hom}_A\simeq\mathsf{Hom}_A(fu,A)\simeq\mathsf{Hom}_B(u,u)$, proving, by Lemma 13.4.3, that $u\colon A\to B$ is fully faithful. □

Exercises.

13.4.i. Exercise. Prove that if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow\lambda" near start] & C \\ B \arrow[ur, "\ell"'] & \end{tikzcd}$$

is a pointwise left extension and k is fully faithful, then λ is an isomorphism.

13.5. Limits and colimits in cartesian closed ∞-cosmoi

In this section we work in an ∞-cosmos $\mathcal{K}$ that is *cartesian closed*, satisfying the extra assumption of Definition 1.2.20. In a cartesian closed ∞-cosmos a functor $f\colon A\to C$ transposes to define an element $f\colon 1\to C^A$ of the ∞-category of functors from A to C and a family of diagrams $f\colon A\times D\to C$ of shape A parametrized by an ∞-category D transposes to a generalized element $f\colon D\to A^C$. The aim is to reinterpret the limits and colimits of §2.3 as pointwise right and left extensions.

13.5.1. Proposition. *Suppose the diagram below-left is a pointwise right extension diagram in a cartesian closed ∞-cosmos.*

$$\begin{tikzcd} A\times D \arrow[r, "f"] \arrow[d, "k\times D"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ B\times D \arrow[ur, "r"'] & \end{tikzcd} \quad\leftrightsquigarrow\quad \begin{tikzcd} & C^B \arrow[d, "C^k"] \\ D \arrow[ur, "r"] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\nu" near end, shift right=3] & C^A \end{tikzcd}$$

Then its transpose above-right defines an absolute right lifting diagram in the homotopy 2-category which moreover is stable under pasting with the square C^ϕ induced from any exact square ϕ.

Proof. The universal property that characterizes the absolute right lifting diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "d"'] \arrow[dr, phantom, "\Downarrow\chi"] & C^B \arrow[d, "C^k"] \\ D \arrow[r, "f"'] & C^A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "d"'] \arrow[d, phantom, "\Downarrow\exists!\zeta", shift left=5] & C^B \arrow[d, "C^k"] \\ D \arrow[r, "f"'] \arrow[ur, "r"] \arrow[r, phantom, "\Downarrow\nu", shift left=3] & C^A \end{tikzcd}$$

transposes to

$$\begin{tikzcd} A\times X \arrow[r, "A\times d"] \arrow[d, "k\times X"'] \arrow[dr, phantom, "\Uparrow\chi"] & A\times D \arrow[r, "f"] & C \\ B\times X \arrow[urr, bend right, "e"'] & & \end{tikzcd} = \begin{tikzcd} A\times X \arrow[r, "A\times d"] \arrow[d, "k\times X"'] & A\times D \arrow[r, "f"] \arrow[d, "k\times D"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\ B\times X \arrow[r, "B\times d"] \arrow[urr, bend right=60, "e"'] & B\times D \arrow[ur, "r"'] \arrow[r, phantom, "\Uparrow\exists!\zeta"] & {} \end{tikzcd}$$

Similarly, the pasted composite of the absolute right lifting diagram with an exponentiated exact square transposes below-left to the diagram below-right

$$\begin{tikzcd} & C^B \arrow[r, "C^h"] \arrow[d, "C^k"] \arrow[dr, phantom, "\Downarrow C^\phi"] & C^E \arrow[d, "C^q"] \\ D \arrow[ur, "r"] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\nu"] & C^A \arrow[r, "C^p"'] & C^F \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} F\times D \arrow[r, "p\times D"] \arrow[d, "q\times D"'] \arrow[dr, phantom, "\Downarrow\phi\times D"] & A\times D \arrow[r, "f"] \arrow[d, "k\times D"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ E\times D \arrow[r, "h\times D"'] & B\times D \arrow[ur, "r"'] & \end{tikzcd}$$

Lemma 13.2.7 and Lemma 13.2.8 prove that the square defining the product $k\times d$ and the square $\phi\times D$ are exact.

Now if ν is a pointwise right extension, then by Lemma 13.2.7 and Corollary 13.3.4 so is $\nu\cdot(A\times d)$. The transposed universal property of this right extension diagram proves that ν defines an absolute right lifting. Since by Corollary 13.3.4, the pasted composite of ν with the exact square $\phi\times D$ gives another pointwise right extension, the same argument shows that the pasted composite of ν with E^ϕ is again an absolute right lifting diagram. $\square$

13.5.2. Proposition. *In a cartesian closed ∞-cosmos any limit as encoded by the absolute right lifting diagram below-right transposes to define a pointwise right extension diagram as below-left:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ 1 \arrow[ur, "\lim f"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & C \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\lim f"] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\nu"] & C^A \end{tikzcd}$$

Conversely, any pointwise right extension diagram of this form transposes to define a limit in C.

Proof. Proposition 13.5.1 proves that pointwise right extension diagrams transpose to define absolute right lifting diagrams. Conversely, Lemma 13.2.1 reveals that if $\nu\colon \Delta\lim f \Rightarrow f$ is absolute right lifting, then $\nu\colon \ell! \Rightarrow f$ is a right extension diagram stable under pasting with pullback squares. Over the terminal ∞-category 1, all comma squares are pullback squares, so by Theorem 13.3.3 this stability property proves that $\nu\colon \ell! \Rightarrow f$ is a pointwise right extension. $\square$

13.5.3. Corollary. *For any pointwise right extension in a cartesian closed ∞-cosmos*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ B \arrow[ur, "r"'] & \end{tikzcd}$$

and any element $b\colon 1 \to B$, the element $rb\colon 1 \to C$ is the limit of the diagram

$$\mathsf{Hom}_B(b,k) \xrightarrow{p_1} A \xrightarrow{f} C$$

Proof. By Theorem 13.3.3(ii), the composite

$$\begin{tikzcd} \mathsf{Hom}_B(b,k) \arrow[r, two heads, "p_1"] \arrow[d, "!"'] \arrow[dr, phantom, "\Nearrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ 1 \arrow[r, "b"'] & B \arrow[ur, "r"'] & \end{tikzcd}$$

is a pointwise right extension. By Proposition 13.5.2 this can be interpreted as saying that $rb\colon 1 \to B$ defines the limit of the restriction of the diagram $f\colon A \to C$ along $p_1\colon \mathsf{Hom}_B(b,k) \twoheadrightarrow A$. ☐

In Definition 2.2.1, initial and terminal elements in A were defined respectively as left or right adjoints to the unique functor $!\colon A \to 1$

$$\begin{tikzcd} A \arrow[r, "!" description] & 1 \arrow[l, bend right=50, "i"', ""{name=U, below}] \arrow[l, bend left=50, "t", ""{name=D, above}] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"] \end{tikzcd}$$

or equivalently, by Definition 2.3.2, as colimits or limits for the empty diagram in $A^{\varnothing} \cong 1$. In a cartesian closed ∞-cosmos, we now show that an initial element may also be characterized as a *limit* and a terminal element may be characterized as a *colimit* of the identity diagram $\mathrm{id}_A\colon A \to A$.

13.5.4. Corollary. *For an ∞-category A in a cartesian closed ∞-cosmos:*

(i) An element $t\colon 1 \to A$ is terminal if and only if it defines a colimit for the identity functor $\mathrm{id}_A\colon A \to A$ in which case the unit for the adjunction $! \dashv t$ defines the colimit cone.

(ii) An element $i\colon 1 \to A$ is initial if and only if it defines a limit for the identity functor $\mathrm{id}_A\colon A \to A$ in which case the counit for $i \dashv !$ defines the limit cone.

Proof. We prove (ii). By Proposition 4.1.1 and Lemma 12.1.17, an element $i\colon 1 \to A$ defines a left adjoint to $!\colon A \to 1$ if and only if the modules $\mathsf{Hom}_A(i, A)$ and $\mathsf{Hom}_1(1, !)$ are isomorphic in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. By Lemma 13.1.3 and Exercise 13.1.i, this is the case if and only if there is a right extension diagram

$$\begin{tikzcd} A \arrow[r, "\mathsf{Hom}_1(1{,}!)", "\vert" marking] \arrow[d, equal] & A \arrow[r, "\mathsf{Hom}_A(A{,}i)", "\vert" marking] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, equal] \\ A \arrow[rr, "\mathsf{Hom}_A"', "\vert" marking] & & A \end{tikzcd}$$

in which case the binary cell corresponds to the counit transformation $\epsilon\colon i! \Rightarrow \mathrm{id}_A$. By Theorem 13.3.3, this is the case if and only if the counit defines a pointwise right extension as below-left, which by Proposition 13.5.2 is equivalent to a limit of the identity diagram as below-right:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "!"'] & A \\ 1 \arrow[ur, "i"'] \arrow[u, phantom, shift left=3, "\Uparrow\epsilon"] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "i"] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A^A \end{tikzcd}$$

☐

Recall from Definition 2.4.6 that a functor $k\colon I \to J$ is **final** if and only if for any ∞-category A, the square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\ A^J \arrow[r, "A^k"'] & A^I \end{tikzcd}$$

preserves and reflects all absolute left lifting diagrams. Dually a functor $k\colon I \to J$ is **initial** if this square preserves and reflects all absolute right lifting diagrams. We can now give a more concise formulation of these notions.

13.5.5. Definition. A functor $k\colon I \to J$ is **final** if and only if the square below-left is exact and **initial** if and only if the square below-right is exact.

$$
\begin{tikzcd}[column sep=small]
 & I \arrow[dl, "!"'] \arrow[dr, "k"] & \\
1 \arrow[dr, equal] & & J \arrow[dl, "!"] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & I \arrow[dl, "k"'] \arrow[dr, "!"] & \\
J \arrow[dr, "!"'] & & 1 \arrow[dl, equal] \\
 & 1 &
\end{tikzcd}
$$

Note that the functor $!\colon J \to 1$ is represented on the right and on the left by the modules $J \overset{J}{\nrightarrow} 1$ and $1 \overset{J}{\nrightarrow} J$. So we see that $k\colon I \to J$ is **final** if and only if $p_0\colon \mathsf{Hom}_J(J,k) \xrightarrow{\sim}\!\!\!\!\!\rightarrow J$ is a trivial fibration, and $k\colon I \to J$ is **initial** if and only if $p_1\colon \mathsf{Hom}_J(k,J) \xrightarrow{\sim}\!\!\!\!\!\rightarrow J$ is a trivial fibration.

To reconcile Definition 13.5.5 with Definition 2.4.6 we must prove:

13.5.6. Proposition. *In a cartesian closed ∞-cosmos, if $k\colon I \to J$ is initial and $f\colon J \to C$ is any diagram, then a limit of f also defines a limit of $fk\colon I \to C$ and conversely if the limit of $fk\colon I \to C$ exists, then it also defines a limit of $f\colon J \to C$.*

Proof. By Proposition 13.5.2, a limit of f defines a pointwise right extension as below left, which by Corollary 13.3.4 and Definition 13.5.5 gives rise to another pointwise right extension below-right.

$$
\begin{tikzcd}
J \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\
1 \arrow[ur, "\ell"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I \arrow[r, "k"] \arrow[d, "!"'] & J \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & C \\
1 \arrow[r, equal] & 1 \arrow[ur, "\ell"'] &
\end{tikzcd}
$$

By Proposition 13.5.2 again, this tells us that ℓ is the limit of fk.

For the converse, suppose we are given a pointwise right extension diagram

$$
\begin{tikzcd}
I \arrow[r, "fk"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\mu" near start] & C \\
1 \arrow[ur, "\ell"'] &
\end{tikzcd}
$$

By Theorem 13.3.3(iii), this universal property tells us that for any composable sequence of modules $E_1 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} E_n$ from 1 to C, composing with $\mu\colon \mathsf{Hom}_!(1,!) \mathbin{\dot{\times}} \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ defines a bijection between cells $E_1 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells below-left:

$$
\begin{tikzcd}
I \arrow[r, "\mathsf{Hom}_1(1{,}!)", "|" marking] \arrow[d, equal] & 1 \arrow[r, "E_1", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] & C \arrow[d, equal] \\
I \arrow[rrr, "\mathsf{Hom}_C(C{,}fk)"', "|" marking] \arrow[urrr, phantom, "\Downarrow"] & & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "\mathsf{Hom}_J(k{,}J)", "|" marking] \arrow[d, equal] & I \arrow[r, "\mathsf{Hom}_1(1{,}!)", "|" marking] & 1 \arrow[r, "E_1", "|" marking] & \cdots \arrow[r, "E_n", "|" marking] & C \arrow[d, equal] \\
{} \arrow[rrrr, "\mathsf{Hom}_C(C{,}f)"', "|" marking] \arrow[urrrr, phantom, "\Downarrow"] & & & & C
\end{tikzcd}
$$

By Corollary 12.4.6, composing with $\mu\colon \mathsf{Hom}_!(1,!) \mathbin{\dot{\times}} \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ also defines a bijection between cells $E_1 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells above-right.

To say that $k\colon I \to J$ is initial means, by Definition 13.5.5, that the map $\mathsf{Hom}_J(k,J) \mathbin{\dot{\times}} \mathsf{Hom}_1(1,!) \Rightarrow \mathsf{Hom}_1(1,!)$ of modules from J to 1 induced by the identity is a composite. Thus composing with

$\mu\colon \mathsf{Hom}_!(1,!) \mathbin{\dot\times} \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,fk)$ also defines a bijection between cells $E_1 \mathbin{\dot\times} \cdots \mathbin{\dot\times} E_n \Rightarrow \mathsf{Hom}_C(C,\ell)$ and the cells

$$\begin{tikzcd} J \arrow[r, "\mathsf{Hom}_1(1{,}!)", "\mid" marking] \arrow[d, equal] & 1 \arrow[r, "E_1", "\mid" marking] & \cdots \arrow[r, "E_n", "\mid" marking] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ J \arrow[rrr, "\mathsf{Hom}_C(C{,}f)"', "\mid" marking] & & & C \end{tikzcd}$$

But this says exactly that the cell $\nu\colon \mathsf{Hom}_1(1,!) \mathbin{\dot\times} \mathsf{Hom}_C(C,\ell) \Rightarrow \mathsf{Hom}_C(C,f)$ that corresponds to μ under this series of bijections displays $1 \overset{\mathsf{Hom}_C(C,\ell)}{\nrightarrow} C$ as the right extension of $J \overset{\mathsf{Hom}_C(C,f)}{\nrightarrow} C$ along $J \overset{\mathsf{Hom}_1(1,!)}{\nrightarrow} 1$; unpacking this definition, we see that $\nu k = \mu$. Thus $\nu\colon \ell! \Rightarrow f$ is a pointwise right extension and by Proposition 13.5.2 we conclude that $\ell\colon 1 \to C$ also defines a limit of $f\colon J \to C$ as claimed. ☐

13.5.7. Definition. In a cartesian closed ∞-cosmos $\mathcal{K}$, an ∞-category E **admits functorial pointwise right extensions** along $k\colon A \to B$ if there is a pointwise right extension

$$\begin{tikzcd} A \times E^A \arrow[r, "\mathrm{ev}"] \arrow[d, "k\times E^A"'] \arrow[dr, phantom, "\Uparrow\epsilon"] & E \\ B \times E^A \arrow[ur, "\mathrm{ran}_k"'] \end{tikzcd}$$

of the evaluation functor along $k \times E^A$.

By Proposition 13.5.1 if E admits functorial pointwise right extensions along $k\colon A \to B$, there is an absolute right lifting diagram

$$\begin{tikzcd} & E^B \arrow[d, "\mathrm{res}"] \\ E^A \arrow[ur, "\mathrm{ran}_k"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & E^A \end{tikzcd}$$

that is stable under pasting with E^ϕ for any exact square ϕ.

13.5.8. Proposition (Beck-Chevalley condition). *For any exact square*

$$\begin{tikzcd} & D \arrow[dl, "q"'] \arrow[dr, "p"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

in a cartesian closed ∞-cosmos and any ∞-category E, whenever E admits functorial pointwise left and right extensions, the Beck-Chevalley condition is satisfied for the induced 2-cell ϕ^* below-left: the mates of ϕ^* below center and below right are both isomorphisms.

$$\begin{tikzcd} & E^A \arrow[dl, "g^*"'] \arrow[dr, "f^*"] & \\ E^C \arrow[dr, "q^*"'] \arrow[rr, phantom, "\overset{\phi^*}{\Leftarrow}"] & & E^B \arrow[dl, "p^*"] \\ & E^D & \end{tikzcd} \qquad \begin{tikzcd} & E^A \arrow[dr, "f^*"] & \\ E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[dr, "q^*"'] \arrow[rr, phantom, "\Downarrow\phi^!"] & & E^B \\ & E^D \arrow[ur, "\mathrm{ran}_p"'] & \end{tikzcd} \qquad \begin{tikzcd} & E^A \arrow[dl, "g^*"'] & \\ E^C \arrow[rr, phantom, "\Uparrow\phi_!"] & & E^B \arrow[ul, "\mathrm{lan}_f"'] \arrow[dl, "p^*"] \\ & E^D \arrow[ul, "\mathrm{lan}_q"] & \end{tikzcd}$$

Proof. By Proposition 13.5.1, the pointwise right extensions defining ran_g and ran_p transpose to define absolute right lifting diagrams

$$\begin{tikzcd} & E^A \arrow[r, "f^*"] \arrow[d, "g^*"] \arrow[dr, phantom, "\Downarrow\phi^*"] & E^B \arrow[d, "p^*"] \\ E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon"] & E^C \arrow[r, "q^*"'] & E^D \end{tikzcd} \qquad \begin{tikzcd} & & E^B \arrow[d, "p^*"] \\ E^C \arrow[r, "q^*"'] & E^D \arrow[ur, "\mathrm{ran}_p"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon"] & E^D \end{tikzcd}$$

Moreover, the mate $\phi^!$ of ϕ^* defines a factorization of the left-hand diagram through the right-hand diagram:

$$\begin{tikzcd} & E^A \arrow[r, "f^*"] \arrow[d, "g^*"] \arrow[dr, phantom, "\Downarrow\phi^*"] & E^B \arrow[d, "p^*"] \\ E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon"] & E^C \arrow[r, "q^*"'] & E^D \end{tikzcd} = \begin{tikzcd} & E^A \arrow[r, "f^*"] \arrow[d, phantom, "\Downarrow\phi^!"] & E^B \arrow[d, "p^*"] \\ E^C \arrow[ur, "\mathrm{ran}_g"] \arrow[r, "q^*"'] & E^D \arrow[ur, "\mathrm{ran}_p"] \arrow[r, equal] \arrow[ur, phantom, "\Downarrow\epsilon"] & E^D \end{tikzcd}$$

Immediately from the universal property of the absolute right liftings of q^* through p^* we have that $\phi^!$ is an isomorphism. The proof for $\phi_!$ is similar using the absolute left lifting diagrams arising from the pointwise left extensions defining lan_f and lan_q □

13.6. Weighted limits and colimits of ∞-categories

Proposition 13.5.2 reveals that in cartesian closed ∞-cosmoi, the notion of limit of a diagram of ∞-categories $f: A \to C$ first introduced in §2.3 is recaptured by the notion of pointwise right extension of the diagram $f: A \to C$ along $!: A \to 1$. With this result in hand, we may now define limits and colimits of diagrams of ∞-categories in arbitrary ∞-cosmoi, not necessarily cartesian closed, as pointwise right or left extensions, respectively. We then generalize these definitions further to define a notion of limit or colimit for a diagram weighted by a module $A \overset{E}{\nrightarrow} B$.

13.6.1. Definition. A **limit** of a diagram of ∞-categories $f: A \to C$ is given by a pointwise right extension as below-left

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ 1 \arrow[ur, dashed, "\lim f"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\rho"] & C \\ 1 \arrow[ur, dashed, "\mathrm{colim}\, f"'] \end{tikzcd}$$

while a **colimit** of a diagram of ∞-categories $f: A \to C$ is given by a pointwise left extension as above-right.

The notions of weighted limit and colimit generalize Definition 13.6.1 — see Exercise 13.6.i — even though they are defined in the virtual equipment of modules, rather than in the ∞-cosmos itself.

13.6.2. Definition. Given a module $A \overset{W}{\nrightarrow} B$ and a functor $f: A \to C$, a functor $\lim_W f: B \to C$ defines the W-**weighted limit** of f if it covariantly represents the right extension of $\mathrm{Hom}_C(C, f)$ along W:

$$\begin{tikzcd} A \arrow[r, "W", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, dashed, "\mathrm{Hom}_C(C, \lim_W f)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ A \arrow[rr, "\mathrm{Hom}_C(C,f)"', "\shortmid"{marking}] & & C \end{tikzcd}$$

Dually, given a module $B \overset{V}{\nrightarrow} A$ and a functor $f\colon A \to C$, a functor $\mathsf{colim}_V f\colon B \to C$ defines the V-**weighed colimit** of f if it contravariantly represents the right lifting of $\mathsf{Hom}_C(f, C)$ through V:

$$\begin{tikzcd} C \arrow[r, dashed, "\mathsf{Hom}_C(\mathrm{colim}_V f, C)"] \arrow[d, equal] & B \arrow[r, "V"] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, equal] \\ C \arrow[rr, "\mathsf{Hom}_C(f, C)"'] & & A \end{tikzcd}$$

By Proposition 13.1.5 we have immediately:

13.6.3. PROPOSITION. *For any functor $f\colon A \to E$ and weights $A \overset{W}{\nrightarrow} B$ and $B \overset{V}{\nrightarrow} C$, the functors $\lim_{W\otimes V} f$ and $\lim_V(\lim_W f)$ are isomorphic whenever these weighted limits and the composite weight exist.* ☐

The following lemma can be used to compute the values of the weighted limit functors $\lim_W f\colon B \to C$ or $\mathsf{colim}_V f\colon B \to C$ at a generalized element $b\colon X \to B$.

13.6.4. LEMMA. *If $f\colon A \to C$ admits a limit $\lim_W f\colon B \to C$ weighted by $A \overset{W}{\nrightarrow} B$ then for any generalized element $b\colon X \to B$, the functor $\lim_W f \circ b\colon X \to C$ is isomorphic to the limit of f weighted by $A \overset{W(b,1)}{\nrightarrow} X$.*

PROOF. Recall from Proposition 12.4.7 that the module $W(b, 1)$ is the composite $W \otimes \mathsf{Hom}_B(b, B)$. Thus, by Proposition 13.1.5, the $W(b, 1)$-weighted limit of f, if it exists, is the functor that right-represents the right extension of the module $\mathsf{Hom}_C(C, \lim_W f)$ along the module $\mathsf{Hom}_B(b, B)$. By Lemma 13.1.4, the right extension of $\mathsf{Hom}_C(C, \lim_W f)$ along $\mathsf{Hom}_B(b, B)$ is given by the restriction $\mathsf{Hom}_C(C, \lim_W f \circ b)$. By Definition 13.6.2, this tells us that $\lim_W f \circ b\colon X \to C$ defines the $W(b, 1)$-weighted limit of $f\colon A \to C$, as claimed. ☐

13.6.5. REMARK. We can use weighted limits to re-express stability of pointwise right extensions under pasting with comma squares

$$\begin{tikzcd} \mathsf{Hom}_B(b, k) \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_0"'] \arrow[dr, phantom, "\nearrow\phi"] & A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\Uparrow\nu"] & C \\ X \arrow[r, "b"'] & B \arrow[ur, "\mathrm{ran}_k f"'] & \end{tikzcd}$$

By Definition 13.6.1, the pointwise right extension $\mathrm{ran}_k f\colon B \to C$ encodes the weighted limit functor $\lim_{\mathsf{Hom}_B(B,k)} f\colon B \to C$; this spoils Exercise 13.6.ii. Now by Lemma 13.6.4, the composite

$$\mathrm{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f,$$

which is to say the value of the pointwise right extension at a generalized element $b\colon X \to B$ is the limit of $f\colon A \to C$ weighted by the module $A \overset{\mathsf{Hom}_B(b,k)}{\nrightarrow} X$. By Proposition 12.3.7, the module $\mathsf{Hom}_B(b, k)$ is the composite $\mathsf{Hom}_A(p_1, A) \otimes \mathsf{Hom}_X(X, p_0)$ of the modules contravariantly and covariantly represented by its legs. Thus

$$\mathrm{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f \cong \lim_{\mathsf{Hom}_X(X,p_0)} \big(\lim_{\mathsf{Hom}_A(p_1,A)} f\big).$$

Finally Lemma 13.1.4 allows us to compute the limit of $f\colon A \to C$ weighted by $\mathsf{Hom}_A(p_1, A)$; this is given by the composite functor $fp_1\colon \mathsf{Hom}_B(b, k) \to C$. So

$$\mathrm{ran}_k f \circ b \cong \lim_{\mathsf{Hom}_B(B,k)} f \cong \lim_{\mathsf{Hom}_B(b,k)} f \cong \lim_{\mathsf{Hom}_X(X,p_0)} \big(\lim_{\mathsf{Hom}_A(p_1,A)} f\big) \cong \lim_{\mathsf{Hom}_X(X,p_0)} (fp_1) \cong \mathrm{ran}_{p_0}(fp_1).$$

Exercises.

13.6.i. Exercise. Determine weights W_{lim} and W_{colim} so that the W_{lim}-weighted limit of a diagram f is the ordinary limit of f, as defined in 13.6.1, and the W_{colim}-weighted colimit is the ordinary colimit.

13.6.ii. Exercise. Express pointwise right extensions and pointwise left extensions as weighted limits and weighted colimits, respectively.

Part IV

Change of model and model independence

CHAPTER 14

Cosmological biequivalences

In this chapter we study a certain class of cosmological functors between ∞-cosmoi that do not merely preserve ∞-categorical structure but also reflect and create it. We refer to these functors as *biequivalences* because the 2-functors they induce between homotopy 2-categories are biequivalences: surjective on objects up to equivalence and defining a local equivalence of hom-categories. Informally, we refer to cosmological biequivalences as "change-of-model functors." For example, the four ∞-cosmoi introduced in Example 1.2.21 whose objects model $(\infty, 1)$-categories are connected by the following biequivalences briefly described in Example 1.3.8:

$$\begin{tikzcd}
\mathcal{CSS} \arrow[rr, "\text{discretization}"] \arrow[dr, shift left, "\text{row}_0"] & & \mathcal{S}egal \arrow[dl, shift left, "\text{row}_0"] \\
& \mathcal{QC}at \arrow[ul, shift left, "\text{cylinder}"] \arrow[ur, shift left, "\text{prism}"] \arrow[d, shift left, "\natural"] & \\
& 1\text{-}\mathcal{C}omp \arrow[u, shift left, "U"] &
\end{tikzcd} \tag{14.0.1}$$

In §14.1, we collect together a number of results about cosmological functors that are scattered throughout the text. In §14.2, we introduce the special class of biequivalences and discuss several examples. Then in §14.3 we establish the basic 2-categorical properties of biequivalences, which will provide the essential ingredient in the proof of the model-independence results in Chapter 15.

14.1. Cosmological functors

Recall from Definition 1.3.1 that a **cosmological functor** is a simplicial functor $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi that preserves

- the specified classes of isofibrations and
- all of the simplicial limits enumerated in 1.2.1(i).

By Lemma 1.3.2, it follows that cosmological functors also preserves the equivalences and the trivial fibrations. By Corollary 7.3.3, cosmological functors preserve all flexible weighted limits, since Proposition 7.3.1 reveals that these can be constructed out of simplicial limits of diagrams of isofibrations of the form listed in 1.2.1(i).

Our aim in this section is to show that cosmological functors preserve all of the ∞-categorical structures we have introduced. In many cases this will not be evident from the original 2-categorical definitions (eg of cartesian fibrations in Definition 5.1.6) but can be deduced rather easier from the accompanying "internal characterization" of each categorical notion (such as given in Theorem 5.1.11(iii)).

First, we orient ourselves to the main examples. Recall Proposition 1.3.3:

14.1.1. Example (cosmological functors).

(i) The nerve embedding defines a cosmological functor $\mathcal{C}at \hookrightarrow \mathcal{QC}at$.

(ii) For any object A in an ∞-cosmos $\mathcal{K}$,

$$\mathsf{Fun}(A,-)\colon \mathcal{K} \to \mathcal{QC}at$$

defines a cosmological functor.

(iii) In particular, each ∞-cosmos has an **underlying quasi-category** functor

$$(-)_0 \coloneqq \mathsf{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at,$$

defined by mapping out of the terminal ∞-category. We use the notation "$(-)_0$" for this functor because the underlying quasi-categories of a complete Segal space or Segal category displayed in (14.0.1) are defined by evaluating these simplicial objects at 0 and because this notation for an "underlying category" is commonly used in enriched category theory (see Definition A.1.5).

(iv) For any ∞-cosmos $\mathcal{K}$ and any simplicial set U, the simplicial cotensor defines a cosmological functor

$$(-)^U\colon \mathcal{K} \to \mathcal{K}.$$

(v) For any object A in a cartesian closed ∞-cosmos $\mathcal{K}$, exponentiation defines a cosmological functor

$$(-)^A\colon \mathcal{K} \to \mathcal{K}.$$

(vi) For any map $f\colon A \to B$ in an ∞-cosmos $\mathcal{K}$, pullback defines a cosmological functor

$$f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}.$$

In the special case of the map $!\colon A \to 1$, the pullback cosmological functor has the form

$$- \times A\colon \mathcal{K} \to \mathcal{K}_{/A}.$$

(vii) For any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map on slices

$$F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$$

defines a cosmological functor.

14.1.2. Example. The inclusion of a replete sub ∞-cosmos described in Proposition 8.2.3 is a cosmological functor. Examples include:

(i) The full subcategory of discrete objects

$$\mathcal{D}isc(\mathcal{K}) \hookrightarrow \mathcal{K}$$

defines a replete sub ∞-cosmos; see Proposition 1.2.25.

(ii) The inclusions

$$\mathcal{K}_\top \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_\perp$$

of the ∞-cosmoi of ∞-categories possessing and functors preserving a terminal or initial element. More generally, the inclusions

$$\mathcal{K}_{\top,J} \hookrightarrow \mathcal{K} \hookleftarrow \mathcal{K}_{\perp,J}$$

of the ∞-cosmoi of ∞-categories possessing and functors preserving limits or colimits of shape J; see Propositions 8.2.4 and 8.2.11.

(iii) The inclusions

$$\mathscr{R}ari(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow \mathcal{L}ari(\mathcal{K})_{/B}$$

or

$$\mathscr{R}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{2} \hookleftarrow \mathcal{L}ari(\mathcal{K})$$

of ∞-cosmoi whose objects are isofibrations admitting a right or left adjoint right inverse; see Corollary 8.2.7 and Proposition 8.2.9.

(iv) The inclusions

$$\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow co\mathcal{C}art(\mathcal{K})_{/B}$$

or

$$\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{2} \hookleftarrow co\mathcal{C}art(\mathcal{K})$$

of the ∞-cosmoi of cartesian or cartesian fibrations and cartesian functors between them; see Proposition 8.2.12.

14.1.3. Observation. Inclusions $\mathcal{K}' \hookrightarrow \mathcal{K}$ and $\mathcal{L}' \hookrightarrow \mathcal{L}$ of replete sub ∞-cosmoi create isofibrations and the simplicial limits of axiom 1.2.1(i). As these inclusions are full on positive-dimensional arrows, a simplicial functor

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] & \mathcal{L} \\ \mathcal{K}' \arrow[u, hook] \arrow[r, dashed, "F"] & \mathcal{L}' \arrow[u, hook] \end{tikzcd}$$

restricts to a simplicial functor on the subcategories if and only if it carries the objects and 0-arrows of the subcategory $\mathcal{K}'$ into the objects and 0-arrows of $\mathcal{L}'$, as will be the case whenever the replete sub ∞-cosmoi are "naturally defined" by the same ∞-categorical property. Whenever this is the case, the restricted functor is cosmological.

For example, by Proposition 5.1.20 and Exercise 5.1.v, pullback $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves cartesian fibrations and cartesian functors. Thus, pullback along any functor $f\colon A \to B$ in $\mathcal{K}$ restricts to define a cosmological functor:

$$\begin{tikzcd} \mathcal{K}_{/B} \arrow[r, "f^*"] & \mathcal{K}_{/A} \\ \mathcal{C}art(\mathcal{K})_{/B} \arrow[u, hook] \arrow[r, dashed, "f^*"] & \mathcal{C}art(\mathcal{K})_{/A} \arrow[u, hook] \end{tikzcd}$$

14.1.4. Example. There are also cosmological functors connecting the ∞-cosmoi of "Reedy fibrant diagrams."

(i) By Proposition 8.1.1, the domain, codomain, and identity functors

$$\begin{tikzcd} \mathcal{K}^{2} \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{K} \arrow[l, "\mathrm{id}" description] \end{tikzcd}$$

are all cosmological.

(ii) By Exercise 12.1.i, the domain and codomain functors

$$\begin{tikzcd} \mathcal{K}^{\lrcorner} \arrow[r, shift left, "\mathrm{dom}"] \arrow[r, shift right, "\mathrm{cod}"'] & \mathcal{K} \end{tikzcd}$$

from the ∞-cosmos of two-sided isofibrations are cosmological.

Cosmological functors may fail dramatically to be surjective on objects — $\mathrm{id}\colon \mathcal{K} \to \mathcal{K}^{\mathbb{2}}$ comes to mind. Thus a 2-categorical property involving a universal quantifier is not obviously preserved. For instance, by Definition 1.2.24 an object $E \in \mathcal{K}$ is **discrete** just when for all $X \in \mathcal{K}$, $\mathsf{Fun}(X, E)$ is a Kan complex. However, by Exercise 1.2.iv, which we belatedly solve, discrete objects admit an internal characterization given in (iv) below:

14.1.5. PROPOSITION. *An object E in an ∞-cosmos $\mathcal{K}$ is* discrete *if the following equivalent conditions are satisfied:*

(i) E is a discrete object in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, that is, every 2-cell with codomain E is invertible.
(ii) For each $X \in \mathcal{K}$, the hom-category $\mathsf{hFun}(X, E)$ is a groupoid.
(iii) For each $X \in \mathcal{K}$, the mapping quasi-category $\mathsf{Fun}(X, E)$ is a Kan complex.
(iv) The isofibration $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$, induced by the inclusion of simplicial sets $\mathbb{2} \hookrightarrow \mathbb{I}$, is a trivial fibration.

PROOF. Here (ii) is an unpacking of (i). The equivalence of (ii) and (iii) is a well-known result of Joyal [**48**, 1.4] reproduced in Corollary 1.1.15. Condition (iv) is equivalent to the assertion that $\mathsf{Fun}(X, E)^{\mathbb{I}} \twoheadrightarrow \mathsf{Fun}(X, E)^{\mathbb{2}}$ is a trivial fibration between quasi-categories for all X. If this is a trivial fibration, then surjectivity on vertices implies that every 1-simplex in $\mathsf{Fun}(X, E)$ is an isomorphism, proving (iii). By Exercise 1.1.iv, $\mathbb{2} \hookrightarrow \mathbb{I}$ is in the class of maps cellularly generated by the outer horn inclusions, so (iii) implies (iv). ☐

Note that if E is discrete, then any equivalent object E' is also discrete. Now the result we want is easy to establish.

14.1.6. COROLLARY. *Cosmological functors preserve discrete objects.*

PROOF. If $E \in \mathcal{K}$ is discrete and $F\colon \mathcal{K} \to \mathcal{L}$ is cosmological, then

$$F(E^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} E^{\mathbb{2}}) \cong (FE)^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} (FE)^{\mathbb{2}}$$

is a trivial fibration in $\mathcal{L}$, proving that FE is discrete by Proposition 14.1.5(iv). ☐

Importantly:

14.1.7. PROPOSITION. *Cosmological functors preserve comma ∞-categories: if $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological functor and the diagram below-left*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ C \arrow[dr, "g"'] & {} & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} & FE \arrow[dl, "Fe_1"'] \arrow[dr, "Fe_0"] \arrow[d, phantom, "\overset{F\epsilon}{\Leftarrow}"] & \\ FC \arrow[dr, "Fg"'] & {} & FB \arrow[dl, "Ff"] \\ & FA & \end{tikzcd} \tag{14.1.8}$$

is a comma cone in $\mathcal{K}$, then the diagram above-right is a comma cone in $\mathcal{L}$.

PROOF. The simplicial pullback (3.4.2) that constructs the comma cone is preserved by any cosmological functor. By Proposition 3.4.11, any comma cone as above left arises from a fibered equivalence $e\colon E \simeq \mathsf{Hom}_A(f, g)$ where $\epsilon = \phi e$, and any fibered equivalence of this form defines a comma cone. Since $Fe\colon FE \simeq F(\mathsf{Hom}_A(f, g)) \cong \mathsf{Hom}_{FA}(Ff, Fg)$, we conclude that the right-hand data again defines a comma cone. ☐

Using Proposition 14.1.7, we can quickly establish the following preservation properties of cosmological functors by arguing along similar lines to Corollary 14.1.6. For ease of reference, we include on this list the preservation properties that have been previously established.

14.1.9. Proposition. *Cosmological functors preserve:*

(i) *Equivalences between ∞-categories.*
(ii) *Adjunctions between ∞-categories, including right adjoint right inverse adjunctions and left adjoint right inverse adjunctions.*
(iii) *Invertible 2-cells and mates.*
(iv) *Homotopy coherent adjunctions and monads.*
(v) *Trivial fibrations and discrete objects.*
(vi) *Flexible weighted limits.*
(vii) *Comma spans and comma cones.*
(viii) *Absolute right and left lifting diagrams.*
(ix) *Limits or colimits of diagrams indexed by a simplicial set J inside an ∞-category.*
(x) *Cartesian and cocartesian fibrations and cartesian functors between them.*
(xi) *Discrete cartesian fibrations and discrete cocartesian fibrations.*
(xii) *Two-sided fibrations and cartesian functors between them and also modules.*
(xiii) *Modules and represented modules.*

Proof. Lemma 1.3.2 proves that cosmological functors preserve equivalences between ∞-categories since these can be characterized as 2-categorical equivalences, which are preserved by any 2-functor. Similarly, as observed in Lemma 2.1.3, adjunctions are preserved by any 2-functor, as are invertible 2-cells and mates. In the same way, homotopy coherent adjunctions are preserved by any simplicial functor. So the preservation properties (ii), (iii) and (iv) hold under much weaker hypotheses.

The preservation of trivial fibrations was also established in Lemma 1.3.2 and Corollary 14.1.6 uses this to prove that discrete objects are also preserved, proving (v). Corollary 7.3.3 proves that cosmological functors preserve all flexible weighted limits as stated in (vi). Proposition 14.1.7 proves that cosmological functors preserve comma spans in the ∞-cosmos and comma cones in the homotopy 2-category as stated in (vii).

The preservation property (viii) follows from Theorem 3.5.3, which characterizes absolute lifting diagrams as fibered equivalences of comma ∞-categories, Proposition 14.1.7, which says that cosmological functors preserve commas, and the fact that cosmological functors preserve equivalences. Now (ix) follows from this by Definition 2.3.7 and the fact that cosmological functors preserve simplicial tensors.

The preservation properties (x) and (xi) also follow from Proposition 14.1.7 and the fact that cosmological functors preserve right or left adjoint right inverse adjunctions, mates, and trivial fibrations via the characterizations of Theorem 5.1.11(iii), Theorem 5.1.19(iii), and Proposition 5.4.6. More details are given in the proof of Corollary 5.1.16, which also observes that cartesian natural transformations are preserved by cosmological functors.

Directly from the internal characterization of Theorem 11.1.4 and the preservation of adjunctions and invertible 2-cells, cartesian functors preserve two-sided fibrations; preservation of cartesian functors between them is left to Exercise 14.1.i. This establishes (xii). By Proposition 1.3.3(vi), a cosmological functor induces a direct image cosmological functor between sliced ∞-cosmoi, which then preserves discrete objects by Corollary 14.1.6. Thus, modules are also preserved. By specializing Proposition 14.1.7 to cospans involving identities, it becomes clear that left and right representable commas

are preserved. Since a module is representable if and only if it is fibered equivalent to one of these, representable modules are preserved as well, completing the proof of (xiii). □

14.1.10. Non-Example. The composite of the underlying quasi-category functor with the homotopy category functor

$$\mathcal{K} \xrightarrow{(-)_0} \mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at$$

defines a simplicial functor that carries an ∞-category A to its homotopy category $\mathsf{h}A$ first introduced in Definition 1.4.12. This functor does preserve isofibrations but does not preserve simplicial cotensors and so is not cosmological. This is a good thing because if it were Proposition 14.1.9(ix) would imply that limits in an ∞-category would also define limits in its homotopy category, which is not generally true.

Exercises.

14.1.i. Exercise. Prove that any cosmological functor preserves cartesian functors between two-sided fibrations. Conclude that any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a cosmological functor $F\colon {}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{F}ib(\mathcal{L})_{/FB}$.

14.2. Cosmological biequivalences as change-of-model functors

Special cosmological functors, called *biequivalences*, preserve and also reflect the category theory developed in this text. Recall Definition 1.3.6:

14.2.1. Definition (cosmological biequivalences). A cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ is a **biequivalence** when it is:

(i) surjective on objects up to equivalence: i.e., if for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$; and
(ii) a local equivalence of quasi-categories: i.e., if for every pair $A, B \in \mathcal{K}$, the map

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$$

is an equivalence of quasi-categories.

More generally, two ∞-cosmoi are **biequivalent** if there exists a finite zig-zag of biequivalences connecting them.

14.2.2. Example (biequivalences between ∞-cosmoi of $(\infty,1)$-categories). In Appendix E, we will show that the functors (14.0.1) are all biequivalences. Except for the functor $\natural\colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$, each of the functors (14.0.1) arises from a simplicially enriched right Quillen equivalence between model categories enriched over the Joyal model structure with all objects cofibrant. Functors of this form are easily seen to encode cosmological biequivalences.

Recall that any ∞-cosmos has an **underlying quasi-category functor**

$$\mathcal{K} \xrightarrow{(-)_0 := \mathsf{Fun}(1,-)} \mathcal{QC}at$$

defined by mapping out of the terminal ∞-category. We now show that for every ∞-cosmos that is biequivalent to $\mathcal{QC}at$, its underlying quasi-category functor is a biequivalence.

14.2.3. Proposition. *If an ∞-cosmos $\mathcal{K}$ is biequivalent to $\mathcal{QC}at$, then the underlying quasi-category functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is a cosmological biequivalence.*

Proof. We will show that in the presence of cosmological biequivalences

$$\mathcal{K} \xleftarrow[\sim]{G} \mathcal{L} \xrightarrow[\sim]{F} \mathcal{Q}\mathcal{C}at$$

the underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{Q}\mathcal{C}at$ is a cosmological biequivalence. By induction, perhaps taking one of these functors to be the identity, the same conclusion holds for any ∞-cosmos connected by a finite zig-zag of biequivalences to $\mathcal{Q}\mathcal{C}at$.

As F is a biequivalence, for each quasi-category Q, there exists an ∞-category $B \in \mathcal{L}$ so that $FB \simeq Q$. Because F and G are both local equivalences preserving the terminal ∞-category 1 (for which we adopt the same notation in each of $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{Q}\mathcal{C}at$), there is then a zig-zag of equivalences of quasi-categories

$$(GB)_0 = \mathsf{Fun}(1, GB) \xleftarrow{\sim} \mathsf{Fun}(1, B) \xrightarrow{\sim} \mathsf{Fun}(1, FB) \cong FB \simeq Q.$$

This proves that there exists an ∞-category $GB \in \mathcal{K}$ whose underlying quasi-category is equivalent to Q.

To show that the underlying quasi-category functor $(-)_0 \colon \mathcal{K} \to \mathcal{Q}\mathcal{C}at$ is a local equivalence, consider a pair of ∞-categories $A, B \in \mathcal{K}$. By essential surjectivity of G, there exist ∞-categories $X, Y \in \mathcal{L}$ so that $GX \simeq A$ and $B \simeq GY$. By pre- and post-composing with these equivalences, Corollary 1.4.9 implies that $\mathsf{Fun}(A, B) \to \mathsf{Fun}(A_0, B_0)$ is equivalent to $\mathsf{Fun}(GX, GY) \to \mathsf{Fun}((GX)_0, (GY)_0)$, so it suffices to prove that the latter map is an equivalence of quasi-categories.

By simplicial functoriality of the local equivalences F and G, there is a commutative diagram whose vertical maps are equivalences

$$\begin{tikzcd}
\mathsf{Fun}(GX, GY) \arrow[r] & \mathsf{Fun}(\mathsf{Fun}(1, GX), \mathsf{Fun}(1, GY)) \arrow[d, "\wr"] \\
 & \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, GY)) \\
\mathsf{Fun}(X, Y) \arrow[uu, "\wr"] \arrow[dd, "\wr"'] \arrow[r] & \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, Y)) \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
 & \mathsf{Fun}(\mathsf{Fun}(1, X), \mathsf{Fun}(1, FY)) \\
\mathsf{Fun}(FX, FY) \arrow[r, "\cong"] & \mathsf{Fun}(\mathsf{Fun}(1, FX), \mathsf{Fun}(1, FY)) \arrow[u, "\wr"']
\end{tikzcd}$$

Any quasi-category is isomorphic to its underlying quasi-category, so the bottom horizontal map is an isomorphism. By the 2-of-3 property, it follows that the top horizontal map is an equivalence, which is what we wanted to show. □

Recall from Proposition 1.3.3(vi) that a cosmological functor $F \colon \mathcal{K} \to \mathcal{L}$ induces a cosmological functor $F \colon \mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ for any $B \in \mathcal{K}$.

14.2.4. Proposition. *If $F \colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence, then for any $B \in \mathcal{K}$ the induced functor $F \colon \mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ is also a cosmological biequivalence.*

Proof. We first argue that the F defines a local equivalence of sliced mapping quasi-categories, as defined in Proposition 1.2.19(ii). Given a pair of isofibration $p \colon E \twoheadrightarrow B$ and $p' \colon E' \twoheadrightarrow B$ in $\mathcal{K}$, the

induced functor on mapping quasi-categories is defined by

$$\begin{tikzcd}
\mathsf{Fun}_B(p,p') \arrow[rr] \arrow[dd, two heads] \arrow[dr, dashed, "\sim"] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathsf{Fun}(E,E') \arrow[dd, two heads] \arrow[dr, "\sim"] & \\
& \mathsf{Fun}_{FB}(Fp,Fp') \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathsf{Fun}(FE,FE') \arrow[dd, two heads] \\
1 \arrow[rr] \arrow[dr, equal] & & \mathsf{Fun}(E,B) \arrow[dr, "\sim"] & \\
& 1 \arrow[rr] \arrow[from=uu, crossing over, two heads] & & \mathsf{Fun}(FE,FB)
\end{tikzcd}$$

As the maps between the cospans in $\mathcal{QC}at$ are equivalences, so is the induced map between the pullbacks.

For surjectivity up to equivalence, consider an isofibration $q\colon L \twoheadrightarrow FB$ in $\mathcal{L}$. As F is surjective on objects up to equivalence, there exists some $A \in \mathcal{K}$ together with an equivalence $i\colon FA \xrightarrow{\sim} L \in \mathcal{L}$. As F defines a local equivalence of mapping quasi-categories, there is moreover a functor $f\colon A \to B$ in $\mathcal{L}$ so that $Ff\colon FA \to FB$ is isomorphic to qi in $\mathfrak{h}\mathcal{L}$. The map f need not be an isofibration, but Lemma 1.2.13 allows us to factor f as $A \xrightarrow{\sim} K \overset{p}{\twoheadrightarrow} B$. Choosing an equivalence inverse $j\colon K \xrightarrow{\sim} A$, this data defines a diagram in $\mathfrak{h}\mathcal{L}$ that commutes up to isomorphism:

$$\begin{tikzcd}
FK \arrow[r, "Fj", "\sim"'] \arrow[dr, two heads, "Fp"'] \arrow[dr, phantom, shift left=3, "\cong"] & FA \arrow[r, "i", "\sim"'] \arrow[d, dashed, "Ff"] & L \arrow[dl, two heads, "q"] \arrow[dl, phantom, shift right=3, "\cong"'] \\
& FB &
\end{tikzcd}$$

Proposition 1.4.10 tells us that isofibrations in ∞-cosmoi define isofibrations in the homotopy 2-category. In particular, we may lift the displayed isomorphism along the isofibration $q\colon L \twoheadrightarrow FB$ to define a commutative triangle:

$$\begin{tikzcd}
FK \arrow[r, "Fj", "\sim"'] \arrow[dr, two heads, "Fp"'] \arrow[dr, phantom, shift left=3, "\cong"] & FA \arrow[r, "i", "\sim"'] \arrow[d, dashed, "Ff"] & L \arrow[dl, two heads, "q"] \arrow[dl, phantom, shift right=3, "\cong"'] \\
& FB &
\end{tikzcd}
=
\begin{tikzcd}
FK \arrow[rr, bend left, "i\cdot Fj", "\sim"'] \arrow[rr, bend right, "\sim", "e"'] \arrow[rr, phantom, "\cong"] \arrow[dr, two heads, "Fp"'] & & L \arrow[dl, two heads, "q"] \\
& FB &
\end{tikzcd}$$

As noted in the proof of Theorem 1.4.7, since e is isomorphic to an equivalence $i \cdot Fj$, it must also define an equivalence. Thus, by Proposition 1.2.19(vii), we have shown that the isofibration $p\colon K \twoheadrightarrow B$ maps under F to an isofibration that is equivalent to our chosen $q\colon L \twoheadrightarrow FB$. ☐

A similar argument proves that a cosmological biequivalence induces a biequivalence between the corresponding ∞-cosmoi of isofibrations of Proposition 8.1.1.

14.2.5. Proposition. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then the induced functor $F\colon \mathcal{K}^{\mathbb{2}} \to \mathcal{L}^{\mathbb{2}}$ is a biequivalence.*

Proof. Exercise 14.2.i. ☐

We establish another family of biequivalences of sliced ∞-cosmoi:

14.2.6. Proposition. *If $f\colon A \xrightarrow{\sim} B$ is an equivalence in $\mathcal{K}$, then the pullback functor $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ is a cosmological biequivalence.*

Proof. To see that $f^* \colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ is essentially surjective, consider an object $r \colon D \twoheadrightarrow A$, factor the composite map $fr \colon D \to B$ as an equivalence followed by an isofibration, and pull the result back along f.

$$\begin{tikzcd}
D \arrow[drr, bend left, "\sim"] \arrow[dr, dashed] \arrow[ddr, bend right, "r"'] & & \\
& P \arrow[r, "\sim"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\
& A \arrow[r, "\sim", "f"'] & B
\end{tikzcd}$$

By Lemma 3.3.2, the pullback of f is an equivalence, so by the 2-of-3 property, r is equivalent to the isofibration $q \colon P \twoheadrightarrow A$, which is in the image of $f^* \colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$.

To show that this simplicial functor is a local equivalence, consider a pair of isofibrations $p \colon E \twoheadrightarrow B$ and $q \colon F \twoheadrightarrow B$. We will show that the quasi-category of functors over B is equivalent to the quasi-category of functors over A between their pullbacks

$$\begin{tikzcd}
f^*E \arrow[rr, "h", "\sim"'] \arrow[dd, two heads, "r"'] \arrow[ddrr, phantom, "\lrcorner" very near start] & & E \arrow[dd, two heads, "p"] & \\
& f^*F \arrow[rr, "k", "\sim"'] \arrow[dl, two heads, "s"'] \arrow[ddrr, phantom, "\lrcorner" very near start] & & F \arrow[dl, two heads, "q"] \\
A \arrow[rr, "\sim", "f"'] & & B &
\end{tikzcd}$$

To define the comparison map $\mathsf{Fun}_B(E, F) \to \mathsf{Fun}_A(f^*E, f^*F)$ consider the following commutative prism

$$\begin{tikzcd}
& \mathsf{Fun}_B(E,F) \arrow[rrr] \arrow[dl, dashed, "\sim"'] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & & \mathsf{Fun}(E,F) \arrow[dl, "h^*"', "\sim"] \arrow[dd, two heads, "q_*"] \\
\mathsf{Fun}_A(f^*E, f^*F) \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & & \mathsf{Fun}(f^*E, f^*F) \arrow[r, "k_*", "\sim"'] \arrow[dd, two heads, "s_*"] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(f^*E, F) \arrow[dd, two heads, "q_*"] & \\
& \mathbb{1} \arrow[rrr, "p"] & & & \mathsf{Fun}(E,B) \arrow[dl, "\sim", "h^*"'] \\
\mathbb{1} \arrow[ur, equal] \arrow[rr, "r"'] & & \mathsf{Fun}(f^*E, A) \arrow[r, "\sim", "f_*"'] & \mathsf{Fun}(f^*E, B) &
\end{tikzcd}$$

The front-right square is a pullback by the simplicial universal property of f^*F, while the front-left square and back face are the pullbacks that define $\mathsf{Fun}_A(f^*E, f^*F)$ and $\mathsf{Fun}_B(E, F)$. The universal property of the composite front pullback rectangle induces the map $\mathsf{Fun}_B(E, F) \to \mathsf{Fun}_A(f^*E, f^*F)$. As this functor is the pullback of the equivalences h^* of Corollary 1.4.9, the induced map defines an equivalence of quasi-categories. □

Exercises.

14.2.i. Exercise. Prove Proposition 14.2.5.

14.2.ii. Exercise. If $F \colon \mathcal{K} \rightsquigarrow \mathcal{L}$ is a biequivalence and $A \in \mathcal{K}$ and $B \in \mathcal{L}$ are so that $FA \simeq B$ prove that the slice ∞-cosmoi $\mathcal{K}_{/A}$ and $\mathcal{L}_{/B}$ are biequivalent.

14.3. Properties of change-of-model functors

We refer to biequivalence between ∞-cosmoi as *change-of-model functors*. In this section, we enumerate their basic properties. First, we observe that cosmological biequivalences descend to biequivalences between homotopy 2-categories, hence the name:

14.3.1. Proposition. *A cosmological biequivalence $F\colon \mathcal{K} \to \mathcal{L}$ induces a biequivalence $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ of homotopy 2-categories: i.e., the 2-functor F is*

(i) surjective on objects up to equivalence and

(ii) defines a local equivalence of categories $\mathsf{hFun}(A,B) \xrightarrow{\sim} \mathsf{hFun}(FA,FB)$ for all $A,B \in \mathcal{K}$.

Proof. This is immediate from the definition: Theorem 1.4.7 observed that ∞-cosmos-level equivalences coincide with 2-categorical equivalences, and by Lemma 1.2.15 the homotopy category functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries equivalences of quasi-categories to equivalences of categories. $\square$

In particular, recalling Definition 1.4.12, it follows immediately that for every ∞-category $A \in \mathcal{K}$, that the biequivalence F induces an equivalence of homotopy categories $\mathsf{h}A \xrightarrow{\sim} \mathsf{h}FA$; see Exercise 14.3.i.

Any biequivalence between 2-categories induces a variety of local and global bijections, as enumerated below, which can be put to use to solve Exercise 14.3.ii.

14.3.2. Corollary. *Consider any cosmological biequivalence $F\colon \mathcal{K} \to \mathcal{L}$.*

(i) The biequivalence F preserves, reflects, and creates equivalences between ∞-categories, and induces a bijection between equivalence classes of objects.

(ii) The biequivalence F induces local bijections between isomorphism classes of functors extending the bijection of (i): choosing any pairs of objects $A,B \in \mathcal{K}$ and $A',B' \in \mathcal{L}$ and equivalences $FA \simeq A'$ and $FB \simeq B'$, the map

$$\mathsf{hFun}(A,B) \xrightarrow{\sim} \mathsf{hFun}(FA,FB) \xrightarrow{\sim} \mathsf{hFun}(A',B') \tag{14.3.3}$$

defines a bijection between isomorphism classes of functors $A \to B$ in $\mathcal{K}$ and isomorphism classes of functors $A' \to B'$ in $\mathcal{L}$.

(iii) The biequivalence F induces local bijections between 2-cells with specified boundary extending the bijections of (i) and (ii): choosing any pairs of objects $A,B \in \mathcal{K}$ and $A',B' \in \mathcal{L}$, equivalences $FA \simeq A'$ and $FB \simeq B'$, functors $f,g\colon A \rightrightarrows B$ and $f',g'\colon A' \rightrightarrows B'$, and natural isomorphisms

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\alpha"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Fg"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\cong\beta"] & FB \arrow[d, "\wr"] \\ A' \arrow[r, "g'"'] & B' \end{tikzcd}$$

the map (14.3.3) *induces a bijection between 2-cells $f \Rightarrow g$ in $\mathcal{K}$ and 2-cells $f' \Rightarrow g'$ in $\mathcal{L}$.*

Proof. For (i), any 2-functor preserves equivalences, so our first take is to show that equivalences are also reflected. If $f\colon A \to B$ is a morphism so that $Ff\colon FA \xrightarrow{\sim} FB$ is an equivalence then by the 2-of-3 property and the commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(X,A) \arrow[r, "f_*"] \arrow[d, "\wr"'] & \mathsf{Fun}(X,B) \arrow[d, "\wr"] \\ \mathsf{Fun}(FX,FA) \arrow[r, "Ff_*", "\sim"'] & \mathsf{Fun}(FX,FB) \end{tikzcd}$$

f_* is an equivalence for all X, proving that $f\colon A \to B$ is an equivalence.

To prove that equivalences are also created — meaning that if $FA \simeq FB$ then $A \simeq B$ — note that the equivalence $\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$ implies that any morphism $FA \to FB$ has a lift $A \to B$ up to isomorphism. In this way, the morphisms of an equivalence $f\colon FA \xrightarrow{\sim} FB$ and $g\colon FB \xrightarrow{\sim} FA$ can be lifted to morphisms $\tilde{f}\colon A \to B$ and $\tilde{g}\colon B \to A$ so that $\tilde{g}\tilde{f}$ and id_A have isomorphic images under the equivalence $\mathsf{Fun}(A,A) \xrightarrow{\sim} \mathsf{Fun}(FA,FA)$. Since an equivalence of quasi-categories induces a bijection on isomorphism classes of vertices, we conclude that $\tilde{g}\tilde{f} \cong \mathrm{id}_A \in \mathsf{Fun}(A,A)$. Similarly, $\tilde{f}\tilde{g}$ and id_B have isomorphic images under $\mathsf{Fun}(B,B) \xrightarrow{\sim} \mathsf{Fun}(FB,FB)$ and thus $\tilde{f}\tilde{g} \cong \mathrm{id}_B$, proving that $A \simeq B$.

Finally, the preservation, reflection, and creation of equivalences together with essential surjectivity of a cosmological biequivalence implies that such functors induces a bijection on equivalence classes of objects.

For (ii), by Corollary 1.4.9, the chosen equivalences $FA \simeq A'$ and $FB \simeq B'$ induce an equivalence of quasi-categories

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB) \xrightarrow{\sim} \mathsf{Fun}(A',B')$$

which descends to the equivalence of homotopy categories (14.3.3). Since equivalences of quasi-categories induce bijections between isomorphism classes of vertices, this yields in particular a bijection between isomorphism classes of functors.

For (iii), the equivalence (14.3.3) is full and faithful, inducing a bijection between cells $f \Rightarrow g$ and $Ff \Rightarrow Fg$. This bijection can be transported along any chosen isomorphisms α and β to yield a bijection between natural transformations $f \Rightarrow g$ in $\mathsf{hFun}(A,B)$ in $\mathcal{K}$ and natural transformations $f' \Rightarrow g'$ in $\mathsf{hFun}(A',B')$ in $\mathcal{L}$. ☐

As an application of Corollary 14.3.2, we now show that the internal hom B^A between ∞-categories A and B in a cartesian closed ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$ is equivalent to the simplicial cotensor B^{A_0} of B with the underlying quasi-category of A. Even if $\mathcal{K}$ is not cartesian closed as an ∞-cosmos, on account of the biequivalence $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ of Proposition 14.2.3, its homotopy 2-category $\mathfrak{h}\mathcal{K}$ is cartesian closed in the bicategorical sense, replacing the natural isomorphisms of Proposition 1.4.5(ii) with natural equivalences: we define $B^A \in \mathcal{K}$ to be any ∞-category whose underlying quasi-category is $B_0^{A_0}$. By the 2-of-3 property for equivalences

$$\begin{tikzcd}
\mathsf{Fun}(X,B^A) \arrow[r, "\sim"] \arrow[d, dashed, "\wr"'] & \mathsf{Fun}(X_0,B_0^{A_0}) \arrow[d, "\cong"] \\
\mathsf{Fun}(X\times A,B) \arrow[r, "\sim"] & \mathsf{Fun}(X_0\times A_0,B_0)
\end{tikzcd}$$

we see that $\mathsf{Fun}(X,B^A) \simeq \mathsf{Fun}(X\times A,B)$ for any X. For this reason, the statement of Proposition 14.3.4 does not require that $\mathcal{K}$ is cartesian closed as an ∞-cosmos; the exponentials can be inferred to exist a posteriori.

14.3.4. Proposition. *Suppose $\mathcal{K}$ is an ∞-cosmos that is biequivalent to $\mathcal{QC}at$. Then for any ∞-categories $A,B \in \mathcal{K}$, the exponential $B^A \simeq B^{A_0}$ is equivalent to the cotensor of B with the underlying quasi-category of A.*

Proof. By Corollary 14.3.2(i), cosmological biequivalences reflect equivalences of ∞-categories. Thus, to prove $B^A \simeq B^{A_0}$, it suffices by Proposition 14.2.3 and Corollary 14.3.2 to prove that B^A and

B^{A_0} have equivalent underlying quasi-categories. The defining universal properties of the exponential and cotensor provide equivalences

$$\mathsf{Fun}(1, B^A) \simeq \mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(A_0, B_0) \cong \mathsf{Fun}(1, B^{A_0})$$

which compose with the local equivalence of the underlying quasi-category functor to provide the desired equivalence. ☐

We now prove that biequivalences reflect and create, as well as preserve, the ∞-categorical structures considered in Section 14.1.

14.3.5. PROPOSITION. *A biequivalence $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi:*

(i) Preserves and reflects invertibility of 2-cells.
(ii) Preserves, reflects, and creates adjunctions between ∞-categories, including right adjoint right inverse adjunctions and left adjoint right inverse adjunctions.
(iii) Preserves and reflects discreteness.
(iv) Preserves, reflects, and creates fibered equivalences.
(v) Preserves and reflects comma and arrow ∞-categories: a cell as on the left of (14.1.8) *is a comma cone in $\mathcal{K}$ if and only if its image is a comma cone in $\mathcal{L}$.*
(vi) Preserves and reflects cartesian and cocartesian fibrations and cartesian functors between them.
(vii) Preserves and reflects discrete cartesian fibrations and discrete cocartesian fibrations.
(viii) Preserves and reflects two-sided fibrations and cartesian functors between them.
(ix) Preserves and reflects modules and represented modules and induces a bijection on equivalence classes of modules between a fixed pair of ∞-categories.
(x) Preserves, reflects, and creates absolute right and left lifting diagrams over a given cospan.
(xi) Preserves and reflects limits or colimits of diagrams indexed by a simplicial set J inside an ∞-category and creates the property of an ∞-category in $\mathcal{K}$ admitting a limit or colimit of a given diagram.

PROOF. Properties (i) and (ii) hold for any biequivalence between 2-categories, such as $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ and are left to Exercise 14.3.ii as a useful exercise to familiarize oneself with the 2-categorical notion of biequivalence.

Property (iii) follows from (i) and Proposition 14.1.5(i): if FE is discrete, then the image under F of any 2-cell in $\mathcal{K}$ with codomain E is invertible, which implies that that 2-cell is invertible in E.

Property (iv) is a special case of Corollary 14.3.2(i) applied to the induced biequivalence of Proposition 14.2.4. Since both $\mathcal{K}$ and $\mathcal{L}$ admit comma ∞-categories and Proposition 3.4.11 shows that comma spans are characterized by a fibered equivalence class of two-sided isofibrations, (v) follows from (iv).

Proposition (vi) follows from (ii) and (i) and (vii) follows from Corollary 14.3.2(ii) which shows that invertibility of the morphism considered in Proposition 5.4.6 is reflected. Property (viii) follows similarly from Theorem 11.1.4(iii) and (ii) and (i). Preservation and reflection of modules now follows from (iii) and the bijection between equivalence classes follows from (iv); we defer full details until Proposition 15.3.1. The final statement of (ix), on representability, combines (iv) with (v), as will be elaborated upon in Proposition 15.3.2.

The reflection properties of (x) and (xi) follow from Theorem 3.5.3 and the creation properties follow from Theorem 3.5.11 and (ix). The precise steps are spelled out further in Theorem 15.1.1. ☐

14.3.6. COROLLARY. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then the following induced cosmological functors are all biequivalences:*

(i) $F\colon \mathcal{D}isc(\mathcal{K}) \to \mathcal{D}isc(\mathcal{L})$

(ii) $F\colon \mathcal{K}_{\top,J} \to \mathcal{L}_{\top,J}$ *and* $F\colon \mathcal{K}_{\perp,J} \to \mathcal{L}_{\perp,J}$
(iii) $F\colon \mathscr{R}\mathit{ari}(\mathcal{K})_{/B} \to \mathscr{R}\mathit{ari}(\mathcal{L})_{/FB}$ *and* $F\colon \mathcal{L}\mathit{ari}(\mathcal{K})_{/B} \to \mathcal{L}\mathit{ari}(\mathcal{L})_{/FB}$
(iv) $F\colon \mathscr{R}\mathit{ari}(\mathcal{K}) \to \mathscr{R}\mathit{ari}(\mathcal{L})$ *and* $F\colon \mathcal{L}\mathit{ari}(\mathcal{K}) \to \mathcal{L}\mathit{ari}(\mathcal{L})$
(v) $F\colon \mathcal{C}\mathit{art}(\mathcal{K})_{/B} \to \mathcal{C}\mathit{art}(\mathcal{L})_{/FB}$ *and* $F\colon \mathit{co}\mathcal{C}\mathit{art}(\mathcal{K})_{/B} \to \mathit{co}\mathcal{C}\mathit{art}(\mathcal{L})_{/FB}$
(vi) $F\colon \mathcal{C}\mathit{art}(\mathcal{K}) \to \mathcal{C}\mathit{art}(\mathcal{L})$ *and* $F\colon \mathit{co}\mathcal{C}\mathit{art}(\mathcal{K}) \to \mathit{co}\mathcal{C}\mathit{art}(\mathcal{L})$
(vii) $F\colon {}_{A\backslash}\mathcal{F}\mathit{ib}(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{F}\mathit{ib}(\mathcal{L})_{/FB}$
(viii) $F\colon {}_{A\backslash}\mathcal{M}\mathit{od}(\mathcal{K})_{/B} \to {}_{FA\backslash}\mathcal{M}\mathit{od}(\mathcal{L})_{/FB}$

Proof. In each case we start with a cosmological biequivalence — perhaps $\mathcal{K}_{/B} \to \mathcal{L}_{/FB}$ or $\mathcal{K}^{\mathbb{2}} \to \mathcal{L}^{\mathbb{2}}$ — and must show that the restricted cosmological functor of Observation 14.1.3 is again a biequivalence between the replete sub ∞-cosmoi. Each of the arguments is similar: we know that $F\colon \mathcal{K} \to \mathcal{L}$ induces a bijection between equivalence classes of objects. Since the sub ∞-cosmoi are replete, the equivalence classes of objects of the replete sub ∞-cosmos are given by a subset of the equivalence classes of objects of $\mathcal{K}$. Since, moreover, Proposition 14.3.5 proves that the property that characterizes the objects of the sub ∞-cosmos is preserved, reflected, and created by any biequivalence, it is clear that the bijection between equivalence classes of objects in $\mathcal{K}$ and $\mathcal{L}$ restricts to define a bijection between equivalence classes of objects in the sub ∞-cosmoi.

Now suppose $A, B \in \mathcal{K}$ are objects of its replete sub ∞-cosmos. We must show the local equivalence

$$\mathsf{Fun}(A,B) \xrightarrow{\sim} \mathsf{Fun}(FA,FB)$$

restricts to a local equivalence of the functor spaces of the sub ∞-cosmoi. Since we know that the biequivalence defines a bijection between isomorphism classes of 0-arrows in the sub ∞-cosmoi, the proof is completed by the following lemma. $\square$

14.3.7. Lemma. *Let $f\colon K \xrightarrow{\sim} L$ be an equivalence of quasi-categories. Let $K' \subset K$ and $L' \subset L$ be* ***replete sub quasi-categories*** *— that is, full and replete up to isomorphism on some collection of vertices — and suppose further that f restricts to define a simplicial functor*

$$\begin{tikzcd} K \arrow[r, "f", "\sim"'] & L \\ K' \arrow[u, hook] \arrow[r, dashed, "f"] & L' \arrow[u, hook] \end{tikzcd}$$

that is bijective on isomorphism classes of vertices. Then $f\colon K' \xrightarrow{\sim} L'$ is an equivalence of quasi-categories.

Proof. Choose any inverse equivalence $g\colon L \xrightarrow{\sim} K$. Since $K' \subset K$ is replete and full on some subset of vertices and $f\colon K \to L$ is bijective on isomorphism classes of vertices, g restricts to define a functor $g\colon L' \to K'$: for each $y \in L'$, $fg(y') \cong y' \in L$ so $g(y')$ must lie in K'. By repleteness and fullness again, the simplicial natural isomorphisms $K \to K^{\mathbb{I}}$ and $L \to L^{\mathbb{I}}$ that witness $gf \cong \mathrm{id}_K$ and $fg \cong \mathrm{id}_L$ restrict to L' and K' to define the desired equivalence. $\square$

Exercise.

14.3.i. Exercise. Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence. Show, for any ∞-category $A \in \mathcal{K}$, that F induces an equivalence of homotopy categories $\mathsf{h}A \xrightarrow{\sim} \mathsf{h}FA$.

14.3.ii. Exercise. Consider a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ that defines a biequivalence in the sense of Proposition 14.3.1. Prove that:

(i) A 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ in $\mathcal{C}$ is invertible if and only if $F\alpha$ is invertible in $\mathcal{D}$.

(ii) $u\colon A \to B$ admits a left adjoint in $\mathcal{C}$ if and only if $Fu\colon FA \to FB$ admits a left adjoint in $\mathcal{D}$ in which case F preserve the adjunction.

14.3.iii. Exercise. Prove that cosmological biequivalences between cartesian closed ∞-cosmoi preserve exponential objects up to equivalence.

14.4. Inverse cosmological biequivalences

Definition 14.2.1 declares two ∞-cosmoi to be **biequivalent** just when they are connected by a finite zig-zag of cosmological biequivalences. In this section, we establish a few useful properties of the "composite" of such a zig-zag, the analysis of which immediately reduces to the base case: describing the inverse $G\colon \mathcal{L} \stackrel{\sim}{\rightsquigarrow} \mathcal{K}$ to a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$. This material will be applied in Chapter 16 to streamline proofs that ∞-categorical structures transfer to biequivalent ∞-cosmoi. The reader might consider skipping this for now and referring back to it with the applications of Chapter 16 in mind.

To explain what to expect, consider the analogous 2-categorical case. By Proposition 14.3.1, a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces a biequivalence $F\colon \mathfrak{h}\mathcal{K} \xrightarrow{\sim} \mathfrak{h}\mathcal{L}$ of homotopy 2-categories, this being a 2-functor that is

- surjective on objects up to equivalence and
- defines a local equivalence of categories $\mathsf{hFun}(A,B) \xrightarrow{\sim} \mathsf{hFun}(FA,FB)$ for all $A,B \in \mathfrak{h}\mathcal{K}$.

From these properties we may attempt to define an inverse biequivalence G as follows:

- For each $C \in \mathfrak{h}\mathcal{L}$, we choose an $A \in \mathfrak{h}\mathcal{K}$ together with a specified equivalence $\epsilon\colon FA \simeq C$ and define $GC := A$.
- For each pair $C,D \in \mathfrak{h}\mathcal{L}$, we define the action of G on hom-categories to be the composite

$$G_{C,D} := \mathsf{hFun}(C,D) \xrightarrow[\sim]{(-\circ\epsilon,\epsilon^{-1}\circ-)} \mathsf{hFun}(FGC,FGD) \xrightarrow[\sim]{{F_{GC,GD}}^{-1}} \mathsf{hFun}(GC,GD)$$

of the equivalence defined by pre- and post-composing with the maps of the specified equivalences $FGC \simeq C$ and $FGD \simeq D$ together with an inverse of the equivalence defined by the action of F.

These choices are suitably unique: the action on objects is well-defined up to equivalence and the action on hom-categories is well-defined up to natural isomorphism. However, these mappings do not assemble into a 2-functor (see Definition B.2.1): for instance, while the triangle on the left commutes on the nose, the composite triangle on the right only commutes up to isomorphism:

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_{FA}"] & \\ \mathsf{hFun}(A,A) \arrow[rr, "F_{A,A}"'] & & \mathsf{hFun}(FA,FA) \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "\mathrm{id}_{FGC}"] \arrow[dr, "\mathrm{id}_{GC}"] \arrow[dl, phantom, "\cong", shift left=4] \arrow[dr, phantom, "\cong", shift right=4] & \\ \mathsf{hFun}(C,C) \arrow[r, "(-\circ\epsilon,\epsilon^{-1}\circ-)"'] & \mathsf{hFun}(FGC,FGC) \arrow[r, "F^{-1}_{GC,GC}"'] & \mathsf{hFun}(GC,GC) \end{tikzcd}$$

Instead, the mapping $G\colon \mathfrak{h}\mathcal{L} \rightsquigarrow \mathfrak{h}\mathcal{K}$ defines a *pseudofunctor* between the homotopy 2-categories, which we now define.

14.4.1. Definition. A **pseudofunctor** $G\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ between 2-categories is given by:

- a mapping on objects $\mathcal{C} \ni x \mapsto Gx \in \mathcal{D}$;

- a mapping on hom-categories $G_{x,y}\colon C(x,y) \to \mathcal{D}(Gx,Gy)$ for each $x,y \in C$;
- an invertible 2-cell

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] \arrow[d, phantom, "\iota_x \Downarrow\cong"] & \\ C(x,x) \arrow[rr, "G_{x,x}"'] & {} & \mathcal{D}(Gx,Gx) \end{tikzcd}$$

  for each $x \in C$, defining an isomorphism $\iota_x\colon\ \mathrm{id}_{Gx} \cong G\,\mathrm{id}_X$ in $\mathcal{D}(Gx,Gx)$; and
- an natural isomorphism

$$\begin{tikzcd} C(y,z) \times C(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{D}(Gy,Gz) \times \mathcal{D}(Gx,Gy) \arrow[d, "\circ"] \\ C(x,z) \arrow[r, "G"'] & \mathcal{D}(Gx,Gz) \end{tikzcd}$$

  for each triple of objects $x,y,z \in C$

so that three pasting diagrams commute:

$$\begin{tikzcd} C(y,z) \times C(x,y) \times C(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\times\mathrm{id}\Downarrow\cong"] & \mathcal{D}(Gy,Gz) \times \mathcal{D}(Gx,Gy) \times \mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ C(x,z) \times C(w,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,x,z}\Downarrow\cong"] & \mathcal{D}(Gx,Gz) \times \mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ C(w,z) \arrow[r, "G"'] & \mathcal{D}(Gw,Gz) \end{tikzcd}$$

$$=$$

$$\begin{tikzcd} C(y,z) \times C(x,y) \times C(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\mathrm{id}\times\alpha^{w,x,y}\Downarrow\cong"] & \mathcal{D}(Gy,Gz) \times \mathcal{D}(Gx,Gy) \times \mathcal{D}(Gw,Gx) \arrow[d, "\circ"] \\ C(y,z) \times C(w,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,y,z}\Downarrow\cong"] & \mathcal{D}(Gy,Gz) \times \mathcal{D}(Gw,Gy) \arrow[d, "\circ"] \\ C(w,z) \arrow[r, "G"'] & \mathcal{D}(Gw,Gz) \end{tikzcd}$$

$$\begin{tikzcd} C(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}_y\times\mathrm{id}"'] \arrow[dr, phantom, "\iota_y\times\mathrm{id}\Downarrow\cong"] & \mathcal{D}(Gx,Gy) \arrow[d, "\mathrm{id}_{Gy}\times\mathrm{id}"] \\ C(x,y) \times C(x,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,x,y}\Downarrow\cong"] & \mathcal{D}(Gx,Gy) \times \mathcal{D}(Gx,Gx) \arrow[d, "\circ"] \\ C(x,y) \arrow[r, "G"'] & \mathcal{D}(Gx,Gy) \end{tikzcd} = \begin{tikzcd} C(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}\times\mathrm{id}_x"'] \arrow[dr, phantom, "\mathrm{id}\times\iota_x\Downarrow\cong"] & \mathcal{D}(Gx,Gy) \arrow[d, "\mathrm{id}\times\mathrm{id}_{Gx}"] \\ C(y,y) \times C(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,y}\Downarrow\cong"] & \mathcal{D}(Gy,Gy) \times \mathcal{D}(Gx,Gy) \arrow[d, "\circ"] \\ C(x,y) \arrow[r, "G"'] & \mathcal{D}(Gx,Gy) \end{tikzcd}$$

where both of these latter composites equal the unit 2-cell $\mathrm{id}_{G_{x,y}}$.

There is a closely related notion of pseudonatural transformation between pseudofunctors.

14.4.2. Definition. For any 2-categories C and $\mathcal{D}$ and pseudofunctors $F,G\colon C \rightsquigarrow \mathcal{D}$, a **pseudonatural transformation** $\phi\colon F \Rightarrow G$ is given by:

- a 1-cell $\phi_x : Fx \to Gx \in \mathcal{D}$ for every object $x \in \mathcal{C}$ and
- an invertible 2-cell in $\mathcal{D}$

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f \Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

for each 1-cell $f : x \to y \in \mathcal{C}$

so that this data

- is natural, in the sense that for each 2-cell $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & y \end{tikzcd}$ in $\mathcal{C}$ the pasted composites are equal

$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_g \Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, bend right, "Gg"'] & Gy \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f \Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\gamma"] & Gy \end{tikzcd}$$

  and
- respects the composition and unit constraints specified by the pseudofunctors

$$\begin{tikzcd} & Fy \arrow[dr, "Fk"] & \\ Fx \arrow[ur, "Ff"] \arrow[rr, bend right, "F(kf)"'] \arrow[rr, phantom, "\alpha^{x,y,z}_{f,g} \Downarrow\cong"] \arrow[d, "\phi_x"'] & & Fz \arrow[d, "\phi_z"] \\ Gx \arrow[rr, bend right, "G(kf)"'] \arrow[urr, phantom, "\phi_{kf} \Downarrow\cong"] & & Gz \end{tikzcd} = \begin{tikzcd} & Fy \arrow[dr, "Fk"] \arrow[dd, "\phi_y"] & \\ Fx \arrow[ur, "Ff"] \arrow[dd, "\phi_x"'] \arrow[r, phantom, "\phi_f \Downarrow\cong"] & & Fz \arrow[dd, "\phi_z"] \arrow[l, phantom, "\phi_k \Downarrow\cong"] \\ & Gy \arrow[dr, "Gk"] & \\ Gx \arrow[ur, "Gf"] \arrow[rr, bend right, "G(kf)"'] \arrow[rr, phantom, "\alpha^{x,y,z}_{f,g} \Downarrow\cong"] & & Gz \end{tikzcd}$$

  and

$$\begin{tikzcd} Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[r, bend right, "F\,\mathrm{id}_x"'] \arrow[r, phantom, "\iota_x \Downarrow\cong"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_{\mathrm{id}_x} \Downarrow\cong"] & Fx \arrow[d, "\phi_x"] \\ Gx \arrow[r, bend right, "G\,\mathrm{id}_x"'] & Gx \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[d, "\phi_x"'] & Fx \arrow[d, "\phi_x"] \\ Gx \arrow[r, bend left, "\mathrm{id}_{Gx}"] \arrow[r, bend right, "G\,\mathrm{id}_x"'] \arrow[r, phantom, "\iota_y \Downarrow\cong"] & Gx \end{tikzcd}$$

As suggested above, one instance in which pseudofunctors naturally arise is as inverses to 2-functors that define biequivalences. The pseudofunctors that arise in this manner are themselves **biequivalences**: surjective on objects up to equivalence and defining local equivalences on hom-categories.

These functors are inverses in the sense that there exist **pseudonatural equivalences** between the composites and the identities, these being pseudonatural transformations that are componentwise equivalences; see Exercise 14.4.ii for an alternate characterization. Collectively, this data defines an equivalence of 2-categories, not in the sense of enriched category theory—cf. Definition A.3.15—but in the sense appropriate to bicategory theory:

14.4.3. Proposition. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a 2-functor between 2-categories $\mathcal{C}$ and $\mathcal{D}$ and a biequivalence then there exists a pseudofunctor $G\colon \mathcal{D} \rightsquigarrow \mathcal{C}$, that is also a biequivalence, and is a pseudoinverse to F in the sense that there exist pseudonatural equivalences $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

Proof. Exercise 14.4.iii. □

Proposition 14.4.3 describes a classical result in bicategory theory, so we feel content to leave its proof to the exercises. We shall also shortly prove a generalization subsuming this classical statement. Recall from Definition 1.2.1 that an ∞-cosmos is, among other things, a category enriched in the cartesian closed category of quasi-categories, in the sense explicated in §A.1–A.2. By Definition 1.4.1 and Proposition 1.4.5, the cartesian closed category of quasi-categories underlies a cartesian closed *2-category* of quasi-categories $\mathfrak{h}\mathcal{QC}at$. While we have been referring to $\mathfrak{h}\mathcal{QC}at$ as the *homotopy 2-category of quasi-categories*, it really is just the natural 2-categorical enrichment: its objects are quasi-categories, its 1-cells are simplicial functors between them, and its 2-cells deserve to be called "natural transformations" between quasi-categories.

The extra dimension in the 2-category $\mathfrak{h}\mathcal{QC}at$ enables us to define *quasi-categorically enriched pseudofunctors* as follows:

14.4.4. Definition. For quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{L}$, a **quasi-categorically enriched pseudofunctor** — a **quasi-pseudofunctor** for short — $G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ is given by:

- a mapping on objects $\mathcal{K} \ni x \mapsto Gx \in \mathcal{L}$;
- a simplicial map of hom quasi-categories $G_{x,y}\colon \mathcal{K}(x,y) \to \mathcal{L}(Gx,Gy)$ for each $x,y \in \mathcal{K}$;
- an invertible 2-cell

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] \arrow[d, phantom, "\iota_x \Downarrow\cong"] & \\ \mathcal{K}(x,x) \arrow[rr, "G_{x,x}"'] & {} & \mathcal{L}(Gx,Gx) \end{tikzcd}$$

  in the homotopy 2-category of quasi-categories for each $x \in \mathcal{K}$;
- an invertible 2-cell

$$\begin{tikzcd} \mathcal{K}(y,z) \times \mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{L}(Gy,Gz) \times \mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\ \mathcal{K}(x,z) \arrow[r, "G"'] & \mathcal{L}(Gx,Gz) \end{tikzcd}$$

  in the homotopy 2-category of quasi-categories for each triple of objects $x,y,z \in \mathcal{K}$

so that three pasting diagrams commute:

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y)\times\mathcal{K}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\times\mathrm{id}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(x,z)\times\mathcal{K}(w,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,x,z}\Downarrow\cong"] & \mathcal{L}(Gx,Gz)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(w,z) \arrow[r, "G"'] & \mathcal{L}(Gw,Gz)
\end{tikzcd}$$

$$= \tag{14.4.5}$$

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y)\times\mathcal{K}(w,x) \arrow[r, "G\times G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\mathrm{id}\times\alpha^{w,x,y}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy)\times\mathcal{L}(Gw,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(y,z)\times\mathcal{K}(w,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{w,y,z}\Downarrow\cong"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gw,Gy) \arrow[d, "\circ"] \\
\mathcal{K}(w,z) \arrow[r, "G"'] & \mathcal{L}(Gw,Gz)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}_y\times\mathrm{id}"'] \arrow[dr, phantom, "\iota_y\times\mathrm{id}\Downarrow\cong"] & \mathcal{L}(Gx,Gy) \arrow[d, "\mathrm{id}_{Gy}\times\mathrm{id}"] \\
\mathcal{K}(x,y)\times\mathcal{K}(x,x) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,x,y}\Downarrow\cong"] & \mathcal{L}(Gx,Gy)\times\mathcal{L}(Gx,Gx) \arrow[d, "\circ"] \\
\mathcal{K}(x,y) \arrow[r, "G"'] & \mathcal{L}(Gx,Gy)
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "G"] \arrow[d, "\mathrm{id}\times\mathrm{id}_x"'] \arrow[dr, phantom, "\mathrm{id}\times\iota_x\Downarrow\cong"] & \mathcal{L}(Gx,Gy) \arrow[d, "\mathrm{id}\times\mathrm{id}_{Gx}"] \\
\mathcal{K}(y,y)\times\mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,y}\Downarrow\cong"] & \mathcal{L}(Gy,Gy)\times\mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\
\mathcal{K}(x,y) \arrow[r, "G"'] & \mathcal{L}(Gx,Gy)
\end{tikzcd}$$

where both of these latter composites equal the unit 2-cell $\mathrm{id}_{G_{x,y}}$.

14.4.6. Remark. For any pair of objects $a, b \in \mathcal{L}$ in a quasi-categorically enriched category,

$$\mathsf{Fun}(\mathbb{1}, \mathcal{L}(a,b)) \cong \mathcal{L}(a,b),$$

and so $\mathsf{hFun}(\mathbb{1}, \mathcal{L}(a,b)) \cong \mathsf{h}\mathcal{L}(a,b)$. Thus 2-cells in the homotopy 2-category of quasi-categories $\mathfrak{h}\mathcal{QC}at$ with domain $\mathbb{1}$ and codomain $\mathcal{L}(a,b)$ correspond to 2-cells in the homotopy 2-category of $\mathcal{L}$ — defined exactly as in Definition 1.4.1 — from a to b.

In particular, the data of the invertible 2-cell ι_x is no more and no less than an invertible 2-cell $\iota_x\colon \mathrm{id}_{Gx} \cong G\,\mathrm{id}_X$ in the homotopy 2-category of $\mathcal{L}$.

14.4.7. Definition. A quasi-pseudofunctor $G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ whose codomain $\mathcal{L}$ is an ∞-cosmos is a **biequivalence** when it is:

(i) surjective on objects up to equivalence: if for all $a \in \mathcal{L}$ there exists $x \in \mathcal{K}$ so that $Fx \simeq a$; and
(ii) a local equivalence of quasi-categories: if for every pair $x, y \in \mathcal{K}$, the map

$$\mathcal{K}(x,y) \xrightarrow[\sim]{G_{x,y}} \mathcal{L}(Gx,Gy)$$

is an equivalence of quasi-categories.

14.4.8. Remark. We find it convenient to assume that $\mathcal{L}$ is an ∞-cosmos in Definition 14.4.7 because that provides us access to the equivalent characterizations of the equivalences in $\mathcal{L}$ given by Theorem 1.4.7. In what follows we will ask that an equivalence $a \simeq b$ in $\mathcal{L}$

- defines an equivalence in the homotopy 2-category of $\mathcal{L}$ and
- induces an equivalence of quasi-categories $\mathcal{L}(x,a) \xrightarrow{\sim} \mathcal{L}(x,b)$ in the homotopy 2-category $\mathfrak{h}\mathcal{QC}at$ that is 2-natural in x.

The latter of these properties implies the former, so if we required a notion of quasi-pseudofunctorial biequivalence between general quasi-categorically enriched categories, we could use this notion of equivalence in Definition 14.4.7. But we will make no use of the concept outside of the context provided by ∞-cosmoi and so prefer the simpler terminology. Note that we permit the domain $\mathcal{K}$ to be merely quasi-categorically enriched.

Similarly:

14.4.9. Definition. For quasi-categorically enriched categories $\mathcal{K}$ and $\mathcal{L}$ and quasi-pseudofunctors $F, G \colon \mathcal{K} \rightsquigarrow \mathcal{L}$, a **quasi-categorically enriched pseudonatural transformation** — a **quasi-pseudonatural transformation** for short — $\phi \colon F \Rightarrow G$ is given by:

- a 0-arrow $\phi_x \colon Fx \to Gx \in \mathcal{L}$ for every object $x \in \mathcal{K}$ and
- an invertible 2-cell

$$\begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, "G_{x,y}"'] \arrow[dr, phantom, "\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fx,Fy) \arrow[d, "\phi_y\circ-"] \\
\mathcal{L}(Gx,Gy) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gy)
\end{tikzcd}$$

in the homotopy 2-category of quasi-categories, for each pair of objects $x, y \in \mathcal{K}$

so that this data respects the composition and unit constraints specified by the quasi-pseudofunctors, as expressed by the following two pasting diagrams

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[r, "F\times F"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\alpha^{x,y,z}\Downarrow\cong"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[d, "\circ"] \\
\mathcal{K}(x,z) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\phi^{x,z}\Downarrow\cong"] & \mathcal{L}(Fx,Fz) \arrow[d, "\phi_z\circ-"] \\
\mathcal{L}(Gx,Gz) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gz)
\end{tikzcd}
\qquad = $$

$$\begin{tikzcd}
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[dr, "(\phi_z\circ-)\times\mathrm{id}"] \arrow[d, phantom, "\phi^{y,z}\times\mathrm{id}\Downarrow\cong"] & \\
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[ur, "F\times F"] \arrow[r, dashed, "G\times F"] \arrow[d, "\circ"'] \arrow[dr, "G\times G"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Fx,Fy) \arrow[r, "(-\circ\phi_y)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times(\phi_y\circ-)"] \arrow[d, phantom, "\mathrm{id}\times\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fy,Gz)\times\mathcal{L}(Fx,Fy) \arrow[dd, bend left, "\circ"] \\
\mathcal{K}(x,z) \arrow[r, phantom, "\alpha^{x,y,z}\Downarrow\cong"] \arrow[dr, "G"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[r, "\mathrm{id}\times(-\circ\phi_x)"'] \arrow[d, "\circ"] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Fx,Gy) \arrow[d, "\circ"] \\
& \mathcal{L}(Gx,Gz) \arrow[r, "-\circ\phi_x"'] & \mathcal{L}(Fx,Gz)
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
& \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Fx}"] \arrow[d, phantom, "\iota_x\Downarrow\cong"] & \\
\mathcal{K}(x,x) \arrow[rr, "F_{x,x}"'] \arrow[d, "G_{x,x}"'] & \phi^{x,y}\Downarrow\cong & \mathcal{L}(Fx,Fx) \arrow[d, "\phi_x\circ-"] \\
\mathcal{L}(Gx,Gx) \arrow[rr, "-\circ\phi_x"'] & & \mathcal{L}(Fx,Gx)
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& \mathbb{1} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{Gx}"] \arrow[d, phantom, "\iota_x\Downarrow\cong"] & \\
\mathcal{K}(x,x) \arrow[rr, "G_{x,x}"'] \arrow[d, "G_{x,x}"'] & & \mathcal{L}(Gx,Gx) \arrow[d, "-\circ\phi_x"] \\
\mathcal{L}(Gx,Gx) \arrow[rr, "-\circ\phi_x"'] & & \mathcal{L}(Fx,Gx)
\end{tikzcd}$$

14.4.10. Remark. Recall the 0-arrows in $\mathcal{K}(x,y)$ correspond to simplicial maps $f\colon \mathbb{1} \to \mathcal{K}(x,y)$. By the argument of Remark 14.4.6, the restriction $\phi^{x,y}f$ defines the component

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, "Gf"'] & Gy
\end{tikzcd}$$

of an invertible 2-cell in the homotopy 2-category of $\mathfrak{h}\mathcal{L}$. This 2-cell is automatically "natural," in the sense that for each 2-cell $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & y \end{tikzcd}$ in the homotopy 2-category of $\mathcal{K}$ the pasted composites are equal

$$\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_g\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend right, "Gg"'] & Gy
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow\cong"] & Fy \arrow[d, "\phi_y"] \\
Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\gamma"] & Gy
\end{tikzcd}$$

in the homotopy 2-category of $\mathcal{L}$, simply because both pasting composites are represented by the horizontal composite

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{K}(x,y) \arrow[r, bend left, "(\phi_y\circ-)\circ F_{x,y}"] \arrow[r, bend right, "(-\circ\phi_x)\circ G_{x,y}"'] \arrow[r, phantom, "\phi^{x,y}\Downarrow\cong"] & \mathcal{L}(Fx,Gy)
\end{tikzcd}$$

in the homotopy 2-category of quasi-categories.

Similarly, the composition and unit diagrams of Definition 14.4.9 imply that the corresponding diagrams displayed in Definition 14.4.2 commute in the homotopy 2-category of $\mathcal{L}$; see Exercise 14.4.iv.

14.4.11. Lemma. *The action on homs of a quasi-pseudofunctor $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ between quasi-categorically enriched categories defines a quasi-pseudonatural transformation*

$$\mathcal{K}(-,-) \xrightarrow{F_{-,-}} \mathcal{L}(F-,F-)$$

between simplicial functors $\mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{QC}at$.

Proof. The 2-cell component of the quasi-pseudonatural transformation associated to a pair of objects (x, y) and (w, z) in $\mathcal{K}^{\mathrm{op}} \times \mathcal{K}$ is given by

$$\begin{tikzcd} \mathcal{K}(y,z) \times \mathcal{K}(w,x) \arrow[r, "\circ"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha^{w,x,y,z}\Downarrow\cong"] & \mathcal{K}(w,z)^{\mathcal{K}(x,y)} \arrow[d, "F_{w,z}\circ -"] \\ \mathcal{L}(Fw,Fz)^{\mathcal{L}(Fx,Fy)} \arrow[r, "-\circ F_{x,y}"'] & \mathcal{L}(Fw,Fz)^{\mathcal{K}(x,y)} \end{tikzcd}$$

where $\alpha^{w,x,y,z}$ is a transpose of the pasted composite of (14.4.5). We leave the verification of the composition and unit axioms to Exercise 14.4.v. $\square$

14.4.12. Definition. A quasi-pseudonatural transformation $\phi\colon F \Rightarrow G$ between quasi-pseudofunctors $F, G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ whose codomain is an ∞-cosmos is a **quasi-pseudonatural equivalence** if each of its components $\phi_x\colon Fx \to Gx$ defines an equivalence in the homotopy 2-category of $\mathcal{L}$.

Again, as noted in Remark 14.4.8, we find it convenient to assume that $\mathcal{L}$ is an ∞-cosmos so we need not be more explicit about the appropriate notion of equivalence in the target category. Our interest in the class of quasi-pseudonatural equivalences stems from the following result, which can be understood as a version of the bicategorical Yoneda lemma in the context of quasi-categorically enriched categories, quasi-pseudofunctors, quasi-pseudonatural transformations, and the "modifications" between them.

14.4.13. Lemma. *If there exists a quasi-pseudonatural equivalence*

$$\phi_x\colon \mathcal{K}(x,a) \rightsquigarrow \mathcal{K}(x,b)$$

between the simplicial functors $\mathcal{K}^{\mathrm{op}} \to \mathcal{QC}at$ represented by a pair of objects a, b in an ∞-cosmos $\mathcal{K}$, then a and b are equivalent in $\mathcal{K}$.

Proof. We will show that the 0-arrow $\phi_a(\mathrm{id}_a)\colon a \to b$ is an equivalence in $\mathcal{K}$. First observe from the diagram

$$\begin{tikzcd} \mathcal{K}(x,a) \arrow[rr, "\circ"] \arrow[dd, "\circ"'] \arrow[ddrr, phantom, "\phi^{x,a}\Downarrow\cong"] & & \mathcal{K}(x,a)^{\mathcal{K}(a,a)} \arrow[dd, "\phi_x\circ -"] \arrow[dr, "\mathrm{ev}_{\mathrm{id}_a}"] & \\ & & & \mathcal{K}(x,a) \arrow[dd, "\phi_x"] \\ \mathcal{K}(x,b)^{\mathcal{K}(a,b)} \arrow[rr, "-\circ\phi_a"'] \arrow[dr, "\mathrm{ev}_\phi"'] & & \mathcal{K}(x,b)^{\mathcal{K}(a,a)} \arrow[dr, "\mathrm{ev}_{\mathrm{id}_a}"] & \\ & \mathcal{K}(x,b) \arrow[rr, equal] & & \mathcal{K}(x,b) \end{tikzcd}$$

that for every $x \in \mathcal{K}$ the component $\phi_x\colon \mathcal{K}(x,a) \rightsquigarrow \mathcal{K}(x,b)$ — the top-right composite in the above diagram — is isomorphic in $\mathcal{QC}at_2$ to the postcomposition map $\phi \circ -$ — the lower-left composite in the above diagram. In particular, the map $\phi \circ -\colon \mathcal{K}(b,a) \rightsquigarrow \mathcal{K}(b,b)$ is an equivalence, so we may choose a 0-arrow $\psi\colon b \to a$ so that $\beta\colon \phi\psi \cong \mathrm{id}_b$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. Now id_a and $\psi\phi$ define a pair of 0-arrows in $\mathcal{K}(a,a)$ whose images under the equivalences $\phi\circ -\colon \mathcal{K}(a,a) \rightsquigarrow \mathcal{K}(a,b)$ are isomorphic via $\beta\phi\colon \phi\psi\phi \cong \phi$. Hence, there exists an isomorphism $\alpha\colon \mathrm{id}_a \cong \psi\phi$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, completing the proof that $a \simeq b$ in $\mathcal{K}$. $\square$

Quasi-pseudonatural equivalences may be constructed as adjoint equivalence inverses of simplicial natural transformations that define componentwise equivalences.

14.4.14. Lemma. *Consider a simplicial natural transformation* $\begin{tikzcd}\mathcal{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{L}\end{tikzcd}$ *between ∞-cosmoi whose 0-arrow components $\phi_x \colon Fx \rightsquigarrow Gx$ all define equivalences in $\mathcal{L}$. Then any choice of adjoint equivalence inverses $\psi_x \colon Gx \rightsquigarrow Fx$ assemble into the components of a quasi-pseudonatural transformation $\psi \colon G \Rightarrow F$.*

Proof. The components of the quasi-pseudonatural transformation ψ are defined by the adjoint equivalence inverse arrows $\psi_x \colon Gx \rightsquigarrow Fx$ and by the pasted composite natural transformation

$$\psi^{x,y} := \qquad \begin{tikzcd}
\mathcal{K}(x,y) \arrow[r, "G_{x,y}"] \arrow[d, "F_{x,y}"'] & \mathcal{L}(Gx,Gy) \arrow[d, "-\circ\phi_x"'] \arrow[dr, equal] \arrow[dr, phantom, "\eta_x\cong"'] & \\
\mathcal{L}(Fx,Fy) \arrow[r, "\phi_y\circ-"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon_y\cong"] & \mathcal{L}(Fx,Gy) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_y\circ-"] & \mathcal{L}(Gx,Fy) \arrow[d, "\psi_y\circ-"] \\
& \mathcal{L}(Fx,Fy) \arrow[r, "-\circ\psi_x"'] & \mathcal{L}(Gx,Fy)
\end{tikzcd}$$

of the unit and counit isomorphisms of the adjoint equivalence.

Since F and G are simplicial functors, the unit condition simplifies to ask only that the component of this pasted natural transformation at the identity arrow $\mathrm{id}_x \colon \mathbb{1} \to \mathcal{K}(x,x)$ is an identity 2-cell id_{ψ_x}. This component is the pasted composite

$$\begin{tikzcd}
Gx \arrow[rr, equal] \arrow[dr, "\psi_x"'] & \arrow[d, phantom, "\eta\cong"] & Gx \arrow[dr, "\psi_x"] & \\
& Fx \arrow[ur, "\phi_x"] \arrow[rr, equal] & \arrow[u, phantom, "\epsilon\cong"] & Fx
\end{tikzcd}$$

which is indeed the identity, since the specified data defines an adjoint equivalence.

Similarly, to verify the composition axiom, we must show that the composite

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[dr, "\circ"] & & & \\
& \mathcal{K}(x,z) \arrow[r, "G_{x,z}"] \arrow[d, "F_{x,z}"'] & \mathcal{L}(Gx,Gz) \arrow[d, "-\circ\phi_x"'] \arrow[dr, equal] \arrow[dr, phantom, "\eta_x\cong"'] & \\
& \mathcal{L}(Fx,Fz) \arrow[r, "\phi_z\circ-"] \arrow[dr, equal] \arrow[dr, phantom, "\epsilon_z\cong"] & \mathcal{L}(Fx,Gz) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_z\circ-"] & \mathcal{L}(Gx,Fz) \arrow[d, "\psi_z\circ-"] \\
& & \mathcal{L}(Fx,Fz) \arrow[r, "-\circ\psi_x"'] & \mathcal{L}(Gx,Fz)
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathcal{K}(y,z)\times\mathcal{K}(x,y) \arrow[r, "G\times G"] \arrow[dr, "F\times G"] \arrow[dd, "F\times F"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[dr, "-\circ\phi_y"] \arrow[drr, equal] \arrow[drr, phantom, "\eta_y\cong"] & & \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Gy) \arrow[r, "\phi_z\circ-"] \arrow[dd, "-\circ\phi_x"] \arrow[ddr, equal] \arrow[ddr, phantom, "\eta_x\cong"'] & \mathcal{L}(Fy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[r, "-\circ\psi_y"'] \arrow[dd, "\psi_z\circ-"] \arrow[dd, phantom, "\epsilon_z\cong"'] & \mathcal{L}(Gy,Gz)\times\mathcal{L}(Gx,Gy) \arrow[dd, "\psi_z\circ-"] \\
\mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[dr, "\phi_y\circ-"] \arrow[ddr, equal, bend right] \arrow[ddr, phantom, "\epsilon_y\cong"] & & & \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Gy) \arrow[r, "-\circ\psi_x"'] \arrow[d, "\psi_y\circ-"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Gz) \arrow[r, "-\circ\psi_y"] \arrow[d, "\psi_y\circ-"] & \mathcal{L}(Gy,Fz)\times\mathcal{L}(Gx,Gy) \arrow[d, "\circ"] \\
& \mathcal{L}(Fy,Fz)\times\mathcal{L}(Fx,Fy) \arrow[r, "-\circ\psi_x"] & \mathcal{L}(Fy,Fz)\times\mathcal{L}(Gx,Fy) \arrow[r, "\circ"'] & \mathcal{L}(Gx,Fz)
\end{tikzcd}$$

The pre- and post-composition maps appearing in this diagram are 2-natural, so for instance the whiskered composite of ϵ_y and $-\circ\psi_x$ can be formed in either order. Using this commutativity property repeatedly to the second pasting diagram and applying the triangle identity $\phi_y\epsilon_y \cdot \eta_y\phi_y = \mathrm{id}_{\phi_y}$, the second pasting diagram reduces to the first one. $\square$

We now show that any biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ of quasi-categorically enriched categories admits a quasi-pseudofunctorial "inverse" $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$ equipped with quasi-pseudonatural equivalences $\eta\colon \mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$.

14.4.15. Proposition. *If $F\colon \mathcal{K} \to \mathcal{L}$ is a quasi-categorically enriched functor between ∞-cosmoi and a biequivalence then there exists a quasi-pseudofunctor $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$, that is also a biequivalence. Moreover G is a quasi-pseudoinverse to F in the sense that there exist quasi-pseudonatural equivalences $\mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}_{\mathcal{L}}$.*

Proof. To coherently define an inverse to a biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$, we "fully specify" its data, choosing:

(β) fully specified adjoint equivalences $\epsilon_a\colon Fx \simeq a$ for each $a \in \mathcal{L}$ and

(γ) fully specified inverse adjoint equivalences of quasi-categories

$$\mathcal{K}(x,y) \xrightarrow[\sim]{F_{x,y}} \mathcal{L}(Fx,Fy)$$

for each pair $x, y \in \mathcal{K}$ whose inverse is quasi-pseudonatural in x and y.

In (γ), we apply Lemma 14.4.14 to the simplicial natural transformation $F_{-,-}\colon \mathcal{K}(-,-) \to \mathcal{L}(F-,F-)$ to observe that the pointwise adjoint equivalences to these maps assemble into a quasi-pseudonatural transformation, which is also a pointwise equivalence.

Now, to define $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$, use ($\beta$) to specify for each $a \in \mathcal{L}$ an object $Ga \in \mathcal{K}$ together with an equivalence $\epsilon_a\colon FGa \simeq a$ in $\mathcal{L}$. This defines the mapping of G on objects and the 0-arrow components of the quasi-pseudonatural transformation ϵ. To define the action of G on hom quasi-categories, use this data and (γ) to define

$$G_{a,b} \coloneqq \mathcal{L}(a,b) \xrightarrow[\sim]{(-\circ\epsilon_a,\epsilon_b^{-1}\circ-)} \mathcal{L}(FGa,FGb) \xrightarrow[\sim]{F^{-1}_{Ga,Gb}} \mathcal{K}(Ga,Gb)$$

For each $a \in \mathcal{L}$ define $\iota_a \colon \mathrm{id}_{Ga} \cong G_{a,a}\,\mathrm{id}_a$ to be the composite

$$\begin{tikzcd} & & \mathbb{1} \arrow[dll, "\mathrm{id}_a"'] \arrow[d, "\mathrm{id}_{FGa}"] \arrow[drr, "\mathrm{id}_{Ga}"] & & \\ \mathcal{L}(a,a) \arrow[rr, "{(-\circ\epsilon_a, \epsilon_a^{-1}\circ -)}"'] \arrow[urr, phantom, "\beta\cong"] & & \mathcal{L}(FGa,FGa) \arrow[rr, "F^{-1}_{Ga,Ga}"'] \arrow[urr, phantom, "\gamma\cong"] & & \mathcal{K}(Ga,Ga) \end{tikzcd}$$

of the isomorphism $\beta_a \colon \epsilon_a^{-1} \circ \epsilon_a \cong \mathrm{id}_{FGa}$ in the homotopy 2-category of $\mathcal{L}$ with the component of the isomorphism $\gamma\mathrm{id}_{Ga} \colon F^{-1}_{Ga,Ga} \circ F_{Ga,Ga} \cong \mathrm{id}_{\mathcal{K}(Ga,Ga)}$ at id_{Ga}. For each $a, b, c \in \mathcal{L}$, define $\alpha^{a,b,c}$ to be the composite

$$\begin{tikzcd} \mathcal{L}(b,c) \times \mathcal{L}(a,b) \arrow[r, "{(-\circ\epsilon_b, \epsilon_c^{-1}\circ -)\times(-\circ\epsilon_a, \epsilon_b^{-1}\circ -)}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGb,FGc) \times \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Gb,Gc}\times F^{-1}_{Ga,Gb}"] \arrow[d, "\circ"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Gb,Gc) \times \mathcal{K}(Ga,Gb) \arrow[d, "\circ"] \\ \mathcal{L}(a,c) \arrow[r, "{(-\circ\epsilon_a, \epsilon_c^{-1}\circ -)}"'] & \mathcal{L}(FGa,FGc) \arrow[r, "F^{-1}_{Ga,Gc}"'] & \mathcal{K}(Ga,Gc) \end{tikzcd}$$

of the canonical natural transformations built from the data of (β) and (γ).

We next verify that these choices make G into a quasi-pseudofunctor. For the unit condition, we must verify that the composite

$$\begin{tikzcd} \mathcal{L}(a,b) \arrow[r, "{(-\circ\epsilon_a, \epsilon_b^{-1}\circ -)}"] \arrow[d, "\mathrm{id}_b \times \mathrm{id}"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"] \arrow[d, "\mathrm{id}_{FGb} \times \mathrm{id}"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Ga,Gb) \arrow[d, "\mathrm{id}_{Gb} \times \mathrm{id}"] \\ \mathcal{L}(b,b) \times \mathcal{L}(a,b) \arrow[r, "{(-\circ\epsilon_b, \epsilon_b^{-1}\circ -)\times(-\circ\epsilon_a, \epsilon_b^{-1}\circ -)}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\beta\cong"] & \mathcal{L}(FGb,FGb) \times \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Gb,Gb}\times F^{-1}_{Ga,Gb}"] \arrow[d, "\circ"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(Gb,Gb) \times \mathcal{K}(Ga,Gb) \arrow[d, "\circ"] \\ \mathcal{L}(a,b) \arrow[r, "{(-\circ\epsilon_a, \epsilon_b^{-1}\circ -)}"'] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"'] & \mathcal{K}(Ga,Gb) \end{tikzcd}$$

is the identity; in fact, each pair of vertical composites is the identity. On the left-hand side, this is on account of one of the triangle equality relations for the adjoint equivalence ϵ_b. On the right-hand side, this is a consequence of quasi-pseudonaturality of the pair $F^{-1}_{Ga,Gb}$ and γ established in Lemma 14.4.14. The right unit constraint and associativity conditions are similar. This completes the proof that $G \colon \mathcal{L} \rightsquigarrow \mathcal{K}$ defines a quasi-pseudofunctor.

By construction, the quasi-pseudofunctor G is a local equivalence, its action on homs defined by composing an equivalence with a map induced by pre- and post-composing with an equivalence in the ∞-cosmos $\mathcal{L}$, which is then an equivalence by Corollary 1.4.9. We use this local equivalence to argue that for each $x \in \mathcal{K}$, there is an equivalence $\eta_x \colon x \xrightarrow{\sim} GFx$, proving essential surjectivity of G. This component is defined by applying the specified inverse adjoint equivalence $F^{-1}_{x,Gfx} \colon \mathcal{L}(Fx, FGFx) \xrightarrow{\sim} \mathcal{K}(x, GFx)$ of (γ) to the inverse of the specified adjoint equivalence $\epsilon^{-1}_{Fx} \colon Fx \to FGFx$ of (β). Since F is a cosmological biequivalence, which caries the map η_x to an equivalence in $\mathcal{L}$, it is easily verified that η_x is itself an equivalence in $\mathcal{K}$. Thus, the quasi-pseudofunctor G is an biequivalence.

It remains only to check quasi-pseudonaturality of η and ϵ. For the latter, we define the component natural isomorphism by the pasting diagram

$$\epsilon^{a,b} := \begin{tikzcd} \mathcal{L}(a,b) \arrow[r, "-\circ\epsilon_a"] \arrow[d, equal] & \mathcal{L}(FGa,b) \arrow[r, "\epsilon_b^{-1}\circ-"] \arrow[rrrd, equal, "\cong\beta"'] & \mathcal{L}(FGa,FGb) \arrow[r, "F^{-1}_{Ga,Gb}"] \arrow[rr, equal, bend right, "\cong\gamma"'] & \mathcal{K}(Ga,Gb) \arrow[r, "F_{Ga,Gb}"] & \mathcal{L}(FGa,FGb) \arrow[d, "\epsilon_b\circ-"] \\ \mathcal{L}(a,b) \arrow[rrrr, "-\circ\epsilon_a"'] & & & & \mathcal{L}(FGa,b) \end{tikzcd}$$

For the former, using the definition $\eta_x := F^{-1}\epsilon_{Fx}^{-1}$ and the quasi-pseudonaturality of $F^{-1}_{-,-}$, we have a pasting diagram

$$\eta^{x,y} := \begin{tikzcd} \mathcal{K}(x,y) \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{L}(Fx,Fy) \arrow[r, "(-\circ\epsilon_{Fx}\text{,}\epsilon_{Fy}^{-1}\circ-)"] \arrow[dr, "\epsilon_{Fy}^{-1}\circ-"'] & \mathcal{L}(FGFx,FGFy) \arrow[r, "F^{-1}_{GFx,GFy}"] \arrow[d, "-\circ\epsilon_{Fx}^{-1}"] \arrow[dl, phantom, "\beta\cong"] \arrow[dr, phantom, "\gamma\cong"] & \mathcal{K}(GFx,GFy) \arrow[d, "-\circ\eta_x"] \\ \mathcal{K}(x,y) \arrow[rrr, bend right, "\eta_y\circ-"'] & & \mathcal{L}(Fx,FGFy) \arrow[r, "F^{-1}_{x,GFy}"] & \mathcal{K}(x,GFy) \end{tikzcd}$$

which defines component natural isomorphism. We leave the verification that these natural transformations satisfy the unit and composition coherence conditions to define quasi-pseudonatural equivalences $\eta\colon \mathrm{id}_{\mathcal{K}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$ to the reader. ☐

It is easily verified that composites of quasi-pseudofunctors are quasi-pseudofunctors and composites of biequivalences are biequivalences. Hence:

14.4.16. Corollary. *Any zig-zag of cosmological biequivalences composes to define a quasi-pseudofunctor $\mathcal{K} \rightsquigarrow \mathcal{L}$ between ∞-cosmoi that is also a biequivalence.* ☐

Moreover, the preservation and reflection properties of cosmological biequivalences established in Proposition 14.3.5 extend to their quasi-pseudofunctorial inverses. For future reference, we prove one result in this vein in detail.

14.4.17. Lemma. *A quasi-pseudofunctor $G\colon \mathcal{L} \rightsquigarrow \mathcal{K}$ defined as an pseudoinverse to a cosmological biequivalence $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ preserves and reflects comma ∞-categories: a cell as on the left is a comma cone in $\mathcal{L}$ if and only if its image is a comma cone in $\mathcal{K}$.*

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_0"] & & & & (GE)' \arrow[dl, "(Ge_1)'"'] \arrow[dr, "(Ge_0)'"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\overset{\epsilon}{\Leftarrow}"] & & B \arrow[dl, "f"] \arrow[r, phantom, "\overset{G}{\rightsquigarrow}"] & {} & GC \arrow[dr, "Gg"'] \arrow[rr, phantom, "\overset{G\epsilon}{\Leftarrow}"] & & GB \arrow[dl, "Gf"] \\ & A & & & & GA & \end{tikzcd}$$

Note there is no reason why the inverse to a cosmological biequivalence should preserve isofibrations, so the map $(Ge_1, Ge_0)\colon GE \to GC \times GB$ should be replaced by an equivalent isofibration $((Ge_1)', (Ge_0)')\colon GE' \twoheadrightarrow GC \times GB$.

Proof. Proposition 14.3.5(v) proves that the right-hand square defines a comma cone in $\mathcal{K}$ if and only if its image under the cosmological biequivalence $F\colon \mathcal{K} \rightsquigarrow \mathcal{L}$ defines an comma cone in $\mathcal{L}$. On account of the quasi-pseudonatural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{L}}$ of Proposition 14.4.15, the image of the

right-hand square is equivalent to the left-hand square. By a mild extension of Propositions 3.4.5 and 3.4.11, the universal property that characterizes the comma square, transfers across this equivalence. □

Exercises.

14.4.i. Exercise. For a fixed pair of 2-categories $\mathcal{C}$ and $\mathcal{D}$, show that the collection of pseudofunctors $\mathcal{C} \rightsquigarrow \mathcal{D}$, pseudonatural transformations, and modifications (see Definition B.2.3) assemble into a 2-category.

14.4.ii. Exercise. For a pseudonatural transformation $\phi\colon F \Rightarrow G$ between pseudofunctors $F, G\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$, show that the following are equivalent.

- Each 1-cell component $\phi_x\colon Fx \rightleftarrows Gx$ is an equivalence in $\mathcal{D}$.
- The 1-cell ϕ defines an equivalence in the 2-category described in Exercise 14.4.i.

14.4.iii. Exercise. Derive a proof of Proposition 14.4.3 from Proposition 14.4.15.

14.4.iv. Exercise. Let $\phi\colon F \Rightarrow G$ be a quasi-pseudonatural transformation between quasi-pseudofunctors $F, G\colon \mathcal{K} \rightsquigarrow \mathcal{L}$. Show that for any pair of 0-arrows $f\colon x \to y$ and $k\colon y \to z$ in the $\mathcal{K}$, the diagrams

$$\begin{tikzcd}
 & Fy \arrow[dr, "Fk"] & \\
Fx \arrow[ur, "Ff"] \arrow[rr, bend right, "F(kf)"'] \arrow[ur, phantom, shift left=6ex, "\alpha^{x,y,z}_{f,g}\Downarrow\cong"'] \arrow[dd, "\phi_x"'] & & Fz \arrow[dd, "\phi_z"] \\
 & \phi_{kf}\Downarrow\cong & \\
Gx \arrow[rr, bend right, "G(kf)"'] & & Gz
\end{tikzcd}
=
\begin{tikzcd}
 & Fy \arrow[dr, "Fk"] \arrow[d, "\phi_y"] & \\
Fx \arrow[ur, "Ff"] \arrow[dd, "\phi_x"'] \arrow[r, phantom, "\phi_f\Downarrow\cong"] & Gy \arrow[dr, "Gk"] \arrow[r, phantom, "\phi_k\Downarrow\cong"] & Fz \arrow[dd, "\phi_z"] \\
 & \alpha^{x,y,z}_{f,g}\Downarrow\cong & \\
Gx \arrow[uur, "Gf"] \arrow[rr, bend right, "G(kf)"'] & & Gz
\end{tikzcd}$$

and

$$\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[r, phantom, "\iota_x\Downarrow\cong"] \arrow[r, bend right, "F\,\mathrm{id}_x"'] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_{\mathrm{id}_x}\Downarrow\cong"] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend right, "G\,\mathrm{id}_x"'] & Gx
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "\mathrm{id}_{Fx}"] \arrow[d, "\phi_x"'] & Fx \arrow[d, "\phi_x"] \\
Gx \arrow[r, bend left, "\mathrm{id}_{Gx}"] \arrow[r, phantom, "\iota_y\Downarrow\cong"] \arrow[r, bend right, "G\,\mathrm{id}_x"'] & Gx
\end{tikzcd}$$

commute in the homotopy 2-category of $\mathcal{L}$.

14.4.v. Exercise. Finish the proof of Lemma 14.4.11.

CHAPTER 15

Proof of model independence

We begin in §15.1 by applying the results of Chapter 14 to give an ad hoc exploration of the model invariance of the fundamental ∞-categorical notions. This discussion previews the line of reasoning that underpins our main theorem in this chapter, which develops a more systematic approach to this model-independence question. Explicitly, in §15.3 we prove that a cosmological biequivalence induces a biequivalence of virtual equipments. In §15.2, we review the construction of the virtual equipment of modules associated to an ∞-cosmos and explain why it describes a suitable context for proving the model-independence of the fundamental ∞-categorical notions.

15.1. Ad hoc model invariance

An ad hoc approach to proving the model-independence of the basic category theory of ∞-categories is given by elaborations of Proposition 14.3.5, as described for instance in the following theorem:

15.1.1. Theorem (model independence of basic category theory I). *The following notions are preserved, reflected, and created by any cosmological biequivalence:*

(i) A left or right adjoint to a functor $u \colon A \to B$.
(ii) A limit or a colimit for a J-indexed diagram $d \colon 1 \to A^J$ in an ∞-category A.

Proof. For (i), if $u \colon A \to B$ admits a left adjoint $f \colon B \to A$, then this adjunction is preserved by any 2-functor, and hence by any cosmological functor $F \colon \mathcal{K} \to \mathcal{L}$. If there merely exists $f \colon B \to A$ in $\mathcal{K}$ so that $Ff \dashv Fu$ in $\mathcal{L}$, then in the case where F is a biequivalence, Corollary 14.3.2(iii) can be used to lift the unit and counit from $\mathcal{L}$ to define 2-cells $\eta \colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon \colon fu \Rightarrow \mathrm{id}_A$ in $\mathcal{K}$. A priori the composites $\epsilon f \cdot f\eta$ and $u\epsilon \cdot \eta u$ need not be identities, but they will be invertible and the standard 2-categorical argument appearing in the proof of Proposition 2.1.11 can be used to modify one of these 2-cells to produce a genuine adjunction $f \dashv u$. A similar line of reasoning can be used in the absence of a candidate left adjoint using Corollary 14.3.2(ii) to lift the left adjoint from $\mathcal{L}$ to a functor $f \colon B \to A$ in $\mathcal{K}$ whose image is isomorphic to the left adjoint to Fu. The rest of the argument proceeds as before.

For (ii), given an absolute lifting diagram in $\mathfrak{h}\mathcal{L}$ as displayed on the right

$$\begin{tikzcd} & FA \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\bar{\ell}"] \arrow[r, "Fd"'] \arrow[ur, phantom, "\Downarrow\bar{\lambda}", shift right=3] & FA^J \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\lambda", shift right=3] & A^J \end{tikzcd}$$

Corollary 14.3.2(ii) and (iii) yield a 1-cell $\ell \colon 1 \to A$ with $F\ell \cong \bar{\ell}$ and a 2-cell $\lambda \colon \Delta\ell \Rightarrow d$, as displayed on the right, so that $F\lambda$ composes with the isomorphism $F\ell \cong \bar{\ell}$ to $\bar{\lambda}$. Our task is to show that that this data defines an absolute right lifting diagram in $\mathcal{K}$, which amounts to showing for all objects X

and for all maps $a\colon X \to A$ that the composition operation with λ induces a bijection

$$\begin{tikzcd} \mathsf{hFun}(X,A)(a,\ell) \arrow[r, "\lambda\circ-"] \arrow[d, phantom, "\cong" rotate] & \mathsf{hFun}(X,A^J)(\Delta a, d!) \arrow[d, phantom, "\cong" rotate] \\ \mathsf{hFun}(FX,FA)(Fa,F\ell) \arrow[r, "F\lambda\circ-"] \arrow[d, phantom, "\cong" rotate] & \mathsf{hFun}(FX,FA^J)(\Delta Fa, Fd!) \arrow[d, phantom, "\cong" rotate] \\ \mathsf{hFun}(FX,FA)(Fa,\bar{\ell}) \arrow[r, "\bar{\lambda}\circ-"] & \mathsf{hFun}(FX,FA^J)(\Delta Fa, Fd!) \end{tikzcd}$$

On account of the isomorphisms of Corollary 14.3.2(ii) and (iii), this follows from the corresponding encoding of the universal property of $\bar{\lambda}$ in $\mathcal{L}$.

This argument also shows that the universal property of absolute lifting diagrams is reflected. To see that absolute lifting diagrams are also preserved, Corollary 14.3.2(i) must be invoked to lift any cone in $\mathcal{L}$ with summit Y over the 2-cell $F\lambda$ to an equivalent one in $\mathcal{K}$ with summit X chosen so that $FX \simeq Y$. As the local bijections Corollary 14.3.2(ii) and (iii) can also be defined relative to specified equivalences of objects, the same style of argument goes through. □

Special features of the homotopy 2-category of a particular ∞-cosmos can also easily be seen to transfer to biequivalent ∞-cosmoi. For instance, Corollary D.4.17 proves that a natural transformation between functors between quasi-categories $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a natural isomorphism if and only if it is a **pointwise isomorphism**,

Exercises.

15.1.i. Exercise. Pick your favorite ∞-categorical notion and give an ad hoc proof of its model independence.

15.2. The context for the model independence theorem

Recall from Chapter 12, that the **virtual double category of modules** $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ in an ∞-cosmos $\mathcal{K}$ consists of

- a category of **objects** and **vertical arrows**, here the ∞-categories and ∞-functors
- for any pair of objects A, B, a class of **horizontal arrows** $A \overset{E}{\twoheadrightarrow} B$, here the modules $(q,p)\colon E \twoheadrightarrow A \times B$ from A to B
- **cells**, with boundary depicted as follows

$$\begin{tikzcd} A_0 \arrow[r, "E_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "E_2", "\shortmid"{marking}] & \cdots \arrow[r, "E_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ B_0 \arrow[rrr, "F"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\Downarrow"] & & & B_n \end{tikzcd}$$

including those whose horizontal source has length zero, in the case $A_0 = A_n$. Here, a cell with the displayed boundary is an isomorphism class of objects in the functor space

$$\begin{tikzcd}
\mathsf{Fun}_{f,g}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, F) \arrow[d] \\
\mathbb{1} \arrow[r] & \mathsf{Fun}(E_1 \underset{A_1}{\times} \cdots \underset{A_{n-1}}{\times} E_n, B_0 \times B_n)
\end{tikzcd}$$

i.e., a fibered isomorphism class of maps of spans over $f \times g$

- a **composite cell**, for any configuration

$$\begin{tikzcd}
A_0 \arrow[r, "E_{11},\ldots,E_{1k_1}", "\mapstochar" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow"] & A_1 \arrow[r, "E_{21},\ldots,E_{2k_2}", "\mapstochar" marking] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, "E_{n1},\ldots,E_{nk_n}", "\mapstochar" marking] \arrow[d, phantom, "\cdots"] \arrow[dr, phantom, "\Downarrow"] & A_n \arrow[d, "f_n"] \\
B_0 \arrow[r, "F_1", "\mapstochar" marking] \arrow[d, "g"'] & B_1 \arrow[r, "F_2", "\mapstochar" marking] \arrow[dr, phantom, "\Downarrow"] & \cdots \arrow[r, "F_n", "\mapstochar" marking] & B_n \arrow[d, "h"] \\
C_0 \arrow[rrr, "G"', "\mapstochar" marking] & & & C_n
\end{tikzcd}$$

- an **identity cell** for every horizontal arrow

$$\begin{tikzcd}
A \arrow[r, "E", "\mapstochar" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_E"] & B \arrow[d, equal] \\
A \arrow[r, "E"', "\mapstochar" marking] & B
\end{tikzcd}$$

so that composition of cells is strictly associative and unital in the usual multi-categorical sense.

The virtual double category of modules is a **virtual equipment** in the sense introduced by Cruttwell and Shulman [**28**, §7], which means that it satisfies the two further properties:

(i) For any module and pair of functors as displayed on the left, there exists a module and cartesian cell as displayed on the right

$$\begin{tikzcd}
A' \arrow[d, "a"'] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "\mapstochar" marking] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A' \arrow[r, "E(b,a)", "\mapstochar" marking] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\rho"] & B' \arrow[d, "b"] \\
A \arrow[r, "E"', "\mapstochar" marking] & B
\end{tikzcd}$$

characterized by the universal property that any cell as displayed below-left factors uniquely through ρ as below-right:

$$\begin{tikzcd}
X_0 \arrow[r, "E_1", "\mapstochar" marking] \arrow[d, "af"'] \arrow[drrr, phantom, "\Downarrow"] & X_1 \arrow[r, "E_2", "\mapstochar" marking] & \cdots \arrow[r, "E_n", "\mapstochar" marking] & X_n \arrow[d, "bg"] \\
A \arrow[rrr, "E"', "\mapstochar" marking] & & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X_0 \arrow[r, "E_1", "\mapstochar" marking] \arrow[d, "f"'] \arrow[drrr, phantom, "\exists!\Downarrow"] & X_1 \arrow[r, "E_2", "\mapstochar" marking] & \cdots \arrow[r, "E_n", "\mapstochar" marking] & X_n \arrow[d, "g"] \\
A' \arrow[rrr, "E(b,a)", "\mapstochar" marking] \arrow[d, "a"'] \arrow[drrr, phantom, "\Downarrow\rho"] & & & B' \arrow[d, "b"] \\
A \arrow[rrr, "E"', "\mapstochar" marking] & & & B
\end{tikzcd}$$

(ii) Every object A admits a *unit* module equipped with a nullary cocartesian cell

$$\begin{tikzcd} A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[d, equal] \\ A \arrow[r, "\shortmid" marking, "\mathsf{Hom}_A"'] & A \end{tikzcd}$$

satisfying the universal property that any cell in the virtual double category of modules whose horizontal source includes the object A, as displayed on the left

$$\begin{tikzcd}[column sep=small] X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking, "E_n"] & A \arrow[r, "\shortmid" marking, "F_1"] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] & Y \arrow[d, "g"] \\ B \arrow[rrrr, "\shortmid" marking, "G"'] & & & & C \end{tikzcd} = \begin{tikzcd}[column sep=small] X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_1}"] & \cdots \arrow[r, "\shortmid" marking, "E_n"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{E_n}"] & A \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\iota"] & A \arrow[r, "\shortmid" marking, "F_1"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_1}"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{id}_{F_m}"] & Y \arrow[d, equal] \\ X \arrow[r, "\shortmid" marking, "E_1"] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking, "E_n"] & A \arrow[r, "\shortmid" marking, "\mathsf{Hom}_A"] \arrow[d, phantom, "\Downarrow\exists!"] & A \arrow[r, "\shortmid" marking, "F_1"] & \cdots \arrow[r, "\shortmid" marking, "F_m"] & Y \arrow[d, "g"] \\ B \arrow[rrrrr, "\shortmid" marking, "G"'] & & & & & C \end{tikzcd}$$

factors uniquely through ι as displayed on the right.

The module in (i) is defined by pulling back a module $A \overset{E}{\nrightarrow} B$ along functors $a\colon A' \to A$ and $b\colon B' \to B$. The simplicial pullback defining $E(b,a)$ induces an equivalence of functor spaces

$$\mathsf{Fun}_{af,bg}(E_1 \times \cdots \times E_n, E) \simeq \mathsf{Fun}_{f,g}(E_1 \times \cdots \times E_n, E(b,a)),$$

which gives rise to the universal property. See Proposition 12.2.1.

The unit module is the arrow ∞-category, given the notation $A \overset{\mathsf{Hom}_A}{\nrightarrow} A$ when considered as a module from A to A. The universal property of (ii) follows from the Yoneda lemma; see Proposition 12.2.4.

The virtual equipment of modules in $\mathcal{K}$ has a lot of pleasant properties, which follow formally from the axiomatization of a virtual equipment. For instance, certain sequences of composable modules can be said to *have composites*, witnessed by cocartesian cells as in (ii) (see 12.3.5, 12.3.11, 12.4.7, and 12.4.8 and the non-formal composite given in 12.3.7). Also, for any functor $f\colon A \to B$, the modules $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ and $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ behave like adjoints is a sense suitable to a virtual double category; more precisely, the module $A \overset{\mathsf{Hom}_B(B,f)}{\nrightarrow} B$ defines a *companion* and the module $B \overset{\mathsf{Hom}_B(f,B)}{\nrightarrow} A$ defines a *conjoint* to $f\colon A \to B$ (see 12.4.1, 12.4.4, and 12.4.6). Another consequence of Theorem 12.2.6 is the following: for any parallel pair of functors there are natural bijections between 2-cells in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

and cells in the virtual equipment of modules:

$$\begin{tikzcd} A \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B(B,f)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ A \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B(B,g)"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\shortmid" marking, "\mathsf{Hom}_A"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B(g,B)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\ B \arrow[r, "\shortmid" marking, "\mathsf{Hom}_B(f,B)"'] & A \end{tikzcd}$$

See Proposition 12.4.10.

As remarked upon in Definition 12.4.11, as a consequence of these results, there are two locally-fully-faithful homomorphisms $\mathfrak{h}\mathcal{K} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K})$ and $\mathfrak{h}\mathcal{K}^{\mathrm{coop}} \hookrightarrow \mathcal{M}od(\mathfrak{h}\mathcal{K})$ — referred to as the *covariant* and *contravariant embeddings*, respectively — embedding the homotopy 2-category into the substructure[1] of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ comprised only of unary cells whose vertical boundaries are identities. The modules in the image of the covariant embedding are the right representables and the modules in the image of the contravariant embedding are the left representables.

The theme of Chapter 13 could be summarized by saying that the virtual equipment of modules in an ∞-cosmos is a robust setting to develop the category theory of ∞-categories. On the one hand, it contains the homotopy 2-category of the ∞-cosmos, which was the setting for the results of Part I. It is also a very natural home to study ∞-categorical properties that are somewhat awkward to express in the homotopy 2-category. For instance, the weak 2-universal property of comma ∞-categories is now encoded by a bijection in Lemma 12.1.16: cells in the virtual equipment whose codomain is a comma module correspond bijectively to natural transformations of a particular form in the homotopy 2-category. Fibered equivalences of modules, as used to express the universal properties of adjunctions, limits, and colimits in Chapter 4, are now vertical isomorphisms in the virtual equipment between parallel modules. The virtual equipment also cleanly encodes the universal property of pointwise left and right Kan extensions.

In this chapter, we show that a cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces a *biequivalence of virtual equipments* $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \xrightarrow{\sim} \mathcal{M}od(\mathfrak{h}\mathcal{L})$ in a suitable sense that we introduce to describe what is true in our setting. We then explain the interpretation of this result: that the category theory of ∞-categories is preserved, reflected, and created by any "change-of-model" functor of this form.

15.3. A biequivalence of virtual equipments

We first elaborate on Proposition 14.3.1(ix).

15.3.1. Proposition. *A cosmological biequivalence $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ preserves, reflects, and creates modules:*

- *(i) An isofibration $E \twoheadrightarrow A \times B$ is a module in $\mathcal{K}$ if and only if $FE \twoheadrightarrow FA \times FB$ is a module in $\mathcal{L}$.*
- *(ii) For every module $G \twoheadrightarrow A' \times B'$ in $\mathcal{L}$ and every pair of ∞-categories A, B in $\mathcal{K}$ with specified equivalences $FA \simeq A'$ and $FB \simeq B'$ there is a module $E \twoheadrightarrow A \times B$ in $\mathcal{K}$ so that FE is equivalent to G over the pair of equivalences.*

Proof. For (i) consider an isofibration $E \twoheadrightarrow A \times B$ in $\mathcal{K}_{/A\times B}$ and the induced biequivalence $F\colon \mathcal{K}_{/A\times B} \xrightarrow{\sim} \mathcal{L}_{/FA\times FB}$ of Proposition 14.2.4. By Proposition 14.3.5(iii), E defines a discrete object in $\mathcal{K}_{/A\times B}$ if and only if FE defines a discrete object in $\mathcal{L}_{/FA\times FB}$. By Theorem 15.1.1(i), $E \to \mathrm{Hom}_B(B,p)$ admits a right adjoint and $E \to \mathrm{Hom}_A(q,A)$ admits a left adjoint in $\mathcal{K}_{/A\times B}$ if and only if $FE \to F\mathrm{Hom}_B(B,p) \cong \mathrm{Hom}_{FB}(FB,Fp)$ admits a right adjoint and $FE \to F\mathrm{Hom}_A(q,A) \cong \mathrm{Hom}_{FA}(Fq,FA)$ admits a left adjoint in $\mathcal{L}_{/FA\times FB}$. By Lemma 11.4.2 these properties characterize modules.

For (ii), fix a pair of equivalences $FA \simeq A'$ and $FB \simeq B'$, defining an equivalence $e\colon A'\times B' \xrightarrow{\sim} FA \times FB$, and consider the composite biequivalence

$$\mathcal{K}_{/A\times B} \xrightarrow[\sim]{F} \mathcal{L}_{/FA\times FB} \xrightarrow[\sim]{e^*} \mathcal{L}_{/A'\times B'}$$

given by Propositions 14.2.4 and 14.2.6. Consider a module $G \twoheadrightarrow A' \times B'$. By essential surjectivity, there is an isofibration $E \twoheadrightarrow A \times B$ whose image under this cosmological functor—the pullback of

[1] This substructure is very nearly a bicategory, with horizontal composites of unary cells constructed as in Definition 12.3.12, except that compatible sequences of modules do not always admit a horizontal composite.

$FE \twoheadrightarrow FA \times FB$ along $e\colon A' \times B' \simeq FA \times FB$—defines an isofibration $(q,p)\colon E' \twoheadrightarrow A' \times B'$ that is equivalent to G in $\mathcal{L}_{/A'\times B'}$. It remains only to argue that E defines a module from A to B, which will follow, essentially as in the proof of (i), from the fact that $E' \simeq G$ defines a module from A' to B'.

As the image E' of E is equivalent to a discrete object, Proposition 14.3.5(iii) tells us E is discrete in $\mathcal{K}_{/A\times B}$. The final step is to argue that the desired right adjoint to $E \to \mathsf{Hom}_B(B,p)$ is present in the image of the biequivalence $\mathcal{K}_{/A\times B} \simeq \mathcal{L}_{/A'\times B'}$, and apply Theorem 15.1.1(i) to deduce its presence in $\mathcal{K}_{/A\times B}$; a similar argument of course applies to the functor $E \to \mathsf{Hom}_A(q,A)$. To see this note that $F\colon \mathcal{K}_{/A\times B} \simeq \mathcal{L}_{/FA\times FB}$ carries $\mathsf{Hom}_B(B,p) \twoheadrightarrow A\times B$ to $\mathsf{Hom}_{FB}(FB,Fp) \twoheadrightarrow FA\times FB$. By (i), it suffices to argue that this functor has a right adjoint over $FA\times FB$. Applying Theorem 15.1.1(i) to the biequivalence $e^*\colon \mathcal{L}_{/FA\times FB} \simeq \mathcal{L}_{/A'\times B'}$, this follows from the fact that $E' \to \mathsf{Hom}_{B'}(B',p')$ has a right adjoint over $A'\times B'$. □

15.3.2. Proposition. *Let $F\colon \mathcal{K} \simeq \mathcal{L}$ be a cosmological biequivalence. Then a module $A \overset{E}{\nrightarrow} B$ in $\mathcal{K}$ is right representable if and only if the module $FA \overset{FE}{\nrightarrow} FB$ is right representable in $\mathcal{L}$, in which case, F carries the representing functor $f\colon A \to B$ in $\mathcal{K}$ to a representing functor $Ff\colon FA \to FB$ in $\mathcal{L}$.*

Proof. To say that $A \overset{E}{\nrightarrow} B$ is right representable in $\mathcal{K}$ is to say that there exists a functor $f\colon A \to B$ together with an equivalence $E \simeq_{A\times B} \mathsf{Hom}_B(B,f)$ of modules over B. If this is the case then any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ carries this to a fibered equivalence $FE \simeq_{FA\times FB} \mathsf{Hom}_{FB}(FB,Ff)$, and hence the module $FA \overset{FE}{\nrightarrow} FB$ is right-represented by $Ff\colon FA \to FB$ in $\mathcal{L}$.

Conversely, if $FA \overset{FE}{\nrightarrow} FB$ is right-represented by some functor $g\colon FA \to FB$, then by Corollary 14.3.2(ii), there exists a functor $f\colon A \to B$ in $\mathcal{K}$ so that $Ff \cong g$ in $\mathcal{L}$. By Proposition 12.4.10, naturally isomorphic functors represent equivalent modules; that is, $\mathsf{Hom}_{FB}(FB,g) \simeq_{FA\times FB} \mathsf{Hom}_{FB}(FB,Ff)$. Thus $FE \simeq_{FA\times FB} \mathsf{Hom}_{FB}(FB,Ff)$. By Corollary 14.3.2(i), this fibered equivalence lifts along the cosmological functor $F\colon \mathcal{K}_{/A\times B} \to \mathcal{L}_{/FA\times FB}$ to a fibered equivalence $E \simeq_{A\times B} \mathsf{Hom}_B(B,f)$, which proves that E is right represented by $f\colon A \to B$ in $\mathcal{K}$. □

Because cosmological functors preserve modules, simplicial pullbacks, and the arrow construction, we see that a functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a functor of virtual equipments $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$ preserving all of the structures described in Theorem 12.2.6.

15.3.3. Theorem (model independence of ∞-category theory). *If $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence, then the induced functor of virtual equipments*

$$F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$$

defines a ***biequivalence of virtual equipments****: i.e., it is*

(i) bijective on equivalence classes of objects;
(ii) locally bijective on isomorphism classes of parallel vertical functors extending the bijection of (i);
(iii) locally bijective on equivalence classes of parallel modules extending the bijection of (ii);
(iv) locally bijective on cells extending the bijections of (i), (ii), and (iii).

Consequently, any ∞-categorical notion that can be encoded as an equivalence-invariant proposition in the virtual equipment of modules is model invariant: preserved, reflected, and created by cosmological biequivalences.

Note further that if two ∞-cosmoi are connected by a finite zig-zag of biequivalences, then the bijections described in Theorem 15.3.3 compose.

Proof. Properties (i) and (ii) are restatements of Corollary 14.3.2(i) and (ii).

The local bijection (iii) follows immediately from Proposition 15.3.1 and the fact that for any pair of equivalences $e\colon A' \times B' \xrightarrow{\sim} FA \times FB$, the composite biequivalence

$$\mathcal{K}_{/A\times B} \xrightarrow[\sim]{F} \mathcal{L}_{/FA\times FB} \xrightarrow[\sim]{e^*} \mathcal{L}_{/A'\times B'}$$

preserves, reflects, and creates equivalences between objects (Corollary 14.3.2(i)). Finally (iv) is an application of Corollary 14.3.2(ii) to this cosmological biequivalence. □

For instance, the presence of an adjunctions between ∞-categories and the existence of limits and colimits inside an ∞-category can both be encoded as an equivalence-invariant proposition in the virtual equipment of modules. Using Theorem 15.3.3, we can reprove Theorem 15.1.1.

Module-theoretic proof of Theorem 15.1.1. By Proposition 4.1.1, $u\colon A \to B$ admits a left adjoint if and only if the module $A \overset{\mathrm{Hom}_B(B,u)}{\twoheadrightarrow} B$ is contravariantly represented. Any representing functor in $\mathcal{L}$ lifts up to isomorphism to define a functor $f\colon B \to A$ in $\mathcal{K}$ and now by Theorem 15.3.3(iii) there is an equivalence $\mathrm{Hom}_A(f, A) \simeq \mathrm{Hom}_B(B, u)$ of modules from A to B in $\mathcal{K}$ if and only if there is an equivalence $\mathrm{Hom}_{FA}(Ff, FA) \simeq \mathrm{Hom}_{FB}(FB, fu)$ of modules from FA to FB in $\mathcal{L}$.

Similarly, a diagram $d\colon 1 \to A^J$ admits a limit if and only if the module $1 \overset{\mathrm{Hom}_{A^J}(\Delta,d)}{\twoheadrightarrow} A$ is covariantly represented, and Proposition 15.3.2 or the argument just given completes the proof in this case as well. □

A biequivalence of virtual equipments preserves, reflects, and creates composites of modules.

15.3.4. Lemma. *Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence.*

(i) Then a composable sequence of modules in $\mathcal{K}$ admits a composite in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ if and only if the image of this sequence admits a composite in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$.

(ii) Hence, cosmological biequivalences preserve and reflect exact squares.

Proof. Via Definition 13.2.2, (ii) follows immediately from (i), so it remains only to show that a biequivalence of virtual equipments $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \xrightarrow{\sim} \mathcal{M}od(\mathfrak{h}\mathcal{L})$ preserves, reflects, and creates composites of modules. To see that an n-ary composite cell $\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$ in $\mathfrak{h}\mathcal{K}$ is preserved, note that by Theorem 15.3.3(iv), any cell in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ is isomorphic to a cell in the image of F: first replace the objects by equivalent ones in the image, then replace the vertical functors by naturally isomorphic ones in the image, then replace the modules by equivalent ones in the image over the specified equivalences between their ∞-categorical sources and targets, and then finally apply the local bijection (iv) to replace the cell in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ by a unique cell in the image of $\mathcal{M}od(\mathfrak{h}\mathcal{K})$ by composing with this data. Now, by local full and faithfulness and essential surjectivity, the universal property of the cocartesian cell $\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$ implies that its image $F\mu\colon FE_1 \times \cdots \times FE_n \Rightarrow FE$ is again a cocartesian cell. Thus composites $E_1 \otimes \cdots \otimes E_n \simeq E$ are preserved by cosmological biequivalences.

Now if $F\mu\colon FE_1 \times \cdots \times FE_n \Rightarrow FE$ is a composite, the fact that $F\colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$ is locally fully faithful implies immediately that $\mu\colon E_1 \times \cdots \times E_n \Rightarrow E$ is a composite; thus composites $FE_1 \otimes \cdots \otimes FE_n \simeq FE$ are reflected by cosmological biequivalences.

Finally, suppose the sequence $FE_1 \times \cdots \times FE_n$ of modules in $\mathcal{M}od(\mathfrak{h}\mathcal{L})$ admits a composite $FA_0 \overset{G}{\twoheadrightarrow} FA_n$; since the composable sequence of modules is in the image of F the source and target ∞-categories of the composite module G are as well. By Theorem 15.3.3(iii) there exists a module $A_0 \overset{E}{\twoheadrightarrow} A_n$ in $\mathfrak{h}\mathcal{K}$ so that $FE \simeq G$ as modules from FA_0 to FA_n. The cocartesian cell $FE_1 \times \cdots \times$

$FE_n \Rightarrow G$ composes with the unary cell of this equivalence to define a cocartesian cell $FE_1 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} FE_n \Rightarrow FE$. By Theorem 15.3.3(iv), this lifts to an n-ary cell $E_1 \mathbin{\dot{\times}} \cdots \mathbin{\dot{\times}} E_n \Rightarrow E$ in $\mathcal{M}od(\mathfrak{h}\mathcal{K})$. As we've just seen that cocartesianness of cells is reflected by biequivalences $F \colon \mathcal{M}od(\mathfrak{h}\mathcal{K}) \to \mathcal{M}od(\mathfrak{h}\mathcal{L})$, this completes the proof that composites are created by cosmological biequivalences. □

Exercises.

15.3.i. Exercise. Prove that cosmological biequivalences between cartesian closed ∞-cosmoi preserve and reflect initial and final functors.

CHAPTER 16

Applications of model-independence

In this chapter, we establish some elementary properties of a certain class of ∞-cosmoi we might call **∞-cosmoi of $(\infty,1)$-categories**, by which we mean ∞-cosmoi that are biequivalent to $\mathcal{QC}at$. By Proposition 14.2.3 an ∞-cosmos $\mathcal{K}$ is an ∞-cosmos of $(\infty,1)$-categories if and only if its underlying quasi-category functor $(-)_0 \coloneqq \mathsf{Fun}(1,-)\colon \mathcal{K} \to \mathcal{QC}at$ is a biequivalence — meaning that every quasi-category is equivalent to the underlying quasi-category of an ∞-category in $\mathcal{K}$ and that for any $A, B \in \mathcal{K}$ the map $\mathsf{Fun}(A,B) \to B_0^{A_0}$ is an equivalence of quasi-categories. Several examples of ∞-cosmoi of this form are established in §E.2.

The aim is to illustrate how the model-independence theorem can be used to combine synthetic and analytic techniques to prove results concerning any family of biequivalent ∞-cosmoi. In what follows we appeal to explicit descriptions of $(\infty,1)$-categories as quasi-categories to supply analytic proofs of certain key results — for instance, that a functor defines an equivalence of quasi-categories just when it is fully faithful and essential surjective in a suitable sense. We then explain how the model-independence theorem can be used to transfer these results to biequivalent ∞-cosmoi. In this way, we conclude that any functor defines an equivalence of $(\infty,1)$-categories just when it is fully faithful and essentially surjective, even though we can't translate the specific proof of the quasi-categorical case of this result. We then apply this result to further develop the synthetic theory of ∞-cosmoi of $(\infty,1)$-categories.

Many of the results that follow could have appeared earlier, but in the presence of the results of Chapters 14 and 15 their conclusions apply more broadly, to all ∞-cosmoi of $(\infty,1)$-categories, not just in the quasi-categorical case. In particular, we discuss some special features of the ∞-cosmos of quasi-categories, proving in particular that universal properties in this ∞-cosmos are determined pointwise, again appealing to model-independence to generalize this result to other ∞-cosmoi of $(\infty,1)$-categories.

16.1. Cores and opposites of $(\infty,1)$-categories

The construction of the co-dual of an ∞-cosmos in Definition 1.2.23 makes use of the construction of the opposite of a simplicial set. We start by exploring the role played by this operation in the ∞-cosmos of quasi-categories and then investigate a related operation on other ∞-cosmoi of ∞-categories.

16.1.1. Recall. There is an identity-on-objects involution $(-)^\circ\colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering of the elements in each ordinal $[n] \in \mathbf{\Delta}$, sending a face map $\delta^i\colon [n-1] \rightarrowtail [n]$ to the face map $\delta^{n-i}\colon [n-1] \rightarrowtail [n]$ and sending the degeneracy map $\sigma^i\colon [n+1] \twoheadrightarrow [n]$ to the degeneracy map $\sigma^{n-i}\colon [n+1] \twoheadrightarrow [n]$. Precomposition with $(-)^\circ$ defines a functor $(-)^{\mathrm{op}}\colon s\mathcal{S}et \to s\mathcal{S}et$ which carries a simplicial set X to its **opposite simplicial set** X^{op}.

16.1.2. Lemma. *If X is a quasi-category, then X^{op} is a quasi-category.*

PROOF. The lifting problem below-left is solved by the lifting problem below-right

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X^{\mathrm{op}} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{n-k}[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

□

16.1.3. DEFINITION. For a quasi-category A, its **opposite quasi-category** A^{op} is the simplicial set defined by "reversing the ordering of the elements in each ordinal."

16.1.4. LEMMA. *The opposite quasi-category construction defines a cosmological functor* $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$ *that acts on functor spaces via a natural isomorphism*

$$\mathsf{Fun}(A,B) \xrightarrow{\cong} \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}.$$

PROOF. The isomorphism $\mathsf{Fun}(A,B) \cong \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}$ is best understood at the level of simplices: the simplicial maps $A \times \Delta[n] \to B$ that define n-simplices in the left-hand functor space correspond bijectively via the isomorphism $(-)^{\mathrm{op}}\colon s\mathcal{S}et \to s\mathcal{S}et$ to simplicial maps $A^{\mathrm{op}} \times \Delta[n]^{\mathrm{op}} \to B^{\mathrm{op}}$, these defining the n-simplices in the right-hand functor space.

By an extension of the proof of Lemma 16.1.2, the opposite of an isofibration is an isofibration. The conical limits in $\mathcal{QC}at$, being defined pointwise in $\mathcal{S}et$, are clearly preserved by restriction along $(-)^{\circ}$. Simplicial cotensors are also preserved: for a quasi-category B and a simplicial set X, $(B^X)^{\mathrm{op}} \cong (B^{\mathrm{op}})^{X^{\mathrm{op}}}$, which accords with the general construction of the cotensor of $B^{\mathrm{op}} \in \mathcal{QC}at^{\mathrm{co}}$ with a simplicial set X as noted in Definition 1.2.23. Alternatively, one can use the natural isomorphism $\mathsf{Fun}(A,B) \cong \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}$ to directly verify that the quasi-category $(B^X)^{\mathrm{op}}$ has the universal property that characterizes the cotensor in $\mathcal{QC}at^{\mathrm{co}}$. □

This extends the usual construction of the opposite of a 1-category and the corresponding 2-functor $(-)^{\mathrm{op}}\colon \mathcal{C}at \to \mathcal{C}at^{\mathrm{co}}$. On account of the explicitness of the construction given in Definition 16.1.3, the opposite of a quasi-category is defined up to isomorphism. By contrast, without any additional hypotheses, we'll only be able to define the opposite of an ∞-category in a biequivalent ∞-cosmos up to equivalence. While at first this may seem undesirable, it is arguably morally correct to give the definition in this manner, since from the model-independent point of view, the ∞-category itself ought only be considered up to equivalence.

16.1.5. DEFINITION. Let A be an ∞-category in an ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$. Define the opposite ∞-category A^{op} to be any ∞-category whose underlying quasi-category is A_0^{op}.

By Proposition 14.2.3 and Corollary 14.3.2, the underlying quasi-category functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is bijective on equivalence classes of objects; hence Definition 16.1.5 is well-defined up to equivalence. This also proves

16.1.6. PROPOSITION. *Let* $\mathcal{K}$ *be an* ∞*-cosmos of* $(\infty,1)$*-categories.*

(i) Any ∞*-category in* $\mathcal{K}$ *has an opposite, well-defined up to equivalence.*
(ii) For any ∞*-categories* $A,B \in \mathcal{K}$*, there is an equivalence* $\mathsf{Fun}(A,B) \simeq \mathsf{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}$ *that is quasi-pseudonatural in* A *and in* B*.*

(iii) *Moreover, any specified choices in (i) and (ii) assemble into a quasi-pseudofunctor* $(-)^{\mathrm{op}}\colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$ *defined as the composite of the zig-zag of cosmological biequivalences*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "(-)_0", "\sim"'] \arrow[d, dashed, "(-)^{\mathrm{op}}"', "\wr"] \arrow[dr, phantom, "\cong"] & \mathcal{QC}at \arrow[d, "\cong"', "(-)^{\mathrm{op}}"] \\ \mathcal{K}^{\mathrm{co}} \arrow[r, "\sim", "(-)_0"'] & \mathcal{QC}at^{\mathrm{co}} \end{tikzcd}$$

Proof. The underlying quasi-category functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is bijective on equivalence classes of objects by Proposition 14.2.3 and Corollary 14.3.2. Since the underlying quasi-category A_0 of any ∞-category $A \in \mathcal{K}$ has an opposite, this proves that A must have an opposite as well, and thus Definition 16.1.5 is well-defined up to equivalence, with A^{op} characterized by the equivalence of quasi-categories

$$\mathrm{Fun}(1, A^{\mathrm{op}}) \simeq \mathrm{Fun}(1, A)^{\mathrm{op}}.$$

For ∞-categories A and B, by construction $(A^{\mathrm{op}})_0 \simeq (A_0)^{\mathrm{op}}$ and $(B^{\mathrm{op}})_0 \simeq (B_0)^{\mathrm{op}}$. Composing with these equivalences, the biequivalence $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ provides local equivalences of quasi-categories:

$$\begin{tikzcd} \mathrm{Fun}(A,B) \arrow[rr, "\sim"] \arrow[d, dashed, "\wr"'] \arrow[drr, phantom, "\cong"] & & \mathrm{Fun}(A_0,B_0) \arrow[d, "\cong"] \\ \mathrm{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "\sim"] & \mathrm{Fun}((A^{\mathrm{op}})_0,(B^{\mathrm{op}})_0)^{\mathrm{op}} \arrow[r, "\sim"] & \mathrm{Fun}((A_0)^{\mathrm{op}},(B_0)^{\mathrm{op}})^{\mathrm{op}} \end{tikzcd}$$

which compose to define the desired equivalence in such a way that the square commutes up to a homotopy coherent isomorphism $\mathrm{Fun}(A,B) \times \mathbb{I} \to \mathrm{Fun}((A_0)^{\mathrm{op}},(B_0)^{\mathrm{op}})^{\mathrm{op}}$.

The final claim is a special case of Corollary 14.4.16, whose proof specializes to recover the constructions in (i) and (ii). This also proves the quasi-pseudonaturality statement of (ii), via Lemma 14.4.11. ☐

On account of the equivalence $\mathrm{Fun}(A,B) \simeq \mathrm{Fun}(A^{\mathrm{op}},B^{\mathrm{op}})^{\mathrm{op}}$, a functor between ∞-categories $f\colon A \to B$ has an opposite functor $f^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$, well-defined up to isomorphism. Furthermore:

16.1.7. Lemma. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty,1)$-categories.*

(i) *For any ∞-category A and simplicial set U, $(A^U)^{\mathrm{op}} \simeq (A^{\mathrm{op}})^{U^{\mathrm{op}}}$.*
(ii) *For any functors $f\colon B \to A$ and $g\colon C \to A$, $\mathrm{Hom}_A(f,g)^{\mathrm{op}} \simeq \mathrm{Hom}_{A^{\mathrm{op}}}(g^{\mathrm{op}}, f^{\mathrm{op}})$ over $B^{\mathrm{op}} \times C^{\mathrm{op}}$.*

Proof. We have a quasi-pseudonatural equivalence:

$$\begin{aligned} \mathrm{Fun}(X,(A^U)^{\mathrm{op}}) &\simeq \mathrm{Fun}(X^{\mathrm{op}},A^U)^{\mathrm{op}} && \text{by } \mathbf{16.1.6}(\mathit{ii}) \\ &\cong (\mathrm{Fun}(X^{\mathrm{op}},A)^U)^{\mathrm{op}} && \text{by (1.2.6)} \\ &\cong (\mathrm{Fun}(X^{\mathrm{op}},A)^{\mathrm{op}})^{U^{\mathrm{op}}} && \text{by } \mathbf{16.1.4} \\ &\simeq \mathrm{Fun}(X,A^{\mathrm{op}})^{U^{\mathrm{op}}} && \text{by } \mathbf{16.1.6}(\mathit{ii}) \\ &\cong \mathrm{Fun}(X,(A^{\mathrm{op}})^{U^{\mathrm{op}}}) && \text{by (1.2.6).} \end{aligned}$$

Hence, by Lemma 14.4.13, $(A^U)^{\mathrm{op}} \simeq (A^{\mathrm{op}})^{U^{\mathrm{op}}}$.

The second statement is an application of Lemma 14.4.17 to the composite quasi-pseudofunctorial biequivalence $(-)^{\mathrm{op}}\colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$ established in Proposition 16.1.6. ☐

We now argue that the opposite of an ∞-category behaves like you'd expect. For instance, Exercise 16.1.i reveals that $\mathsf{h}(A^{\mathrm{op}}) \simeq (\mathsf{h}A)^{\mathrm{op}}$. More importantly, the following result provides another perspective on "appeals to duality" where facts about colimits of diagrams in $\mathcal{K}$ were deduced from corresponding proofs about limits in $\mathcal{K}^{\mathrm{co}}$, and similarly results about cartesian fibrations were interpreted in $\mathcal{K}^{\mathrm{co}}$ to conclude the corresponding results about cocartesian fibrations in $\mathcal{K}$.

16.1.8. Proposition. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty,1)$-categories.*

(i) A J-shaped family of diagrams in A has a colimit if and only if the transposed J^{op}-shaped family of diagrams in A^{op} has a limit.

(ii) A functor $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ defines a cocartesian fibration.

Note that if $p\colon E \twoheadrightarrow B$ is an isofibration, it is always possible to choose a functor $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ that is again an isofibration, perhaps by changing the choice of total space E^{op}.

Proof. By Lemma 16.1.7, a J-shaped family of diagrams $d\colon D \to A^J$ transposes to define a J^{op}-shaped family of diagrams $d^{\mathrm{op}}\colon D^{\mathrm{op}} \to (A^{\mathrm{op}})^{J^{\mathrm{op}}}$. By Proposition 4.3.1, d admits a colimit in A if and only if there is an equivalence of comma ∞-categories

$$\mathsf{Hom}_{A^J}(d,\Delta) \simeq_{A\times D} \mathsf{Hom}_A(c,A),$$

in which case the representing functor $c\colon D \to A$ defines the colimit functor.

By Lemma 16.1.7, such an equivalence exists if and only if

$$\mathsf{Hom}_{(A^{\mathrm{op}})^{J^{\mathrm{op}}}}(\Delta, d^{\mathrm{op}}) \simeq_{D^{\mathrm{op}}\times A^{\mathrm{op}}} \mathsf{Hom}_{A^{\mathrm{op}}}(A^{\mathrm{op}}, c^{\mathrm{op}}),$$

an equivalence which, by Proposition 4.3.1, characterizes the limit functor $c^{\mathrm{op}}\colon D^{\mathrm{op}} \to A^{\mathrm{op}}$.

The second statement is proven similarly. By Theorem 5.1.11, $p\colon E \twoheadrightarrow B$ defines a cartesian fibration if and only if the induced functor $k\colon E^{\mathbb{2}} \to \mathsf{Hom}_B(B,p)$ admits a right adjoint whose counit is an isomorphism. By applying the quasi-pseudofunctorial biequivalence $(-)^{\mathrm{op}}\colon \mathcal{K} \to \mathcal{K}^{\mathrm{co}}$, this adjunction exists if and only if the opposite functor admits a left adjoint whose unit is an isomorphism, and Lemma 16.1.7 identifies this opposite functor as $k^{\mathrm{op}}\colon (E^{\mathrm{op}})^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_{B^{\mathrm{op}}}(p^{\mathrm{op}}, B^{\mathrm{op}})$. By the dual of Theorem 5.1.11, such an adjunction exists if and only if $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ is a cocartesian fibration. $\square$

16.1.9. Remark (on dual co/cartesian fibrations). In Part V, we'll see that cartesian fibrations $p\colon E \twoheadrightarrow B$ in an ∞-cosmos of $(\infty,1)$-categories in $\mathcal{K}$, correspond to homotopy coherent diagrams indexed by the underlying quasi-category of B^{op} and valued in the quasi-category of $(\infty,1)$-categories in $\mathcal{K}$. The action on objects carries an element $b\colon 1 \to B$ to ∞-category E_b defined as the fiber of p over b. Similarly, cocartesian fibrations correspond to covariant homotopy coherent diagrams indexed by the underlying quasi-category of the base ∞-category and valued in the quasi-category of $(\infty,1)$-categories in $\mathcal{K}$.

Of course, a homotopy coherent diagram indexed by B_0^{op} could equally be regarded as a *contravariant* diagram indexed by the underlying quasi-category of B or as a *covariant* diagram indexed by the opposite of the underlying quasi-category of B. However, as observed by Barwick, Glasman, and Nardin in [4], the homotopy coherent diagram encoded by the cocartesian fibration $p\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ is *not* the same as the homotopy coherent diagram encoded by the cartesian fibration $p\colon E \twoheadrightarrow B$. Rather, the former is defined by postcomposing the latter with the simplicial functor $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$.[1]

[1]Or, more accurately, with the $(\infty,1)$-categorical core of this simplicial functor, to be introduced momentarily.

We now turn our attention to the construction of the Kan complex core of a quasi-category and discuss its analogue in other ∞-cosmoi of $(\infty, 1)$-categories.

16.1.10. DEFINITION. For a quasi-category A, its **groupoid core** is the largest sub Kan complex $\mathsf{core}(A) \subset A$, which may be constructed as the simplicial subset containing

- all of the vertices of A,
- only those edges that define isomorphisms in A, in the sense of Definition 1.1.13,
- every higher simplex whose edges are all isomorphisms.

16.1.11. LEMMA. *The simplicial set defined in the manner described in Definition 16.1.10 is a Kan complex, and indeed is the largest Kan complex contained in the quasi-category A.*

PROOF. The inclusion $\mathsf{core}(A) \subset A$ constructed in Definition 16.1.10 is full on simplices of all dimensions except dimension. Thus, to see that $\mathsf{core}(A)$ is a quasi-category, we need only argue that it admits extensions along the horn $\Lambda^1[2] \hookrightarrow \Delta[2]$. By construction, a horn $\Lambda^1[2] \to \mathsf{core}(A)$ picks outs two isomorphisms in A. The filler $\Delta[2] \to A$ witnesses a composition relation in the homotopy category $\mathsf{h}(A)$; thus the composite edge is also an isomorphism, and by fullness this filler lifts to $\Delta[2] \to \mathsf{core}(A)$.

By construction $\mathsf{h}(\mathsf{core}(A))$ is a groupoid; indeed, it is the maximal subgroupoid contained in $\mathsf{h}A$. So by Corollary 1.1.15, $\mathsf{core}(A)$ is a Kan complex.

Finally, a large simplicial subset of $\mathsf{core}(A) \subsetneq K \subset A$ would necessarily contain an additional edge $f \colon x \to y$. If K were a Kan complex, then it would have to admit fillers for $\Lambda^0[2]$- and $\Lambda^2[2]$-horns whose 2nd or 0th faces, respectively, were the 1-simplex f, and whose remaining face is degenerate. The fillers would construct left and right inverses to f in $\mathsf{h}(A)$. Hence, f is an isomorphism in A and already lives in $\mathsf{core}(A)$. □

The inclusion defines a cosmological functor $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$, as an instance of Proposition 1.2.25. Functors of quasi-categories preserve isomorphisms, so a functor $f \colon A \to B$ restricts to a functor $f \colon \mathsf{core}(A) \to \mathsf{core}(B)$. In this way the groupoid core construction acts functorially on the underlying category of $\mathcal{Q}Cat$ and, as an unenriched functor, is right adjoint to the inclusion $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$. Note, however, as discussed in Example 1.3.5, that the core construction is not simplicial, at least not with respect to the usual quasi-categorical enrichment of $\mathcal{Q}Cat$. Indeed, a natural transformation between functors of quasi-categories will only restrict to groupoid cores if each of its components is invertible.

The groupoid core does, however, define a simplicial functor with respect to a new enrichment that we now introduce. An ∞-cosmos is a type of $(\infty, 2)$-category since it is a category enriched over a model of $(\infty, 1)$-categories. We now introduce the **$(\infty, 1)$-categorical core** of an ∞-cosmos. In the following definition, note that since $\mathsf{core}(-) \colon \mathcal{Q}Cat \to \mathcal{K}an$ is an (unenriched) right adjoint, it preserves products so we may apply it to the functor spaces of a quasi-categorically enriched category to construct a Kan complex enriched subcategory that we now introduce:

16.1.12. DEFINITION ($(\infty, 1)$-core of an ∞-cosmos). For any ∞-cosmos $\mathcal{K}$, write $\mathsf{core}_*\mathcal{K} \subset \mathcal{K}$ for the subcategory with the same objects and with homs defined to be the groupoid cores of the functor spaces of $\mathcal{K}$. We refer to $\mathsf{core}_*\mathcal{K}$ as the **$(\infty, 1)$-core** of $\mathcal{K}$ and think of it as being the core $(\infty, 1)$-category inside this $(\infty, 2)$-category.

16.1.13. REMARK. The $(\infty, 1)$-categorical core is not an ∞-cosmos in the strict sense that we've axiomatized in Definition 1.2.1. It inherits its class of isofibrations and the conical limits from the original ∞-cosmos, but simplicial cotensors exist only weakly: the cotensor of an ∞-category A in $\mathsf{core}_*\mathcal{K}$

by a simplicial set U is constructed by the cotensor in $\mathcal{K}$ of A by a Kan complex replacement $\tilde{U}$ of U, defined by "freely inverting" its edges and adding fillers for horns. This results in an equivalence $\mathsf{core}(\mathsf{Fun}(X,A))^U \simeq \mathsf{core}(\mathsf{Fun}(X,A^{\tilde{U}}))$ in place of the usual isomorphism. Alternatively, Exercise 16.1.ii suggests an alternate approach to defining the enrichment of an ∞-cosmos in such a way that the $(\infty,1)$-core remains an ∞-cosmos.

16.1.14. Lemma. *The natural inclusion $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ factors through the inclusion $\mathsf{core}_*\mathcal{Q}Cat \subset \mathcal{Q}Cat$ and this latter functor admits a simplicially enriched right adjoint left inverse, namely the functor that sends each quasi-category to its groupoid core.*

$$\begin{tikzcd} \mathcal{K}an \arrow[r, hook, bend left] \arrow[r, phantom, "\perp"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[l, dashed, bend left, "\mathsf{core}"] \end{tikzcd}$$

Proof. If K and L are Kan complexes, then so is $\mathsf{Fun}(K,L)$. Hence the natural inclusion $\mathcal{K}an \hookrightarrow \mathcal{Q}Cat$ factors through the $(\infty,1)$-categorical core.

The right adjoint $\mathsf{core}\colon \mathsf{core}_*\mathcal{Q}Cat \to \mathcal{Q}Cat$ acts on objects by the construction of Definition 16.1.10. To define its action on functor spaces, we must supply a canonical map

$$\mathsf{core}(\mathsf{Fun}(A,B)) \to \mathsf{Fun}(\mathsf{core}(A),\mathsf{core}(B)),$$

for any pair of quasi-categories A and B. By Corollary D.4.17, the isomorphism in $\mathsf{Fun}(A,B)$ are simplicial maps $\alpha\colon A \times \Delta[1] \to B$ whose components $\alpha_a\colon \Delta[1] \to B$, indexed by vertices a of A, define isomorphisms in B. Combining this observation with Definition 16.1.10, we see that an n-simplex in $\mathsf{core}(\mathsf{Fun}(A,B))$ is a simplicial map $\phi\colon A \times \Delta[n] \to B$ with the property that upon restriction to any vertex of A and any edge of $\Delta[n]$, the resulting edge in B is an isomorphism. When A is restricted to its Kan complex core, the edges of $\mathsf{core}(A)$ are also isomorphisms. It follows that $\phi\colon \mathsf{core}(A) \times \Delta[n] \to B$ carries every edge of the domain to an isomorphism in B, and hence factors through $\mathsf{core}(B) \hookrightarrow B$, since this inclusion is full on the invertible edges.[2] Thus the n-simplex ϕ restricts to define an n-simplex $\phi\colon \mathsf{core}(A) \times \Delta[n] \to \mathsf{core}(B)$. This defines the canonical map.

Now for a Kan complex K and quasi-category A, the simplicial natural isomorphism

$$\mathsf{core}(\mathsf{Fun}(K,A)) \cong \mathsf{Fun}(K,\mathsf{core}(A))$$

is easily verified. The correspondence on vertices expresses the unenriched adjunction, while the correspondence on higher simplices follows for the reason just discussed and the isomorphism $\mathsf{core}(K) \cong K$. □

16.1.15. Corollary. *If A and B are equivalent quasi-categories, then $\mathsf{core}(A)$ and $\mathsf{core}(B)$ are equivalent Kan complexes.*

Proof. An equivalence of quasi-categories is specified by a pair of 0-arrows together a pair of invertible 1-arrows. As such it is contained in the $(\infty,1)$-categorical core $\mathsf{core}_*\mathcal{Q}Cat \hookrightarrow \mathcal{Q}Cat$ and preserved by the simplicial functor $\mathsf{core}\colon \mathsf{core}_*\mathcal{Q}Cat \to \mathcal{K}an$. □

The core of an ∞-category in a general ∞-cosmos of $(\infty,1)$-categories can be defined in a similar manner to Definition 16.1.5, but this notion also has an up-to-equivalence universal property that we prefer to use as the definition.

[2]In the language of marked simplicial sets, a map in $\mathsf{core}(\mathsf{Fun}(A,B))$ is a marked map $A^\natural \times \Delta[n]^\sharp \to B^\natural$. Upon restriction along $\mathsf{core}(A)^\sharp \hookrightarrow A^\natural$, the domain $\mathsf{core}(A)^\sharp \times \Delta[n]^\sharp$ is maximally marked, and hence factors through the maximally marked core $\mathsf{core}(B)^\sharp \hookrightarrow B^\natural$. See §D.4.

16.1.16. Definition. Let $\mathcal{K}$ be an ∞-cosmos of $(\infty, 1)$-categories and let A be an ∞-category in $\mathcal{K}$. Its **groupoid core** is an ∞-category $\mathsf{core}(A)$ equipped with a map $\iota\colon \mathsf{core}(A) \to A$ so that

- $\mathsf{core}(A)$ is a discrete ∞-category, meaning that $\mathsf{Fun}(X, A)$ is a Kan complex for all X
- if G is a discrete ∞-category, then ι defines an equivalence

$$\mathsf{Fun}(G, \mathsf{core}(A)) \xrightarrow[\sim]{\iota\circ -} \mathsf{core}(\mathsf{Fun}(G, A))$$

In practice, an ∞-cosmos of $(\infty, 1)$-categories frequently comes with an explicit core functor.

16.1.17. Proposition. *Let $\mathcal{K}$ be an ∞-cosmos of $(\infty, 1)$-categories.*

(i) Any ∞-category in $\mathcal{K}$ has a core, well-defined up to equivalence.
(ii) For any ∞-categories $A, B \in \mathcal{K}$, there is an map $\mathsf{core}(\mathsf{Fun}(A, B)) \to \mathsf{Fun}(\mathsf{core}(A), \mathsf{core}(B))$ that is quasi-pseudonatural in A and in B as objects of $\mathsf{core}_*\mathcal{K}$.
(iii) Moreover, any specified choices in (i) and (ii) assemble into a quasi-pseudofunctor $\mathsf{core}\colon \mathsf{core}_*\mathcal{K} \to \mathcal{D}isc(\mathcal{K})$ defined as the composite of the zig-zag of simplicial functors

$$\begin{tikzcd} \mathsf{core}_*\mathcal{K} \arrow[r, "(-)_0", "\sim"'] \arrow[d, dashed, "\mathsf{core}(-)"'] \arrow[dr, phantom, "\cong"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[d, "\mathsf{core}(-)"] \\ \mathcal{D}isc(\mathcal{K}) \arrow[r, "\sim", "(-)_0"'] & \mathcal{D}isc(\mathcal{Q}Cat) \cong \mathcal{K}an \end{tikzcd}$$

Implicitly in the statement of (iii) we have asserted that a cosmological biequivalence $\mathcal{K} \rightsquigarrow \mathcal{L}$ descends to a cosmological biequivalence $\mathcal{D}isc(\mathcal{K}) \rightsquigarrow \mathcal{D}isc(\mathcal{L})$ and a simplicially enriched biequivalence $\mathsf{core}_*\mathcal{K} \rightsquigarrow \mathsf{core}_*\mathcal{L}$. The first statement follows from Proposition 14.3.5(iii) while the second follows from Corollary 16.1.15. We leave the details to Exercise 16.1.iii.

Proof. By Proposition 14.4.15 the inverse to the cosmological biequivalence $(-)_0\colon \mathcal{D}isc(\mathcal{K}) \rightsquigarrow \mathcal{K}an$, defines a quasi-pseudofunctor and biequivalence $\mathcal{K}an \rightsquigarrow \mathcal{D}isc(\mathcal{K})$, which composes with the simplicial functor $\mathsf{core}((-)_0)\colon \mathsf{core}_*\mathcal{K} \to \mathcal{K}an$ to define the quasi-pseudofunctor $\mathsf{core}\colon \mathsf{core}_*\mathcal{K} \to \mathcal{D}isc(\mathcal{K})$ claimed in (iii).

By Lemma 14.4.11, the action on homs of this quasi-pseudofunctor defines a quasi-pseudonatural transformation

$$\mathsf{core}(\mathsf{Fun}(A, B)) \to \mathsf{Fun}(\mathsf{core}(A), \mathsf{core}(B)),$$

as required in (ii).

It remains only to verify that the action on objects of the quasi-pseudofunctor satisfies the conditions of Definition (16.1.16). By construction, $\mathsf{core}(A)$ is a discrete ∞-category for any $A \in \mathcal{K}$. The map $\iota\colon \mathsf{core}(A) \to A$ is defined by composing whiskering the corresponding inclusion of the Kan complex core of quasi-category with the underlying quasi-category functor and its quasi pseudofunctorial inverse:

$$\begin{tikzcd} \mathsf{core}_*\mathcal{K} \arrow[r, "(-)_0", "\sim"'] & \mathsf{core}_*\mathcal{Q}Cat \arrow[r, bend left, "\mathsf{core}"] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\iota"] & \mathsf{core}_*\mathcal{Q}Cat \arrow[r, "(-)_0^{-1}", "\sim"'] & \mathsf{core}_*\mathcal{K} \end{tikzcd}$$

Now if G is a discrete ∞-category, then $G_0 = \mathsf{Fun}(1, G)$ is a Kan complex, so by Lemma 16.1.14 $\iota_{A_0} \circ -\colon \mathsf{Fun}(G_0, \mathsf{core}(A_0)) \rightsquigarrow \mathsf{core}\mathsf{Fun}(G_0, \mathsf{core}(A_0))$ is an equivalence. By construction $\mathsf{core}(A)$ is defined so that $\mathsf{core}(A)_0 \simeq \mathsf{core}(A_0)$. Note that since $(-)_0\colon \mathcal{K} \rightsquigarrow \mathcal{Q}Cat$ is a biequivalence, this shows

that the core of an ∞-category in an ∞-cosmos of $(\infty,1)$-categories is well-defined up to equivalence. Since the simplicial functor $(-)_0\colon \mathsf{core}_*\mathcal{K} \to \mathsf{core}_*\mathcal{QC}at$ is an equivalence on homs, the functor defined by post-composition with ι_A is equivalent to this functor

$$\begin{tikzcd} \mathsf{Fun}(G,\mathsf{core}(A)) \arrow[r, "\iota_A\circ-"] \arrow[d, "(-)_0"', "\wr"] \arrow[dr, phantom, "\cong"] & \mathsf{coreFun}(G,A) \arrow[d, "\wr"', "(-)_0"] \\ \mathsf{Fun}(G_0,\mathsf{core}(A_0)) \arrow[r, "\iota_{A_0}\circ-", "\sim"'] & \mathsf{coreFun}(G_0,A_0) \end{tikzcd}$$

Thus post-composition with ι_A induces the equivalence $\mathsf{Fun}(G,\mathsf{core}(A)) \simeq \mathsf{coreFun}(G,A)$ required by Definition (16.1.16). This completes the proof of (i). ☐

The core of an ∞-category can be thought of as an infinite-dimensional extension of the "fundamental groupoid" of the ∞-category A. Recall from Definition 1.4.12 that the **homotopy category** of an ∞-category A is the 1-category defined by $\mathsf{h}A \coloneqq \mathsf{hFun}(1,A) \coloneqq \mathsf{h}(\mathsf{Fun}(1,A))$. Similarly:

16.1.18. Definition. The **fundamental groupoid** of an ∞-category A is the 1-groupoid defined by

$$\pi_0 A \coloneqq \mathsf{hFun}(1,\mathsf{core}A) \cong \mathsf{core}(\mathsf{h}(A)).$$

Exercises.

16.1.i. Exercise. Prove that the homotopy category of the opposite of an ∞-category A is equivalent to the opposite of the homotopy category of A.

16.1.ii. Exercise. In consultation with §D.4 and §D.5:
(i) Redefine the notion of an ∞-cosmos from Definition 1.2.1 to be a category enriched over marked simplicial sets, whose functor spaces are naturally marked quasi-categories.
(ii) Describe the construction of the groupoid core of a naturally marked quasi-category and of the $(\infty,1)$-categorical core of an ∞-cosmos with this enrichment.
(iii) Show that $(\infty,1)$-categorical cores are cotensored over simplicial sets, although these cotensors are not preserved by the inclusion $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$.
(iv) Show that the $(\infty,1)$-categorical core of an ∞-cosmos is an ∞-cosmos, although the functor $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$ is not cosmological.

16.1.iii. Exercise. Let $F\colon \mathcal{K} \to \mathcal{L}$ be a cosmological functor. Prove that F induces
(i) a cosmological functor $F\colon \mathcal{D}isc(\mathcal{K}) \to \mathcal{D}isc(\mathcal{L})$ and
(ii) a simplicial functor $F\colon \mathsf{core}_*\mathcal{K} \to \mathsf{core}_*\mathcal{L}$

and show moreover that both functors are biequivalences if the original functor is.

16.2. Pointwise universal properties

In ∞-cosmoi of $(\infty,1)$-categories, the terminal ∞-category 1 plays a special role which can be summarized by the slogan that "universal properties are detected pointwise." In this section, we collect together a number of results that encapsulate this slogan, which are proven through a combination of synthetic and analytic techniques.

For instance, Corollary D.4.17 proves that a natural transformation between functors between quasi-categories $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a natural isomorphism if and only if it is a **pointwise isomorphism**,

meaning that each of its components

$$\begin{tikzcd} 1 \arrow[r, "x"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

is invertible. Consequently:

16.2.1. LEMMA. *In an ∞-cosmos of $(\infty, 1)$-categories, a natural transformation is a natural isomorphism if and only if it is a pointwise isomorphism.*

Put another way, a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ between functors between $(\infty, 1)$-categories is an isomorphism if and only if each of its components αx defines an isomorphism in the homotopy category of A; see Definition 1.4.12.

PROOF. It's clear that any natural isomorphism is a pointwise isomorphism. For the converse, we use the biequivalence $(-)_0 \colon \mathfrak{h}\mathcal{K} \rightsquigarrow \mathfrak{h}\mathcal{QC}at$ of Propositions 14.2.3 and 14.3.1. Suppose $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$ is a pointwise natural isomorphism in $\mathfrak{h}\mathcal{K}$ and consider the underlying natural transformation between underlying quasi-categories $\begin{tikzcd} X_0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "\Downarrow\alpha_0"] & A_0 \end{tikzcd}$. By construction, vertices of $X_0 \cong \mathsf{Fun}(1, X)$ correspond bijectively to elements of X, so the underlying natural transformation α_0 is a pointwise natural isomorphism in $\mathfrak{h}\mathcal{QC}at$ as well. Thus, Corollary D.4.17 applies to prove that α_0 admits an inverse $\begin{tikzcd} X_0 \arrow[r, bend left, "g_0"] \arrow[r, bend right, "f_0"'] \arrow[r, phantom, "\Downarrow\alpha_0^{-1}"] & A_0 \end{tikzcd}$. By the full and faithfulness of the local equivalence $\mathsf{hFun}(X, A) \rightsquigarrow \mathsf{hFun}(X_0, A_0)$ established in Proposition 14.3.1, this 2-cell lifts to define an inverse natural transformation $\begin{tikzcd} X \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha^{-1}"] & A \end{tikzcd}$ witnessing the invertibility of α. $\square$

We took advantage of a special feature of the cosmological biequivalence $(-)_0 \colon \mathcal{K} \rightsquigarrow \mathcal{QC}at$ to simplify the proof of Lemma 16.2.1 that is worth calling attention to.

16.2.2. OBSERVATION (on the elements of the underlying quasi-category). By Corollary 14.3.2(ii), a cosmological biequivalence $F \colon \mathcal{K} \rightsquigarrow \mathcal{L}$ induces a bijection between the isomorphism class of elements of an ∞-category $A \in \mathcal{K}$ and the isomorphism class of elements of $FA \in \mathcal{L}$, and in fact induces an equivalence of homotopy categories $\mathsf{h}A \rightsquigarrow \mathsf{h}FA$, the objects of which are exactly these elements; see Exercise 14.3.i. In particular, any element $x \colon 1 \to FA$ is naturally isomorphic to an element $Fa \colon 1 \to FA$ that is the image of an element $a \colon 1 \to A$. Since "pointwise" ∞-categorical notions — invertibility of the components of a natural isomorphism, possession of a terminal element in the fibers of a cocartesian fibration — are invariant under isomorphism, if A satisfies some pointwise criterion in $\mathcal{K}$, then FA will satisfy the corresponding pointwise criterion in $\mathcal{L}$.

But in the case of ∞-cosmoi of $(\infty, 1)$-categories, Proposition 14.2.3 supplies a cosmological biequivalence $(-)_0 \colon \mathcal{K} \to \mathcal{QC}at$ that acts bijectively on elements of ∞-categories. By construction, vertices of $A_0 \cong \mathsf{Fun}(1, A)$ correspond bijectively to elements of A. Thus, every element of the underlying quasi-category A_0 of an ∞-category A comes from some element of A. Consequently, as we saw in the proof of Lemma 16.2.1, "pointwise" properties may be transferred even more readily.

Using Lemma 16.2.1, we can show that an isofibration in an ∞-cosmos of $(\infty, 1)$-categories is a discrete object of the slice ∞-cosmos if and only if its fibers are discrete ∞-categories. It follows that cocartesian, cartesian, or two-sided fibrations of $(\infty, 1)$-categories are discrete if and only if they have discrete fibers.

16.2.3. Proposition. *Let $p \colon E \twoheadrightarrow B$ be an isofibration in an ∞-cosmos $\mathcal{K}$ of $(\infty, 1)$-categories. Then p is discrete as an object of $\mathcal{K}_{/B}$ if and only if the fibers of p are discrete ∞-categories in $\mathcal{K}$.*

Proof. Any element $b \colon 1 \to B$ induces a cosmological functor $b^* \colon \mathcal{K}_{/B} \to \mathcal{K}$ which preserves discrete objects by Corollary 14.1.6. So in fact for any ∞-cosmos $\mathcal{K}$ if $p \colon E \twoheadrightarrow B$ is a discrete object in $\mathcal{K}_{/B}$, then its fibers are discrete ∞-categories.

For the converse we assume that $\mathcal{K}$ is an ∞-cosmos of $(\infty, 1)$-categories and appeal to Lemma 16.2.1. To show that $p \colon E \twoheadrightarrow B$ is discrete in $\mathcal{K}_{/B}$ we must argue that the quasi-category defined by the pullback

$$\begin{tikzcd} \mathsf{Fun}_B(f \colon X \to B, p \colon E \twoheadrightarrow B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(X, E) \arrow[d, two heads, "p_*"] \\ 1 \arrow[r, "f"] & \mathsf{Fun}(X, B) \end{tikzcd}$$

is a Kan complex. By Corollary 1.1.15, it suffices to show that its homotopy category is a groupoid. To that end, consider a 1-simplex $\alpha \colon e \to e'$ in $\mathsf{Fun}_B(f, p) \hookrightarrow \mathsf{Fun}(X, E)$, representing a natural transformation $\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow\alpha"] & E \end{tikzcd}$ in the homotopy 2-category. The 1-simplex α has the property that the whiskered composite $p\alpha \in \mathsf{Fun}(X, B)$ is the degenerate 1-simplex at f.[3] By Lemma 3.1.5, there is a smothering functor

$$\mathsf{hFun}_B(f \colon X \to B, p \colon E \twoheadrightarrow B) \to \mathbb{1} \underset{\mathsf{hFun}(X,B)}{\times} \mathsf{hFun}(X, E),$$

which in particular, reflects isomorphisms, so our task is to show that α defines a natural isomorphism. By Lemma 16.2.1 α is invertible if and only if its components αx are invertible for every $x \colon 1 \to X$. Since the 1-simplex $p\alpha$ is degenerate in $\mathsf{Fun}(X, B)$, the 1-simplex $p\alpha x$ is degenerate at the vertex $fx \in \mathsf{Fun}(1, B)$, which says that αx lies in the fiber over $fx \colon 1 \to B$. Since the fibers of $p \colon E \twoheadrightarrow B$ are discrete, this tells us that αx is invertible, so we conclude by Lemma 16.2.1 that α is invertible as claimed. $\square$

16.2.4. Proposition. *A cocartesian fibration $q \colon E \twoheadrightarrow A$ of quasi-categories admits a right adjoint right inverse $t \colon A \to E$ if and only if for each $a \colon 1 \to A$ the fiber E_a has a terminal element.*

Proof. By Corollary 3.6.11, a right adjoint right inverse to q can be interpreted as defining a terminal element in $q \colon E \twoheadrightarrow A$, considered as an object in the sliced ∞-cosmos $\mathcal{QC}at_{/A}$. By Lemma

[3] This implies, but is stronger than, the property that the whiskered composite in the homotopy 2-category equals the 2-cell id_f; see §3.6 for a discussion.

3.6.7(i), this fibered adjunction may be pulled back along any element $a\colon 1 \to A$ to define a terminal element in the fiber E_a.

For the converse, let $ta\colon 1 \to E_a$ denote a chosen terminal element in the fiber E_a over $a\colon 1 \to A$. Lemma F.3.1 characterizes those isofibrations between quasi-categories that admit a right adjoint right inverse in terms of a lifting property. In this case, it suffices to show that any lifting problem

$$\begin{tikzcd} 1 \arrow[r, "\{n\}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r, "y"] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r, "x"'] \arrow[ur, dashed] & A \end{tikzcd}$$

for $n \geq 1$ has a solution. To that end, consider the simplicial map $k\colon \Delta[n] \times \Delta[1] \to \Delta[n]$ defined on vertices by

$$k(i,0) \coloneqq i \qquad \text{and} \qquad k(i,1) \coloneqq n.$$

The composite $xk\colon \Delta[n] \times \Delta[1] \to A$ restricts to define a map $xk\colon \partial\Delta[n] \times \Delta[1] \to A$ which represents a 2-cell whose codomain, defined by evaluating at the vertex $\{1\}$ of $\Delta[1]$, is constant at a and whose domain factors through q along y. This yields a new lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "y"] \arrow[d, hook, "i_0"'] & E \arrow[d, two heads, "q"] \\ \partial\Delta[n] \times \Delta[1] \arrow[r, "xk"'] \arrow[ur, dashed, "z"'] & A \end{tikzcd}$$

which Lemma F.4.9 enables us to solve. By Lemma F.4.9, the lift z represents a q-cocartesian lift ζ of the 2-cell κ represented by the restriction of xk:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "y"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\kappa"] & E \arrow[d, two heads, "q"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd} = \begin{tikzcd} \partial\Delta[n] \arrow[r, bend left, "y"] \arrow[r, phantom, "\Downarrow\zeta"] \arrow[r, dashed, bend right, "u"'] \arrow[d, "!"'] & E \arrow[d, two heads, "q"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd}$$

By construction, the codomain functor of the q-cocartesian lift displayed above right lands in the fiber over a. Now the component of κ at the final vertex $\{n\}\colon 1 \to \partial\Delta[n]$ is id_a, so by Lemma 5.1.4(ii), the component $z\{n\}$ representing the 2-cell $\zeta\{n\}$ is an isomorphism. In particular, the element $u\{n\}\colon 1 \to E$ is isomorphic to the terminal element $y\{n\} = ta$ of E_a, so we may apply the universal property of Proposition F.1.1(v) to extend u to a simplex:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E_a \\ \Delta[n] \arrow[ur, dashed, "v"'] \end{tikzcd}$$

This data defines a new lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\iota_0"] \arrow[d, hook] \arrow[rr, bend left, "y"] & \partial\Delta[n]\times\Delta[1]\cup\Delta[n]\times\{1\} \arrow[r, "z\cup v"] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ \Delta[n] \arrow[r, hook, "\iota_0"] \arrow[rr, bend right, "x"'] & \Delta[n]\times\Delta[1] \arrow[r, "xk"] \arrow[ur, dashed] & A \end{tikzcd}$$

which we solve inductively by choosing lifts of the $n+1$ $(n+1)$-simplices in $\Delta[n]\times\Delta[1]$ not present in $\partial\Delta[n]\times\Delta[1]\cup\Delta[n]\times\{1\}$, starting from the $n+1$-simplex that contains the face $\Delta[n]\times\{1\}$. All but the last of these can be lifted by means of lifting inner horns against the isofibration q. For the final simplex, we must solve an outer horn lifting problem

$$\begin{tikzcd} \Delta[1] \arrow[r, hook, "\{n,n+1\}"'] \arrow[rr, bend left, "z\{n\}"] & \Lambda^{n+1}[n+1] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ & \Delta[n+1] \arrow[r] & A \end{tikzcd}$$

but in this case the final edge of the outer horn is the isomorphism $z\{n\}$, so Proposition 1.1.14 permits its solution as well. Now this lift restricts to define the sought-for solution to the original lifting problem, proving that $q\colon E \twoheadrightarrow A$ admits a right adjoint right inverse. □

16.2.5. Remark (on duals). Via the cosmological isomorphism $(-)^{\mathrm{op}}\colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$ of Lemma 16.1.4, the proof of Proposition 16.2.4 dualizes to prove that a cartesian fibration of quasi-categories admits a left adjoint right inverse if and only if each fiber has an initial element. See Exercise 16.2.i.

The proof of Proposition 16.2.4 relied heavily on "analytic" techniques. Nonetheless its conclusion transfers to any ∞-cosmos that is biequivalent to the ∞-cosmos of quasi-categories.

16.2.6. Proposition. *In an ∞-cosmos of $(\infty,1)$-categories a cocartesian fibration $q\colon E \twoheadrightarrow A$ of admits a right adjoint right inverse $t\colon A \to E$ if and only if for each $a\colon 1 \to A$ the fiber E_a has a terminal element.*

Proof. The argument that the fibers of an isofibration with right adjoint right inverse admit terminal elements is the same as given in the proof of Proposition 16.2.4. For the converse, suppose $q\colon E \twoheadrightarrow A$ is a cocartesian fibration in an ∞-cosmos $\mathcal{K}$ that is biequivalent to $\mathcal{QC}at$ with the property that for each element $a\colon 1 \to A$ of the base, the fiber E_a has a terminal element. By Proposition 14.2.3, we may use the biequivalence underlying quasi-category functor $(-)_0\colon \mathcal{K} \rightsquigarrow \mathcal{QC}at$ to conclude that q admits a right adjoint right inverse.

Cosmological functors preserve cocartesian fibrations, so the underlying map $q_0\colon E_0 \twoheadrightarrow A_0$ defines a cocartesian fibration of quasi-categories. By construction, vertices of $A_0 \cong \mathsf{Fun}(1,A)$ correspond bijectively to elements of A. Put another way, while Corollary 14.3.2(ii) tells us that a cosmological biequivalence

By Observation 16.2.2, elements of the underlying quasi-category A_0 correspond bijectively to elements of the ∞-category A. By hypothesis, for every $a\colon 1 \to A$ the ∞-category E_a admits a terminal element, so by Proposition 14.1.9 the underlying quasi-category $(E_a)_0$ does as well. In this way, we

see that every fiber of the cocartesian fibration of quasi-categories $q_0 \colon E_0 \twoheadrightarrow A_0$ admits a terminal element. By Proposition 16.2.4, q_0 admits a right adjoint right inverse. Now by Proposition 14.3.5(ii), we may conclude that $q \colon E \twoheadrightarrow A$ admits a right adjoint right inverse in $\mathcal{K}$, as desired. □

An important special case of Proposition 16.2.6 proves a result promised in the discussion surrounding Proposition 4.1.5: in an ∞-cosmos of $(\infty, 1)$-categories a functor $f \colon B \to A$ admits a right adjoint just when for each element $a \colon 1 \to A$, the ∞-category $\mathsf{Hom}_A(f, a)$ admits a terminal element. This describes the components of the counit of an adjunction as a "universal arrow" from the functor f to the element a, in the terminology used by Mac Lane [**64**, §III.1].

16.2.7. Corollary. *In an ∞-cosmos of $(\infty, 1)$-categories, a functor $f \colon B \to A$ admits a right adjoint if and only if for each element $a \colon 1 \to A$, the comma ∞-category $\mathsf{Hom}_A(f, a)$ admits a terminal element. Dually, $f \colon B \to A$ admits a left adjoint if and only if for every a, $\mathsf{Hom}_A(a, f)$ admits an initial element.*

Proof. Proposition 4.1.5 demonstrates that in any ∞-cosmos, $f \colon B \to A$ admits a right adjoint if and only if $\mathsf{Hom}_A(f, A)$ "admits a terminal element over A," meaning that $p_1 \colon \mathsf{Hom}_A(f, A) \twoheadrightarrow A$ admits a right adjoint right inverse. By Corollary 5.4.12, this functor is a cocartesian fibration, so Proposition 16.2.6 tells us that p_1 admits a right adjoint right inverse if and only if each fiber $\mathsf{Hom}_A(f, a)$ admits a terminal element.

The dual result follows from the dual of Proposition 16.2.6, whose proof is discussed in Exercises 16.2.i and 16.2.ii. □

Proposition 16.2.6 implies that modules between $(\infty, 1)$-categories admit an analogous "pointwise" representability condition, characterizing those modules that are covariantly or contravariantly represented by a functor in the sense of Definition 11.4.8.

16.2.8. Corollary. *In an ∞-cosmos of $(\infty, 1)$-categories, a module $A \overset{E}{\nrightarrow} B$ is covariantly represented if and only if for each $a \colon 1 \to A$, the module $1 \overset{E(1,a)}{\nrightarrow} B$ is covariantly represented by some element $b \colon 1 \to B$, which is the case if and only if each ∞-category $E(1, a)$ admits a terminal element.*

Proof. By Proposition 12.4.9, a module $A \overset{E}{\nrightarrow} B$ encoded by a span $(q, p) \colon E \twoheadrightarrow A \times B$ is covariantly represented if and only if its left leg $q \colon E \twoheadrightarrow A$ admits a right adjoint right inverse $r \colon A \to E$, in which case $E \simeq \mathsf{Hom}_B(B, pr)$. By Lemma 11.4.3, the left left $q \colon E \twoheadrightarrow A$ defines a cocartesian fibration, so by Proposition 16.2.6, q admits a right adjoint right inverse if and only if the fiber E_a over each element $a \colon 1 \to A$ admits a terminal element. For each $a \colon 1 \to A$, the module $1 \overset{E(1,a)}{\nrightarrow} B$ is given by the pullback along $a \times \mathrm{id} \colon 1 \times B \to A \times B$ and hence is isomorphic to the module $1 \overset{E_a}{\nrightarrow} B$. Applying Proposition 12.4.9 again, this module is covariantly represented by some element $b \colon 1 \to B$ if and only if $! \colon E_a \cong E(1, a) \twoheadrightarrow 1$ admits a right adjoint right inverse, which is the case if and only if this ∞-category admits a terminal element. □

16.2.9. Theorem. *In any ∞-cosmos of $(\infty, 1)$-categories:*

(i) A functor $g \colon C \to A$ admits an absolute right lifting through a functor $f \colon B \to A$ if and only if for all $c \colon 1 \to C$, the comma ∞-category $\mathsf{Hom}_A(f, gc)$ admits a terminal element.

(ii) A triangle as below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ 1 \arrow[ur, "rc"] \arrow[r, "gc"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho c"] & A \end{tikzcd}$$

displays r as an absolute right lifting of g through f if and only if for all $c\colon 1 \to C$, the restricted triangle as above-right displays rc as an absolute right lifting of gc through f.

(iii) A functor $g\colon C \to A$ admits an absolute right lifting through a functor $f\colon B \to A$ if and only if for all $c\colon 1 \to C$ there exists an absolute right lifting of gc through f as below-left

$$\begin{tikzcd} & B \arrow[d, "f"] \\ 1 \arrow[ur, "r_c"] \arrow[r, "gc"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho_c"] & A \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A \end{tikzcd}$$

in which case the components of the absolute right lifting of g through f are isomorphic to the corresponding pointwise absolute liftings: $r_c \cong rc$ and hence $\rho_c \cong \rho c$.

In the ∞-cosmos of quasi-categories, Corollary D.6.6 provides an equivalence over A between the comma quasi-category $\mathsf{Hom}_A(A, gc)$ and Joyal's slice quasi-category $A_{/gc}$, which pulls back to an equivalence $\mathsf{Hom}_A(f, gc) \simeq f_{/gc}$.

Proof. Theorems 3.5.7 and 3.5.11 demonstrate that in any ∞-cosmos, a functor $g\colon C \to A$ admits a right lifting through $f\colon B \to A$ if and only if the codomain-projection functor $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a right adjoint right inverse. By Corollary 5.4.12 this functor is a cocartesian fibration. Now Proposition 16.2.6 proves that in an ∞-cosmos of $(\infty,1)$-categories, $p_1\colon \mathsf{Hom}_A(f,g) \twoheadrightarrow C$ admits a right adjoint right inverse if and only if each fiber $\mathsf{Hom}_A(f,gc)$ over an element $c\colon 1 \to C$ admits a terminal element. This proves the first statement.

Since absolute lifting diagrams restrict, it is immediately clear that any absolute right lifting diagram as above-left, restricts to define a pointwise absolute right lifting diagram as above-right. For the converse, suppose $(rc, \rho c)$ defines an absolute right lifting of gc through f for any $c\colon 1 \to C$. By Theorem 3.5.11, it follows that each comma ∞-category $\mathsf{Hom}_A(f, gc)$ has a terminal element and so by (i), $g\colon C \to A$ must admit an absolute right lifting (s,σ) through $f\colon B \to A$. By its universal property, the pair (r,ρ) factors through (s,σ) via a 2-cell $\tau\colon r \Rightarrow s$ so that $\rho = \sigma \cdot f\tau$. Since $(rc,\rho c)$ and $(sc,\sigma c)$ are both absolute right lifting diagrams, we know that each component τc is an isomorphism. Hence, by Lemma 16.2.1, τ is an isomorphism, and thus (r,ρ) is also an absolute right lifting digram, as desired.

The third statement is a convenient summary of the first two. If gc admits an absolute right lifting through f then $\mathsf{Hom}_A(f, gc)$ has a terminal element and (i) guarantees the existence of an absolute right lifting of g through f. The component of (r,ρ) at c defines a second absolute right lifting of gc through f inducing the claimed isomorphisms. □

As a corollary, we may resolve the question raised in Remark 2.3.8.

16.2.10. Corollary. *In an ∞-cosmos of $(\infty,1)$-categories if A is an ∞-category that admits limits of every diagram $d\colon 1 \to A^J$ of shape J then A admits all limits of shape J: that is, the constant diagram functor admits*

a right adjoint

$$\begin{tikzcd} A^J \arrow[r, leftarrow, "\Delta" description] & A \arrow[l, bend right, "\mathrm{colim}"', "\perp"] \arrow[l, bend left, leftarrow, "\mathrm{lim}", "\perp"'] \end{tikzcd}$$

PROOF. By Theorem 16.2.9(iii), the functor $\mathrm{id}\colon A^J \to A^J$ admits an absolute right lifting through $\Delta\colon A \to A^J$ if and only if each $d\colon 1 \to A^J$ admits an absolute right lifting through $\Delta\colon A \to A^J$. By Definition 2.3.7 the latter condition encodes what it means for A to admit a limit of the diagram d, while by Lemma 2.3.6 the former condition encodes what it means for $\Delta\colon A \to A^J$ to admit a right adjoint. $\square$

Another proof of Theorem 16.2.9(ii) is possible. By Theorem 3.5.7, a 2-cell $\rho\colon fr \Rightarrow g$ defines an absolute right lifting if and only if the induced functor $\ulcorner\rho\urcorner\colon \mathrm{Hom}_B(B,r) \to \mathrm{Hom}_A(f,g)$ is an equivalence over $C \times B$. As we shall now discover, equivalences between co/cartesian fibrations or modules can be detected fiberwise in ∞-cosmoi of $(\infty,1)$-categories.

16.2.11. PROPOSITION. *A cartesian functor*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

between cocartesian fibrations of quasi-categories is a fibered equivalence if and only if it is a fiberwise equivalence, meaning that for each $b\colon 1 \to B$, the induced functor between fibers $g_b\colon E_b \to F_b$ is an equivalence.

Note the subtle difference in terminology between "fibered equivalences" — equivalences over B — and "fiberwise equivalences" — maps inducing equivalences on fibers over elements of B. This result and Proposition 16.2.12 to follow show that in fact these two notions coincide for ∞-cosmoi of $(\infty,1)$-categories.

PROOF. Fibered equivalences are stable under pullback to fibres, so the content is in the converse implication: that any cartesian functor between cocartesian fibrations that induces a fiberwise equivalence is necessarily an equivalence.

The cartesian functor g can be factored in the slice $\mathcal{QC}at_{/B}$ as an equivalence followed by an isofibration. By Corollary 5.1.17, the intermediate object of that factorization is again a cocartesian fibration and the isofibration from it to $q\colon F \twoheadrightarrow B$ is again a cartesian functor. Replacing $p\colon E \twoheadrightarrow B$ by the equivalence cocartesian fibration, it therefore suffices to assume that $g\colon E \twoheadrightarrow B$ is an isofibration and a cartesian functor and postulate that each induced map $g_b\colon E_b \xrightarrow{\sim}\!\!\!\!\!\rightarrow F_b$ is a trivial fibration. Under these assumptions, we must show that g is itself is a trivial fibration.

To that end, suppose that we are given a lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "g"] \\ \Delta[n] \arrow[r, "f"'] & F \end{tikzcd}$$

over $b\colon \Delta[n] \to B$. Consider the retract diagram

$$\begin{array}{ccccc} \Delta[n] & \xhookrightarrow{\mathrm{id}\times\{0\}} & \Delta[n]\times\Delta[1] & \xrightarrow{\quad r \quad} & \Delta[n] \\ i & \longmapsto & (i,0) & & \\ & & (i,j) & \longmapsto & \begin{cases} i & \text{if } j = 0 \\ n & \text{if } j = 1 \end{cases} \end{array}$$

and choose a pointwise p-cocartesian lift

$$\begin{tikzcd} \Delta[n]\times\{0\} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n]\times\Delta[1] \arrow[r, "br"'] \arrow[ur, dashed, "\chi"] & B \end{tikzcd}$$

as permitted by Lemma F.4.9. Applying g we obtain a hollow cylinder

$$\partial\Delta[n]\times\Delta[1] \xrightarrow{\chi} E \overset{g}{\twoheadrightarrow} F$$

and since g is a cartesian functor and χ is pointwise p-cocartesian it follows that $g\chi$ is pointwise q-cocartesian. Now by construction the simplex $f\colon \Delta[n]\times\{0\} \to F$ agrees with $g\chi\colon \Delta[n]\times\Delta[1] \to F$ on the subset $\Delta[n]\times\{0\}$ where they are both defined. It follows that they combine to give a well defined simplicial map on the union of their domains and so provide us with a second lifting problem:

$$\begin{tikzcd} \Delta[n]\times\{0\} \cup \partial\Delta[n]\times\Delta[1] \arrow[r, "f\cup g\chi"] \arrow[d, hook] & F \arrow[d, two heads, "q"] \\ \Delta[n]\times\Delta[1] \arrow[r, "br"'] \arrow[ur, dashed, "\rho"] & B \end{tikzcd}$$

which can again be solved to give a pointwise q-cocartesian lift ρ by Lemma F.4.9. Note now that the retraction $r\colon \Delta[n]\times\Delta[1] \to \Delta[n]$ was constructed to map the subset $\Delta[n]\times\{1\}$ onto the vertex $\{n\}$, from which it follows that the n-simplex

$$\Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{br} B$$

is a degenerate image of the final vertex $b_n \coloneqq b\cdot\{n\}$. Now observe that the cylinders χ and ρ were defined to lie over $br\colon \Delta[n]\times\Delta[1] \to B$, so it follows that the restricted maps

$$\Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{\chi} E \qquad \Delta[n]\times\{1\} \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{\rho} F$$

land in the fibers E_{b_n} and F_{b_n} of p and q respectively. Consequently we obtain a lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "\chi|_{\{1\}}"] \arrow[d, hook] & E_{b_n} \arrow[d, two heads, "g_{b_n}", "\wr"'] \\ \Delta[n] \arrow[r, "\rho|_{\{1\}}"'] \arrow[ur, dashed, "\gamma"] & F_{b_n} \end{tikzcd}$$

which we may solve since the map of fibers on the right is, by assumption, a trivial fibration. Now the upper left triangle tells us that χ and γ agree on the subset $\Delta[n]\times\{1\}$ where they are both defined.

Thus, these maps combine to give the well defined simplicial map on the union of their domains depicted as the upper-horizontal in the lifting problem on the right of the following diagram:

$$\begin{tikzcd} \partial\Delta[n]\times\{0\} \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "e"] & \Delta[n]\times\{1\}\cup\partial\Delta[n]\times\Delta[1] \arrow[r, dashed, "\gamma\cup\chi"] \arrow[d, hook] & E \arrow[d, two heads, "g"] \\ \Delta[n]\times\{0\} \arrow[r, hook] \arrow[rr, bend right, "f"'] & \Delta[n]\times\Delta[1] \arrow[r, "\rho"] \arrow[ur, dashed] & F \end{tikzcd}$$

A standard argument shows that the lifting problem in the right-hand square can be solved by filling a sequence of inner horns and a single outer horn of shape $\Lambda^{n+1}[n+1]$ whose final edge is the cocartesian lift of the degeneracy at b_n to a simplex with domain e_n. Lemma 5.1.4 observes that cocartesian lifts of degenerate simplices are isomorphisms, so this last horn is actually a "special outer horn" with first edge invertible. Consequently, by Theorem D.4.18 it can therefore be lifted against the isofibration p. This construction fills the sphere at the domain end of the hollow cylinder, solving the original lifting problem. □

16.2.12. PROPOSITION (equivalences of co/cartesian fibrations are determined fiberwise). *In an ∞-cosmos of $(\infty,1)$-categories, a cartesian functor*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & B & \end{tikzcd}$$

between cocartesian fibrations is a fibered equivalence if and only if it is a fiberwise equivalence, meaning that for each $b\colon 1\to B$, the induced functor between fibers $g_b\colon E_b\to F_b$ is an equivalence.

PROOF. Again, it is clear that an equivalence of cocartesian fibrations is necessarily a fiberwise equivalence, so we need only prove the converse. By Propositions 14.2.4 and 14.2.3 if $\mathcal{K}$ is an ∞-cosmos of $(\infty,1)$-categories, then the underlying quasi-category functor induces a cosmological biequivalence $(-)_0\colon \mathcal{K}_{/B}\to \mathcal{QC}at_{/B_0}$. By Corollary 14.3.2, this cosmological biequivalence reflects equivalences, and by Proposition 14.3.5 it preserves cocartesian fibrations and cartesian functors between them. Hence, to show that a cartesian functor and fiberwise equivalence $g\colon E\to F$ is an equivalence over B, it suffices to show that $g_0\colon E_0\to F_0$ is an equivalence over B_0, which we do by verifying that this functor satisfies the hypotheses of Proposition 16.2.11.

By Observation 16.2.2, elements of the underlying quasi-category B_0 correspond bijectively to elements of the ∞-category B, and the cosmological functor $(-)_0\colon\mathcal{K}\to\mathcal{QC}at$ preserves both fibers and equivalences. In this way we see that $g_0\colon E_0\to F_0$ is a fiberwise equivalence for all $b\colon 1\to B_0$. By Proposition 16.2.11 this functor defines a fibered equivalence, and hence g does as well. □

Proposition 16.2.12 has the following important corollary:

16.2.13. COROLLARY (equivalences of modules are determined fiberwise). *In an ∞-cosmos of $(\infty,1)$-categories, a map*

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "{(q,p)}"'] & & F \arrow[dl, two heads, "{(s,r)}"] \\ & A\times B & \end{tikzcd}$$

between modules E and F from A to B is an equivalence if and only if it is a fiberwise equivalence, meaning that for each $a\colon 1 \to A$ and $b\colon 1 \to B$, the induced functor between the fibers $g_{a,b}\colon E(b,a) \to F(b,a)$ is an equivalence.

Proof. Recall from Lemma 11.4.3 that the left-hand legs $q\colon E \twoheadrightarrow A$ and $s\colon F \twoheadrightarrow A$ are cocartesian fibrations and the functor g defines a cartesian functor between them. It follows, by Proposition 16.2.12, that g is an equivalence if and only if for each $a\colon 1 \to A$, the pullback $g_a\colon E(1,a) \to F(1,a)$ is an equivalence. Now each of these ∞-categories defines a module from 1 to B, so by Lemma 11.4.3 again, the pulled back projections $p\colon E(1,a) \twoheadrightarrow B$ and $r\colon F(1,a) \to B$ are cartesian fibrations and g_a defines a cartesian functor between them. By the dual of Proposition 16.2.12, proven in Exercise 16.2.ii, this functor is an equivalence if and only if for each $b\colon 1 \to B$, the induced action $g_{a,b}\colon E(b,a) \to F(a,b)$ on fibers is an equivalence. The stated result follows immediately. $\square$

Corollary 16.2.13 may be applied to characterize fully faithful functors and equivalences between $(\infty,1)$-categories. In [77], Rezk refers to this result as "the fundamental theorem of quasi-category theory."

16.2.14. Proposition. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is fully faithful if and only if for all elements $a,a'\colon 1 \to A$, its action on internal mapping spaces $f_{a,a'}\colon \mathsf{Hom}_A(a,a') \to \mathsf{Hom}_B(fa,fa')$ defines an equivalence of discrete ∞-categories.*

Proof. Corollary 3.5.6 defines a functor $f\colon A \to B$ to be fully faithful if and only if the induced functor $A^{\mathbb{2}} \to \mathsf{Hom}_B(f,f)$ between modules from A to A is an equivalence. By Corollary 16.2.13, this is the case if and only if this map defines a fiberwise equivalence, which means exactly that for all elements $a,a'\colon 1 \to A$, its action on internal mapping spaces $f_{a,a'}\colon \mathsf{Hom}_A(a,a') \to \mathsf{Hom}_B(fa,fa')$ defines an equivalence of discrete ∞-categories. $\square$

There is a closely related notion defined on the point-set level: a simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is **homotopically fully-faithful** if its action $F_{x,y}\colon \mathcal{S}(x,y) \to \mathcal{T}(Fx,Fy)$ on each hom-space is an equivalence of Kan complexes.

16.2.15. Lemma. *A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is homotopically fully faithful if and only if $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$ defines a fully faithful functor of quasi-categories.*

Proof. By Corollary 6.5.12, for any fixed pair of objects $x,y \in \mathcal{S}$, there is a natural equivalence of Kan complexes as displayed vertically below,

$$\begin{tikzcd} \mathcal{S}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, tail, "\wr"] & \mathcal{T}(Fx,Fy) \arrow[d, tail, "\wr"] \\ \mathsf{Hom}_{\mathfrak{N}\mathcal{S}}(x,y) \arrow[r, "\ulcorner \mathfrak{N}F \urcorner"] & \mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,Fy) \end{tikzcd}$$

Thus by Proposition 16.2.14 and the 2-of-3 property, the simplicial functor F is homotopically fully faithful if and only if $\mathfrak{N}F$ is a fully faithful functor of quasi-categories. $\square$

Combining these results, we show that equivalences of $(\infty,1)$-categories are precisely those functors that are pointwise fully faithful and essentially surjective in a suitable sense. Our proof mixes synthetic and analytic techniques:

16.2.16. Theorem. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is an equivalence if and only if it is*

(i) ***fully faithful***: *in the sense that for all elements $a, a' \colon 1 \to A$, the induced map*

$$\mathsf{Hom}_A(a, a') \to \mathsf{Hom}_B(fa, fa')$$

is an equivalence and

(ii) ***essentially surjective*** *in the sense that for all $b \colon 1 \to B$ there exists $a \colon 1 \to A$ and an isomorphism $fa \cong b$ in the homotopy category of B.*

Proof. It is clear that an equivalence of ∞-categories is both fully faithful and essentially surjective. To prove the converse, we start by factoring f as an equivalence followed by an isofibration. Both factors are easily seen to be pointwise fully faithful and essentially surjective, so it suffices to assume that $f \colon A \twoheadrightarrow B$ is an isofibration. Our task is now to show that f is a trivial fibration. In an ∞-cosmos $\mathcal{K}$ of $(\infty,1)$-categories, the cosmological biequivalence $(-)_0 \colon \mathcal{K} \xrightarrow{\sim} \mathcal{QC}at$ of Proposition 14.2.3 preserves trivial fibration and reflects equivalences. So to show that an isofibration $f \colon A \twoheadrightarrow B$ is a trivial fibration, it suffices to show that the underlying isofibration $f_0 \colon A_0 \twoheadrightarrow B_0$ is a trivial fibration, i.e., that we can solve lifting problems of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & \mathsf{Fun}(1, A) \arrow[d, two heads, "f_0"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(1, B) \end{tikzcd} \tag{16.2.17}$$

for $n \geq 0$.

When $f_0 \colon A_0 \twoheadrightarrow B_0$ is an isofibration, the hypothesis that $f \colon A \twoheadrightarrow B$ is essentially surjective in fact implies that $f_0 \colon A_0 \twoheadrightarrow B_0$ is surjective on vertices. Recall that the homotopy category of B is defined to be $\mathsf{h}B \coloneqq \mathsf{hFun}(1, B) \coloneqq \mathsf{h}(B_0)$. Now Corollary 1.1.16 implies that any isomorphism in the homotopy category of B can be represented by a homotopy coherent isomorphism $\mathbb{I} \to \mathsf{Fun}(1, B)$. By definition, a map $\Delta[0] \to B_0$ is an element $b \colon 1 \to B$. A choice of $a \colon 1 \to A$ and a homotopy coherent isomorphism $\beta \colon \mathbb{I} \to \mathsf{Fun}(1, B)$ representing $fa \cong b$ defines a lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{1} \arrow[r, "a"] \arrow[d, hook] & \mathsf{Fun}(1, A) \arrow[d, two heads, "f_0"] \\ \mathbb{1} \arrow[r, hook] \arrow[rr, bend right, "b"'] & \mathbb{I} \arrow[r, "\beta"] \arrow[ur, dashed] & \mathsf{Fun}(1, B) \end{tikzcd}$$

which can be solved by lifting the isomorphism along the isofibration. This solves the lifting problems (16.2.17) in the case $n = 0$.

Applying Proposition 3.4.5 to the commutative diagram below-left, we see that the induced map between modules is an isofibration.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, two heads, "f"] & A \arrow[d, equal] \\ A \arrow[r, two heads, "f"'] & B & A \arrow[l, two heads, "f"] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} & A^2 \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "p_0"] \arrow[d, dashed, two heads] & \\ A & \mathsf{Hom}_f(A, A) \arrow[d, dashed, two heads] & A \\ & \mathsf{Hom}_A(f, f) \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_0"'] & \end{tikzcd}$$

By Proposition 16.2.14 the hypothesis that f is pointwise fully faithful, inducing equivalences between fibers $f_{a,a'} \colon \mathsf{Hom}_A(a, a') \xrightarrow{\sim} \mathsf{Hom}_B(fa, fa')$, implies that the induced map $\mathsf{Hom}_A(f, f) \colon A^2 \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow \mathsf{Hom}_B(f, f)$ is an equivalence and hence under present hypotheses a trivial fibration. The cosmological

functor carries this map to a trivial fibration between quasi-categories, which then enjoys the lifting property below-left for $n \geq 0$:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & (A_0)^{\mathbb{2}} \arrow[d, two heads, "\mathsf{Hom}_{f_0}(A_0,A_0)", "\wr"'] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathsf{Hom}_{B_0}(f_0, f_0) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\partial\Delta[1]}{\cup} \Delta[n]\times\partial\Delta[1] \arrow[r] \arrow[d, hook] & A_0 \arrow[d, two heads, "f_0"] \\ \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B_0 \end{tikzcd}$$

Via the description of the comma construction as a weighted limit giving in Example 7.1.17, the lifting property above-left transposes across the Leibniz version of the weighted limit two-variable adjunction of Definition 7.1.7 to the lifting property displayed above-right, again for $n \geq 0$.

We have already shown that $f_0 \colon A_0 \twoheadrightarrow B_0$ also possesses the right lifting property with respect to the inclusion $\emptyset \hookrightarrow \Delta[0]$. However, this map together with the Leibniz product inclusions $(\partial\Delta[n] \hookrightarrow \Delta[n]) \widehat{\times} (\partial\Delta[1] \hookrightarrow \Delta[1])$ generate the class of monomorphisms of simplicial sets under transfinite composition, pushout, and retract, it follows now from the fact that the map $\mathsf{Hom}_{f_0}(A_0, A_0)$ is a trivial fibration that $f_0 \colon A_0 \overset{\sim}{\twoheadrightarrow} B_0$ is a trivial fibration. Hence $f \colon A \overset{\sim}{\twoheadrightarrow} B$ is a trivial fibration, which is what we wanted to show. ☐

Theorem 16.2.16 can be used to give a direct proof of a result promised in Remark 6.3.15.

16.2.18. Proposition. *Let $F \colon \mathcal{S} \to \mathcal{T}$ be a DK-equivalence of Kan complex enriched categories: a simplicial functor that defines a local equivalence of Kan complexes and is essentially surjective on objects in the homotopy categories of $\mathcal{S}$ and $\mathcal{T}$. Then $\mathfrak{N}F \colon \mathfrak{N}\mathcal{S} \xrightarrow{\sim} \mathfrak{N}\mathcal{T}$ is an equivalence of quasi-categories.*

Proof. Theorem 6.3.13 proves that $\mathfrak{N}\mathcal{S}$ and $\mathfrak{N}\mathcal{T}$ are quasi-categories, so by Theorem 16.2.16 it suffices to verify that $\mathfrak{N}F$ is fully faithful and essentially surjective. By Exercise 6.3.iii that the homotopy category of the quasi-category $\mathfrak{N}\mathcal{S}$ characterized in Lemma 1.1.12 is equivalent to the homotopy category constructed in Digression ??. Thus, since F is essentially surjective, $\mathfrak{N}F$ is as well. Since DK-equivalences are homotopically fully faithful, Lemma 16.2.15 shows that $\mathfrak{N}F$ is a fully faithful functor of quasi-categories. The conclusion follows. ☐

We close with another application of the pointwise universal property results in this section, by describing a common abstract setting which gives rise to adjunctions between quasi-categories defined as homotopy coherent nerves.

16.2.19. Definition. A simplicial functor $U \colon \mathcal{A} \to \mathcal{B}$ between Kan complex enriched categories **admits a pointwise left biadjoint** if for each $b \in \mathcal{B}$ there exists an object $Fb \in \mathcal{A}$ together with a quasi-pseudonatural equivalence

$$\mathcal{A}(Fb, a) \simeq \mathcal{B}(b, Ua)$$

for all $a \in \mathcal{A}$; see Definition 14.4.9.

We now show that the functors considered in Definition 16.2.19 define genuine adjoint functors of quasi-categories upon applying the homotopy coherent nerve.

16.2.20. Proposition. *If $U \colon \mathcal{A} \to \mathcal{B}$ is a simplicial functor between Kan complex enriched categories that admits a pointwise left biadjoint, then its homotopy coherent nerve $\mathfrak{N}U \colon \mathfrak{N}\mathcal{A} \to \mathfrak{N}\mathcal{B}$ admits a left adjoint as a functor between quasi-categories.*

PROOF. By Lemma 2.3.6, $\mathfrak{N}U$ admits a left adjoint if and only if there exists an absolute left lifting of the identity through $\mathfrak{N}U\colon \mathfrak{N}\mathcal{A} \to \mathfrak{N}\mathcal{B}$. By Theorem 16.2.9, this exists if and only if for all objects $b\colon 1 \to \mathfrak{N}\mathcal{B}$ of $\mathcal{B}$ there exists an absolute left lifting of of b through $\mathfrak{N}U$. We define a candidate absolute left lifting diagram

$$\begin{tikzcd} & \mathfrak{N}\mathcal{A} \arrow[d, "\mathfrak{N}U"] \\ 1 \arrow[ur, dashed, "Fb"] \arrow[r, "b"'] \arrow[ur, phantom, shift right=3, "\Uparrow\eta_b"] & \mathfrak{N}\mathcal{B} \end{tikzcd}$$

by choosing the object $Fb \in \mathcal{A}$ for the lift and the 0-arrow $\eta_b\colon b \to UFb$ in $\mathcal{B}$, the image of id_{Fb} under the quasi-pseudonatural equivalence $\mathcal{A}(Fb, a) \simeq \mathcal{B}(b, Ub)$, as a representative for the 2-cell.

To prove that this data defines an absolute right lifting, we appeal to Theorem 3.5.3, which tells us this is the case if and only if η_b induces a fibered equivalence

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{N}\mathcal{A}}(Fb, \mathfrak{N}\mathcal{A}) \arrow[rr, "\ulcorner \mathfrak{N}U(-)\cdot\eta_b \urcorner"] \arrow[dr, two heads] & & \mathrm{Hom}_{\mathfrak{N}\mathcal{B}}(b, \mathfrak{N}U) \arrow[dl, two heads] \\ & \mathfrak{N}\mathcal{A} & \end{tikzcd}$$

By Corollary 16.2.13, this map is an equivalence if and only if for all objects $a\colon 1 \to \mathfrak{N}\mathcal{A}$ of $\mathcal{A}$, the component

$$\mathrm{Hom}_{\mathfrak{N}\mathcal{A}}(Fb, a) \xrightarrow{\ulcorner \mathfrak{N}U(-)\cdot\eta_b \urcorner_a} \mathrm{Hom}_{\mathfrak{N}\mathcal{B}}(b, Ua)$$

defines an equivalence of Kan complexes. By Corollary 6.5.12, this map is equivalent to the component

$$\mathcal{A}(Fb, a) \xrightarrow{\sim} \mathcal{B}(b, Ua)$$

of the quasi-pseudonatural equivalence, which proves the claim. □

Exercises.

16.2.i. EXERCISE. Use Lemma 16.1.4 to prove the duals of Propositions 16.2.4 and 16.2.11.

16.2.ii. EXERCISE. State and prove the duals of Proposition 16.2.6 and 16.2.12.

16.2.iii. EXERCISE. Use Corollary 16.2.13 to give a second proof of Theorem 16.2.9(ii).

16.3. Strongly generating functors

In Proposition 4.3.12 we discovered that any fully faithful functor $f\colon A \to B$ between ∞-categories reflects any limits or colimits that exist in B. In this section, we prove a companion result, that shows that any functor that is both fully faithful and strongly generating, a notion we define presently, preserves limits of any diagram in A known to also have a limit in B.

16.3.1. DEFINITION (strongly generating). A functor of ∞-categories $f\colon A \to B$ is **strongly generating** if it satisfies the property that a 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

is invertible whenever the functor $f \downarrow \beta\colon \mathrm{Hom}_B(f, h) \to \mathrm{Hom}_B(f, k)$ induced by the construction of Observation 3.4.8 is an equivalence.

16.3.2. PROPOSITION. *In an ∞-cosmos of $(\infty,1)$-categories, a functor $f\colon A \to B$ is strongly generating if and only if an arrow $\beta\colon b \to b'$ in the homotopy category of B is invertible precisely when its action by post-composition on the intneral mapping spaces $fa \downarrow \beta\colon \mathsf{Hom}_B(fa,b) \to \mathsf{Hom}_B(fa,b')$ is an equivalence of discrete ∞-categories for all elements $a\colon 1 \to A$.*

PROOF. Given functors $h,k\colon X \to B$ and a 2-cell $\beta\colon h \Rightarrow k$ as in Definition 16.3.1, the fibers of the induced map $f \downarrow \beta\colon \mathsf{Hom}_B(f,h) \to \mathsf{Hom}_B(f,k)$ over a pair of elements $x\colon 1 \to X$ and $a\colon 1 \to A$ is the action of the component $\beta x\colon hx \Rightarrow kx$ on the internal mapping spaces $fa \downarrow \beta x\colon \mathsf{Hom}_B(fa,hx) \to \mathsf{Hom}_B(fa,kx)$. So we know, by Corollary 16.2.13, that $f\downarrow\beta$ is an equivalence if and only if $fa\downarrow\beta x$ is an equivalence for all elements $a\colon 1 \to A$ and $x\colon 1 \to X$. By Lemma 16.2.1, β is an invertible 2-cell if and only if for each element $x\colon 1 \to X$ its component $\beta x\colon hx \Rightarrow kx$ is an isomorphism in the homotopy category $\mathsf{h}B$. This proves the stated result. □

16.3.3. OBSERVATION. The characterization of the Proposition 16.3.2 reveals that a functor $f\colon A \to B$ between $(\infty,1)$-categories is strongly generating if and only if the set of elements $\{fa\colon 1 \to B \mid a\colon 1 \to A\}$ has the property that mapping out of the elements in this set detects isomorphisms with codomain B. In particular, this characterization says nothing about the rest of the structure of the functor f. In this setting, we say that some set G of elements in an $(\infty,1)$-category B is **strongly generating** if it has this isomorphism detection property. It follows from Proposition 16.3.2 that a functor $f\colon A \to B$ is strongly generating if and only if it maps surjectively onto some strongly generating set of elements.

We now show that fully faithful and strongly generating functors of quasi-categories defined via the homotopy coherent nerve construction can be identified by analogous properties at the point-set level.

16.3.4. DEFINITION. A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is **homotopically strongly generating** if a 0-arrow $e\colon a \to b$ in $\mathcal{T}$ is an equivalence if and only if for all $x \in \mathcal{S}$ the map $e \circ -\colon \mathcal{T}(Fx,a) \to \mathcal{T}(Fx,b)$ is an equivalence of Kan complexes.

16.3.5. EXAMPLE. Recall that a 0-arrow $e\colon a \to b$ in a Kan complex enriched category $\mathcal{T}$ is an **equivalence** if there exists a 0-arrow $f\colon b \to a$ and a pair of 1-arrows $\alpha\colon \mathrm{id}_a \to fe$ and $\beta\colon \mathrm{id}_b \to ef$. By the Kan complex enriched analog of Lemma 1.2.15, left as Exercise 16.3.i, $e\colon a \to b$ is an equivalence in $\mathcal{T}$ if and only if for all $z \in \mathcal{T}$, $e \circ -\colon \mathcal{T}(z,a) \to \mathcal{T}(z,b)$ is an equivalence of Kan complexes, which is to say that the identity functor is always homotopically strongly generating.

16.3.6. LEMMA. *A simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ between Kan complex enriched categories is homotopically strongly generating if and only if $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$ is a strongly generating functor of quasi-categories.*

PROOF. Any 0-arrow $e\colon a \to b$ in a Kan complex enriched category $\mathcal{T}$ gives rise to an essentially commutative square of Kan complexes whose verticals are the canonical anodyne extensions of Corollary 6.5.12

$$\begin{tikzcd} \mathcal{T}(Fx,a) \arrow[r, "e\circ-"] \arrow[d, tail, "\wr"'] \arrow[dr, phantom, "\cong"] & \mathcal{T}(Fx,b) \arrow[d, tail, "\wr"] \\ \mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,a) \arrow[r, "e\circ-"'] & \mathsf{Hom}_{\mathfrak{N}\mathcal{T}}(Fx,b) \end{tikzcd}$$

where in this case $x \in \mathcal{S}$. It follows that the top horizontal is an equivalence if and only if the bottom horizontal is an equivalence.

By the characterization of the equivalences mentioned in Example 16.3.5 and proven in Exercise 16.3.i, a 0-arrow $e\colon a \to b$ in $\mathcal{T}$ is an equivalence if and only if the corresponding 1-simplex $e\colon a \to b$ defines an isomorphism in the quasi-category $\mathfrak{N}\mathcal{T}$. Hence by Proposition 16.3.2 the condition that characterizes a homotopically strongly generating simplicial functor $F\colon \mathcal{S} \to \mathcal{T}$ is equivalent to the condition that characterizes a strongly generating functor of quasi-categories $\mathfrak{N}F\colon \mathfrak{N}\mathcal{S} \to \mathfrak{N}\mathcal{T}$. $\square$

By Observation 16.3.3, in an ∞-cosmos of $(\infty,1)$-categories we may sensibly refer to a strongly generating set of elements and need not specify the corresponding functor. As we shall discover later, there is a functor whose image is spanned by the elements of the following lemma, namely the internal Yoneda embedding of Proposition 18.2.3. The following result is then generalized by Proposition 18.2.5.

16.3.7. Lemma. *For any ∞-category A in an ∞-cosmos of $(\infty,1)$-categories $\mathcal{K}$, the set of representable modules*
$$\{p_0\colon \mathsf{Hom}_A(A,a) \twoheadrightarrow A \mid a \in A\}$$
is strongly generating in the quasi-category ${}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A} := \mathfrak{N}({}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A})$ of modules from 1 to A.

Proof. By Lemma 16.3.6 it suffices to show that the collection of covariantly represented modules is homotopically strongly generating in the ∞-cosmos ${}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A} \cong \mathcal{D}isc(\mathcal{C}art(\mathcal{K})_{/A}$. To that end, consider a map
$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & F \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd}$$
with the property that for all elements $a\colon 1 \to A$, the map of functor spaces
$$\begin{tikzcd} \mathsf{Fun}_A(\mathsf{Hom}_A(A,a),E) \arrow[r, "g\circ-", "\sim"'] \arrow[d, "\mathsf{ev}_{\ulcorner \mathrm{id}_a \urcorner}"', "\wr"] & \mathsf{Fun}_A(\mathsf{Hom}_A(A,a),F) \arrow[d, "\mathsf{ev}_{\ulcorner \mathrm{id}_a \urcorner}", "\wr"'] \\ E_a \arrow[r, "g"'] & E_b \end{tikzcd}$$
is an equivalence of Kan complexes. By the external Yoneda lemma for discrete cartesian fibrations, this is equivalent to the action $g\colon E_a \to F_a$ of g on the fibers over a, so this tells us that g is a fiberwise equivalence. Now Corollary 16.2.13 tells us that g is an equivalence, as desired. $\square$

16.3.8. Proposition. *Suppose $f\colon A \to B$ is a functor between ∞-categories that is both fully faithful and strongly generating. Then f preserves limits of any family of diagrams whose limits exist in both A and B.*

Proof. Consider a family of J-shaped diagrams $d\colon D \to A^J$ so that d admits a limit in A and $f^J d$ admits a limit in B, as presented by absolute right lifting diagrams (ℓ,ρ) and (ℓ',ρ') as displayed below. By the universal property of the latter, we have a factorization
$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow\rho" near start] & B \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] & A^J \arrow[r, "f^J"'] & B^J \end{tikzcd} = \begin{tikzcd} & B \arrow[d, "\Delta"] \arrow[dl, phantom, "\Downarrow\rho'" near start] \\ D \arrow[ur, bend left=50, "f\ell"] \arrow[ur, "\ell'"'] \arrow[r, "f^J d"'] & B^J \end{tikzcd}$$
with $\Downarrow\zeta$ between $f\ell$ and ℓ'.

and the claim follows if we can show that ζ is invertible. Rearranging this pasting equality as

$$\begin{tikzcd} D \arrow[r, "\ell"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow\rho"] & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, equal] \\ D \arrow[r, "d"] \arrow[d, equal] & A^J \arrow[d, "f^J"] & A \arrow[l, "\Delta"'] \arrow[d, equal] \\ D \arrow[r, "f^J d"'] & B^J & A \arrow[l, "f^J\Delta"] \end{tikzcd} = \begin{tikzcd} D \arrow[r, "\ell"] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "f"] & A \arrow[d, equal] \\ D \arrow[r, "f\ell"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow\zeta"] & B \arrow[d, equal] & A \arrow[l, "f"'] \arrow[d, equal] \\ D \arrow[r, "\ell'"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow\rho'"] & B \arrow[d, "\Delta"] & A \arrow[l, "f"'] \arrow[d, equal] \\ D \arrow[r, "f^J d"'] & B^J & A \arrow[l, "\Delta f"] \end{tikzcd}$$

we see by Observation 3.4.8 that the natural transformations ρ, ρ', and ζ induce functors between the comma ∞-categories constructed from the horizontal cospans, defining a fibered diagram that commutes up to fibered natural isomorphism:

$$\begin{tikzcd} & \mathsf{Hom}_A(A,\ell) \arrow[rr, "\mathrm{id}\downarrow\mathrm{id}"] \arrow[dl, "\mathrm{id}\downarrow\rho"'] & & \mathsf{Hom}_B(f,f\ell) \arrow[dr, "\mathrm{id}\downarrow\zeta"] & \\ \mathsf{Hom}_{A^J}(\Delta,d) \arrow[drr, "\mathrm{id}\downarrow\mathrm{id}"'] & & \cong & & \mathsf{Hom}_B(f,\ell') \arrow[dll, "\mathrm{id}\downarrow\rho'"] \\ & & \mathsf{Hom}_{B^J}(f^J\Delta, f^J d) = \mathsf{Hom}_{B^J}(\Delta f, f^J d) & & \end{tikzcd}$$

Since (ℓ,ρ) is an absolute right lifting diagram, Theorem 3.5.3 tells us that the functor $\mathrm{id}\downarrow\rho$ is an equivalence. Similarly, since (ℓ',ρ') is absolute right lifting, $\mathrm{id}\downarrow\rho' \colon \mathsf{Hom}_B(B,\ell') \to \mathsf{Hom}_{B^J}(\Delta, f^J d)$ is an equivalence, which pulls back along f to define the equivalence $\mathrm{id}\downarrow\rho'$ in the above pentagon. Since f is fully faithful, Corollary 3.5.6 tells us that $\mathrm{id}\downarrow\mathrm{id}\colon \mathsf{Hom}_A(A,A) \to \mathsf{Hom}_B(f,f)$ is an equivalence, which pulls back to define along ℓ to define the equivalence appearing as the top-horizontal edge of the above pentagon. As observed in the proof of Proposition 4.3.12, since $f\colon A \to B$ is fully faithful, $f^J\colon A^J \to B^J$ is as well, so we conclude similarly that the lower-left functor $\mathrm{id}\downarrow\mathrm{id}\colon \mathsf{Hom}_{A^J}(\Delta,d) \to \mathsf{Hom}_{B^J}(\Delta f^J, f^J d)$ is an equivalence as well.

Thus, we conclude that the functor $\mathrm{id}\downarrow\zeta\colon \mathsf{Hom}_B(f,f\ell) \to \mathsf{Hom}_B(f,\ell')$ is an equivalence. Since f is strongly generating, we conclude that ζ is invertible as desired. ☐

Proposition 16.3.8 will be used in Part V to prove that the internal Yoneda embedding for an $(\infty,1)$-category A preserves all limits that exist in A.

Exercises.

16.3.i. Exercise. Prove that in a Kan complex enriched category $\mathcal{S}$ the following are equivalent, characterizing when a 0-arrow $e\colon a \to b$ is an **equivalence** in $\mathcal{S}$:

(i) There exists a 0-arrow $f\colon b \to a$ together with a pair of 1-arrows $\alpha\colon \mathrm{id}_a \to fe$ and $\beta\colon \mathrm{id}_b \to ef$.
(ii) For all $x \in \mathcal{S}$, the map $e\circ - \colon \mathcal{S}(x,a) \rightsquigarrow \mathcal{S}(x,b)$ is an equivalence of Kan complexes.
(iii) The 1-simplex $e\colon a \to b$ in $\mathfrak{N}\mathcal{S}$ corresponding to the 0-arrow e is an isomorphism in the quasi-category $\mathfrak{N}\mathcal{S}$.

16.4. Complete and cocomplete $(\infty, 1)$-categories

Definition 7.4.1 introduces the notion of *flexible weighted homotopy limits* in a quasi-categorically enriched category, which satisfy the universal property that characterizes flexible weighted limits but up to an equivalence, rather than an isomorphism, of quasi-categories. In this section, we'll prove a companion result to Corollary 7.3.3, demonstrating that the $(\infty, 1)$-core of an ∞-cosmos admits flexible weighted *homotopy* limits. We will then use these results in conjunction with Theorem 7.4.2 to construct examples of complete and cocomplete $(\infty, 1)$-categories modeled as homotopy coherent nerves of Kan complex enriched categories of various provenances.

There is one scenario where the weakened universal property is not necessary. When $W : \mathcal{A} \to s\mathcal{S}et$ Is a weight and $F : \mathcal{A} \to \mathsf{core}_*\mathcal{K}$ is a diagram valued in the $(\infty, 1)$-core of a quasi-categorically enriched category, then cones $\lambda : W \to \mathsf{core}\mathcal{K}(X, F-)$ in $\mathsf{core}_*\mathcal{K}$ correspond to cones in $\mathcal{K}$ whose components $\lambda_a : Wa \to \mathcal{K}(X, Fa)$ send every edge in Wa to an isomorphism in the quasi-category $\mathcal{K}(X, Fa)$. This condition is vacuous if all the edges in WA happen to be invertible, so it follows that the subcategory $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$ is closed on flexible weighted limits whose weights are valued in Kan complexes:

16.4.1. Lemma. *Consider a diagram $F : \mathcal{A} \to \mathsf{core}_*\mathcal{K}$ valued in the $(\infty, 1)$-core of a quasi-categorically enriched category and a flexible weight $W : \mathcal{A} \to s\mathcal{S}et$ that takes values in Kan complexes. Then if a flexible weighted limit cone exists in $\mathcal{K}$, it also defines a W-weighted limit cone in $\mathsf{core}_*\mathcal{K}$.*

Proof. Combining Definitions 7.1.4 and 7.1.7, the latter in the form of Corollary 7.1.10, to say that F admits a W-weighted limit in $\mathcal{K}$ is to supply a natural isomorphism

$$\begin{aligned} \mathcal{K}(X, \lim_W F) &\cong \lim_W \mathcal{K}(X, F-) \\ &\cong \int_{a \in \mathcal{A}} \mathcal{K}(X, Fa)^{W(a)} := \mathrm{eq}\left(\prod_{a \in \mathcal{A}} \mathcal{K}(X, Fa)^{W(a)} \rightrightarrows \prod_{a, b \in \mathcal{A}} \mathcal{K}(X, Fb)^{\mathcal{A}(a,b) \times W(a)} \right) \end{aligned}$$

of quasi-categories. The core functor $\mathsf{core} : \mathcal{QC}at \to \mathcal{K}an$ is an unenriched right adjoint and thus

$$\mathsf{core}\mathcal{K}(X, \lim_W F) \cong \left(\prod_{a \in \mathcal{A}} \mathsf{core}(\mathcal{K}(X, Fa)^{W(a)}) \rightrightarrows \prod_{a, b \in \mathcal{A}} \mathsf{core}(\mathcal{K}(X, Fb)^{\mathcal{A}(a,b) \times W(a)}) \right). \tag{16.4.2}$$

Since the simplicial sets $W(a)$ are Kan complexes, Lemma 16.1.14 tells us that

$$\mathsf{core}(\mathcal{K}(X, Fa)^{W(a)}) \cong (\mathsf{core}\mathcal{K}(X, Fa))^{W(a)},$$

and similarly $\mathsf{core}(\mathcal{K}(X, Fb)^{\mathcal{A}(a,b) \times W(a)}) \cong \left(\mathsf{core}\left(\mathcal{K}(X, Fb)^{\mathcal{A}(a,b)}\right)\right)^{W(a)}$. If the hom spaces $\mathcal{A}(a, b)$ are not Kan complexes, then $(\mathsf{core}\mathcal{K}(X, Fb))^{\mathcal{A}(a,b)}$ will likely define a proper subspace of the Kan complex $\mathsf{core}(\mathcal{K}(X, Fb)^{\mathcal{A}(a,b)})$. However, the hypothesis that the diagram F is valued in $\mathsf{core}_*\mathcal{K} \hookrightarrow \mathcal{K}$ implies that the diagram $\mathcal{K}(X, F-)$ is valued in $\mathsf{core}_*\mathcal{QC}at \hookrightarrow \mathcal{QC}at$

$$\begin{tikzcd} \mathcal{A}(a,b) \arrow[r, "F_{a,b}"] \arrow[dr, dashed] & \mathcal{K}(Fa, Fb) \arrow[rr, "\circ"] & & \mathcal{K}(X, Fb)^{\mathcal{K}(X,Fa)} \\ & \mathsf{core}\mathcal{K}(Fa, Fb) \arrow[u, hook] \arrow[r, "\circ"'] & \mathsf{core}\mathcal{K}(X, Fb)^{\mathsf{core}\mathcal{K}(X,Fa)} \arrow[r, hook] & \mathsf{core}\left(\mathcal{K}(X, Fb)^{\mathcal{K}(X,Fa)}\right) \arrow[u, hook] \end{tikzcd}$$

This tells us that in the equalizer diagram (16.4.2), the images of the parallel pair of maps factor through the subobject

$$(\mathrm{core}\mathcal{K}(X, Fb))^{\mathcal{A}(a,b)} \subset \mathrm{core}(\mathcal{K}(X, Fb)^{\mathcal{A}(a,b)}).$$

Hence,

$$\mathrm{core}\mathcal{K}(X, \lim_W F) \cong \left(\prod_{a\in\mathcal{A}} (\mathrm{core}\mathcal{K}(X, Fa))^{W(a)} \rightrightarrows \prod_{a,b\in\mathcal{A}} (\mathrm{core}\mathcal{K}(X, Fb))^{\mathcal{A}(a,b)\times W(a)} \right)$$

which says that the flexible weighted limit $\lim_W F$ defines a W-weighted limit in $\mathrm{core}_*\mathcal{K}$ as well as in $\mathcal{K}$. □

16.4.3. Corollary. *The core functor* $\mathrm{core}\colon \mathcal{QC}at \to \mathcal{K}an$ *preserves flexible weighted limits of diagrams* $F\colon \mathcal{A} \to \mathrm{core}_*\mathcal{QC}at$ *valued in the* $(\infty,1)$*-core of* $\mathcal{QC}at$ *weighted by weights* $W\colon \mathcal{A} \to s\mathcal{S}et$ *that take values in Kan complexes: i.e., the canonical map*

$$\mathrm{core}(\lim_W F) \to \lim_W \mathrm{core}(F)$$

is an isomorphism of Kan complexes.

Proof. By Lemma 16.4.1, the weighted limit $\lim_W F$ exists in $\mathrm{core}_*\mathcal{QC}at$ under these hypothesis, constructed by applying the formula of Definition 7.1.4 in $s\mathcal{S}et$. By Lemma 16.1.14, the functor $\mathrm{core}\colon \mathrm{core}_*\mathcal{QC}at \to \mathcal{K}an$ is simplicially enriched and hence preserves weighted limits. Thus, the canonical morphism $\mathrm{core}(\lim_W F) \to \lim_W \mathrm{core}(F)$ is an isomorphism in $\mathcal{K}an$ as claimed. □

Lemma 16.4.1 inspires our approach to building flexible weighted homotopy limits in $(\infty,1)$-cores of Kan complex enriched categories: we complete a general flexible weight $W \in s\mathcal{S}et^{\mathcal{A}}$ to a levelwise equivalent flexible weight $W' \in s\mathcal{S}et^{\mathcal{A}}$ valued in Kan complexes and argue that the W'-weighted limit in $\mathcal{K}$ defines a W-weighted *homotopy* limit in $\mathrm{core}_*\mathcal{K}$.

The following lemma records the homotopical properties of the construction of the levelwise Kan complex replacement of a flexible weight.

16.4.4. Lemma.

(i) Any weight $W \in s\mathcal{S}et^{\mathcal{A}}$ *admits a levelwise Kan complex replacement* W'*, related via a projective anodyne extension* $W \hookrightarrow W'$*. Moreover, if the weight* W *is flexible so is* W'*.*

(ii) Given any projective anodyne extension $U \hookrightarrow V$ *of weights in* $s\mathcal{S}et^{\mathcal{A}}$ *and any diagram* $F\colon \mathcal{A} \to \mathcal{K}an$ *valued in Kan complexes, then the induced map* $\lim_V F \xrightarrow{\sim\!\!\!\!\to} \lim_U F$ *on weighted limits is a trivial fibration.*

Proof. By applying the small object argument to the set of **projective horns**

$$\{\Lambda^k[n] \times \mathcal{A}(a,-) \hookrightarrow \Delta[n] \times \mathcal{A}(a,-)\}_{a\in\mathcal{A}, n\geq 0, 0\leq k\leq n},$$

we obtain a weak factorization system on the category of weights $s\mathcal{S}et^{\mathcal{A}}$ which factors each simplicial natural transformation $w\colon U \to V$ of weights into a composite $w = v \circ u$ where

- the map u is a **projective anodyne extension**: a countable composite of pushouts of coproducts of projective horns, and
- the map v is a levelwise Kan fibration.

Applying this factorization to the map $W \to 1$ constructs a projective anodyne extension $u: W \hookrightarrow W'$ whose codomain is a pointwise Kan complex. Since horn inclusions are built as composites of pushouts of sphere boundary inclusions, any projective anodyne extension is a projective cell complex, as in Definition 7.2.1. Consequently, if W is a flexible weight, then so is W'.

For the second claim observe, as in the proof of Proposition 7.3.1, that the map

$$\lim_{\Delta[n]\times\mathcal{A}(a,-)} F \to \lim_{\Lambda^k[n]\times\mathcal{A}(a,-)} F$$

induced on weighted limits by the projective horn $\Lambda^k[n] \times \mathcal{A}(a,-) \hookrightarrow \Delta[n] \times \mathcal{A}(a,-)$ is the trivial fibration $Fa^{\Delta[n]} \overset{\sim}{\twoheadrightarrow} Fa^{\Lambda^k[n]}$. □

16.4.5. Proposition. *Consider a diagram $F: \mathcal{A} \to \mathsf{core}_*\mathcal{K}$ valued in the $(\infty,1)$-core of a quasi-categorically enriched category $\mathcal{K}$, and let $W: \mathcal{A} \to s\mathcal{S}et$ be a flexible weight with pointwise Kan complex replacement $u: W \hookrightarrow W'$. Then any cone $\lambda': W' \to \mathcal{K}(L, F-)$ which displays F as a W'-weighted homotopy limit in $\mathcal{K}$ restricts to define a cone*

$$\begin{tikzcd} W \arrow[r, hook, "u"] \arrow[d, dashed, "\lambda"'] & W' \arrow[d, "\lambda'"] \\ \mathsf{core}\mathcal{K}(L, F-) \arrow[r, hook] & \mathcal{K}(L, F-) \end{tikzcd}$$

which displays L as a W-weighted homotopy limit in $\mathsf{core}_*\mathcal{K}$.

Proof. Since each component $W'a$ is a Kan complex, each cone leg $\lambda'_a: W'a \to \mathcal{K}(L, Fa)$ factors through $\mathsf{core}\mathcal{K}(L, Fa)$; restricting these components further along $u_a: Wa \hookrightarrow Wa'$ defines the cone $\lambda: W \to \mathcal{K}(L, F-)$.

To see that λ has the postulated universal property, consider the composite

$$\mathsf{core}\mathcal{K}(X, L) \xrightarrow{\sim} \mathsf{core}(\lim_{W'} \mathcal{K}(X, F-)) \xrightarrow{\cong} \lim_{W'} \mathsf{core}\mathcal{K}(X, F-) \overset{\sim}{\twoheadrightarrow} \lim_W \mathsf{core}\mathcal{K}(X, F-)$$

The left-hand map is constructed by applying the core functor $\mathsf{core}: \mathcal{QC}at \to \mathcal{K}an$ to the equivalence that encodes the fact that (L, λ') defines a W'-homotopy limit cone. The middle isomorphism is an application of Corollary 16.4.3 to the Kan complex valued weight W' and the diagram $\mathcal{K}(X, F-): \mathcal{A} \to \mathsf{core}_*\mathcal{QC}at$, which takes values in $\mathsf{core}_*\mathcal{QC}at \hookrightarrow \mathcal{QC}at$ as discussed in the proof of Lemma 16.4.1. The right-hand trivial fibration arises by application of Lemma 16.4.4 to $u: W \hookrightarrow W'$ and the Kan-complex valued diagram $\mathsf{core}\mathcal{K}(X, F-)$. This composite is simply the map obtained by post-composing with the cone λ in $\mathsf{core}_*\mathcal{K}$ and thus displays the universal property of a W-weighted homotopy limit cone. □

Proposition 16.4.5 tells us how flexible weighted homotopy limits in the $(\infty,1)$-core of a quasi-categorically enriched category may be constructed, which also lets us identify criteria under which such limits are guaranteed to exist.

16.4.6. Corollary. *The $(\infty,1)$-core of an ∞-cosmos admits flexible homotopy limits. Indeed, the $(\infty,1)$-core of any quasi-categorically enriched category which admits flexible weighted homotopy limits admits flexible weighted homotopy limits as well.*

Proof. By Corollary 7.3.3, ∞-cosmoi admit flexible weighted limits of any shape. Hence Proposition 16.4.5 implies that the $(\infty,1)$-core of any ∞-cosmos admits flexible weighted homotopy limits, constructed by applying Lemma 16.4.4 to take a flexible levelwise Kan complex replacement of the desired weight. The second statement follows from the observation that this argument requires only

that the quasi-categorically enriched category admits flexible weighted homotopy limits; strict flexible weighted limits are not actually required. □

The second statement will enable us to construct flexible homotopy *colimits* in the $(\infty, 1)$-cores of those ∞-cosmoi that arise in the manner described in Proposition E.1.1. The opposites of the ∞-cosmoi constructed in this fashion are not themselves ∞-cosmoi, but they do admit flexible weighted homotopy limits, which is to say that the original ∞-cosmoi admits flexible weighted homotopy colimits.

By construction, the $(\infty, 1)$-core of an ∞-cosmos is a Kan complex enriched category. Theorem 7.4.2 demonstrates that particular classes of flexible weighted homotopy limits and flexible weighted homotopy limits in Kan complex enriched categories model limits and colimits of corresponding diagrams in the quasi-category defined by taking the homotopy coherent nerve. Combining these results, we can now demonstrate that various $(\infty, 1)$-categories defined in this manner are complete and/or cocomplete. Along the way, we will also use results proven earlier in this chapter to prove the converse to Theorem 7.4.2, appearing as Theorem 16.4.20 below.

16.4.7. Definition. For any ∞-cosmos $\mathcal{K}$, the **quasi-category $\mathsf{Cat}_\infty\mathcal{K}$ of ∞-categories** in $\mathcal{K}$ is defined to be the homotopy coherent nerve of the $(\infty, 1)$-core of $\mathcal{K}$,

$$\mathsf{Cat}_\infty\mathcal{K} := \mathfrak{N}\mathsf{core}_*\mathcal{K}.$$

16.4.8. Lemma. *If $\mathcal{K}$ and $\mathcal{L}$ are biequivalent ∞-cosmoi, then $\mathsf{Cat}_\infty\mathcal{K} \simeq \mathsf{Cat}_\infty\mathcal{L}$ are equivalent quasi-categories. Hence, there is a well-defined quasi-category of $(\infty, 1)$-categories, which we denote by $\mathsf{Cat}_{(\infty,1)}$.*

Proof. It suffices to show that a biequivalence $\mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces an equivalence of quasi-categories $\mathsf{Cat}_\infty\mathcal{K} \xrightarrow{\sim} \mathsf{Cat}_\infty\mathcal{L}$. A biequivalence $\mathcal{K} \xrightarrow{\sim} \mathcal{L}$ restricts to define a DK-equivalence of Kan-complex enriched categories $\mathsf{core}_*\mathcal{K} \xrightarrow{\sim} \mathsf{core}_*\mathcal{L}$. Proposition 16.2.18 then implies that the induced map on homotopy coherent nerves defines an equivalence of quasi-categories. □

By passing through a change-of-model functor, the quasi-category $\mathsf{Cat}_{(\infty,1)}$ can be regarded as an ∞-category in any ∞-cosmos of $(\infty, 1)$-categories.

16.4.9. Proposition (completeness of the quasi-category of ∞-categories in an ∞-cosmos). *For any ∞-cosmos $\mathcal{K}$, the quasi-category $\mathsf{Cat}_\infty\mathcal{K}$ of ∞-categories in $\mathcal{K}$ is complete. Moreover, any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ defines a limit preserving functor $F\colon \mathsf{Cat}_\infty\mathcal{K} \to \mathsf{Cat}_\infty\mathcal{L}$.*

Proof. By Corollary 16.4.6 the $(\infty, 1)$-core $\mathsf{core}_*\mathcal{K}$ of an ∞-cosmos possesses all flexible weighted homotopy limits and in particular possesses pseudo homotopy limits, modeled by weighted limits formed by a levelwise Kan complex replacement of the flexible weight. It follows then from Corollary 7.3.3 that cosmological functors, which preserve flexible weighted limits, preserve flexible weighted homotopy limits in $(\infty, 1)$-cores.

Now consider any diagram $f\colon J \to \mathsf{Cat}_\infty\mathcal{K}$ indexed by a small simplicial set J. By Theorem 7.4.2, a pseudo homotopy limit of the corresponding diagram $F\colon \mathfrak{C}J \to \mathsf{core}_*\mathcal{K}$ gives rise to a limit in the quasi-category $\mathsf{Cat}_\infty\mathcal{K}$, and we have just argued that such limits exist and are preserved by the cosmological functor F. This gives the desired result. □

We will be interested in many special cases of Proposition 16.4.9, beyond $\mathsf{Cat}_{(\infty,1)}$. For instance:

16.4.10. EXAMPLE. In Proposition 8.2.12, we demonstrated that for any ∞-cosmos $\mathcal{K}$ and ∞-category $B \in \mathcal{K}$, there are ∞-cosmoi of cartesian and cocartesian fibrations over B and cosmological functors

$$\mathcal{C}art(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B} \hookleftarrow co\mathcal{C}art(\mathcal{K})_{/B}$$

Consequently, by Proposition 16.4.9 the corresponding quasi-categories are complete and closed under limits in the quasi-category $\mathsf{Cat}_\infty\mathcal{K}_{/B}$.

16.4.11. DEFINITION. For any ∞-cosmos $\mathcal{K}$, the **quasi-category $\mathsf{Sp}\mathcal{K}$ of discrete ∞-categories** in $\mathcal{K}$ is defined to be the homotopy coherent nerve of the full subcategory $\mathcal{D}isc(\mathcal{K}) \hookrightarrow \mathcal{K}$ of discrete objects

$$\mathsf{Sp}\mathcal{K} \coloneqq \mathfrak{N}\mathcal{D}isc(\mathcal{K}).$$

Recall from Proposition 1.2.25 that $\mathcal{D}isc(\mathcal{K})$ is an ∞-cosmos that is Kan complex enriched.

16.4.12. REMARK. By Proposition 16.2.18, the quasi-category Sp of "∞-groupoids," by which we mean discrete objects in an ∞-cosmos of $(\infty,1)$-categories is well-defined. On account of Grothendieck's homotopy hypothesis, we refer to Sp as the **quasi-category of spaces**. This motivates our choice of notation in Definition 16.4.7.

It is not necessary to consider Sp exclusively as a quasi-category. Any ∞-cosmos of $(\infty,1)$-categories admits an object whose underlying quasi-category is equivalent to Sp so where convenient we regard Sp as an $(\infty,1)$-category in any of the models appearing in §E.2.

16.4.13. PROPOSITION (completeness of the quasi-category of discrete ∞-categories in an ∞-cosmos). *For any ∞-cosmos $\mathcal{K}$, the quasi-category $\mathsf{Sp}\mathcal{K}$ of discrete ∞-categories in $\mathcal{K}$ is complete. Moreover, any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ defines a limit preserving functor $F\colon \mathsf{Sp}\mathcal{K} \to \mathsf{Sp}\mathcal{L}$.*

PROOF. By Proposition 1.2.25 this is a special case of Proposition 16.4.9. $\square$

16.4.14. EXAMPLE. For any ∞-categories A and B in an ∞-cosmos $\mathcal{K}$, the modules from A to B are defined in 11.4.1 to be the discrete objects in the ∞-cosmos ${}_{A\backslash}\mathcal{F}ib(\mathcal{K})_{/B}$ of two-sided fibrations from A to B. Hence, by Proposition 16.4.13, the quasi-category of modules from A to B in $\mathcal{K}$ is complete.

As an application of these results we prove:

16.4.15. THEOREM. *The inclusion $\mathsf{Sp} \hookrightarrow \mathsf{Cat}_{(\infty,1)}$ admits both left and right adjoints*

$$\begin{tikzcd} \mathsf{Sp} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Cat}_{(\infty,1)} \arrow[l, bend right, "\mathsf{invert}"'] \arrow[l, bend left, "\mathsf{core}"] \end{tikzcd}$$

and is monadic and comonadic.

PROOF. In the models introduced in Definitions 16.4.7 and 16.4.11, $\mathsf{Cat}_{(\infty,1)}$ and Sp are defined as homotopy coherent nerves of the Kan complex enriched categories of Lemma 16.1.14. By Example 2.1.5 the adjunction constructed there descends to an adjunction of quasi-categories, defining right adjoint to the inclusion $\mathsf{Sp} \hookrightarrow \mathsf{Cat}_{(\infty,1)}$.

We construct the left adjoint to the inclusion by appealing to a different point-set level model. By Example D.7.12, the inclusion $\mathsf{Sp} \hookrightarrow \mathsf{Cat}_{(\infty,1)}$ can also be understood as the inclusion of 0-complicial

sets into 1-complicial sets. More precisely, the maximal marking functor of Definition D.1.2

$$
\begin{tikzcd}
s\mathcal{S}et \supset \mathcal{K}an \arrow[rr, bend right, "(-)^\sharp"'] \arrow[rr, phantom, "\perp"] & & 1\text{-}\mathcal{C}omp \subset s\mathcal{S}et^+ \arrow[ll, bend right, "U"']
\end{tikzcd}
$$

carries Kan complexes to 0-complicial sets, a subcategory of 1-complicial sets. By the same argument given in Lemma 16.1.14, this defines a simplicially enriched functor $(-)^\sharp \colon \mathcal{K}an \to \mathsf{core}_* 1\text{-}\mathcal{C}omp$.

The left adjoint to $(-)^\sharp \colon s\mathcal{S}et \to s\mathcal{S}et^+$, which is just the functor that forgets the markings, does not restrict to define a functor from 1-complicial sets to Kan complexes. However it can be used to construct a pointwise left biadjoint, in the sense of Definition 16.2.19, to the simplicial functor $(-)^\sharp \colon \mathcal{K}an \to \mathsf{core}_* 1\text{-}\mathcal{C}omp$. The pointwise left biadjoint carries $A \in \mathsf{core}_* 1\text{-}\mathcal{C}omp$ to the Kan complex replacement of the simplicial set UA. By Proposition 16.2.20, this defines a left adjoint to a functor that is isomorphic to $\mathsf{Sp} \hookrightarrow \mathsf{Cat}_{(\infty,1)}$, which we call $\mathsf{invert} \colon \mathsf{Cat}_{(\infty,1)} \to \mathsf{Sp}$.

Having constructed the adjunctions of the statement, it remains to prove co/monadicity, and in fact by Proposition 10.4.13, it suffices to demonstrate either that these adjunctions are monadic or comonadic. We pick the latter.

By Exercise 16.3.i, a functor between Kan complexes is an isomorphism in Sp if and only if it defines an equivalence of Kan complexes, which is the case if and only if it defines an equivalence of quasi-categories, which is the case if and only if its image defines an isomorphism in $\mathsf{Cat}_{(\infty,1)}$. This proves that the inclusion $\mathsf{Sp} \to \mathsf{Cat}_{(\infty,1)}$ is conservative. By Proposition 1.2.25, $\mathsf{Sp} \to \mathsf{Cat}_{(\infty,1)}$ underlies a cosmological functor $\mathcal{K}an \hookrightarrow \mathcal{Q}\mathcal{C}at$, with flexible weighted limits of Kan complexes created in the ∞-cosmos of quasi-categories. By Proposition 16.4.9, the ∞-categories Sp and $\mathsf{Cat}_{(\infty,1)}$ have and the inclusion creates all limits. This verifies the conditions of the dual of Theorem 10.4.12, so we conclude that $\mathsf{Sp} \to \mathsf{Cat}_{(\infty,1)}$ is comonadic. Proposition 10.4.13 then implies that $\mathsf{Sp} \to \mathsf{Cat}_{(\infty,1)}$ is also monadic. ☐

A similar argument provides the proof of the following general result.

16.4.16. Proposition (co/completeness of the quasi-category of fibrant-cofibrant objects in a simplicial model category). *If $\mathcal{M}$ is a simplicial model category, the quasi-category $\mathfrak{N}\mathcal{M}_{cf}$ of fibrant-cofibrant objects in $\mathcal{M}$ is complete and cocomplete.*

Barnea, Harpaz, and Horel prove more generally that the quasi-category presented by any model category is complete and cocomplete [?, 2.5.9] using the theory of weak fibration categories.

Proof. In a simplicial model category $\mathcal{M}$, the full subcategory $\mathcal{M}_{cf}$ of fibrant-cofibrant objects in Kan complex enriched. By the proof of Proposition 7.3.1, strictly-defined flexible weighted limits exist in the simplicial subcategory $\mathcal{M}_f$ of fibrant objects. So for any flexible weight $W \colon \mathcal{A} \to s\mathcal{S}et$ and diagram $F \colon \mathcal{A} \to \mathcal{M}_{cf}$, we may form the weighted limit $\lim_W F \in \mathcal{M}_f$ and then take its cofibrant replacement $e \colon L \xrightarrow{\sim} \lim_W F$ to obtain an object of $\mathcal{M}_{cf}$. For any $X \in \mathcal{M}_{cf}$ there is a composite equivalence of Kan complexes

$$\mathcal{M}(X, L) \xrightarrow[\sim]{e\circ-} \mathcal{M}(X, \lim_W F) \cong \lim_W \mathcal{M}(X, F-)$$

which shows that the restriction of the W-weighted limit cone along e defines a W-weighed homotopy limit cone in $\mathcal{M}_{cf}$ with summit L. A dual construction gives flexible weighted homotopy colimits in $\mathcal{M}_{cf}$.

In particular, $\mathcal{M}_{cf}$ has pseudo homotopy limits and pseudo homotopy colimits, so Theorem 7.4.2 and its dual prove that any diagram $J \to \mathfrak{N}\mathcal{M}_{cf}$ indexed by a small simplicial set J has a limit and a colimit. □

16.4.17. Example. When $\mathcal{C}$ is a small Kan complex enriched category, the projective model structure, relative to Quillen model's structure for Kan complexes, turns $s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ into a simplicial model category whose weak equivalences and fibrations are defined pointwise and whose cofibrations are generated by the projective cells $\{\partial\Delta[n] \times \mathcal{C}(-, c) \hookrightarrow \Delta[n] \times \mathcal{C}(-, c)\}_{c \in \mathcal{C}, n \geq 0}$ as in Definition 7.2.1. Write $\hat{\mathcal{C}}$ for the subcategory of fibrant-cofibrant objects in $s\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$, the projectively cofibrant and pointwise Kan complex-valued presheaves. By Proposition 16.4.16, the quasi-category $\mathfrak{N}\hat{\mathcal{C}}$ is complete and cocomplete.

Each representable functor in $\mathcal{C}(-, c)$ in Example 16.4.17 is projectively cofibrant by definition and pointwise fibrant since the homs in $\mathcal{C}$ are Kan complexes, so the Yoneda embedding restricts to define a simplicial functor よ: $\mathcal{C} \to \hat{\mathcal{C}}$. By Lemma 16.4.18, this functor enjoys a few special properties:

16.4.18. Lemma. *Let $\mathcal{C}$ be a small Kan complex enriched category and consider the restricted Yoneda embedding よ: $\mathcal{C} \to \hat{\mathcal{C}}$ valued in the full subcategory of projectively cofibrant and pointwise Kan-complex valued presheaves. This functor is simplicially fully faithful and homotopically strongly generating: a simplicial natural transformation $\alpha\colon F \to G \in \hat{\mathcal{C}}$, e is an equivalence in $\hat{\mathcal{C}}$ if and only if for all $c \in \mathcal{C}$, the map $\alpha_*\colon \hat{\mathcal{C}}(\mathcal{C}(-, c), F) \to \hat{\mathcal{C}}(\mathcal{C}(-, c), G)$ is an equivalence of Kan complexes.*

Proof. By the simplicially-enriched Yoneda lemma, Theorem A.3.11, よ: $\mathcal{C} \to \hat{\mathcal{C}}$ is fully faithful. By the same result, there is a natural isomorphism of Kan complexes $\hat{\mathcal{C}}(\mathcal{C}(-, c), F) \cong Fc$, so the map $\alpha_*\colon \hat{\mathcal{C}}(\mathcal{C}(-, c), F) \to \hat{\mathcal{C}}(\mathcal{C}(-, c), G)$ is isomorphic to the component $\alpha_c\colon Fc \to Gc$. As the equivalences in $\hat{\mathcal{C}}$ are the pointwise equivalences, this proves that よ is homotopically strongly generating. □

By Lemmas 16.2.15 and 16.3.6, these properties of the simplicial functor よ: $\mathcal{C} \to \hat{\mathcal{C}}$ pass to the map of quasi-categories defined by homotopy coherent nerves. Writing $C \coloneqq \mathfrak{N}\mathcal{C}$ to streamline our notation, the functor よ: $C \to \hat{C}$ so-produced is an instance of an **internal Yoneda embedding**, as will be discussed in Chapter 18. Consequently:

16.4.19. Corollary. *For any quasi-category C defined as the homotopy coherent nerve of a small Kan complex enriched category $\mathcal{C}$, the internal Yoneda embedding よ: $C \to \hat{C}$ constructed in Example 16.4.17 is fully faithful and strongly generating.* □

Putting it all together, we can now prove the converse to Theorem 7.4.2.

16.4.20. Theorem. *For any Kan complex enriched category $\mathcal{C}$ and small simplicial set X, a homotopy coherent diagram $f\colon X \to \mathfrak{N}\mathcal{C}$ admits a limit or colimit if and only if the transposed diagram $F\colon \mathfrak{C}X \to \mathcal{C}$ admits a pseudo homotopy limit or pseudo homotopy colimit.*

Proof. One half of this result was proven in Theorem 7.4.2, so we need only argue the converse that if $f\colon X \to \mathfrak{N}\mathcal{C}$ admits a limit, then $F\colon \mathfrak{C}X \to \mathcal{C}$ admits a pseudo homotopy limit; the dual result for colimits follows by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$.

By the proof of Proposition 16.4.16 applied to Example 16.4.17, the diagram $よF\colon \mathfrak{C}X \to \hat{C}$ admits a pseudo homotopy limit cone $\Lambda\colon W_X \to \hat{C}(L, よF-)$. By Theorem 7.4.2, this data defines a limit cone λ with summit L in the quasi-category $\mathfrak{N}\hat{C}$.

Let ℓ denote the limit of the diagram $f\colon X \to \mathfrak{N}C$. By Corollary 16.4.19, $よ\colon \mathfrak{N}C \hookrightarrow \mathfrak{N}\hat{C}$ is fully faithful and strongly generating and by Example 16.4.17 $\mathfrak{N}\hat{C}$ is complete. Thus, by Proposition 16.3.8, this functor preserves the limit ℓ, which is to say that $よ\ell$ and L both define limits for the diagram $よf\colon X \to \mathfrak{N}\hat{C}$. In particular, the induced comparison map $e\colon よ\ell \to L$ defines an isomorphism in $\mathfrak{N}\hat{C}$, or equivalently, by Exercise 16.3.i, an equivalence in $\hat{C}$. Since $よ\colon C \hookrightarrow \hat{C}$ is simplicially fully faithful, we may define a W_X-shaped cone over the diagram F in C with summit ℓ by restricting the pseudo-cone Λ along e to obtain a cone in the image of the Yoneda embedding:

$$\begin{tikzcd} W_X \arrow[r, "\Lambda"] \arrow[d, dashed, "\Gamma"'] & \hat{C}(L, よF-) \arrow[d, "-\circ e", "\wr"'] \\ C(\ell, F-) \arrow[r, hook, "よ", "\cong"'] & \hat{C}(よ\ell, よF-) \end{tikzcd}$$

Since by construction the cone Γ is equivalent to the cone Λ, it satisfies the universal property

$$\begin{tikzcd} C(X,\ell) \arrow[r, "\Gamma\circ-"] \arrow[d, phantom, "\wr"] & \lim_{W_X} C(X, F-) \arrow[d, phantom, "\wr"] \\ \hat{C}(よX, L) \arrow[r, "\Lambda\circ-", "\sim"'] & \lim_{W_X} \hat{C}(よX, よF-) \end{tikzcd}$$

to define a pseudo homotopy limit cone over F in C as required. □

Part V

Yoneda stuff

In classical category theory, one of the first things one defines, for an ordinary locally-small 1-category C, is the co- and contravariant Yoneda embeddings arising from the hom-bifunctor

$$C \xhookrightarrow{よ} \mathcal{S}et^{C^{\mathrm{op}}} \qquad C^{\mathrm{op}} \xhookrightarrow{よ} \mathcal{S}et^{C} \qquad C^{\mathrm{op}} \times C \xrightarrow{\mathsf{Hom}_C} \mathcal{S}et$$

One of the central aims of this part will be to construct analogous ∞-functors for an $(\infty, 1)$-category A, a task which is considerably more complicated.

One necessary modification is relatively straightforward: the arrows between a fixedpair of elements $x, y \colon 1 \to A$ assemble into a space rather than a set, so the corresponding ∞-functors will have the form

$$A \xhookrightarrow{よ} \mathsf{Sp}^{A^{\mathrm{op}}} \qquad A^{\mathrm{op}} \xhookrightarrow{よ} \mathsf{Sp}^{A} \qquad A^{\mathrm{op}} \times A \xrightarrow{\mathsf{Hom}_A} \mathsf{Sp}$$

where Sp is the $(\infty, 1)$-category of spaces introduced in Remark 16.4.12. The action on objects of the latter bifunctor is clear from the notation: the pair (x, y) maps to the internal mapping space $\mathsf{Hom}_A(x, y)$ of Definition 3.4.9.

The complexity arises in establishing the ∞-functoriality of this mapping, which must describe the functorial actions of arrows in A in each dimension. If A were a strict topologically enriched category, we'd have continuous functions $\mathsf{Hom}_A(y, z) \times \mathsf{Hom}_A(x, y) \xrightarrow{\circ} \mathsf{Hom}_A(x, z)$ that are strictly associative and unital but this strict structure doesn't exist in general.

To explore what structure does exist, consider the problem of describing the covariant action of a 1-arrow $f \colon y \to z$ of A on the spaces of arrows with domain x. The relevant hom-spaces are both described as pullbacks of the left fibration $p_1 \colon \mathsf{Hom}_A(x, A) \twoheadrightarrow A$ appearing below-left

$$\begin{tikzcd}
\mathsf{Hom}_A(x,y) \arrow[rrr, bend left] \arrow[ddd] \arrow[ddr, dotted, "f_*"'] \arrow[ddr, phantom, very near start, "\lrcorner"] & & & \\
& & & \mathsf{Hom}_A(x,A) \arrow[ddd, two heads, "p_1"] \\
& \mathsf{Hom}_A(x,z) \arrow[urr] \arrow[ddd, two heads] \arrow[r, phantom, "\lrcorner"] & \arrow[l, Rightarrow, dotted, "\chi_f"] & \\
1 \arrow[rrr, "y"] \arrow[dr, equal] & & & \\
& & & A \\
& 1 \arrow[uurr, "z"'] \arrow[uu, phantom, "\Leftarrow f"] & &
\end{tikzcd}$$

The arrow f then represents a 2-cell with codomain A. Because $p_1 \colon \mathsf{Hom}_A(x, A) \twoheadrightarrow A$ is a discrete cocartesian fibration, there exists an essentially uniquely lift χ_f whose domain is the inclusion $\mathsf{Hom}_A(x, y) \to \mathsf{Hom}_A(x, A)$ and the codomain of this natural transformation then factors through the pullback defining $\mathsf{Hom}_A(x, z)$. This defines the required functor $f_* \colon \mathsf{Hom}_A(x, y) \to \mathsf{Hom}_A(y, z)$. Note, however, that essential uniqueness doesn't make this construction strictly functorial. Instead, Proposition 5.4.3 provides pseudofunctoriality. For composable arrows f and g, $(g \circ f)_*$ will be naturally isomorphic to g_* and f_*. But this pseudofunctoriality isn't enough to define the Yoneda embedding either: we must specify natural isomorphisms $(g \circ f)_* \cong g_* \circ f_*$ which then must satisfy further coherence conditions involving the 2-arrows of A and so on. And we haven't yet addressed the contravariant functoriality of the hom bifunctor in the domain variable, which involves further actions which must commute with the ones described above.

In this part we will discover that the two-sided discrete fibration $(p_1, p_0) \colon A^2 \twoheadrightarrow A \times A$ provided for any ∞-category A in an ∞-cosmos $\mathcal{K}$, whose fibers are the internal mapping spaces $\mathsf{Hom}_A(x, y)$, to some extent already encodes the co- and contravariant Yoneda embeddings by virtue of defining a

discrete cocartesian fibration, displayed below-left, in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/A}$, as well as a discrete cartesian fibration, displayed below-right, in $co\mathcal{C}art(\mathcal{K})_{/A}$:

$$\begin{tikzcd} A^2 \arrow[rr, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_0"'] & & A \times A \arrow[dl, two heads, "\pi_0"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A^2 \arrow[rr, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_1"'] & & A \times A \arrow[dl, two heads, "\pi_1"] \\ & A & \end{tikzcd}$$

The key to this is provided in Chapter 17 which proves that any cocartesian fibration $p: E \twoheadrightarrow B$ gives rise to a homotopy coherent diagram indexed by the underlying quasi-category $\mathsf{Fun}(1, B)$ of B and valued in the ∞-cosmos $\mathcal{K}$, defined on objects by mapping an element $b: 1 \to B$ to the fiber E_b. When p is a discrete cocartesian fibration, these fibers E_b are discrete ∞-categories, so this homotopy coherent diagram is valued in the subcosmos $\mathcal{D}isc(\mathcal{K})$. These homotopy coherent diagrams define maps of quasi-categories $c_p: \mathsf{Fun}(1, B) \to \mathsf{Cat}_\infty\mathcal{K}$ or $c_p: \mathsf{Fun}(1, B) \to \mathsf{Sp}(\mathcal{K})$ which we refer to as the *comprehension* functor associated to a cocartesian or discrete cocartesian fibration p. When $\mathcal{K}$ is an ∞-cosmos of $(\infty, 1)$-categories, these may be regarded as functors $c_p: B \to \mathsf{Cat}_{(\infty,1)}$ or $c_p: B \to \mathsf{Sp}$ in the ∞-cosmos $\mathcal{K}$ itself.

In Chapter 18, we specialize the construction of the comprehension functor to the discrete co/cartesian fibrations encoded by $(p_1, p_0): A^2 \twoheadrightarrow A \times A$ to define what we refer to as the "external" co- and contravariant Yoneda embeddings for an ∞-category A. In the covariant case, this takes the form of a homotopy coherent diagram indexed by the underlying quasi-category of A and valued in the ∞-cosmos ${}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A} \cong \mathcal{D}isc\mathcal{C}art(\mathcal{K})_{/A}$ of discrete cartesian fibrations over A. In the remaining chapters, our aim is to explain how to internalize this homotopy coherent diagram, in the case where $\mathcal{K}$ is an ∞-cosmos of $(\infty, 1)$-categories, to an ∞-functor よ$: A \hookrightarrow \mathsf{Sp}^{A^{\mathrm{op}}}$. What this amounts to is proving a "straightening-unstraightening equivalence" a la **[61]** between the ∞-category $\mathsf{Sp}^{A^{\mathrm{op}}}$ and the ∞-category of discrete cartesian fibrations over A.

CHAPTER 17

The comprehension construction

Recall from Chapter 5 that a cocartesian fibration $p\colon E \twoheadrightarrow B$ is meant to encode a covariantly functorial action of the base ∞-category B on the fibers over p, which may be expressed as a mapping from the homotopy category of B to the homotopy 2-category of $\mathcal{K}$. The action on objects carries an element $b\colon 1 \to B$ to the fiber $E_b \in \mathcal{K}$, while the action on morphisms is defined by factoring the codomain of the p-cocartesian lift of a 2-cell β as below-left to define a functor $\beta_*\colon E_a \to E_b$ as displayed below-right:

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & E \arrow[d, two heads, "p"] \\ 1 \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[rr, phantom, "\Downarrow\beta"] & & B \end{tikzcd} \quad = \quad \begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[rr, phantom, bend left=20, "\Downarrow\chi_\beta"] \arrow[dr, dashed, "\beta_*"'] \arrow[rr, dotted, "\beta_*(\ell_a)"] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_b \arrow[ur, dashed, "\ell_b"'] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & \\ 1 \arrow[rr, bend right, "b"'] & & B \end{tikzcd} \tag{17.0.1}$$

This action is not functorial on the nose: the uniqueness property of cocartesian lifts implies that the functors $\gamma_* \cdot \beta_*$ and $(\gamma \circ \beta)_*$ are fiberwise isomorphic, but they may not be identical. In this section, we will show that the assignment of an ∞-category E_b to each element b of B extends to a *homotopy coherent diagram* indexed by the underlying quasi-category of B and valued in the quasi-category of ∞-categories in $\mathcal{K}$. Passing to a quotient, this shows that the data described in Exercise 5.1.iii coherently defines a pseudofunctor $E\colon \mathsf{h}B \rightsquigarrow \mathfrak{h}\mathcal{K}$ in the case of a cocartesian fibration or $E\colon \mathsf{h}B^{\mathrm{op}} \rightsquigarrow \mathfrak{h}\mathcal{K}$ in the case of a contravariant fibration, with both diagrams valued in the $(2,1)$-core of the homotopy 2-category.

In any ∞-cosmos of $(\infty,1)$-categories, this homotopy coherent diagram may be internalized to the ∞-cosmos to define a "straightening" $\mathsf{st}(p)\colon B \to \mathsf{Cat}_{(\infty,1)}$ of the cocartesian fibration p. We construct the straightening of a cartesian or cocartesian fibration as a special case of the more general *comprehension construction*, which we now describe.

17.1. Extending cocartesian cones

We work with a fixed cocartesian fibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$.

The mechanics of the comprehension construction involve a global lifting property enjoyed by cocartesian fibrations, that extends the local lifting properties of §5.3 and §F.4, which we first introduce in its simplest form. Suppose given a pair of simplicial functors

$$\mathfrak{C}\Lambda^n[n] \xrightarrow{E} \mathcal{K} \qquad \text{and} \qquad \mathfrak{C}\Delta[n] \xrightarrow{B} \mathcal{K}$$

connected by a simplicial natural transformation

$$
\begin{tikzcd}
\mathfrak{C}\Lambda^n[n] \arrow[r, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K}
\end{tikzcd}
$$

whose final component $p_n \colon E_n \twoheadrightarrow B_n$ is a cocartesian fibration. Under conditions that we presently enumerate, this data may be extended to a simplicial natural transformation

$$
\begin{tikzcd}
\mathfrak{C}\Lambda^n[n] \arrow[dr, bend left, "E"] \arrow[dr, bend right, "B"'] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow p"] & \\
\mathfrak{C}\Delta[n] \arrow[r, dotted, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K}
\end{tikzcd}
\tag{17.1.1}
$$

the data of which is constructed by “lifting” along $p_n \colon E_n \twoheadrightarrow B_n$.

We illustrate this process in low dimensions:

17.1.2. Observation (low dimensional outer horn extensions).

(i) In the case $n = 1$, the data given in (17.1.1) defines a cocartesian fibration $p_1 \colon E_1 \twoheadrightarrow B_1$ and a functor $B_0 \to B_1$, which may be extended to a simplicial natural transformation of shape $\mathfrak{C}\Delta[1]$ by forming the pullback

$$
\begin{tikzcd}
E_0 \arrow[r, dotted] \arrow[d, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \arrow[d, two heads] \\
B_0 \arrow[r] & B_1
\end{tikzcd}
$$

(ii) In the case $n = 2$, the data given in (17.1.1) defines a cocartesian fibration $p_2 \colon E_2 \twoheadrightarrow B_2$ together with the solid arrows of the diagram appearing below:

$$
\begin{tikzcd}
E_0 \arrow[rr] \arrow[dr, dotted] \arrow[dd, "p_0"'] & & E_2 \arrow[dd, two heads, "p_2"] \\
& E_1 \arrow[ur] \arrow[dd, two heads, "p_1"'] & \\
B_0 \arrow[rr] \arrow[dr] & & B_2 \\
& B_1 \arrow[ur] &
\end{tikzcd}
$$

where the 1-arrow in the functor space $\mathsf{Fun}(B_0, B_2)$ defined as the image of the 1-arrow in the functor space $\mathfrak{C}\Delta[2](0,2) \coloneqq \Delta[1]$ is depicted as a solid-arrow 2-cell. Under the additional hypothesis that the front naturality square is a pullback, this data closely resembles the diagram appearing on the right-hand side of (17.0.1) and the desired lift may be constructed for similar reasons: the lifted 1-arrow in $\mathsf{Fun}(E_0, E_2)$ is chosen to be a p_2-cocartesian lift of the downstairs

1-arrow, and its codomain functor may be factored through E_1 by the universal property of the pullback, defining the dotted 0-arrow $E_0 \to E_1$.

These low-dimensional observations inspire the statement of the following result.

17.1.3. Proposition. *For $n \geq 1$ suppose we are given simplicial functors and a simplicial natural transformation*

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n] \arrow[rr, bend left, "E"] \arrow[dr, bend right] & \Downarrow p & \mathcal{K} \\ & \mathfrak{C}\Delta[n] \arrow[ur, bend right, "B"'] & \end{tikzcd}$$

with the property that

(i) the component $p_n \colon E_n \twoheadrightarrow B_n$ is a cocartesian fibration,

(ii) if $n \geq 2$, the naturality square

$$\begin{tikzcd} E_{n-1} \arrow[r, "e_{n-1,n}"] \arrow[d, two heads, "p_{n-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_n \arrow[d, two heads, "p_n"] \\ B_{n-1} \arrow[r, "b_{n-1,n}"'] & B_n \end{tikzcd}$$

is a pullback, and

(iii) if $n \geq 3$, then for each $0 \leq k \leq n-2$ the 1-arrow ϵ_k

$$\begin{tikzcd} E_k \arrow[rr, bend left] \arrow[dr] \arrow[dd, two heads, "p_k"'] & \Downarrow \epsilon_k & E_n \arrow[dd, two heads, "p_n"] \\ & E_{n-1} \arrow[ur, bend right] \arrow[dd, two heads, "p_{n-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \\ B_k \arrow[rr, bend left] \arrow[dr] & \Downarrow \beta_k & B_n \\ & B_{n-1} \arrow[ur, bend right] & \end{tikzcd}$$

is p_n-cocartesian.

Then we may extend E and p as depicted in (17.1.1).

Proof. The constructions in the cases $n = 1$ and $n = 2$ are given in Observation 17.1.2, so we assume $n \geq 3$. By Lemma 6.3.7, the inclusion $\mathfrak{C}\Lambda^n[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is the identity except on the function complexes

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n](0,n-1) \arrow[r, hook] \arrow[d, phantom, "\wr\Vert"] & \mathfrak{C}\Delta[n](0,n-1) \arrow[d, phantom, "\wr\Vert"] & \mathfrak{C}\Lambda^n[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\Vert"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\Vert"] \\ \partial\square^{n-2} \arrow[r, hook] & \square^{n-2} & \sqcap_0^{n-1,n-1} \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

reprising the notation for the simplicial subsets of the cubes introduced in Notation 6.3.6. Since this inclusion is the identity on objects, all of the components $p_0, \ldots, p_n$ in the extended simplicial natural transformation have already been defined, so all that is required is to extend the simplicial functor E to these new hom-spaces in a way that ensures functoriality with respect to composition with the

0-arrow $e_{n-1,n}\colon E_{n-1} \to E_n$ and naturality with respect to the components p_0, p_{n-1}, and p_n. This can be achieved in two steps, first by solving the lifting problem

$$\begin{tikzcd}
\mathfrak{C}\Lambda^n[n](0,n) \arrow[rr, "E"] \arrow[d, hook] & & \mathsf{Fun}(E_0,E_n) \arrow[d, two heads, "p_n\circ-"] \\
\mathfrak{C}\Delta[n](0,n) \arrow[urr, dashed, "E"] \arrow[r, "B"'] & \mathsf{Fun}(B_0,B_n) \arrow[r, "-\circ p_0"'] & \mathsf{Fun}(E_0,B_n)
\end{tikzcd}$$

and then by solving the following compound lifting problem

$$\begin{tikzcd}
& \mathsf{Fun}(B_0,B_{n-1}) \arrow[r, "-\circ p_0"] & \mathsf{Fun}(E_0,B_{n-1}) \arrow[dr, "b_{n-1,n}\circ-"] & \\
\mathfrak{C}\Delta[n](0,n-1) \arrow[ur, "B"] \arrow[r, dashed, "E"] \arrow[dr, hook] & \mathsf{Fun}(E_0,E_{n-1}) \arrow[ur, "p_{n-1}\circ-"] \arrow[dr, "e_{n-1,n}\circ-"] \arrow[r, phantom, "\lrcorner"] & & \mathsf{Fun}(E_0.B_n) \\
& \mathfrak{C}\Delta[n](0,n) \arrow[r, "E"'] & \mathsf{Fun}(E_0,E_n) \arrow[ur, "p_n\circ-"'] &
\end{tikzcd}$$

which is achieved automatically by the outer commutative hexagon and the right-hand pullback square.

It remains only to solve the first lifting problem. By Lemma 6.3.7 and Notation 6.3.6, the inclusion $\mathfrak{C}\Lambda^n[n](0,n) \hookrightarrow \mathfrak{C}\Delta[n](0,n)$ is isomorphic to the Leibniz product

$$(\partial\square^{n-2} \hookrightarrow \square^{n-2})\hat{\times}(\Delta[0] \overset{0}{\hookrightarrow} \Delta[1]).$$

Since $p_n\colon E_n \twoheadrightarrow B_n$ is a cocartesian fibration in $\mathcal{K}$, by Proposition 5.3.1, $p_n \circ -\colon \mathsf{Fun}(E_0,E_n) \twoheadrightarrow \mathsf{Fun}(E_0,B_n)$ is a cocartesian fibration of quasi-categories. Thus, we may use Lemma F.4.9 to solve the first lifting problem once we verify that the cylinder

$$\partial\square^{n-2} \times \Delta[1] \hookrightarrow \partial\square^{n-2} \times \Delta[1] \cup \square^{n-2} \times \{0\} \cong \mathfrak{C}\Lambda^n[n](0,n) \xrightarrow{E} \mathsf{Fun}(E_0,E_n)$$

is **pointwise** $p_n \circ -$**-cocartesian** as defined in Definition F.4.7.

Explicitly we must show that for each vertex $(a_1,\ldots,a_{n-2})$ of $\partial\square^{n-2}$, the corresponding 1-simplex component of $\partial\square^{n-2} \times \Delta[1]$ maps to a cocartesian 1-arrow in $\mathsf{Fun}(E_0,E_n)$. Under the isomorphism $\square^{n-2} \times \Delta[1] \cong \mathfrak{C}\Delta[n](0,n)$ given in the proof of Lemma 6.3.7, this 1-simplex corresponds to the 1-arrow

$$\{0,n\} \cup \{i \mid a_i = 1\} \subset \{0,n-1,n\} \cup \{i \mid a_i = 1\}.$$

This 1-arrow may be expressed as a whiskered composite of the 1-arrow

$$\{k,n\} \subset \{k,n-1,n\}$$

in $\mathfrak{C}\Delta[n](k,n)$, where k is the maximum index i with $a_i = 1$ or $k = 0$ otherwise, with the 0-arrow $\{0\} \cup \{i \mid a_i = 1\}$ in $\mathfrak{C}\Delta[n](0,k)$. By hypothesis, the image of the 1-arrow $\{k,n\} \subset \{k,n-1,n\}$ is the 1-arrow ϵ_k, which is assumed to be p_n-cocartesian, so the whiskered composite is also p_n-cocartesian. This proves that the cylinder is $p_n \circ -$-cocartesian and thus the required lift exists by Lemma F.4.9. $\square$

The outer horn inclusions $\Lambda^n[n] \hookrightarrow \Delta[n]$ of Proposition 17.1.3 are isomorphic to the maps obtained by joining a sphere boundary inclusion $\partial\Delta[n-1] \hookrightarrow \Delta[n-1]$ with a "cocone vertex" $\Delta[0]$ on the right. Our aim now is to generalize this global lifting property, extending natural transformations along an inclusion of homotopy coherent diagrams of shape $X^{\triangleright} \hookrightarrow Y^{\triangleright}$, where $X^{\triangleright} := X \star \Delta[0]$ and $X \hookrightarrow Y$ is an arbitrary inclusion of simplicial sets. As in Proposition 17.1.3, this requires a hypothesis

on the given $\mathfrak{C}(X \star \Delta[0])$-shaped simplicial natural transformation: namely that it defines what we shall call a *cocartesian cocone*. For economy of language, we define this notion by imposing the conditions (i) and (i) uniformly, but as we shall comment in the proof of Proposition 17.1.10 less is required to construct the desired extensions.

The source and target of a cocartesian cone are simplicial functors $\mathfrak{C}(X \star \Delta[0]) \to \mathcal{K}$ that we refer to as *lax cocones*, terminology we now explain. By the dual of Lemma 7.2.14, the homotopy coherent realization of the simplicial set $X^{\triangleright} := X \star \Delta[0]$ defines a relative simplicial computad $\mathfrak{C}X + \mathbb{1} \hookrightarrow \mathfrak{C}(X^{\triangleright})$. By the dual of Proposition 7.2.8, this inclusion encodes the **collage** — in the sense dual to Definition 7.2.7 — of a weight $M_{X^{\mathrm{op}}} \colon \mathfrak{C}X^{\mathrm{op}} \to s\mathcal{S}et$; explicitly

$$M_{X^{\mathrm{op}}}(x) := \mathfrak{C}(X^{\triangleright})(x, \perp),$$

where x is a vertex in X and $\perp$ is the unique object not in the image of the inclusion $\mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleright})$. By the dual to Corollary 7.2.9, a simplicial functor $\mathfrak{C}(X^{\triangleright}) \to \mathcal{K}$ encodes a $M_{X^{\mathrm{op}}}$-shaped cone under a homotopy coherent diagram of shape X in $\mathcal{K}$.

In fact, we've seen this weight before: $M_{X^{\mathrm{op}}}$ is naturally isomorphic to the weight $W_{X^{\mathrm{op}}}$ introduced in Definition 7.2.15. If we were primarily interested in $M_{X^{\mathrm{op}}}$-shaped cones in a Kan complex enriched category, we would follow the terminology introduced there and refer to $M_{X^{\mathrm{op}}} \cong W_{X^{\mathrm{op}}}$ as the **weight for pseudo colimits** of homotopy coherent diagrams of shape X. However, at present $\mathcal{K}$ is a quasi-category enriched category, so we follow Remark 7.2.16 and refer to $M_{X^{\mathrm{op}}}$ as the **weight for oplax colimits** of homotopy coherent diagrams of shape X. Confusingly, due to the principle that a W-weighted colimit in an enriched category should coincide with a W-weighted limit in the opposite category, oplax colimits represent lax cones under a diagram, as explained in Exercise 17.1.i, so we might also describe a simplicial functor $\mathfrak{C}(X^{\triangleright}) \to \mathcal{K}$ as a **lax cone** under a homotopy coherent diagram of shape X. We opt for this simpler terminology in the following definition.

17.1.4. Definition (lax cocones). For a simplicial set X and an ∞-cosmos $\mathcal{K}$, a **lax cocone** of shape X is given by a simplicial functor

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[dl, hook'] \arrow[dr, "{\langle B_\bullet, B \rangle}"] & \\ \mathfrak{C}(X^{\triangleright}) \arrow[rr, "\ell"'] & & \mathcal{K} \end{tikzcd}$$

The **base** of a lax cocone refers to the restricted homotopy coherent diagram $B_\bullet \colon \mathfrak{C}X \to \mathcal{K}$, while the object $B \colon \mathbb{1} \to \mathcal{K}$ defines its **nadir**.

Lax cocones can be whiskered in two distinct ways, as can easily be verified directly from the definition:

17.1.5. Lemma (whiskering lax cocones). *Consider a lax cocone of shape X*

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[dl, hook'] \arrow[dr, "{\langle B_\bullet, B \rangle}"] & \\ \mathfrak{C}(X^{\triangleright}) \arrow[rr, "\ell"'] & & \mathcal{K} \end{tikzcd}$$

with base diagram $B_\bullet \colon \mathfrak{C}X \to \mathcal{K}$ and nadir B.

(i) For any functor $f\colon B \to A$ in $\mathcal{K}$, there is a whiskered lax cocone of shape X

$$\begin{tikzcd}
& \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle B_\bullet, A\rangle"] & \\
\mathfrak{C}(X^\triangleright) \arrow[rr, "f\cdot\ell"'] & & \mathcal{K}
\end{tikzcd}$$

with the same base diagram and whose components from $x \in X$ to $\top$ are defined by whiskering with f:

$$\mathfrak{C}X^\triangleright(x,\top) \xrightarrow{\ell_{x,\top}} \mathsf{Fun}(B_x, B) \xrightarrow{f\circ -} \mathsf{Fun}(B_x, A)$$

(ii) For any map of simplicial sets $k\colon Y \to X$, there is a whiskered lax cocone of shape Y

$$\begin{tikzcd}
& \mathfrak{C}Y + \mathbb{1} \arrow[dl] \arrow[rr, "\mathfrak{C}k+\mathbb{1}"] & & \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle B_\bullet, B\rangle"] & \\
\mathfrak{C}Y^\triangleright \arrow[rr, "\mathfrak{C}k^\triangleright"'] & & \mathfrak{C}(X^\triangleright) \arrow[rr, "\ell"'] & & \mathcal{K}
\end{tikzcd}$$

with restricted base diagram and with the same nadir.

Proof. Exercise 17.1.ii. □

17.1.6. Definition (cocartesian cocones). For any simplicial set X and ∞-cosmos $\mathcal{K}$, a simplicial natural transformation

$$\begin{tikzcd}
\mathfrak{C}(X^\triangleright) \arrow[r, bend left, "E"{name=U}] \arrow[r, bend right, "B"'{name=D}] & \mathcal{K} \arrow[from=U, to=D, phantom, "\Downarrow p"]
\end{tikzcd}$$

defines a **cocartesian cocone** if

(i) the nadir of the natural transformation, that being its component at the cone element $\top$ is a cocartesian fibration $p\colon E \twoheadrightarrow B$,

(ii) for all 0-simplices $x \in X$, the naturality square

$$\begin{tikzcd}
E_x \arrow[r, "e_x"] \arrow[d, two heads, "p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
B_x \arrow[r, "b_x"'] & B
\end{tikzcd}$$

is a pullback, and

(iii) for all non-degenerate 1-simplices $f\colon x \to y \in X$, the 1-arrow

$$\begin{tikzcd}
E_x \arrow[rr, bend left] \arrow[dr, "E_f"'] \arrow[dd, "p_x"'] & \Downarrow\epsilon_f & E \arrow[dd, two heads, "p"] \\
& E_y \arrow[ur] \arrow[dd, two heads, "p_y"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
B_x \arrow[rr, bend left] \arrow[dr, "B_f"'] & \Downarrow\beta_f & B \\
& B_y \arrow[ur] &
\end{tikzcd}$$

is p-cocartesian.

17.1.7. LEMMA. *Cosmological functors $F\colon \mathcal{K} \to \mathcal{L}$ preserve cocartesian cocones: if*

$$\begin{tikzcd} \mathfrak{C}(X^\triangleright) \arrow[r, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

is a cocartesian cocone in $\mathcal{K}$, then the whiskered simplicial natural transformation

$$\begin{tikzcd} \mathfrak{C}(X^\triangleright) \arrow[r, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

is a cocartesian cocone in $\mathcal{L}$.

PROOF. Cosmological functors preserve cocartesian fibrations, pullbacks, cocartesian functors, and cocartesian 1-arrows. $\square$

17.1.8. LEMMA. *The class of cocartesian cocones is stable under re-indexing: a cocartesian cocone of shape Y restricts along any simplicial map $f\colon X \to Y$ to a cocartesian cone of shape X.*

$$\begin{tikzcd} \mathfrak{C}(X^\triangleright) \arrow[r, "\mathfrak{C}(f^\triangleright)"] & \mathfrak{C}(Y^\triangleright) \arrow[r, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

PROOF. This is immediate from Definition 17.1.6, since the defining conditions for cocartesian cones are given pointwise. $\square$

Lemma 17.1.8 concerns "pre-whiskering" of cocartesian cocones along a simplicial map that changes the indexing shape. The following lemma, which we record now for later use, can be understood as a "post-whiskering" result that changes the nadir cocartesian fibration of a cocartesian cone.

17.1.9. LEMMA. *Consider a pullback diagram*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in which q and p are cocartesian fibrations.

(i) *Then any cocartesian cocone with nadir q can be whiskered with (g, f) to define a cocartesian cone with nadir p.*

(ii) *Conversely any cocartesian cocone with nadir p whose codomain diagram is defined by whiskering a diagram $A\colon \mathfrak{C}(X^\triangleright) \to \mathcal{K}$ with the functor $f\colon A \to B$ can be pulled back along f to define a cocartesian cocone with nadir q and codomain A.*

PROOF. First consider a cocartesian cocone

$$\begin{tikzcd} \mathfrak{C}(X^\triangleright) \arrow[r, bend left, "F"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow q"] & \mathcal{K} \end{tikzcd}$$

with nadir-compoment $q\colon F \twoheadrightarrow A$. Then we may whisker the domain and codomain diagrams with g and f respectively in such a way that the base diagrams $\mathfrak{C}X \to \mathcal{K}$ are unchanged, the nadirs F and A are replaced with E and B, respectively, and the action of these functors on the homs from x in X to the cone point $\top$ are defined by whiskering; e.g., for the codomain

$$\mathfrak{C}(X^{\triangleright})(x,\top) \xrightarrow{A} \mathsf{Fun}(A_x, A) \xrightarrow{f\circ-} \mathsf{Fun}(A_x, B)$$

We then define a new simplicial natural transformation between these new whiskered diagrams with nadir component $p\colon E \twoheadrightarrow B$ and with other components identical to the original. Since the square from q to p is a pullback, it is easily verified that this new simplicial natural transformation is again a cocartesian cocone.

Conversely, suppose given a cocartesian cocone

$$\begin{tikzcd} \mathfrak{C}(X^{\triangleright}) \arrow[r, bend left, "E"] \arrow[r, bend right, "f\circ A"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

whose nadir component is p and whose codomain diagram is obtained from whiskering along the functor f as described above. The domain $F\colon \mathfrak{C}(X^{\triangleright}) \to \mathcal{K}$ of the pulled back cocone is defined to have the same base diagram $\mathfrak{C}X \to \mathcal{K}$ as E and to have nadir F. The remaining components of this simplicial functor are induced by the pullback diagram

$$\begin{tikzcd} \mathfrak{C}(X^{\triangleright})(x,\top) \arrow[dr, dashed, "F"] \arrow[drr, bend left, "E"] \arrow[ddr, bend right, "A"'] & & \\ & \mathsf{Fun}(E_x, F) \arrow[r, "g\circ-"] \arrow[d, two heads, "q\circ-"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(E_x, E) \arrow[d, two heads, "p\circ-"] \\ & \mathsf{Fun}(E_x, A) \arrow[r, "f\circ-"'] & \mathsf{Fun}(E_x, B) \end{tikzcd}$$

The simplicial natural transformation $q\colon F \Rightarrow A$ is defined to have components $q_x \coloneqq p_x$ and $q_\top \coloneqq q$, with naturality following from the universal property of the pullback depicted above.

It remains only to verify that the simplicial natural transformation so-constructed is a cocartesian cocone. Axiom 17.1.6(i) follows from pullback stability of cocartesian fibrations, while 17.1.6(ii) follows from right cancelation of pullback squares. The final property 17.1.6(iii) is a consequence of Proposition 5.1.20 and Lemma 5.3.5, since the relevant 1-arrow components of the cocone F whisker with g to the corresponding 1-arrow compoments of the cocone E, which are known to be p-cocartesian. □

17.1.10. Proposition. *For any inclusion of simplicial sets $X \hookrightarrow Y$, homotopy coherent diagram $B\colon \mathfrak{C}Y^{\triangleright} \to \mathcal{K}$, and cocartesian cocone of shape X whose codomain is defined by restricting this diagram,*

$$\begin{tikzcd} \mathfrak{C}X^{\triangleright} \arrow[dr, bend left, "E"] \arrow[dr, bend right, "B"'] \arrow[dr, phantom, "\Downarrow p"] \arrow[d, hook] & \\ \mathfrak{C}Y^{\triangleright} \arrow[r, dotted, bend left, "E"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

this cocartesian cocone may be extended to a cocartesian cocone of shape Y with the same nadir component.

PROOF. The inclusion of simplicial sets $X \hookrightarrow Y$ may be presented as a transfinite composite of pushouts of maps of the form $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$. So by induction, it suffices to consider extensions defined by "attaching a cell," as in the following pushout diagram:

$$\begin{tikzcd} \mathfrak{C}\Lambda^{n+1}[n+1] \cong \mathfrak{C}\partial\Delta[n]^{\triangleright} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \mathfrak{C}X^{\triangleright} \arrow[d, hook] \\ \mathfrak{C}\Delta[n+1] \cong \mathfrak{C}\Delta[n]^{\triangleright} \arrow[r] & \mathfrak{C}(X \cup_{\partial\Delta[n]} \Delta[n])^{\triangleright} \end{tikzcd}$$

By Lemma 17.1.8, the given cocartesian cocone of shape X restricts along the attaching map to define a cocartesian cocone of shape $\partial\Delta[n]$, which can be extended to a cocartesian cocone of shape $\Delta[n]$ by Proposition 17.1.3. By the universal property of the pushout, this data combines to define a simplicial natural transformation of shape $(X \cup_{\partial\Delta[n]} \Delta[n])^{\triangleright}$, which is easily seen to be a cocartesian cocone. ☐

Exercises.

17.1.i. EXERCISE. For any 2-categories $\mathcal{C}$ and $\mathcal{D}$ and 2-functors $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$, a **oplax natural transformation $\phi \colon F \Rightarrow G$** is given by

- a 1-cell $\phi_x \colon Fx \to Gx \in \mathcal{D}$ for every object $x \in \mathcal{C}$ and
- an 2-cell in $\mathcal{D}$

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\phi_f\Downarrow"] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

  for each 1-cell $f \colon x \to y \in \mathcal{C}$

satisfying the coherence conditions of Definition 14.4.2. A **lax natural transformation** $\phi \colon F \Rightarrow G$ is defined similarly but with the definition of the 2-cells ϕ_f reversed.

(i) For a 1-category A, show that the weight

$$A \xrightarrow{L_A} \mathcal{C}at$$
$$a \longmapsto A_{/a}$$

represents lax cones over a diagram $F \colon A \to \mathcal{C}at$ with summit the terminal category: i.e., that there is a natural isomorphism between natural transformations $L_A \Rightarrow F$ and lax cones $\Delta\mathbb{1} \Rightarrow F$. Thus, L_A-weighted limits are called **lax limits**.

(ii) Show that the corresponding weight

$$A^{\mathrm{op}} \xrightarrow{L_{A^{\mathrm{op}}}} \mathcal{C}at$$
$$a \longmapsto A^{\mathrm{op}}{}_{/a} \cong ({}^{a/}A)^{\mathrm{op}}$$

represents oplax cones under a diagram $F \colon A \to \mathcal{C}at$ with nadir the terminal category: i.e., that there is a natural isomorphism between natural transformations $L_{A^{\mathrm{op}}} \Rightarrow F$ and oplax cones $F \Rightarrow \Delta\mathbb{1}$. On account of the terminological convention enforced in (i), $L_{A^{\mathrm{op}}}$-weighted colimits are called **lax colimits**.

(iii) For a 1-category A, show that the weight

$$\begin{aligned} A &\xrightarrow{M_A} \mathcal{C}at \\ a &\longmapsto (A_{/a})^{\mathrm{op}} \end{aligned}$$

represents oplax cones over a diagram $F\colon A \to \mathcal{C}at$ with summit the terminal category. Thus, M_A-weighted limits are called **oplax limits**.

(iv) Show that the corresponding weight

$$\begin{aligned} A^{\mathrm{op}} &\xrightarrow{M_{A^{\mathrm{op}}}} \mathcal{C}at \\ a &\longmapsto (A^{\mathrm{op}}{}_{/a})^{\mathrm{op}} \cong ({}^{a/}A) \end{aligned}$$

represents lax cones under a diagram $F\colon A \to \mathcal{C}at$ with nadir the terminal category. On account of the terminological convention enforced in (iii), $M_{A^{\mathrm{op}}}$-weighted colimits are called **oplax colimits**.

17.1.ii. Exercise. Prove Lemma 17.1.5.

17.2. A global reinterpretation of the global lifting property

In this section we take a more global view of the "global lifting property" of Proposition 17.1.10, given a reformulation that re-expresses that result as a genuine lifting property. Recall from Proposition 8.2.12 that for any ∞-cosmos $\mathcal{K}$ there is an ∞-cosmos $co\mathcal{C}art(\mathcal{K})$ defined as a replete sub ∞-cosmos of the ∞-cosmos $\mathcal{K}^{\mathbb{2}}$ of isofibrations in $\mathcal{K}$. We now characterize this quasi-categorically enriched category more explicitly.

17.2.1. Lemma. *For any ∞-cosmos $\mathcal{K}$, the ∞-cosmos $co\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{\mathbb{2}}$ is the replete subcategory whose*

- *objects are the cocartesian fibrations in $\mathcal{K}$ and*
- *0-arrows are cartesian functors in $\mathcal{K}$*

with positive-dimensional arrows created by the inclusion into the ∞-cosmos of isofibrations.

Proof. The notion of replete subcategory introduced in Definition 8.2.1 demands fullness on positive-dimensional arrows. In light of Proposition 8.2.12 it remains only to characterize the 0-arrows in $co\mathcal{C}art(\mathcal{K})$. In the proof of that result, the inclusion $co\mathcal{C}art(\mathcal{K}) \hookrightarrow \mathcal{K}^{\mathbb{2}}$ is characterized as the pullback of the non-full inclusion $\mathcal{L}ari(\mathcal{K}) \hookrightarrow \mathcal{K}^{\mathbb{2}}$ along the simplicial functor $K\colon \mathcal{K}^{\mathbb{2}} \to \mathcal{K}^{\mathbb{2}}$ that sends $p\colon E \twoheadrightarrow B$ to the isofibration $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(p, B)$.

Proposition 8.2.9 observes that the 0-arrows in the functor spaces of $\mathcal{L}ari(\mathcal{K})$ are commutative squares between isofibrations admitting a left adjoint right inverse so that the mate of the identity 2-cell induces an isomorphism in the corresponding square involving the left adjoints. By Theorem 5.1.19, this condition pulls back along the functor K to tell us that 0-arrows in $co\mathcal{C}art(\mathcal{K})$ are commutative squares between cocartesian fibrations that define cartesian functors in the sense of Definition 5.1.18. $\square$

Combining Propositions 8.1.1 and 8.2.12, the codomain-projection functor $\mathsf{cod}\colon \mathcal{coCart}(\mathcal{K}) \to \mathcal{K}$ is cosmological. In this section, we study its properties. The fibers

$$\begin{tikzcd} \mathcal{coCart}(\mathcal{K})_{/B} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{coCart}(\mathcal{K}) \arrow[d, "\mathrm{cod}"] \\ \mathbb{1} \arrow[r, "B"] & \mathcal{K} \end{tikzcd}$$

are the sub ∞-cosmoi $\mathcal{coCart}(\mathcal{K})_{/B} \hookrightarrow \mathcal{K}_{/B}$ of cocartesian fibrations with a fixed base. The functor-spaces in these fibers featured in the external Yoneda lemma of Theorem 5.5.4, so we reprise the notation of (5.5.3) and write

$$\mathsf{Fun}^{\mathrm{cart}}(p,q) \subset \mathsf{Fun}(p,q) \coloneqq \mathsf{Fun}(E,F) \underset{\mathsf{Fun}(E,A)}{\times} \mathsf{Fun}(B,A)$$

for the functor space in $\mathcal{coCart}(\mathcal{K})$ between a pair of cocartesian fibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow A$, defined by restricting to those n-arrows in the functor space of $\mathcal{K}^{\mathbb{2}}$ whose vertices are squares that define cartesian functors from p to q.

17.2.2. Proposition. *For any pair of cocartesian fibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow A$, the functor defined by the action of the codomain-projection functor*

$$\mathsf{Fun}^{\mathrm{cart}}(p,q) \xrightarrow{\mathrm{cod}_{p,q}} \mathsf{Fun}(B,A)$$

is a cocartesian fibration of quasi-categories, whose cocartesian arrows are created from the class of q-cocartesian arrows by the functor $\mathsf{dom}_{p,q}\colon \mathsf{Fun}^{\mathrm{cart}}(p,q) \to \mathsf{Fun}(E,F)$.

Proof. The codomain projection functor factors as

$$\begin{tikzcd} \mathsf{Fun}^{\mathrm{cart}}(p,q) \arrow[dr, hook] \arrow[drr, bend left, "\mathrm{dom}_{p,q}"] \arrow[ddr, bend right, "\mathrm{cod}_{p,q}"'] & & \\ & \mathsf{Fun}(p,q) \arrow[r, "\mathrm{dom}"] \arrow[d, two heads, "\mathrm{cod}"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E,F) \arrow[d, two heads, "q\circ-"] \\ & \mathsf{Fun}(B,A) \arrow[r, "-\circ p"'] & \mathsf{Fun}(E,A) \end{tikzcd}$$

By Proposition 5.3.1, the functor $q \circ -\colon \mathsf{Fun}(E,F) \twoheadrightarrow \mathsf{Fun}(E,A)$ is a cocartesian fibration of quasi-categories. By Proposition 5.1.20 its pullback $\mathsf{cod}\colon \mathsf{Fun}(p,q) \to \mathsf{Fun}(B,A)$ is as well, and moreover cocartesian arrows are preserved and reflected by the functor $\mathsf{dom}\colon \mathsf{Fun}(p,q) \to \mathsf{Fun}(E,F)$.

Thus, it remains only to show that $\mathsf{cod}_{p,q}$ remains a cocartesian fibration when its domain is restricted to $\mathsf{Fun}^{\mathrm{cart}}(p,q) \subset \mathsf{Fun}(p,q)$ by demonstrating that the required cocartesian lifts that make $\mathsf{cod}\colon \mathsf{Fun}(p,q) \twoheadrightarrow \mathsf{Fun}(B,A)$ into a cocartesian fibration are present in $\mathsf{Fun}^{\mathrm{cart}}(p,q)$. Their universal property is then automatically inherited in this full sub-quasi-category. For this it suffices, to argue that for any cod-cocartesian 1-arrow $\chi\colon \Delta[1] \to \mathsf{Fun}(p,q)$ whenever its domain vertex is a cartesian functor then so is its codomain vertex.

From the construction of the functor space $\mathsf{Fun}(p,q)$ in $\mathcal{K}$ we know that the arrow χ comprises a pair of 1-arrows (χ_1, χ_0) of $\mathcal{K}$ as depicted in the following diagram:

$$\begin{tikzcd} E \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\chi_1"] \arrow[d, two heads, "p"'] & F \arrow[d, two heads, "q"] \\ B \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\chi_0"] & A \end{tikzcd}$$

satisfying the compatibility condition $q\chi_1 = \chi_0 p$. By the creation of cocartesian arrows by pullbacks, we know that χ is cocartesian for cod if and only if its component χ_1 is cocartesian for $q \circ -$. By Lemma 5.3.5, this is the case if and only if the 2-cell represented by χ_1 is q-cocartesian.

To complete our proof all we need do is show that the pair (g', f') is a cartesian functor under the assumption that (g, f) is a cartesian functor. So suppose that $\begin{tikzcd} X \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is a p-cocartesian natural transformation, and observe that $g\zeta$, $\chi_1 v$ and $\chi_1 u$ are all q-cocartesian, the first because (g, f) is a cartesian functor, and the last two because the q-cocartesian property of χ_1 is stable under restriction. By naturality of whiskering, the horizontal composite of ζ and χ_1 equals

$$\chi_1 v \cdot g\zeta = g'\zeta \cdot \chi_1 u.$$

Applying the composition and cancelation results of Lemma 5.1.24, we infer that $g'\zeta$ is q-cocartesian. In other words, composition with g' carries p-cocartesian 2-cells to q-cocartesian 2-cells and so (g', f') is a cartesian functor as required. $\square$

17.2.3. LEMMA. *Given cocartesian fibrations p, q, and r and a cartesian functor (g, f)*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, two heads, "p"'] & F \arrow[d, two heads, "q"] & G \arrow[d, two heads, "r"] \\ B \arrow[r, "f"'] & A & C \end{tikzcd}$$

then pre- and post-composition with (g, f) define commutative squares

$$\begin{tikzcd} \mathsf{Fun}^{\mathrm{cart}}(r,p) \arrow[r, "(g,f)\circ-"] \arrow[d, two heads, "\mathrm{cod}_{r,p}"'] & \mathsf{Fun}^{\mathrm{cart}}(r,q) \arrow[d, two heads, "\mathrm{cod}_{r,q}"] & \mathsf{Fun}^{\mathrm{cart}}(q,r) \arrow[r, "-\circ(g,f)"] \arrow[d, two heads, "\mathrm{cod}_{q,r}"'] & \mathsf{Fun}^{\mathrm{cart}}(p,r) \arrow[d, two heads, "\mathrm{cod}_{p,r}"] \\ \mathsf{Fun}(C,B) \arrow[r, "f\circ-"'] & \mathsf{Fun}(C,A) & \mathsf{Fun}(A,C) \arrow[r, "-\circ f"'] & \mathsf{Fun}(B,C) \end{tikzcd}$$

which are both cartesian functors between cocartesian fibrations of quasi-categories.

PROOF. By Proposition 17.2.2, the cocartesian 1-arrows in $\mathsf{Fun}^{\mathrm{cart}}(r,q)$ are created in $\mathsf{Fun}(G,F)$, so to show that the left-hand square defines a cartesian functor, it suffices to argue that the functor $g \circ - \colon \mathsf{Fun}(G,E) \to \mathsf{Fun}(G,F)$ carries $p \circ -$-cocartesian arrows to $q \circ -$-cocartesian arrows. Lemma 5.3.5 observes that $p \circ -$-cocartesian 1-arrows represent p-cocartesian 2-cells in $\mathcal{K}$. Since the pair (g, f) defines a cartesian functor, this preservation property holds.

Similarly, to show that the right-hand square defines a cartesian functor, it suffices to argue that

$$\begin{tikzcd} \mathsf{Fun}(F,G) \arrow[r, "-\circ g"] \arrow[d, two heads, "r\circ -"'] & \mathsf{Fun}(E,G) \arrow[d, two heads, "r\circ -"] \\ \mathsf{Fun}(F,C) \arrow[r, "-\circ g"'] & \mathsf{Fun}(E,C) \end{tikzcd}$$

defines a cartesian functor. This is true without any hypotheses on g by Definition 5.1.6(ii): cocartesian natural transformations are preserved by restriction along any functor. $\square$

Proposition 8.2.12 tells us that limits in the ∞-cosmos of cocartesian fibrations and cartesian functors are created by the inclusion $\mathcal{coCart}(\mathcal{K}) \hookrightarrow \mathcal{K}^{\mathbb{2}}$. The following lemma follows formally, but for sake of concreteness we give a direct verification as well.

17.2.4. Lemma. *Given any pullback square*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & F \arrow[d, two heads, "q"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

between cocartesian fibrations, then for any cocartesian fibration $r\colon G \twoheadrightarrow C$ the square

$$\begin{tikzcd} \mathsf{Fun}^{\mathrm{cart}}(r,p) \arrow[r, "(g{,}f)\circ -"] \arrow[d, two heads, "\mathrm{cod}_{r,p}"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}^{\mathrm{cart}}(r,q) \arrow[d, two heads, "\mathrm{cod}_{r,q}"] \\ \mathsf{Fun}(C,B) \arrow[r, "f\circ -"'] & \mathsf{Fun}(C,A) \end{tikzcd}$$

is a pullback

Proof. Recall that the mapping spaces in $\mathcal{K}$ are defined by pullback squares, as displayed on the front and back faces of the cube below:

$$\begin{tikzcd} \mathsf{Fun}(r,p) \arrow[rr, "\mathrm{dom}"] \arrow[dr, "(g{,}f)\circ -"] \arrow[dd, two heads, "\mathrm{cod}"'] & & \mathsf{Fun}(G,E) \arrow[dr, "g\circ -"] \arrow[dd, two heads, "p\circ -" near start] & \\ & \mathsf{Fun}(r,q) \arrow[rr, "\mathrm{dom}" near start] \arrow[dd, two heads, "\mathrm{cod}" near start] & & \mathsf{Fun}(G,F) \arrow[dd, two heads, "q\circ -"] \\ \mathsf{Fun}(C,B) \arrow[rr, "-\circ r" near end] \arrow[dr, "f\circ -"'] & & \mathsf{Fun}(G,B) \arrow[dr, "f\circ -"] & \\ & \mathsf{Fun}(C,A) \arrow[rr, "-\circ r"'] & & \mathsf{Fun}(G,A) \end{tikzcd}$$

By the simplicial universal property of the pullback defining E, the right-hand face is also a pullback square. Hence by composition and cancelation the left-hand face is as well.

To conclude that the restricted square of the statement defines a pullback square, we must prove that a square from r to p defines a cartesian functor if and only if the composite square from r to q does. Since p-cocartesian arrows are created from q-cartesian arrows by composition with g, this is clear. $\square$

We'll interpret Lemma 17.2.4 as defining a class of cocartesian 0-arrows for the simplicial functor $\mathsf{cod}\colon \mathcal{coCart}(\mathcal{K}) \to \mathcal{K}$ in the sense given by the following definition.

A **local isofibration** between quasi-categorically enriched categories is a simplicial functor $P\colon \mathcal{E} \to \mathcal{B}$ whose action on homs for each pair of objects $P\colon \mathcal{E}(E_0, E_1) \twoheadrightarrow \mathcal{B}(PE_0, PE_1)$ is via an isofibration of quasi-categories.

17.2.5. Definition. Given a local isofibration of quasi-categorically enriched categories $P\colon \mathcal{E} \to \mathcal{B}$, a 0-arrow $g\colon E_0 \to E_1$ in $\mathcal{E}$ is P-**cartesian** if for all objects $X \in \mathcal{E}$, the commutative square

$$\begin{tikzcd} \mathcal{E}(X,E_0) \arrow[r, "g\circ-"] \arrow[d, two heads, "P"'] & \mathcal{E}(X,E_1) \arrow[d, two heads, "P"] \\ \mathcal{B}(PX,PE_0) \arrow[r, "Pg\circ-"'] & \mathcal{B}(PX,PE_1) \end{tikzcd}$$

is a **homotopy pullback** in the sense that the induced map

$$\mathcal{E}(X,E_0) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow \mathcal{E}(X,E_1) \underset{\mathcal{B}(PX,PE_1)}{\times} \mathcal{B}(PX,PE_0)$$

is an trivial fibration of quasi-categories.

17.2.6. Lemma. *Let $P\colon \mathcal{E} \to \mathcal{B}$ define a local isofibration of quasi-categorically enriched categories.*

(i) For any pair of P-cartesian 0-arrows $g\colon E' \to E$ and $h\colon E'' \to E$, there exists an equivalence $w\colon E' \xrightarrow{\sim} E''$ so that $hw = g$ and $Pw = \mathrm{id}$.
(ii) If $g\colon E' \to E$ and $h\colon E'' \to E'$ are P-cartesian 0-arrows, then $h \circ g\colon E'' \to E$ is a P-cartesian 0-arrow.
(iii) If $g\colon E' \to E$ is a P-cartesian 0-arrow and Pg is an equivalence in $\mathcal{B}$ then g is an equivalence in $\mathcal{E}$.

Proof. Exercise 17.2.i. $\square$

17.2.7. Definition. A local isofibration of quasi-categorically enriched categories $P\colon \mathcal{E} \to \mathcal{B}$ is a **cartesian fibration of quasi-categorically enriched categories** if

(i) For each pair of objects $E_0, E_1 \in \mathcal{E}$, the functor $P\colon \mathcal{E}(E_0,E_1) \twoheadrightarrow \mathcal{B}(PE_0,PE_1)$ is a cocartesian fibration of quasi-categories.
(ii) For each 0-arrow $g\colon E_0 \to E_1 \in \mathcal{E}$ and all objects X of $\mathcal{E}$ the squares

$$\begin{tikzcd} \mathcal{E}(X,E_0) \arrow[r, "g\circ-"] \arrow[d, two heads, "P"'] & \mathcal{E}(X,E_1) \arrow[d, two heads, "P"] \\ \mathcal{B}(PX,PE_0) \arrow[r, "Pg\circ-"'] & \mathcal{B}(PX,PE_1) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(E_1,X) \arrow[r, "-\circ g"] \arrow[d, two heads, "P"'] & \mathcal{E}(E_0,X) \arrow[d, two heads, "P"] \\ \mathcal{B}(PE_1,PX) \arrow[r, "-\circ Pg"'] & \mathcal{B}(PE_0,PX) \end{tikzcd}$$

define cartesian functors between cocartesian fibrations.
(iii) For each 0-arrow $f\colon B_0 \to B_1$ in $\mathcal{B}$ and object $E_1 \in \mathcal{E}$ with $PE_1 = B_1$ there exists a P-cartesian 0-arrow $g\colon E_1 \to E_1$ with $Pg = f$.

The motivating example of such a structure is:

17.2.8. Proposition. *For any ∞-cosmos $\mathcal{K}$, the codomain projection functor $\mathsf{cod}\colon \mathcal{coCart}(\mathcal{K}) \to \mathcal{K}$ is a cartesian fibration of quasi-categorically enriched categories.*

PROOF. The three axioms follow immediately from Proposition 17.2.2, Lemma 17.2.3, and Lemma 17.2.4, respectively. □

17.2.9. DEFINITION. Let $P\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of quasi-categorically enriched categories.

- A homotopy coherent 1-simplex $g\colon \mathfrak{C}\Delta[1] \to \mathcal{E}$ is P-**cartesian** if its image $g\colon E_0 \to E_1$ defines a P-cartesian 0-arrow in $\mathcal{E}$.
- A homotopy coherent 2-simplex $g\colon \mathfrak{C}\Delta[2] \to \mathcal{E}$ is P-**cartesian** if the 1-arrow
$$g\colon \mathfrak{C}\Delta[2](0,2) \coloneqq \Delta[1] \to \mathcal{E}(E_0, E_2)$$
is a cocartesian 1-arrow for the cocartesian fibration $P\colon \mathcal{E}(E_0, E_2) \twoheadrightarrow \mathcal{B}(PE_0, PE_1)$.

17.2.10. REMARK. For each $n \geq 3$, every homotopy coherent n-simplex $\mathfrak{C}\Delta[n] \to \mathcal{E}$ in a quasi-categorically enriched category is an equivalence in a suitable sense. As we explain in the next section, the homotopy coherent nerve of a quasi-categorically enriched category canonically defines a 2-complicial set, the $(\infty, 2)$-categorical analogue of a quasi-category introduced in Definition D.7.11. Consequently, it is also natural to extend Definition 17.2.9 and regard these diagrams as being P-cartesian where necessary.

We are now ready to present our reinterpretation of Proposition 17.1.3, interpreting the extension property described there as a lifting property for the cartesian fibration $\mathrm{cod}\colon \mathcal{c}o\mathcal{C}art(\mathcal{K}) \to \mathcal{K}$ of quasi-categorically enriched categories. To contextualize its statement, we first describe this lifting property in low dimensions; compare with Observation 17.1.2.

To begin, observe that a $P\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of quasi-categorically enriched categories admits lifts
$$\begin{tikzcd} \mathfrak{C}\Lambda^1[1] \arrow[r, "E_1"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}\Delta[1] \arrow[r, "f"'] \arrow[ur, dashed, "g"] & \mathcal{B} \end{tikzcd}$$
constructed by choosing $g\colon E_0 \to E_1$ to be a P-cartesian 0-arrow lifting $f\colon B_0 \to PE_1$ as depicted below:
$$\begin{tikzcd} E_0 \arrow[r, dotted, "g"] \arrow[d, mapsto, dotted] \arrow[dr, phantom, "\downarrow"] & E_1 \arrow[d, mapsto] \\ B_0 \arrow[r, "f"'] & PE_1 \end{tikzcd}$$

Similarly, a cartesian fibration of quasi-categorically enriched categories admits lifts
$$\begin{tikzcd} \mathfrak{C}\Lambda^2[2] \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}\Delta[2] \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd}$$

provided that the 0-arrow $u_{1,2} \colon \mathfrak{C}\Lambda^2[2](1,2) \cong \Delta[0] \to \mathcal{E}(E_1, E_2)$ is a P-cartesian 0-arrow. The lifting problem provides the following data in $\mathcal{E}$ and $\mathcal{B}$:

$$\begin{tikzcd}
E_0 \arrow[rr, bend left, "u_{0,2}"] \arrow[dr, dotted, "e"'] \arrow[dd, mapsto] & \arrow[d, phantom, "\Downarrow\chi"] & E_2 \arrow[dd, mapsto] \\
& E_1 \arrow[ur, "u_{1,2}"'] \arrow[dd, mapsto] & \\
PE_0 \arrow[rr, bend left, "Pu_{0,2}"] \arrow[dr, "v_{0,1}"'] & \arrow[d, phantom, "\Downarrow\beta"] & PE_2 \\
& PE_1 \arrow[ur, "Pu_{1,2}"'] &
\end{tikzcd}$$

Start by lifting the 1-arrow β along the cocartesian fibration

$$\begin{tikzcd}
\Delta[0] \arrow[r, "u_{0,2}"] \arrow[d, hook, "0"'] & \mathcal{E}(E_0, E_2) \arrow[d, two heads, "P"] \\
\Delta[1] \arrow[r, "\beta"'] \arrow[ur, dashed, "\chi"] & \mathcal{B}(PE_0, PE_2)
\end{tikzcd}$$

to define a 1-arrow $\chi \colon u_{0,2} \Rightarrow k$ in $\mathcal{E}(E_0, E_2)$. Then lift the vertex $(k, v_{0,1})$ along the trivial fibration

$$\begin{tikzcd}
& \mathcal{E}(E_0, E_1) \arrow[d, two heads, "\wr"] \\
\Delta[0] \arrow[r, "{(k,v_{0,1})}"'] \arrow[ur, dashed, "e"] & \mathcal{E}(E_0, E_2) \underset{\mathcal{B}(PE_0, PE_2)}{\times} \mathcal{B}(PE_0, PE_1)
\end{tikzcd}$$

to define a 0-arrow $e \colon E_0 \to E_1$ with $Pe = v_{0,1}$ and $u_{1,2} \circ e = k$. This data defines the homotopy coherent diagram $w \colon \mathfrak{C}\Delta[2] \to \mathcal{E}$.

17.2.11. Remark. For later use, note that if the 1-arrow $\beta \in \mathcal{B}(PE_0, PE_2)$ is an isomorphism, then so is the 1-arrow $\chi \in \mathcal{E}(E_0, E_2)$ by Corollary 5.1.25. If moreover the 0-arrow $v_{0,1} \in \mathcal{B}(PE_0, PE_1)$ is an equivalence, in the sense that there exists a 0-arrow $v_{0,1}^{-1} \in \mathcal{B}(PE_1, PE_0)$ and isomorphisms $\mathrm{id} \cong v_{0,1}^{-1} \cdot v_{0,1} \in \mathcal{B}(PE_0, PE_0)$ and $\mathrm{id} \cong v_{0,1} \cdot v_{0,1}^{-1} \in \mathcal{B}(PE_1, PE_1)$, then from this data we can produce a second lifting problem

$$\begin{tikzcd}
E_1 \arrow[rr, bend left, "u_{1,2}"] \arrow[dr, dotted, "e^{-1}"'] \arrow[dd, mapsto] & \arrow[d, phantom, "\Downarrow\zeta"] & E_2 \arrow[dd, mapsto] \\
& E_0 \arrow[ur, "u_{0,2}"'] \arrow[dd, mapsto] & \\
PE_1 \arrow[rr, bend left, "Pu_{1,2}"] \arrow[dr, "v_{0,1}^{-1}"'] & \arrow[d, phantom, "\Downarrow\gamma"] & PE_2 \\
& PE_0 \arrow[ur, "Pu_{0,2}"'] &
\end{tikzcd}$$

where γ is defined by composing β^{-1} with the isomorphism $\mathrm{id} \cong v_{0,1} \cdot v_{0,1}^{-1}$. If the 0-arrow $u_{0,2}$ is also a P-cartesian 0-arrow, then the construction just given can be used to define a cocartesian cell ζ and the lifted 0-arrow e^{-1}. By Definition 17.2.7(ii) and Lemma **??**, the composite 1-arrows $\zeta e \cdot \chi$ and $\chi e^{-1} \cdot \zeta$ are cocartesian, so it follows easily from Lemma 5.1.5 that e and e^{-1} are inverse equivalences in the same sense.

In general:

17.2.12. Proposition. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of quasi-categorically enriched categories. A lifting problem*

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n] \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}\Delta[n] \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd}$$

admits a solution whenever

(i) the 0-arrow
$$u_{n-1,n}\colon \mathfrak{C}\Lambda^n[n](n-1,n) \coloneqq \Delta[0] \to \mathcal{E}(E_{n-1}, E_n)$$
is a P-cartesian 0-arrow, and

(ii) for each $0 \leq k \leq n-2$ the 1-arrow $\{k,n\} \subset \{k, n-1, n\}$ in $\mathfrak{C}\Lambda^n[n](k,n)$ is mapped by $u\colon \mathfrak{C}\Lambda^n[n](k,n) \to \mathcal{E}(E_k, E_n)$ to a cocartesian arrow for the cocartesian fibration
$$P\colon \mathcal{E}(E_k, E_n) \twoheadrightarrow \mathcal{B}(PE_k, PE_n).$$

Proof. By Lemma 6.3.7 expressed in the notation introduced in 6.3.6, the inclusion $\mathfrak{C}\Lambda^n[n] \hookrightarrow \mathfrak{C}\Delta[n]$ is the identity except on the function complexes

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n](0,n-1) \arrow[r, hook] \arrow[d, phantom, "\wr\Vert"] & \mathfrak{C}\Delta[n](0,n-1) \arrow[d, phantom, "\wr\Vert"] \\ \partial\square^{n-2} \arrow[r, hook] & \square^{n-2} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}\Lambda^n[n](0,n) \arrow[r, hook] \arrow[d, phantom, "\wr\Vert"] & \mathfrak{C}\Delta[n](0,n) \arrow[d, phantom, "\wr\Vert"] \\ \sqcap_0^{n-1,n-1} \arrow[r, hook] & \square^{n-1} \end{tikzcd}$$

so all that is required is to extend the simplicial functor u to these new hom-spaces, lifting the corresponding data defined by the simplicial functor v along P, in a way that ensures functoriality with respect to composition with the 0-arrow $u_{n-1,n}\colon E_{n-1} \to E_n$. This can be achieved in two steps, first by solving the lifting problem

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n](0,n) \arrow[r, "u"] \arrow[d, hook] & \mathcal{E}(E_0, E_n) \arrow[d, two heads, "P"] \\ \mathfrak{C}\Delta[n](0,n) \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B}(PE_0, PE_n) \end{tikzcd}$$

and then by solving the following compound lifting problem

$$\begin{tikzcd} & & \mathcal{B}(PE_0, PE_{n-1}) \arrow[dr, "Pu_{n-1,n}\circ-"] & \\ \mathfrak{C}\Delta[n](0,n-1) \arrow[r, dashed, "w"] \arrow[urr, bend left, "v"] \arrow[dr, hook] & \mathcal{E}(E_0, E_{n-1}) \arrow[ur, "P"'] \arrow[dr, "e_{n-1,n}\circ-"] & & \mathcal{B}(PE_0.PE_n) \\ & \mathfrak{C}\Delta[n](0,n) \arrow[r, "w"'] & \mathcal{E}(E_0, E_n) \arrow[ur, "P"'] & \end{tikzcd}$$

Here commutativity of the outer pentagon induces a unique map into the pullback of the right-hand cospan, which may then be lifted along the trivial fibration

$$\begin{tikzcd} & \mathcal{E}(E_0, E_{n-1}) \arrow[d, two heads, "\wr"] \\ \mathfrak{C}\Delta[n](0, n-1) \arrow[r] \arrow[ur, dashed, "w"] & \mathcal{E}(E_0, E_n) \underset{\mathcal{B}(PE_0, PE_n)}{\times} \mathcal{B}(PE_0, PE_{n-1}) \end{tikzcd}$$

that encodes the universal property of the P-cartesian 0-arrow $u_{n-1,n}$.

It remains only to solve the first lifting problem. By Lemma 6.3.7 and Notation 6.3.6, the inclusion $\mathfrak{C}\Lambda^n[n](0, n) \hookrightarrow \mathfrak{C}\Delta[n](0, n)$ is isomorphic to the Leibniz product

$$(\partial\Box^{n-2} \hookrightarrow \Box^{n-2})\hat{\times}(\Delta[0] \overset{0}{\hookrightarrow} \Delta[1]).$$

Since $P\colon \mathcal{E}(E_0, E_n) \twoheadrightarrow \mathcal{B}(PE_0, PE_n)$ is a cocartesian fibration of quasi-categories, we may use Lemma F.4.9 to solve the first lifting problem once we verify that the cylinder

$$\partial\Box^{n-2} \times \Delta[1] \hookrightarrow \partial\Box^{n-2} \times \Delta[1] \cup \Box^{n-2} \times \{0\} \cong \mathfrak{C}\Lambda^n[n](0, n) \xrightarrow{u} \mathcal{E}(E_0, E_n)$$

is **pointwise P-cocartesian** as defined in Definition F.4.7.

Explicitly we must show that for each vertex $(a_1, \ldots, a_{n-2})$ of $\partial\Box^{n-2}$, the corresponding 1-simplex component of $\partial\Box^{n-2} \times \Delta[1]$ maps to a cocartesian 1-arrow in $\mathcal{E}(E_0, E_n)$. Under the isomorphism $\Box^{n-2} \times \Delta[1] \cong \mathfrak{C}\Delta[n](0, n)$ given in the proof of Lemma 6.3.7, this 1-simplex corresponds to the 1-arrow

$$\{0, n\} \cup \{i \mid a_i = 1\} \subset \{0, n-1, n\} \cup \{i \mid a_i = 1\}.$$

This 1-arrow may be expressed as a whiskered composite of the 1-arrow

$$\{k, n\} \subset \{k, n-1, n\}$$

in $\mathfrak{C}\Delta[n](k, n)$, where k is the maximum index i with $a_i = 1$ or $k = 0$ otherwise, with the 0-arrow $\{0\} \cup \{i \mid a_i = 1\}$ in $\mathfrak{C}\Delta[n](0, k)$. By hypothesis, the image of the 1-arrow $\{k, n\} \subset \{k, n-1, n\}$ is mapped by u to a cocartesian arrow for the cocartesian fibration $P\colon \mathcal{E}(E_k, E_n) \twoheadrightarrow \mathcal{B}(PE_k, PE_n)$. Hence, by axiom (ii) of Definition 17.2.7, the whiskered composite is a cocartesian arrow for the cartesian fibration $P\colon \mathcal{E}(E_0, E_n) \twoheadrightarrow \mathcal{B}(PE_0, PE_n)$. This proves that the cylinder is P-cocartesian and thus the required lift exists by Lemma F.4.9. $\Box$

The lifting property of Proposition 17.2.12 extends to more general cocone shapes, for which we introduce the following definition.

17.2.13. Definition (P-cocartesian cocones). Let $P\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of quasi-categorically enriched categories and let X be a simplicial set. A simplicial functor $u\colon \mathfrak{C}X^{\triangleright} \to \mathcal{E}$ defines a **P-cocartesian cocone** if

(i) for all 0-simplices $x \in X$, the 0-arrow

$$u_x\colon \mathfrak{C}X^{\triangleright}(x, \top) \coloneqq \Delta[0] \to \mathcal{E}(E_x, E)$$

is a P-cartesian 0-arrow, and

(ii) for all non-degenerate 1-simplices $f\colon x \to y \in X$, the atomic 1-arrow in $\mathfrak{C}X^{\triangleright}(x, \top)$ represented by the 2-simplex

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "f"'] & & \top \\ & y \arrow[ur] & \end{tikzcd}$$

as described in Theorem 6.3.10 is mapped by $u\colon \mathfrak{C}X^{\triangleright}(x, \top) \to \mathcal{E}(E_x, E)$ to a cocartesian arrow for the cocartesian fibration

$$P\colon \mathcal{E}(E_x, E) \twoheadrightarrow \mathcal{B}(PE_x, PE).$$

17.2.14. Lemma. *The class of P-cocartesian cocones is stable under re-indexing: a cocartesian cocone of shape Y restricts along any simplicial map $f\colon X \to Y$ to a cocartesian cone of shape X.*

Proof. Exercise 17.2.ii. □

17.2.15. Proposition. *Let $P\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of quasi-categorically enriched categories and let $X \hookrightarrow Y$ be an inclusion of simplicial sets. If $u\colon \mathfrak{C}X^{\triangleright} \to \mathcal{E}$ is a P-cocartesian cocone then any lifting problem*

$$\begin{tikzcd} \mathfrak{C}X^{\triangleright} \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}Y^{\triangleright} \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd}$$

admits a solution $w\colon \mathfrak{C}Y^{\triangleright} \to \mathcal{E}$, which again defines a P-cocartesian cocone.

Proof. We reprise the argument given in Proposition 17.1.10. By induction, it suffices to consider lifting problems against inclusions formed "attaching a cell," as in the following pushout diagram:

$$\begin{tikzcd} \mathfrak{C}\Lambda^{n+1}[n+1] \cong \mathfrak{C}\partial\Delta[n]^{\triangleright} \arrow[r] \arrow[d, hook] & \mathfrak{C}X^{\triangleright} \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}\Delta[n+1] \cong \mathfrak{C}\Delta[n]^{\triangleright} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathfrak{C}(X \cup_{\partial\Delta[n]} \Delta[n])^{\triangleright} \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd}$$

By Lemma 17.2.14, the given P-cocartesian cocone of shape X restricts along the attaching map to define a P-cocartesian cocone of shape $\partial\Delta[n]$, which can be extended to a cocone of shape $\Delta[n]$ by Proposition 17.2.12, defining the desired functor w by the universal property of the pushout. It remains only to argue that the cocone w is P-cocartesian.

We need only check condition (i) in the case where $n = 0$ for otherwise the inclusion $X \hookrightarrow X \cup_{\partial\Delta[n]} \Delta[n]$ is bijective on vertices. In this case, the extension along $\mathfrak{C}\Lambda^1[1] \hookrightarrow \mathfrak{C}\Delta[1]$ is constructed by choosing a P-cartesian 0-arrow lifting the specified 0-arrow in $\mathcal{B}$; hence w carries the 0-arrow indexed by the newly attached vertex to a P-cartesian 0-arrow as required.

Similarly, we need only check condition (ii) in the case where $n = 1$ for otherwise the inclusion $X \hookrightarrow X \cup_{\partial\Delta[n]} \Delta[n]$ is bijective on 1-simplices. In this case, the extension along $\mathfrak{C}\Lambda^2[2] \hookrightarrow \mathfrak{C}\Delta[2]$ is constructed in such a way that the 1-arrow of interest is sent to a P-cocartesian 1-arrow lifting the specified 1-arrow in $\mathcal{B}$, exactly as required. Thus, the extended functor is again a P-cocartesian cocone and the constructed in completed by induction. □

Exercises.

17.2.i. Exercise. Prove Lemma 17.2.6.

17.2.ii. Exercise. Prove Lemma 17.2.14.

17.3. Homotopical uniqueness of extended homotopy coherent cones

17.3.1. Lemma. *For any cartesian fibration of quasi-categorically enriched categories $P\colon \mathcal{E} \to \mathcal{B}$, the induced map on homotopy coherent nerves $P\colon \mathfrak{N}\mathcal{E} \to \mathfrak{N}\mathcal{B}$ defines a isofibration of 2-complicial sets.*

Proof. To appear. □

We now reap the payoff for this work, using homotopy coherent nerves to define the spaces of extensions of cocartesian cocones as constructed in Proposition 17.1.10 or more generally the spaces of lifts along a cartesian fibration of quasi-categorically enriched categories as given in Proposition 17.2.15 with the aim of proving a homotopical uniqueness result.

17.3.2. Definition. For any quasi-categorically enriched category $\mathcal{E}$ and marked simplicial set X, we refer to a simplicial map $\mathfrak{C}X^{\triangleright} \to \mathcal{E}$ as a **lax cocone** under a homotopy coherent diagram of shape X in $\mathcal{E}$; here $X^{\triangleright} := X \star \Delta[0]$.

17.3.3. Lemma. *For a quasi-categorically enriched category $\mathcal{E}$ and marked simplicial set X, lax cocones of shape X in $\mathcal{E}$ with nadir e correspond to vertices in a 2-complicial set*

$$\mathsf{cocone}_{\mathcal{E}}(X, e) := (N\mathcal{E}_{/\!\!/e})^X,$$

and the accompanying diagram-projection functor

$$\mathsf{cocone}_{\mathcal{E}}(X, e) := (\mathfrak{N}\mathcal{E}_{/\!\!/e})^X \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{N}\mathcal{E})^X =: \mathsf{diag}_{\mathcal{E}}(X)$$

is an isofibration of 2-complicial sets.

Proof. By the adjunction of Proposition D.8.6, a lax cocone $\mathfrak{C}X^{\triangleright} \to \mathcal{E}$ with nadir e, transposes to a map of marked simplicial sets $X \star \Delta[0] \to \mathfrak{N}\mathcal{E}$ that caries the cocone point to e, which transposes again to a map of marked simplicial sets $X \to \mathfrak{N}\mathcal{E}_{/\!\!/e}$ by Lemma D.2.20. By Corollary D.7.16(v) and Theorem D.8.11 $\mathfrak{N}\mathcal{E}_{/\!\!/e}$ is a 2-complicial set, so by Corollary D.7.16(ii) $\mathsf{cocone}_{\mathcal{E}}(X, e) := (N\mathcal{E}_{/\!\!/e})^X$ is as well. By Corollary D.7.16(iii) the domain projection $\mathfrak{N}\mathcal{E}_{/\!\!/e} \twoheadrightarrow \mathfrak{N}\mathcal{E}$ is an isofibration of complicial sets, so by Corollary D.3.12 the map $\mathsf{cocone}_{\mathcal{E}}(X, e) := (\mathfrak{N}\mathcal{E}_{/\!\!/e})^X \twoheadrightarrow (\mathfrak{N}\mathcal{E})^X =: \mathsf{diag}_{\mathcal{E}}(X)$ is as well. □

Recall from Definition 17.2.13 that in the presence of a cartesian fibration of quasi-categorically enriched categories $P\colon \mathcal{E} \to \mathcal{B}$, a lax cocone $\mathfrak{C}X^{\triangleright} \to \mathcal{E}$ is a P-**cocartesian cocone** just when

(i) for all 0-simplices $x \in X$, the 0-arrow

$$u_x\colon \mathfrak{C}X^{\triangleright}(x, \top) := \Delta[0] \to \mathcal{E}(E_x, E)$$

is a P-cartesian 0-arrow, and

(ii) for all non-degenerate 1-simplices $f\colon x \to y \in X$, the atomic 1-arrow in $\mathfrak{C}X^{\triangleright}(x, \top)$ represented by the 2-simplex

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "f"'] & & \top \\ & y \arrow[ur] & \end{tikzcd}$$

is mapped by $u\colon \mathfrak{C}X^{\triangleright}(x, \top) \to \mathcal{E}(E_x, E)$ to a cocartesian arrow for the cocartesian fibration $P\colon \mathcal{E}(E_x, E) \twoheadrightarrow \mathcal{B}(PE_x, PE)$.

Accordingly, we extend the notation introduced in Lemma 17.3.3 and let

$$\mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e) \subset \mathsf{cocone}_{\mathcal{E}}(X,e) \coloneqq (N\mathcal{E}_{/\!/e})^X$$

denote the simplicial subset whose n-simplices are P-cartesian cocones of shape $X \times \Delta[n]$ in $\mathcal{E}$ with nadir e. We refer to $\mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e)$ as the **marked simplicial set of cartesian cocones of shape** X.

These structures allow us to establish the uniqueness of lifts of lax cocones in $\mathcal{B}$ to cartesian cocones in $\mathcal{E}$ with a given nadir.

17.3.4. Proposition. *For any cartesian fibration $P\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ of quasi-categorically enriched categories and simplicial set X, the marked simplicial set of cocartesian cocones of shape X is a 2-complicial set and the isofibration*

$$\mathsf{cocone}_{\mathcal{E}}(X,e) \xrightarrow{P} \mathsf{cocone}_{\mathcal{B}}(X,Pe)$$

of 2-complicial sets restricts to define a trivial fibration

$$\mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e) \xrightarrow[\sim]{P} \mathsf{cocone}_{\mathcal{B}}(X,Pe)$$

The fibers of a trivial fibration of complicial sets define (maximally marked) contractible Kan complexes. Thus, on account of the trivial fibration, the space of lifts of a lax cocone in $\mathcal{B}$ to a cocartesian cocone in $\mathcal{E}$ with nadir e is a contractible Kan complex.

Proof. Recall from Lemma 17.3.3 that $\mathsf{cocone}_{\mathcal{E}}(X,e) \coloneqq (N\mathcal{E}_{/\!/e})^X$. Thus, we see that the map $P\colon \mathsf{cocone}_{\mathcal{E}}(X,e) \twoheadrightarrow \mathsf{cocone}_{\mathcal{B}}(X,Pe)$ is defined as an exponentiation of a slice of the induced map $P\colon \mathfrak{N}\mathcal{E} \twoheadrightarrow \mathfrak{N}\mathcal{B}$, which Lemma 17.3.1 tells us is an isofibration of 2-complicial sets. Thus, by Corollaries D.7.16(iii) and D.3.12, this map is an isofibration of 2-complicial sets as claimed.

We argue that the restricted map $P\colon \mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e) \twoheadrightarrow \mathsf{cocone}_{\mathcal{B}}(X,Pe)$ is a trivial fibration of 2-complicial sets, which will imply that $\mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e)$ is a 2-complicial set, as claimed. Note that $\mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(X,e) \cong \mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(\Delta[0],e)^X$ and again the restricted map is defined by applying $(-)^X$ to $P\colon \mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(\Delta[0],e) \twoheadrightarrow \mathsf{cocone}_{\mathcal{B}}(\Delta[0],Pe)$ so it suffices to verify that this map is a trivial fibration in the case $X = \Delta[0]$.

For this, we must verify the following lifting properties

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & \mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(\Delta[0],e) \arrow[d, "P"] \\ \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathsf{cocone}_{\mathcal{B}}(\Delta[0],Pe) \end{tikzcd} \quad \text{for } n \geq 0 \quad \text{and} \quad \begin{tikzcd} \Delta[n] \arrow[r, "u"] \arrow[d, hook] & \mathsf{cocone}^{\mathrm{cart}}_{\mathcal{E}}(\Delta[0],e) \arrow[d, "P"] \\ \Delta[n]_t \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathsf{cocone}_{\mathcal{B}}(\Delta[0],Pe) \end{tikzcd} \quad \text{for } n \geq 1$$

which transpose to the lifting properties:

$$\begin{tikzcd} \mathfrak{C}\partial\Delta[n]^{\triangleright} \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}\Delta[n]^{\triangleright} \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd} \quad \text{for } n \geq 0 \quad \text{and} \quad \begin{tikzcd} \mathfrak{C}^{+}\Delta[n]^{\triangleright} \arrow[r, "u"] \arrow[d, hook] & \mathcal{E} \arrow[d, "P"] \\ \mathfrak{C}^{+}\Delta[n]^{\triangleright}_t \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \mathcal{B} \end{tikzcd} \quad \text{for } n \geq 1$$

where the simplicial functors u and w define P-cocartesian cocones. The left-hand lifting property was established in Proposition 17.2.12, so it remains only to consider the right-hand lifting property. The only simplices that are marked in $\Delta[n]^{\triangleright}_t \coloneqq \Delta[n]_t \star \Delta[0]$ are the $n+1$-simplex and its $n+1$th face. In the enriched categories these give rise to an n-arrow and an $n-1$-arrow that are marked in $\mathfrak{C}^{+}\Delta[n]^{\triangleright}_t$ but not in $\mathfrak{C}^{+}\Delta[n]^{\triangleright}$. Since the hom-spaces of $\mathcal{E}$ are quasi-categories with all simplices above

dimension 2 marked, this lifting property is automatic for $n > 2$, so we need only consider the cases where $n = 1$ and $n = 2$. Both remaining cases amount to some sort of "conservativity" statement for data defined by cocartesian cocones lifting lax cocones.

The lifting property in the $n = 1$ case is established by Remark 17.2.11. In the case $n = 2$, we need only argue that the image of the 1-arrow in the hom-space $\mathfrak{C}\Delta[2]^{\triangleright}(0,2)$ is marked in $\mathcal{E}$, where we know that the image of the corresponding 1-arrow in the hom-space $\mathfrak{C}^{+}\Delta[2]^{\triangleright}_{t}(0,2)$ is marked in $\mathcal{B}$. Since the cocone $u\colon \mathfrak{C}\Delta[2]^{\triangleright} \to \mathcal{E}$ is P-cocartesian, we are given a trivial fibration of 1-complicial sets

$$\mathcal{E}(e_0, e_2) \xrightarrow{\sim} \mathcal{E}(e_0, e_3) \underset{\mathcal{B}(Pe_0, Pe_3)}{\times} \mathcal{B}(Pe_0, Pe_2)$$

which reflects markings: any 1-simplex in the domain that is marked in the codomain is marked in the domain. Here we have a 1-arrow in $\mathcal{E}(e_0, e_2)$ whose image in $\mathcal{B}(Pe_0, Pe_2)$ and hence $\mathcal{B}(Pe_0, Pe_3)$ are marked. By the construction of the solution w to the lifting problem in in Proposition 17.2.12, the 1-arrow in $\mathcal{E}(e_0, e_3)$ is a cocartesian lift of the 1-arrow in $\mathcal{B}(Pe_0, Pe_3)$ and hence is an isomorphism as well. Thus, we conclude that the 1-arrow in $\mathcal{E}(e_0, e_2)$ is an isomorphism, as desired. ☐

17.4. The comprehension construction

The *comprehension construction* takes as input a cocartesian fibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ and produces a homotopy coherent diagram $c_p\colon \mathfrak{C}B_0 \to \mathcal{K}$ indexed by the underlying quasi-category $B_0 := \mathsf{Fun}(1, B)$ of the ∞-category B. This diagram acts on objects by carrying an element $b\colon 1 \to B$ to the fiber E_b and acts on arrows in the manner displayed in (17.0.1). The *comprehension functor* c_p is defined as the horizontal composite in the diagram below:

$$\begin{tikzcd}
& \mathfrak{C}\varnothing^{\triangleright} \arrow[r, "p"] \arrow[d, hook] & coCart(\mathcal{K}) \arrow[r, "\mathrm{dom}"] \arrow[d, "\mathrm{cod}"] & \mathcal{K} \\
\mathfrak{C}B_0 \arrow[r, hook] & \mathfrak{C}B_0^{\triangleright} \arrow[r, "k^B"'] \arrow[ur, dashed] & \mathcal{K} &
\end{tikzcd}$$

the domain component of the underlying diagram of a cocartesian cocone with nadir p defined by lifting a canonical lax cocone k^B to be described below. As a consequence of Proposition 17.3.4, the comprehension functor is essentially unique.

We begin this section by describing the canonical lax cocone k^B and then formally introducing the comprehension functor. We explore the functoriality of this construction and establish its essential uniqueness. We then achieve several interesting and important generalizations and specializations by making particular choices of cocartesian fibration and varying the ∞-cosmos in which we work.

17.4.1. Lemma. *Given objects $A, B \in \mathcal{E}$ in a simplicially enriched category $\mathcal{E}$ and a simplicial map $f\colon X \to \mathcal{E}(A, B)$ there is a lax cocone $k^f\colon \mathfrak{C}X^{\triangleright} \to \mathcal{E}$ of shape X whose base diagram $\mathfrak{C}X \to \mathcal{E}$ is constant at the object A and whose nadir is B.*

Proof. Recall from Example 6.1.5 the simplicial category $\mathbb{2}[X]$ with with two objects "$-$" and "$+$" and with hom-spaces defined by

$$\mathbb{2}[X](+,-) := \varnothing, \quad \mathbb{2}[X](+,+) := \mathbb{1} =: \mathbb{2}[X](-,-), \quad \mathbb{2}[X](-,+) := X.$$

From this construction it is clear that simplicial functor $\hat{f}\colon \mathbb{2}[X] \to \mathcal{E}$ with $\hat{f}(-) = A$ and $\hat{f}(+) = B$ stand in bijective correspondence with simplicial maps $f\colon X \to \mathcal{E}(A, B)$. The lax cocone of the

statement is defined as the composite

$$k^f \colon \mathfrak{C}X^{\triangleright} \xrightarrow{\tau^X} \mathbb{2}[X] \xrightarrow{\hat{f}} \mathcal{E}$$

of the simplicial functor $\hat{f}$ with the component at X of a natural transformation defined in Lemma 6.5.10 by regarding $\mathfrak{C}X^{\triangleright}$ and $\mathbb{2}[X]$ as pointed simplicial computads under the cone point $\top \colon \mathbb{1} \to \mathfrak{C}X^{\triangleright}$ and $+ \colon \mathbb{1} \to \mathbb{2}[X]$, and then forming the left Kan extension along the Yoneda embedding of an explicitly defined natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend left, "\mathfrak{C}(\Delta[\bullet]^{\triangleright})"] \arrow[r, bend right, "\mathbb{2}[\Delta[\bullet]]"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{1}/s\mathcal{S}et\text{-}\mathcal{C}ptd \end{tikzcd}$$

Since the hom-spaces in $\mathfrak{C}\Delta[n+1]$ and $\mathbb{2}[\Delta[n]]$ are both nerves of posets, all that is required to define $\tau^n \colon \mathfrak{C}\Delta[n+1] \to \mathbb{2}[\Delta[n]]$ is to specify a mapping on objects and an ordering-preserving mapping on atomic 0-arrows. On objects, τ^n maps $0, \ldots, n$ to $-$ and maps $n+1$ to $+$. We leave the details of the specification of the mapping on atomic 0-arrows to the proof of Lemma 6.5.10.

Since the simplicial functor τ^n is constant on the image of the inclusion $\mathfrak{C}\Delta[n] \hookrightarrow \mathfrak{C}\Delta[n]^{\triangleright}$, the left Kan extended simplicial functor $\tau^X \colon \mathfrak{C}X^{\triangleright} \to \mathbb{2}[X]$ is similarly constant on the image of the inclusion $\mathfrak{C}X \hookrightarrow \mathfrak{C}X^{\triangleright}$. Thus the composite diagram $k^f \colon \mathfrak{C}X^{\triangleright} \to \mathcal{E}$ acts as advertised. □

17.4.2. Theorem (comprehension). *For any cocartesian fibration $p \colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$, there is an associated homotopy coherent diagram $c_p \colon \mathfrak{C}B_0 \to \mathcal{K}$ indexed by the underlying quasi-category of B that maps an element $b \colon 1 \to B$ to the fiber E_b and defined on 1-arrows $\beta \colon a \Rightarrow b$ by factoring the codomain component of a p-cocartesian lift*

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[rr, phantom, "\Downarrow\chi_\beta"] \arrow[dr, dashed, "\beta_*"'] \arrow[rr, dotted, bend right=15, "\beta_*(\ell_a)"'] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_b \arrow[ur, dashed, "\ell_b"'] \arrow[dl] & \\ 1 \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[rr, phantom, "\Downarrow\beta"] & & B \end{tikzcd}$$

The comprehension functor is constructed as the domain component of the underlying diagram of a cocartesian cocone with nadir p defined by lifting a canonical lax cocone k^B

$$\begin{tikzcd} & \mathfrak{C}\varnothing^{\triangleright} \arrow[r, "p"] \arrow[d, hook] & co\mathcal{C}art(\mathcal{K}) \arrow[r, "\mathrm{dom}"] \arrow[d, "\mathrm{cod}"] & \mathcal{K} \\ \mathfrak{C}B_0 \arrow[r, hook] & \mathfrak{C}B_0^{\triangleright} \arrow[r, "k^B"] \arrow[ur, dashed] & \mathcal{K} & \end{tikzcd} \tag{17.4.3}$$

that carries each element of B to the terminal ∞-category 1 and carries the cone point to B. Any two comprehension functors defined in this manner are connected by a homotopy coherent natural isomorphism $\mathfrak{C}(B_0 \times \mathbb{I}) \to \mathcal{K}$.

Proof. To define the comprehension functor c_p, we need only define the lax cocone $k^B \colon \mathfrak{C}B_0^{\triangleright} \to \mathcal{K}$. By Lemma 17.4.1, such a lax cocone is determined by a simplicial map $B_0 \to \mathsf{Fun}(1, B)$, which we take to be the identity.

To prove the uniqueness statement, we appeal to Proposition 17.3.4 which tells us that the space of cocartesian cocones with nadir p lifting k^B is contractible, defined as the fiber of a trivial fibration of 2-complicial sets

$$\begin{tikzcd} \mathsf{cocone}^{\mathsf{cart}}_{coCart(\mathcal{K})}(B_0,p)_{k^B} \arrow[r] \arrow[d, two heads, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{cocone}^{\mathsf{cart}}_{coCart(\mathcal{K})}(B_0,p) \arrow[d, two heads, "\wr"] \\ \Delta[0] \arrow[r, "k^B"] & \mathsf{cocone}_{\mathcal{K}}(B_0,B) \end{tikzcd}$$

Any pair of vertices in this contractible space are connected by a homotopy coherent isomorphism $\mathbb{I} \to \mathsf{cocone}^{\mathsf{cart}}_{coCart(\mathcal{K})}(B_0,p)_{k^B}$, which unwinds to a homotopy coherent diagram $\mathfrak{C}(B_0\times\mathbb{I})^{\triangleright} \to coCart(\mathcal{K})$, which composes to define the claimed homotopy coherent natural isomorphism

$$\mathfrak{C}(B_0\times\mathbb{I}) \hookrightarrow \mathfrak{C}(B_0\times\mathbb{I})^{\triangleright} \longrightarrow coCart(\mathcal{K}) \xrightarrow{\mathrm{dom}} \mathcal{K}$$

between two choices of comprehension functor. □

17.4.4. Observation. In the terminology of Definition 17.1.4, the cocartesian cocone constructed in (17.4.3) defines a lax cocone

$$\begin{tikzcd} & \mathfrak{C}B_0 + \mathbb{1} \arrow[dl, hook'] \arrow[dr, "\langle c_p, E\rangle"] & \\ \mathfrak{C}B_0^{\triangleright} \arrow[rr, "\ell^E"'] & & \mathcal{K} \end{tikzcd}$$

whose base diagram is the comprehension functor and whose nadir is the ∞-category E, the domain of the cocartesian fibration $p\colon E \twoheadrightarrow B$. In Corollary 19.1.7, we will show that these lax colimit cones and colimit cocones. In particular, E can be recovered up to equivalence as the oplax colimit of the comprehension functor.

17.4.5. Proposition (comprehension and change of universe). *Suppose $G\colon \mathcal{K} \to \mathcal{L}$ is a functor of ∞-cosmoi, $p\colon E \twoheadrightarrow B$ is a cocartesian fibration in $\mathcal{K}$, and $q\colon F \twoheadrightarrow C$ is the cocartesian fibration in $\mathcal{L}$ obtained by applying G to p. Then the diagram*

$$\begin{tikzcd} \mathfrak{C}B_0 \arrow[r, "c_p"] \arrow[d, "\mathfrak{C}g"'] \arrow[dr, phantom, "\cong"] & \mathcal{K} \arrow[d, "G"] \\ \mathfrak{C}C_0 \arrow[r, "c_q"'] & \mathcal{L} \end{tikzcd}$$

commutes up to a homotopy coherent isomorphism, where

$$g \coloneqq B_0 = \mathsf{Fun}(1,B) \xrightarrow{G_{1,B}} \mathsf{Fun}(1,C) = C_0$$

Proof. In Theorem 17.4.2, the comprehension functors are defined by lifting the canonically-defined lax cocones of Lemma 17.4.1 along the codomain fibration of quasi-categorically enriched

categories. The lax cocones k^B and k^C are defined by the horizontal composites in the diagram below

$$\begin{tikzcd}
\mathfrak{C}B_0^{\triangleright} \arrow[r, "\tau^B"] \arrow[d, "\mathfrak{C}g"'] \arrow[rr, bend left, "k^B"] & \mathbb{2}[B_0] \arrow[r, "\hat{\mathrm{id}}"] \arrow[d, "\mathbb{2}[g]"] & \mathcal{K} \arrow[d, "G"] \\
\mathfrak{C}C_0^{\triangleright} \arrow[r, "\tau^C"] \arrow[rr, bend right, "k^C"'] & \mathbb{2}[C_0] \arrow[r, "\hat{\mathrm{id}}"] & \mathcal{L}
\end{tikzcd}$$

The left hand square commutes by simplicial naturality of τ, while the right-hand square commutes by inspection.

In this way, we see that the diagram $c_q \circ \mathfrak{C}g$ is the base of the restricted cocartesian cocone

$$\begin{tikzcd}
\mathfrak{C}B_0^{\triangleright} \arrow[r, "\mathfrak{C}(g^{\triangleright})"] & \mathfrak{C}C_0^{\triangleright} \arrow[r, bend left, "\ell^F"] \arrow[r, bend right, "k^C"'] \arrow[r, phantom, "\Downarrow q"] & \mathcal{L}
\end{tikzcd}$$

which is a cocartesian cocone by Lemma 17.1.8, and similarly $G \circ c_p$ is the base of the whiskered cocartesian cocone

$$\begin{tikzcd}
\mathfrak{C}B_0^{\triangleright} \arrow[r, bend left, "\ell^E"] \arrow[r, bend right, "k^B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \arrow[r, "G"] & \mathcal{L}
\end{tikzcd}$$

which is a cocartesian cocone by Lemma 17.1.7. By the commutative diagram above, both cocartesian cocones have the same base diagram. Consequently, Proposition 17.3.4 supplies an isomorphism between $c_q \circ \mathfrak{C}g$ and $G \circ c_p$, as claimed. $\square$

17.4.6. Proposition (comprehension and change of base). *For any pullback diagram*

$$\begin{tikzcd}
F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

between cocartesian fibrations in an ∞-cosmos $\mathcal{K}$, there exists a homotopy coherent isomorphism

$$\begin{tikzcd}
\mathfrak{C}A_0 \arrow[rr, "\mathfrak{C}f_0"] \arrow[dr, "c_q"'] & \cong & \mathfrak{C}B_0 \arrow[dl, "c_p"] \\
& \mathcal{K} &
\end{tikzcd}$$

Proof. As in the previous proof it suffices, by Proposition 17.3.4, to argue that c_q and $c_p \circ \mathfrak{C}f_0$ define the base diagrams for the domain cocones of a pair of cocartesian cocones with common nadir $q\colon F \twoheadrightarrow A$ and common codomain cocone. By naturality of the simplicial natural transformation τ

appearing in Lemma 17.4.1, the left-hand square in the diagram below commutes:

$$\begin{tikzcd} \mathfrak{C}A_0^{\triangleright} \arrow[r, "\tau^A"] \arrow[d, "\mathfrak{C}f_0^{\triangleright}"'] & \mathbb{2}[A_0] \arrow[d, "\mathbb{2}[f_0]"'] \arrow[dr, "\ulcorner \mathrm{id} \urcorner"] \arrow[dr, phantom, "\Downarrow f\circ -" near start] & \\ \mathfrak{C}B_0^{\triangleright} \arrow[r, "\tau^B"'] & \mathbb{2}[B_0] \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathcal{K} \end{tikzcd}$$

while the right hand triangle commutes up to a simplicial natural transformation defined by whiskering with f.

From this and Lemma 17.1.8, we see that the cocartesian cocone (ℓ^E, k^B, p) restricts along the functor $\mathfrak{C}f_0^{\triangleright} \colon \mathfrak{C}A_0^{\triangleright} \to \mathfrak{C}B_0^{\triangleright}$ to define a cocartesian cocone of shape A_0 whose codomain is the lax cocone whose base is the constant functor at $\mathbb{1}$ and whose nadir is B. By Lemma 17.1.5, this codomain lax cocone is the whiskered composite of the lax cocone whose base is the constant functor at $\mathbb{1}$ and whose nadir is A with $f \colon A \to B$.

Now Lemma 17.1.9 tells us that the cocartesian cocone

$$\begin{tikzcd} \mathfrak{C}A_0^{\triangleright} \arrow[r, "\mathfrak{C}f_0^{\triangleright}"] & \mathfrak{C}B_0 \arrow[r, bend left, "\ell^E"] \arrow[r, bend right, "k^B"'] \arrow[r, phantom, "\Downarrow p"] & \mathcal{K} \end{tikzcd}$$

pulls back along f to define a cocartesian cocone with codomain ℓ^A and nadir q. Both of these cocartesian cocones define vertices in the space of q-cocartesian lifts of the lax cocone ℓ^A. By Proposition 17.3.4, this provides the desired isomorphism $c_q \cong c_p \circ \mathfrak{C}f_0$. $\square$

17.4.7. Theorem (comprehension for cartesian and cocartesian fibrations).

(i) For any cocartesian fibration $p \colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$, there is an associated homotopy coherent diagram $c_{p,A} \colon \mathfrak{C}\mathsf{Fun}(A,B) \to coCart(\mathcal{K})_{/A}$ defined on 0-arrows by mapping a functor $b \colon A \to B$ to the pullback

$$\begin{tikzcd} E_b \arrow[r, "\ell_b"] \arrow[d, two heads, "p_b"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "b"'] & B \end{tikzcd}$$

and defined on 1-arrows $\beta \colon a \Rightarrow b$ by factoring the codomain component of a p-cocartesian lift

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\ell_a"] \arrow[rr, phantom, "\Downarrow\chi_\beta"] \arrow[dr, dotted, "\beta_*"'] \arrow[rr, dotted, "\beta_*(\ell_a)"] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_b \arrow[ur, dashed, "\ell_b"'] \arrow[dl] & \\ A \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[rr, phantom, "\Downarrow\beta"] & & B \end{tikzcd}$$

(ii) For any cartesian fibration $p \colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$, there is an associated homotopy coherent diagram $c_{p,A} \colon \mathfrak{C}\mathsf{Fun}(A,B)^{\mathrm{op}} \to Cart(\mathcal{K})_{/A}^{\mathrm{co}}$ defined on 0-arrows by mapping a functor $b \colon A \to B$

to the pullback

$$\begin{tikzcd} E_b \arrow[r, "\ell_b"] \arrow[d, two heads, "p_b"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "b"'] & B \end{tikzcd}$$

and defined on 1-arrows $\beta\colon a \Rightarrow b$ by factoring the domain component of a p-cartesian lift

$$\begin{tikzcd} E_b \arrow[rr, bend left, "\ell_b"] \arrow[rr, phantom, "\Uparrow\chi_\beta"] \arrow[rr, dotted, bend right=15, "\beta^*(\ell_b)"] \arrow[dr, dashed, "\beta^*"'] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_a \arrow[ur, dashed, "\ell_a"'] \arrow[dl] & \\ A \arrow[rr, bend left, "b"] \arrow[rr, bend right, "a"'] \arrow[rr, phantom, "\Uparrow\beta"] & & B \end{tikzcd}$$

Proof. We prove (i) and then deduce the dual statement (ii) as a corollary. For any ∞-category A, there is a cosmological functor $- \times A\colon \mathcal{K} \to \mathcal{coCart}(\mathcal{K})_{/A}$ that carries an ∞-category C to the projection $\pi\colon C \times A \twoheadrightarrow A$. This functor carries a cocartesian fibration $p\colon E \twoheadrightarrow B$ to a cocartesian fibration $p \times E\colon A \twoheadrightarrow B \times A$ in $\mathcal{coCart}(\mathcal{K})_{/A}$ to which we may apply the comprehension construction of Theorem 17.4.2 to define a functor

$$c_{p,A}\colon \mathfrak{C}\mathsf{Fun}(A,B) \cong \mathfrak{C}\mathsf{Fun}_A^{\mathsf{cart}}(A \xrightarrow{\mathrm{id}} A, E \times A \xrightarrow{p\times A} B \times A) \xrightarrow{c_{p\times A}} \mathcal{coCart}(\mathcal{K})_{/A}.$$

Thus, we may simply define $c_{p,A}$ to be the comprehension functor $c_{p\times A}$ associated to the cocartesian fibration $p \times A$ in the ∞-cosmos $\mathcal{coCart}(\mathcal{K})_{/A}$.

The dual statement follows by implementing the construction just given in the dual ∞-cosmos $\mathcal{K}^{\mathrm{co}}$ of Definition 1.2.23, which has the same objects as $\mathcal{K}$ but opposite functor spaces $\mathsf{Fun}(A,B)^{\mathrm{op}}$. A cartesian fibration $p\colon E \twoheadrightarrow B$ in $\mathcal{K}$ defines a cocartesian fibration in $\mathcal{K}^{\mathrm{co}}$. Since $\mathcal{coCart}(\mathcal{K}^{\mathrm{co}})_{/A} \cong \mathcal{Cart}(\mathcal{K})^{\mathrm{co}}_{/A}$, the comprehension functor $c_{p,A}$ defined in (i) now takes the form $c_{p,A}\colon \mathsf{Fun}(A,B) \to \mathcal{Cart}(\mathcal{K})^{\mathrm{co}}_{/A}$ as claimed. □

Recall from Lemma 5.4.5 that the classes of discrete cartesian fibrations and discrete cocartesian fibrations are pullback stable. By Corollary ??, the ∞-cosmoi $\mathcal{Disc}(\mathcal{Cart}(\mathcal{K})_{/A})$ and $\mathcal{Disc}(\mathcal{coCart}(\mathcal{K})_{/A})$ define full subcategories of $\mathcal{K}_{/A}$. Hence:

17.4.8. Corollary (comprehension for discrete fibrations).

(i) For any discrete cocartesian fibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ and any ∞-category A, the comprehension functor defines a homotopy coherent diagram

$$c_{p,A}\colon \mathfrak{C}\mathsf{Fun}(A,B) \to \mathcal{Disc}(\mathcal{coCart}(\mathcal{K})_{/A}) \subset \mathcal{K}_{/A},$$

where the codomain is the full subcategory of discrete cocartesian fibrations over A.

(ii) For any discrete cartesian fibration $p\colon E \twoheadrightarrow B$ in an ∞-cosmos $\mathcal{K}$ and any ∞-category A, the comprehension functor defines a homotopy coherent diagram

$$c_{p,A}\colon \mathfrak{C}\mathsf{Fun}(A,B)^{\mathrm{op}} \to \mathcal{Disc}(\mathcal{Cart}(\mathcal{K})_{/A})^{\mathrm{co}} \subset \mathcal{K}^{\mathrm{co}}_{/A},$$

where the codomain is the co-dual of the full subcategory of discrete cartesian fibrations over A. □

Discrete co/cartesian fibrations over the terminal ∞-category are discrete objects in the ∞-cosmos $\mathcal{K}$. Hence:

17.4.9. COROLLARY (comprehension).

(i) *For any discrete cocartesian fibration* $p\colon E \twoheadrightarrow B$ *in an* ∞*-cosmos* $\mathcal{K}$*, there is an associated homotopy coherent diagram* $c_p\colon \mathfrak{C}B_0 \to \mathcal{D}isc(\mathcal{K})$ *indexed by the underlying quasi-category of* B *that maps an element* $b\colon 1 \to B$ *to the fiber* E_b.

(ii) *For any discrete cartesian fibration* $p\colon E \twoheadrightarrow B$ *in an* ∞*-cosmos* $\mathcal{K}$*, there is an associated homotopy coherent diagram* $c_p\colon \mathfrak{C}{B_0}^{\mathrm{op}} \to \mathcal{D}isc(\mathcal{K})^{\mathrm{co}}$ *indexed by the underlying quasi-category of* B *that maps an element* $b\colon 1 \to B$ *to the fiber* E_b.

In Chapter 18 we will discover something more surprising, that an external version of the Yoneda embedding can be defined as a special case of the comprehension construction implemented in a strategically chosen ∞-cosmos.

CHAPTER 18

The external and internal Yoneda embeddings

18.1. The external Yoneda embedding

Any ∞-category A in any ∞-cosmos $\mathcal{K}$ has an *external Yoneda embedding*, defined as a functor from the underlying quasi-category of A to the quasi-category of cartesian fibrations over A in $\mathcal{K}$. This map can be constructed as a special case of the comprehension construction of Chapter 17 as we now explain.

The comprehension construction can be applied in any ∞-cosmos $\mathcal{K}$ to any cocartesian fibration $p\colon E \twoheadrightarrow B$ and any ∞-category A to produce a functor

$$c_{p,A}\colon \mathfrak{C}\mathsf{Fun}(A,B) \to \mathcal{coCart}(\mathcal{K})_{/A}.$$

In the case where p is a discrete cocartesian fibration, the comprehension functor has the form

$$c_{p,A}\colon \mathfrak{C}\mathsf{Fun}(A,B) \to \mathcal{Disc}(\mathcal{coCart}(\mathcal{K})_{/A}).$$

In what follows we frequently condense our notation, writing c_E in place of $c_{p,A}$.

Recall from Proposition 11.4.6 that for any ∞-category A, the span $(p_1,p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A\times A$ defines a module, that is a two-sided discrete fibration, from A to A. In particular, by the definition of two-sided fibration encoded in Theorem 11.1.4, this map defines a discrete cocartesian fibration in $\mathcal{Cart}(\mathcal{K})_{/A}$ over the projection to the right-hand copy of A as well as a discrete cartesian fibration in $\mathcal{coCart}(\mathcal{K})_{/A}$ over the projection to the left-hand copy of A. The covariant and contravariant Yoneda embeddings are defined by restricting the comprehension functor associated to these maps.

18.1.1. Definition (covariant external Yoneda embedding). Consider the comprehension construction associated to the discrete cocartesian fibration

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[rr, two heads, "(p_1{,}p_0)"] \arrow[dr, two heads, "p_0"'] & & A\times A \arrow[dl, two heads, "\pi_0"] \\ & A & \end{tikzcd}$$

and the terminal object $\mathrm{id}_A\colon A \to A$ in $\mathcal{Cart}(\mathcal{K})_{/A}$. By Corollary 17.4.9, the comprehension functor has the form

$$\mathfrak{C}\mathsf{Fun}_{/A}\left(\begin{array}{c}A\\ \downarrow{\scriptstyle\mathrm{id}_A}\\ A\end{array},\ \begin{array}{c}A\times A\\ \downarrow{\scriptstyle\pi_0}\\ A\end{array}\right) \xrightarrow{c_{A^{\mathbb{2}}}} \mathcal{Disc}(\mathcal{coCart}((\mathcal{Cart}(\mathcal{K})_{/A}))_{/\mathrm{id}_A}) \cong \mathcal{DiscCart}(\mathcal{K})_{/A} \cong {}_{1\backslash}\mathcal{Mod}(\mathcal{K})_{/A}.$$

Note the codomain is an ∞-cosmos of discrete objects, so the comprehension functor transposes to define a functor of quasi-categories

$$\mathsf{Fun}_{/A}(A \xrightarrow{\mathrm{id}_A} A, A\times A \xrightarrow{\pi_0} A) \xrightarrow{c_{A^{\mathbb{2}}}} {}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

valued in ${}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \coloneqq \mathfrak{N}({}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A})$, the quasi-category of modules from 1 to A in $\mathcal{K}$.

The domain of the comprehension functor receives a map $\mathsf{Fun}(1, A) \to \mathsf{Fun}_{/A}(\mathrm{id}_A, \pi_0)$ from the underlying quasi-category of A defined by the action of the functor $- \times A \colon \mathcal{K} \to \mathcal{K}_{/A}$. This sends an element $a \colon 1 \to A$ of A to the map $A \cong 1 \times A \xrightarrow{a \times A} A \times A$ over A. The **covariant external Yoneda embedding** $よ \colon \mathsf{Fun}(1, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is defined to be the composite functor

$$よ\colon \mathsf{Fun}(1,A) \xrightarrow{-\times A} \mathsf{Fun}_{/A}(A \overset{\mathrm{id}_A}{\twoheadrightarrow} A, A \times A \overset{\pi_0}{\twoheadrightarrow} A) \xrightarrow{c_{A^2}} {}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A},$$

from the underlying quasi-category of A to the quasi-category of modules from 1 to A in $\mathcal{K}$, which acts by sending $a \colon 1 \to A$ to the module $\mathsf{Hom}_A(A, a)$.

The contravariant Yoneda embedding can be constructed as an instance of Definition 18.1.1 applied in $\mathcal{K}^{\mathrm{co}}$.

18.1.2. Definition (contravariant external Yoneda embedding). The comprehension functor associated to the discrete cartesian fibration

$$\begin{tikzcd} A^2 \arrow[rr, two heads, "(p_1,p_0)"] \arrow[dr, two heads, "p_1"'] & & A \times A \arrow[dl, two heads, "\pi_1"] \\ & A & \end{tikzcd}$$

and the terminal object $\mathrm{id}_A \colon A \to A$ in $\mathcal{C}oCart(\mathcal{K})_{/A}$ has the form

$$\mathfrak{C}\mathsf{Fun}_{/A}\left(\begin{matrix} A & A\times A \\ \downarrow \mathrm{id}_A, & \downarrow \pi_1 \\ A & A \end{matrix}\right)^{\mathrm{op}} \xrightarrow{c_{A^2}} \mathcal{D}isc(\mathcal{C}art((\mathcal{C}oCart(\mathcal{K})_{/A}))_{/\mathrm{id}_A})^{\mathrm{co}} \cong \mathcal{D}isc(\mathcal{C}oCart(\mathcal{K})_{/A})^{\mathrm{co}} \cong {}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A}{}^{\mathrm{co}}.$$

transposing to define a functor of quasi-categories

$$\mathsf{Fun}_{/A}(A \overset{\mathrm{id}_A}{\twoheadrightarrow} A, A \times A \overset{\pi_1}{\twoheadrightarrow} A)^{\mathrm{op}} \xrightarrow{c_{A^2}} {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/1}{}^{\mathrm{co}}$$

valued in the co-dual of ${}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/1} \coloneqq \mathfrak{N}({}_{A\backslash}\mathcal{M}od(\mathcal{K})_{/1})$, the quasi-category of modules from A to 1 in $\mathcal{K}$.

The domain of the comprehension functor receives a map $\mathsf{Fun}(1, A) \to \mathsf{Fun}_{/A}(\mathrm{id}_A, \pi_1)$ from the underlying quasi-category of A defined by the action of the functor $A \times 1 \colon \mathcal{K} \to \mathcal{K}_{/A}$. This sends an element $a \colon 1 \to A$ of A to the map $A \cong A \times 1 \xrightarrow{A \times a} A \times A$ over A. The **contravariant external Yoneda embedding** $よ \colon \mathsf{Fun}(1, A)^{\mathrm{op}} \to {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/1}{}^{\mathrm{co}}$ is defined to be the composite functor

$$よ\colon \mathsf{Fun}(1,A)^{\mathrm{op}} \xrightarrow{A\times -} \mathsf{Fun}_{/A}(A \overset{\mathrm{id}_A}{\twoheadrightarrow} A, A \times A \overset{\pi_1}{\twoheadrightarrow} A)^{\mathrm{op}} \xrightarrow{c_{A^2}} {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/1}{}^{\mathrm{co}},$$

from the underlying quasi-category of A to the quasi-category of modules from A to 1 in $\mathcal{K}$, which acts by sending $a \colon 1 \to A$ to the module $\mathsf{Hom}_A(a, A)$.

The co- and contravariant external Yoneda embeddings can be generalized to define functors

$$よ\colon \mathsf{Fun}(D,A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \qquad \text{and} \qquad よ\colon \mathsf{Fun}(D,A)^{\mathrm{op}} \to {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/D}{}^{\mathrm{co}}$$

with the case $D = 1$ coinciding with the external Yoneda embeddings of Definition 18.1.1 and 18.1.2.

18.1.3. Definition (generalized external Yoneda embeddings). The comprehension functor associated to the discrete cocartesian fibration

$$\begin{tikzcd} A^2 \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_0"'] & & A \times A \arrow[dl, two heads, "\pi_0"] \\ & A & \end{tikzcd}$$

and the projection $\pi_0 \colon D \times A \twoheadrightarrow A$ in $\mathcal{C}art(\mathcal{K})_{/A}$ has the form

$$\mathfrak{C}\mathsf{Fun}_{/A}\left(\begin{array}{c} D \times A \\ \downarrow\!\!\downarrow \pi_0 \\ A \end{array}, \begin{array}{c} A \times A \\ \downarrow\!\!\downarrow \pi_0 \\ A \end{array}\right) \xrightarrow{c_{A^2}} \mathcal{D}isc(co\mathcal{C}art((\mathcal{C}art(\mathcal{K})_{/A}))_{/\pi_0}) \cong {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}.$$

transposing to define a functor of quasi-categories

$$\mathsf{Fun}_{/A}(D \times A \xrightarrow{\pi_0} A, A \times A \xrightarrow{\pi_0} A) \xrightarrow{c_{A^2}} {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

valued in ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \coloneqq \mathfrak{N}({}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A})$, the quasi-category of modules from D to A in $\mathcal{K}$.

The domain of the comprehension functor receives a map $\mathsf{Fun}(D, A) \to \mathsf{Fun}_{/A}(\pi_0, \pi_0)$ defined by the action of the functor $- \times A \colon \mathcal{K} \to \mathcal{K}_{/A}$. This sends an element $a \colon D \to A$ of A to the map $A \cong D \times A \xrightarrow{a \times A} A \times A$ over A. The **covariant generalized external Yoneda embedding** $よ \colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is defined to be the composite functor

$$よ \colon \mathsf{Fun}(D, A) \xrightarrow{-\times A} \mathsf{Fun}_{/A}(D \times A \xrightarrow{\pi_0} A, A \times A \xrightarrow{\pi_0} A) \xrightarrow{c_{A^2}} {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A},$$

from the quasi-category of functors from D to A to the quasi-category of modules from D to A in $\mathcal{K}$, which acts by sending $a \colon D \to A$ to the module $\mathsf{Hom}_A(A, a)$. The **contravariant generalized external Yoneda embedding**

$$よ \colon \mathsf{Fun}(D, A)^{\mathrm{op}} \xrightarrow{A\times -} \mathsf{Fun}_{/A}(A \times D \xrightarrow{\pi_1} A, A \times A \xrightarrow{\pi_1} A)^{\mathrm{op}} \xrightarrow{c_{A^2}} {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/D}{}^{\mathrm{co}},$$

is defined dually.

Our main results about the external Yoneda embeddings extend to their generalized variants. To streamline our language, we typically drop the term "generalized" henceforth.

18.1.4. Theorem (external Yoneda embeddings are fully faithful). *For any ∞-categories D and A the Yoneda embeddings*

$$よ \colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \qquad \text{and} \qquad よ \colon \mathsf{Fun}(D, A)^{\mathrm{op}} \to {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/D}{}^{\mathrm{co}}$$

are fully faithful as functors of quasi-categories.

Proof. For now see [**85**, 7.2.22]. □

18.1.5. Corollary. *Every quasi-category is equivalent to the homotopy coherent nerve of a Kan complex enriched category.*

PROOF. Let A be a quasi-category. By Theorem 18.1.4, the covariant external Yoneda embedding is fully faithful

$$よ: A \cong \mathsf{Fun}(1, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A},$$

except in this context the moniker "external" isn't appropriate, since A is isomorphic to its underlying quasi-category $\mathsf{Fun}(1, A)$.

The Yoneda embedding acts by sending an element $a: 1 \to A$ to the module $\mathsf{Hom}_A(A, a)$ so accordingly we may restrict the codomain to the full sub quasi-category of ${}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A}$ spanned by the covariantly represented modules. More precisely, let $\mathcal{A} \subset {}_{1\backslash}\mathcal{M}od(\mathcal{QC}at)_{/A} \subset \mathcal{QC}at_{/A}$ denote the full subcategory of $\mathcal{QC}at_{/A}$ spanned by the covariant representables $p_0: \mathsf{Hom}_A(A, a) \twoheadrightarrow A$. By Example 5.4.4, these objects are discrete, so $\mathcal{A}$ is a Kan complex enriched category.

The restricted Yoneda embedding $よ: A \to \mathfrak{N}\mathcal{A}$ is then fully faithful and surjective on objects, so by Theorem 16.2.16 defines an equivalence of quasi-categories, proving the result claimed in the statement. □

By Theorem 16.2.16, a fully faithful functor between quasi-categories defines an equivalence onto its essential image. Thus Theorem 18.1.4 gives us the following equivalences:

18.1.6. COROLLARY. *For any ∞-categories D and A the Yoneda embeddings*

$$よ: \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \qquad \textit{and} \qquad よ: \mathsf{Fun}(D, A)^{\mathrm{op}} \to {}_{A\backslash}\mathsf{Mod}(\mathcal{K})_{/D}{}^{\mathrm{co}}$$

restrict to define equivalences between $\mathsf{Fun}(D, A)$ and the full subcategory of modules from D to A that are covariantly represented by a functor $D \to A$ and between $\mathsf{Fun}(D, A)^{\mathrm{op}}$ and the co-dual of the full subcategory of modules from A to D that are contravariantly represented by a functor from $D \to A$.

PROOF. By Theorem 16.2.16, a fully faithful functor between quasi-categories defines an equivalence onto its essential image, which we may calculate as the full sub quasi-category spanned by the elements isomorphic to elements in the image. By Definition 18.1.3, the image of a functor $f: D \to A$ under $よ: \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is the module

$$\begin{tikzcd} \mathsf{Hom}_A(A, f) \arrow[r] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "{(p_1,p_0)}"] \\ D \times A \arrow[r, "f \times A"] & A \times A \end{tikzcd}$$

constructed by the pullback. By Exercise 16.3.i, elements in the quasi-category ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ defined as the homotopy coherent nerve of a Kan complex enriched category are isomorphic if and only if the corresponding objects are equivalent in the ∞-cosmos ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$. Thus, by Definition 11.4.8, the essential image is the full sub quasi-category of covariantly represented modules, as claimed. □

18.1.7. PROPOSITION. *For any cosmological functor $F: \mathcal{K} \to \mathcal{L}$ and ∞-categories A and D in $\mathcal{K}$, the Yoneda embeddings fit into an essentially commutative square*

$$\begin{tikzcd} \mathsf{Fun}(D, A) \arrow[r, "よ"] \arrow[d, "F_{D,A}"'] \arrow[dr, phantom, "\cong"] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d, "F"] \\ \mathsf{Fun}(FD, FA) \arrow[r, "よ"'] & {}_{FD\backslash}\mathsf{Mod}(\mathcal{L})_{/FA} \end{tikzcd}$$

Proof. Any cosmological functor preserves (co)cartesian fibrations, (co)cartesian arrows, cartesian functors, and modules — all of the structure used in the comprehension construction. Furthermore, F preserves the cocartesian fibration $(p_1, p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ over A used to define the Yoneda embedding. Thus, this result is an instance of Proposition 17.4.5, restricted along the natural map $- \times A\colon \mathsf{Fun}(D, A) \to \mathsf{Fun}(D \times A \overset{\pi_0}{\twoheadrightarrow} A, A \times A \overset{\pi_0}{\twoheadrightarrow} A)$. □

We call attention to a particular special case of Proposition 18.1.7

18.1.8. Corollary. *Let A be an ∞-category and $k\colon C \to D$ a functor in $\mathcal{K}$. Then there exists an essentially commutative square of generalized Yoneda embeddings*

$$\begin{tikzcd} \mathsf{Fun}(D, A) \arrow[r, "\text{よ}"] \arrow[d, "k\circ-"'] \arrow[dr, phantom, "\cong"] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d, "(k\times A)^*"] \\ \mathsf{Fun}(C, A) \arrow[r, "\text{よ}"'] & {}_{C\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \end{tikzcd}$$

where $(k \times A)^*\colon {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \to {}_{C\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ acts by pullback. □

18.2. The internal Yoneda lemma

The external Yoneda embedding of Definition 18.1.1 defines a functor

$$\text{よ}\colon \mathsf{Fun}(1, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

from the underlying quasi-category of an ∞-category A in $\mathcal{K}$ to the quasi-category of modules in $\mathcal{K}$ from 1 to A. When A is a quasi-category $A \cong \mathsf{Fun}(1, A)$ and so the domain of this "external" Yoneda embedding is A itself, and we might regard this functor as internal to the ∞-cosmos $\mathcal{QC}at$ — with one important caveat: the quasi-category of modules from 1 to A is a large simplicial set.

To accommodate maps such as the Yoneda embedding for a small quasi-category A, we pass to a larger Grothendieck universe and let $\mathcal{QC}at$ now denote the ∞-cosmos of *large* quasi-categories; this contains a full subcategory of *small* quasi-categories, the old ∞-cosmos $\mathcal{QC}at$. The objects of the expanded ∞-cosmos $\mathcal{QC}at$ now include the large quasi-categories constructed in §16.4 as homotopy coherent nerves of large Kan complex enriched categories of small objects. Of course in this new notation ${}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A}$ is even larger, so accordingly we let $PA \subset {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A}$ denote the full subcategory of discrete cartesian fibrations over A with small fibers. These include the representable modules $p_0\colon \mathsf{Hom}_A(A, a) \twoheadrightarrow A$ associated to each element of A.[1] Thus the external Yoneda embedding factors through this subcategory

$$\begin{tikzcd} A \cong \mathsf{Fun}(1, A) \arrow[rr, hook, "\text{よ}"] \arrow[dr, hook, dashed, "y"'] & & {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A} \\ & PA \arrow[ur, hook] & \end{tikzcd}$$

and since よ is fully faithful, $y\colon A \hookrightarrow PA$ is as well. The map $y\colon A \hookrightarrow PA$ is a functor in $\mathcal{QC}at$ that we refer to as the **internal Yoneda embedding** associated to a small quasi-category A.

In the same way, when $\mathcal{K}$ is an ∞-cosmos of $(\infty, 1)$-categories, the external Yoneda embeddings of Definitions 18.1.1 and 18.1.2 internalize to define functors in an the ∞-cosmos $\mathcal{K}$ of large ∞-categories,

[1] Thus PA is the quasi-category ${}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A}$ defined for the original ∞-cosmos of small quasi-categories.

which we refer to as the internal Yoneda embeddings. In this section, we give an abstract characterization of what it means for a fully faithful functor $y\colon A \to PA$ in an ∞-cosmos $\mathcal{K}$ to define an internal Yoneda embedding.

Before giving the definition, we explain the idea. By Corollary 18.1.6, the generalized Yoneda embedding $よ\colon \mathsf{Fun}(D, A) \hookrightarrow {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ defines an equivalence between the quasi-category $\mathsf{Fun}(D, A)$ and its essential image: the full subcategory of covariantly represented modules from D to A.

18.2.1. Definition. For any functor $y\colon A \to PA$ in $\mathcal{K}$ we may mimic 18.1.3 and define comprehension functor associated to the discrete cocartesian fibration

$$\begin{tikzcd} \mathsf{Hom}_{PA}(y, PA) \arrow[rr, two heads, "{(p_1,p_0)}"] \arrow[dr, two heads, "p_0"'] & & PA \times A \arrow[dl, two heads, "\pi_0"] \\ & A & \end{tikzcd}$$

and the projection $\pi_0\colon D \times A \twoheadrightarrow A$ in $\mathcal{C}art(\mathcal{K})_{/A}$, which has the form

$$\mathfrak{C}\mathsf{Fun}_{/A}\left(\begin{array}{cc} D\times A & PA\times A \\ \downarrow\!\!\!\downarrow \pi_0 \quad , & \downarrow\!\!\!\downarrow \pi_0 \\ A & A \end{array}\right) \xrightarrow{c_{\mathsf{Hom}_{PA}(y,PA)}} \mathcal{D}isc(co\mathcal{C}art((\mathcal{C}art(\mathcal{K})_{/A}))_{/\pi_0}) \cong {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}.$$

transposing to define a functor of quasi-categories

$$\mathsf{Fun}_{/A}(D \times A \overset{\pi_0}{\twoheadrightarrow} A, PA \times A \overset{\pi_0}{\twoheadrightarrow} A) \xrightarrow{c_{\mathsf{Hom}_{PA}(y,PA)}} {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

whose domain receives a map $\mathsf{Fun}(D, PA) \to \mathsf{Fun}_{/A}(\pi_0, \pi_0)$ defined by the action of the functor $- \times A\colon \mathcal{K} \to \mathcal{K}_{/A}$.

A fully faithful functor $y\colon A \to PA$ defines a **internal Yoneda embedding** in $\mathcal{K}$ if the composite functor

$$\Phi\colon \mathsf{Fun}(D, PA) \xrightarrow{-\times A} \mathsf{Fun}_{/A}(D \times A \overset{\pi_0}{\twoheadrightarrow} A, PA \times A \overset{\pi_0}{\twoheadrightarrow} A) \xrightarrow{c_{\mathsf{Hom}_{PA}(y,PA)}} {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A},$$

is fully faithful.

18.2.2. Lemma. *If $y\colon A \to PA$ is an internal Yoneda embedding, then there is an essentially commutative diagram*

$$\begin{tikzcd} \mathsf{Fun}(D, A) \arrow[rr, "y\circ -"] \arrow[dr, "よ"'] & \cong & \mathsf{Fun}(D, PA) \arrow[dl, "\Phi"] \\ & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} & \end{tikzcd}$$

Proof. There is a pullback diagram

$$\begin{tikzcd} \mathsf{Hom}_{PA}(y, y) \arrow[r] \arrow[d, two heads, "{(p_1,p_0)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_{PA}(y, PA) \arrow[d, two heads, "{(p_1,p_0)}"] \\ A \times A \arrow[r, "y\times A"'] & PA \times A \end{tikzcd}$$

in the ∞-cosmos $\mathcal{C}art(\mathcal{K})_{/A}$. Since $y\colon A \to PA$ is fully faithful, the left-hand vertical is equivalent to $(p_1, p_0)\colon A^2 \twoheadrightarrow A \times A$. Applying Proposition 17.4.6 to this pair of discrete cocartesian fibrations in

$\mathcal{C}art(\mathcal{K})_{/A}$ we conclude that the comprehension functor associated to the right-hand vertical restricts along $y \times A$ to the comprehension functor associated to the left-hand vertical, giving rise to the essentially commutative diagram below-right:

$$\begin{tikzcd}
\mathsf{Fun}(D,A) \arrow[r, "-\times A"] \arrow[d, "y\circ-"'] & \mathsf{Fun}_{/A}(D\times A \overset{\pi_0}{\twoheadrightarrow} A, A\times A \overset{\pi_0}{\twoheadrightarrow} A) \arrow[r, "c_{A^2}"] \arrow[d, "y\times A\circ-"'] \arrow[dr, phantom, "\cong"] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d, equal] \\
\mathsf{Fun}(D,PA) \arrow[r, "-\times A"] & \mathsf{Fun}_{/A}(D\times A \overset{\pi_0}{\twoheadrightarrow} A, PA\times A \overset{\pi_0}{\twoheadrightarrow} A) \arrow[r, "c_{\mathsf{Hom}_{PA}(y,PA)}"'] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}
\end{tikzcd}$$

As the top horizontal composite is よ and the bottom horizontal composite is Φ, this proves the statement. ☐

Our next task is to show that for every small quasi-category the external Yoneda embedding of Definition 18.1.1 defines an internal Yoneda embedding, satisfying the condition of Definition 18.2.1.

18.2.3. Proposition. *For any small quasi-category A, let $PA \subset {}_{1\backslash}\mathsf{Mod}(\mathcal{Q}Cat)_{/A}$ denote the full subcategory of discrete cartesian fibrations over A with small fibers. Then the codomain restriction*

$$\begin{tikzcd}
A \cong \mathsf{Fun}(1,A) \arrow[r, hook, "よ"] \arrow[dr, hook, dashed, "y"'] & {}_{1\backslash}\mathsf{Mod}(\mathcal{Q}Cat)_{/A} \\
& PA \arrow[u, hook]
\end{tikzcd}$$

of the covariant external Yoneda embedding defines an internal Yoneda embedding $y\colon A \hookrightarrow PA$.

Proof. To appear. ☐

To begin, note that Theorem 18.1.4 proves that $よ\colon A \to {}_{1\backslash}\mathsf{Mod}(\mathcal{Q}Cat)_{/A}$ is fully faithful. Proposition 16.2.14 gives a "pointwise" characterization of fully faithful functors of quasi-categories: a functor that induces equivalences on internal mapping spaces for every pair of elements of the domain. Since the internal mapping spaces in a full subcategory of a quasi-category coincide with the internal mapping spaces in the original quasi-category, it follows that the codomain restriction y of a fully faithful functor よ remains fully faithful.

Proposition 18.2.3 can be leveraged into a proof of its generalization. We say an ∞-category A in an ∞-cosmos of $(\infty,1)$-categories is **small** just when its underlying quasi-category A_0 is equivalent to a small simplicial set.[2]

18.2.4. Theorem. *Every small ∞-category in an ∞-cosmos of $(\infty,1)$-categories admits an internal Yoneda embedding.*

Proof. If A is a small ∞-category in an ∞-cosmos of $(\infty,1)$-categories then by definition its underlying quasi-category $A_0 \cong \mathsf{Fun}(1,A)$ is small. By Proposition 18.2.3, A_0 has an internal Yoneda embedding $y_0\colon A_0 \to PA_0$ that we may lift up to isomorphism along the cosmological biequivalence $(-)_0\colon \mathcal{K} \rightsquigarrow \mathcal{Q}Cat$ to define a functor $y\colon A \to PA$ with domain A in $\mathcal{K}$. Our task is to show that this map defines an internal Yoneda embedding for A.

Cosmological biequivalences reflect fully faithful functors, so $y\colon A \to PA$ is fully faithful and our task is only to show that the associated functor $\Phi\colon \mathsf{Fun}(D,PA) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is as well. Applying

[2]The biequivalences of (14.0.1) hold for $(\infty,1)$-categories defined in any Grothendieck universe. An $(\infty,1)$-category is small, or perhaps more accurately "essentially small," if it is equivalent to an ∞-category in the ∞-cosmos that is biequivalent to the ∞-cosmos of small quasi-categories.

Proposition 17.4.5 to the cosmological functor $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$, we get an essentially commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}(D,PA) \arrow[r, "-\times A"] \arrow[d, "(-)_0"', "\wr"] & \mathsf{Fun}_{/A}(D\times A \overset{\pi_0}{\twoheadrightarrow} A, PA\times A \overset{\pi_0}{\twoheadrightarrow} A) \arrow[r, "c_{\mathsf{Hom}_{PA}(y,PA)}"] \arrow[d, "(-)_0"', "\wr"] \arrow[dr, phantom, "\cong"] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d, "\wr"', "(-)_0"] \\
\mathsf{Fun}(D_0,PA_0) \arrow[r, "-\times A_0"'] & \mathsf{Fun}_{/A_0}(D_0\times A_0 \overset{\pi_0}{\twoheadrightarrow} A_0, PA_0\times A_0 \overset{\pi_0}{\twoheadrightarrow} A_0) \arrow[r, "c_{\mathsf{Hom}_{PA_0}(y_0,PA_0)}"'] & {}_{D_0\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A_0}
\end{tikzcd}$$

Since $(-)_0\colon \mathcal{K} \to \mathcal{QC}at$ is a biequivalence, the induced cosmological functors $(-)_0\colon \mathcal{C}art(\mathcal{K})_{/A} \to \mathcal{C}art(\mathcal{QC}at)_{/A_0}$ and $(-)_0\colon {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A} \to {}_{D_0\backslash}\mathcal{M}od(\mathcal{QC}at)_{/A_0}$ are biequivalences as well. Hence all three vertical arrows in the diagram above are equivalences; the left and middle ones as the action on functor spaces of a cosmological biequivalence, and the right map as the induced map on homotopy coherent nerves of a cosmological biequivalence, which is an equivalence by Proposition 16.2.18. Thus the top horizontal composite is equivalent to the bottom horizontal composite, which is a fully faithful functor of quasi-categories. This proves that Φ is fully faithful as well. Thus $y\colon A \to PA$ defines an internal Yoneda embedding as claimed. $\square$

The following result generalizes Lemma 16.3.7, which shows that the representable modules define a strong generator in the quasi-category of modules from $\mathbb{1}$ to A.

18.2.5. Proposition (internal Yoneda embeddings are strong generators). *Any internal Yoneda embedding $y\colon A \to PA$ is fully faithful and strongly generating.*

Proof. By assumption, internal Yoneda embeddings are fully faithful, so it remains to verify that they are strongly generating. By construction, the functor $\Phi\colon \mathsf{Fun}(D,PA) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ carries a functor $h\colon D \to PA$ to the module $\mathsf{Hom}_{PA}(y,h)$ and carries a 1-cell $\alpha\colon h \to k \in \mathsf{Fun}(D,PA)$ to the induced map

$$\begin{tikzcd}
\mathsf{Hom}_{PA}(y,h) \arrow[rrr, bend left=10] \arrow[drr, dashed, "y\downarrow\alpha"] \arrow[dd, two heads] \arrow[rr, phantom, shift right=4ex, "\Downarrow\chi_{\alpha\times A}"] & & & \mathsf{Hom}_{PA}(y,PA) \arrow[dd, two heads, "{(p_1,p_0)}"] \\
& & \mathsf{Hom}_{PA}(y,k) \arrow[ur, bend right=10] \arrow[dd, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \\
D\times A \arrow[rrr, bend left=5] \arrow[drr, equal] \arrow[rr, phantom, shift right=4ex, "\Downarrow\alpha\times A"] & & & PA\times A \\
& & D\times A \arrow[ur, bend right=10] &
\end{tikzcd}$$

The cocartesian lift $\chi_{\alpha\times A}$ in $\mathcal{C}art(\mathcal{K})_{/A}$ can be defined by 2-cell induction, this functor $\Phi(\alpha)$ coincides with the functor $y\downarrow\alpha\colon \mathsf{Hom}_{PA}(y,h) \to \mathsf{Hom}_{PA}(y,k)$ constructed in Observation 3.4.8.

Consulting Definition 16.3.1 we see that we must show that whenever $y\downarrow\alpha$ is an equivalence, then α is invertible. That is, we must show that $\Phi\colon \mathsf{Fun}(D,PA) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is a conservative functor of quasi-categories. However, every fully faithful functor is conservative, and since Definition 18.2.1 requires Φ to be fully faithful, we conclude that Φ is conservative, so y is strongly generating. $\square$

By Proposition 16.3.8, any fully faithful and strongly generating functor preserves whatever limits exist in both the domain and the codomain. In an ∞-cosmos of $(\infty,1)$-categories, Theorem 18.2.4 gives an explicit construction of the codomain of the internal Yoneda embedding, which allows us to sharpen this result.

18.2.6. COROLLARY (Yoneda preserves limits). *For every small ∞-category A in an ∞-cosmos of $(\infty, 1)$-categories, the internal Yoneda embedding $y\colon A \to PA$ preserves all limits which exist in A. admits an internal Yoneda embedding.*

PROOF. In the proof of Theorem 18.2.4, the codomain of the internal Yoneda embedding is defined to be an ∞-category PA whose underlying quasi-category is the full subcategory of

$$_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A_0} \simeq {}_{1\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

of modules with small fibers. We will argue that PA is a complete ∞-category. Then by Propositions 16.3.8 and 18.2.5, the fully faithful and strongly generating functor $y\colon A \to PA$ preserves all limits that exist in A.

By model independence, PA is complete just when its underlying quasi-category is. The underlying quasi-category of PA is defined as the homotopy coherent nerve of the Kan complex enriched category of discrete cartesian fibrations of quasi-categories over A_0 with small fibers. This enriched category is the Kan complex enriched category of discrete objects in the ∞-cosmos $\mathcal{C}art(\mathcal{QC}at)_{/A}$, so by Proposition 16.4.9 its underlying quasi-category PA_0 is complete. This completes the proof. $\square$

18.3. Preservation of limits by the external Yoneda embedding

Corollary 18.2.6 demonstrates that for a small quasi-category A, the internal Yoneda embedding $y\colon A \to PA$, defined as the codomain restriction of the external Yoneda embedding よ$\colon A \cong \mathsf{Fun}(1, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/A}$, preserves all limits that exist in A. Our aim in this section is to use this result to prove a vastly more general and considerably more subtle result: that for any ∞-categories D and A in any ∞-cosmos, not necessarily an ∞-cosmos of $(\infty, 1)$-categories, the generalized external Yoneda embedding よ$\colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ preserves all families of limits in the quasi-category $\mathsf{Fun}(D, A)$ that arise from limits in the ∞-category A in a sense that we now make precise. This result, which appears as Proposition 18.3.6, will then be applied in §18.4 to develop formulae for the construction of limits and colimits in generic ∞-categories in arbitrary ∞-cosmoi.

To explain which limits in $\mathsf{Fun}(D, A)$ arise from limits in A note that for any ∞-categories A and D in $\mathcal{K}$ and simplicial sets J and K transposition under the cotensoring natural isomorphism $\mathsf{Fun}(D, (A^K)^J) \cong \mathsf{Fun}(K, \mathsf{Fun}(D, A)^J)$ provides a bijection between the 2-cells

$$\begin{tikzcd} & A^K \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & (A^K)^J \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \mathsf{Fun}(D, A) \arrow[d, "\Delta"] \\ K \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] & \mathsf{Fun}(D, A)^J \end{tikzcd} \tag{18.3.1}$$

in the homotopy 2-categories $\mathfrak{h}\mathcal{K}$ and $\mathfrak{h}\mathcal{QC}at$, respectively. Moreover, as verified by Exercise 18.3.i, the triangle on the left is a right lifting diagram in $\mathcal{K}$ if and only if the triangle on the right is a right lifting diagram in $\mathcal{QC}at$.

While the properties of the triangles in (18.3.1) defining right lifting diagrams transpose, in general the properties of defining *absolute* right lifting diagrams do not. As we shall discover in Lemma 18.3.5, absolute right lifting in $\mathcal{K}$ corresponds to an additional stability property in $\mathcal{QC}at$ under precomposition with functors $f\colon C \to D$, while absolute right lifting in $\mathcal{QC}at$ corresponds to an additional stability in $\mathcal{K}$ under restriction along simplicial maps $g\colon L \to K$.

18.3.2. DEFINITION. For ∞-categories A and D and simplicial sets K and J, an absolute right lifting diagram (18.3.1), which presents a K-indexed family of limits of diagrams of shape J in $\mathsf{Fun}(D, A)$,

is **stable under precomposition** in $\mathcal{K}$ if precomposition along any functor $f\colon C \to D$ preserves this family of limits: i.e., if

$$\begin{tikzcd} & \mathsf{Fun}(D,A) \arrow[r, "-\circ f"] \arrow[d, "\Delta"] & \mathsf{Fun}(C,A) \arrow[d, "\Delta"] \\ K \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & \mathsf{Fun}(D,A)^J \arrow[r, "-\circ f"'] & \mathsf{Fun}(C,A)^J \end{tikzcd}$$

is an absolute right lifting diagram in $\mathcal{QC}at$.

18.3.3. Definition. For ∞-categories A and D and simplicial sets K and J, an absolute right lifting diagram (18.3.1), which presents a D-indexed family of limits of diagrams of shape J in A^K, **displays a pointwise limit** in A if precomposition along any map $g\colon L \to K$ preserves this family of limits: i.e., if

$$\begin{tikzcd} & A^K \arrow[r, "-\circ g"] \arrow[d, "\Delta"] & A^L \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & (A^K)^J \arrow[r, "-\circ g"'] & (A^L)^J \end{tikzcd}$$

is an absolute right lifting diagram in $\mathcal{K}$.

The terminology "pointwise limit in A" is justified by the following observation:

18.3.4. Observation. A diagram

$$\begin{tikzcd} & A^K \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & (A^K)^J \end{tikzcd}$$

displays a pointwise limit, satisfying the condition of Definition 18.3.3 if and only if for all vertices $k\colon \Delta[0] \to K$, the diagram

$$\begin{tikzcd} & A^K \arrow[r, "-\circ k"] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & (A^K)^J \arrow[r, "-\circ k"'] & A^J \end{tikzcd}$$

is an absolute right lifting diagram in $\mathcal{K}$. The fact that pointwise absolute right lifting diagrams have this property is a tautology from Definition 18.3.3. The converse implication, that if for each vertex k the restricted diagram is absolute lifting in A, then the original diagram in A^K is also, is proven in Proposition 4.3.15.

Using these definitions:

18.3.5. Lemma. *Consider a transposed pair of triangles as in* (18.3.1):

$$\begin{tikzcd} & A^K \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & (A^K)^J \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \mathsf{Fun}(D,A) \arrow[d, "\Delta"] \\ K \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\rho"] & \mathsf{Fun}(D,A)^J \end{tikzcd}$$

The triangle on the left defines a pointwise limit in A if and only if the transposed triangle on the right is a limit in $\mathsf{Fun}(D, A)$ that is stable under precomposition in $\mathcal{K}$.

The point is that the triangle on the left defines an absolute right lifting diagram in $\mathcal{K}$ with the stability property of Definition 18.3.3 if and only if the triangle on the right defines an absolute right lifting diagram in $\mathcal{QC}at$ with the stability property of Definition 18.3.2.

PROOF. The simplicially enriched cotensor/hom adjunction descends to a 2-adjunction between the homotopy 2-categories $\mathfrak{h}\mathcal{K}$ and $\mathfrak{h}\mathcal{QC}at$. The remaining details are a straightforward exercise in adjoint transposition across a 2-adjunction, left as Exercise 18.3.ii. □

We can now state and prove our main result.

18.3.6. PROPOSITION (preservation of limits by generalized Yoneda). *The generalized Yoneda embedding*

$$\text{よ}\colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

preserves any family of limits which is stable under precomposition in $\mathcal{K}$.

Proposition 18.3.6 asserts that those limit diagrams in $\mathsf{Fun}(D, A)$ that arise from corresponding limit diagrams in A are preserved by the generalized Yoneda embedding, despite its "external" nature.

PROOF. A functor $f\colon C \to D$ in $\mathcal{K}$ gives rise to a pair of cosmological functors we may form an essentially commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}(D,A) \arrow[rr, "\text{よ}"] \arrow[d] \arrow[dd, bend right=60, "-\circ f"'] \arrow[drr, phantom, "\cong"] & & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d] \arrow[dd, bend left=60, "\mathsf{Fun}(C,-)_f"] \\
\mathsf{Fun}(\mathsf{Fun}(C,D),\mathsf{Fun}(C,A)) \arrow[rr, "\text{よ}"] \arrow[d] \arrow[drr, phantom, "\cong"] & & {}_{\mathsf{Fun}(C,D)\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)} \arrow[d] \\
\mathsf{Fun}(C,A) \cong \mathsf{Fun}(1,\mathsf{Fun}(C,A)) \arrow[rr, "\text{よ}"] & & {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}
\end{tikzcd}$$

in which the top square arises by application of the cosmological functor $\mathsf{Fun}(C, -)\colon \mathcal{K} \to \mathcal{QC}at$ and the bottom square results from pulling back along $f\colon 1 \to \mathsf{Fun}(C, D)$. By inspection, the left-hand vertical composite is the precomposition functor $- \circ f\colon \mathsf{Fun}(D, A) \to \mathsf{Fun}(C, A)$.

By Corollary 18.2.6, the lower horizontal functor $\text{よ}\colon \mathsf{Fun}(C, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$ preserves all limits that exist in $\mathsf{Fun}(C, A)$. By Definition 18.3.2, any family of limits in $\mathsf{Fun}(D, A)$ that is stable under precomposition in $\mathcal{K}$ is preserved by $- \circ f\colon \mathsf{Fun}(D, A) \to \mathsf{Fun}(C, A)$. Consequently, such families of limits are preserved by the composite functors $\mathsf{Fun}(D, A) \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$. To prove that such families of limits are preserved by the external Yoneda embedding $\text{よ}\colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$, we will demonstrate that they are jointly reflected by the right-hand vertical composites, which we now describe, as the functor $f\colon C \to D$ with codomain D varies.

The right-hand vertical composite is the homotopy coherent nerve of the composite cosmological functor

$$\mathsf{Fun}(C, -)_f\colon {}_{D\backslash}\mathcal{M}od(\mathcal{QC}at)_{/A} \xrightarrow{\mathsf{Fun}(C,-)} {}_{\mathsf{Fun}(C,D)\backslash}\mathcal{M}od(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)} \xrightarrow{f^*} {}_{1\backslash}\mathcal{M}od(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$$

To explain our notation for the composite functor, note that, by construction, this functor carries a module $(q,p)\colon E \twoheadrightarrow D \times A$ to the module constructed by the pullback of quasi-categories

$$\begin{tikzcd} \mathsf{Fun}(C,E)_f \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(C,E) \arrow[d, two heads, "(q\circ -, p\circ -)"] \\ 1 \times \mathsf{Fun}(C,A) \arrow[r, "f\times\mathrm{id}"'] & \mathsf{Fun}(C,D) \times \mathsf{Fun}(C,A) \end{tikzcd}$$

Here the total space $\mathsf{Fun}(C,E)_f$ is the quasi-category of maps $C \to E$ that compose with q to f. Correspondingly, the vertical composite functor $\mathsf{Fun}(C,-)_f\colon {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$ carries a module map

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "(q,p)"'] & & F \arrow[dl, two heads, "(s,r)"] \\ & D \times A & \end{tikzcd}$$

to the induced map on fibers $\mathsf{Fun}(C,E)_f \to \mathsf{Fun}(C,F)_f$.

To complete the proof, we argue that limits in ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ are jointly reflected by the functors $\mathsf{Fun}(C,-)_f\colon {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$ as $f\colon C \to D$ ranges over all functors with codomain D. Since $\mathsf{Fun}(C,-)_f\colon {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$ underlies a cosmological functor, by Proposition 16.4.13 the quasi-categories of modules possess and this functor preserves all limits. Consequently, to show that this family of functors jointly reflects limits, it suffices to show that the functors $\mathsf{Fun}(C,-)_f\colon {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \to {}_{1\backslash}\mathsf{Mod}(\mathcal{QC}at)_{/\mathsf{Fun}(C,A)}$ are jointly conservative.

To that end, consider the module map g displayed above. Since equivalences in an ∞-cosmos are defined representably, g is an equivalence if and only if $g_*\colon \mathsf{Fun}(C,E) \to \mathsf{Fun}(C,F)$ is an equivalence of quasi-categories for all $C \in \mathcal{K}$. Since g defines a cartesian functor between the cocartesian fibrations q and s in $\mathcal{K}$, applying the cosmological functor $\mathsf{Fun}(C,-)\colon \mathcal{K} \to \mathcal{QC}at$ we obtain a cartesian functor

$$\begin{tikzcd} \mathsf{Fun}(C,E) \arrow[rr, "g_*"] \arrow[dr, two heads, "q_*"'] & & \mathsf{Fun}(C,F) \arrow[dl, two heads, "s_*"] \\ & \mathsf{Fun}(C,D) & \end{tikzcd}$$

By Proposition 16.2.11, g_* is an equivalence if and only if it defines an equivalence fiberwise, when pulled back along each $f\colon 1 \to \mathsf{Fun}(C,D)$. This pullback is exactly the module map

$$\begin{tikzcd} \mathsf{Fun}(C,E)_f \arrow[rr, "g\circ -"] \arrow[dr, two heads, "q_*"'] & & \mathsf{Fun}(C,F)_f \arrow[dl, two heads, "s_*"] \\ & \mathsf{Fun}(C,A) & \end{tikzcd}$$

defined by applying the cosmological functor $\mathsf{Fun}(C,-)_f$ to g. So we conclude that g is an equivalence in ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ if and only if for all $f\colon C \to D$, the corresponding map $g \circ -\colon \mathsf{Fun}(C,E)_f \to \mathsf{Fun}(C,F)_f$ is an equivalence in ${}_{1\backslash}\mathcal{M}od(\mathcal{QC}at)_{/\mathsf{Fun}(C,D)}$. This proves that the functors $\mathsf{Fun}(C,-)_f$ are jointly conservative, completing the proof that よ$\colon \mathsf{Fun}(D,A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ preserves any family of limits which is stable under precomposition in $\mathcal{K}$. $\square$

Exercises.

18.3.i. EXERCISE. Verify that the triangle on the left-hand side of (18.3.1) defines a right lifting diagram in $\mathfrak{h}\mathcal{K}$ if and only if the triangle on the right-hand side of (18.3.1) defines a right lifting diagram in $\mathfrak{h}\mathcal{Q}\mathcal{C}at$.

18.3.ii. EXERCISE. Prove Lemma 18.3.5.

18.4. Limit and colimit constructions

A classical result of great importance in 1-category theory is that arbitrary co/limits may be built from co/products and co/equalizers. Since equalizers can be constructed from pullbacks and products (see Exercise 18.4.ii), it is also possible to build arbitrary limits from products and pullbacks.

In this section, we will prove that an ∞-category A has limits indexed by arbitrary small simplicial sets provided it has pullbacks and small products; a dual theorem tells us that pushouts and coproducts suffice to build all simplicial set indexed colimits. Our proof constructs limits indexed by an arbitrary simplicial set X inductively using the canonical skeletal decomposition of the simplicial set X, in which the cells attached at stage n are indexed by the set $L_nX \subset X_n$ of non-degenerate n-simplices (see Example C.4.22):

$$\begin{tikzcd} & & & \coprod_{L_nX} \partial\Delta[n] \arrow[r, hook] \arrow[d] & \coprod_{L_nX} \Delta[n] \arrow[d] \arrow[dl, phantom, "\ulcorner"] & \\ \emptyset \arrow[r, hook] & \mathrm{sk}_0\, X \arrow[r, hook] & \mathrm{sk}_1\, X \arrow[r, dotted, no head] & \mathrm{sk}_{n-1}\, X \arrow[r, hook] & \mathrm{sk}_n\, X \arrow[r, dotted, no head] & \mathrm{colim}_n\, \mathrm{sk}_n\, X \cong X \end{tikzcd} \tag{18.4.1}$$

The skeletal description gives rise to a presentation of the diagram ∞-category A^X as the limit in the ∞-cosmos of a countable tower of restriction functors, each of which is a pullback of a product of maps of the form $A^{\Delta[n]} \twoheadrightarrow A^{\partial\Delta[n]}$. We will argue that limits of X-indexed diagrams in A can be defined inductively provided that A admits products, pullbacks, and sequential inverse limits—though since sequential inverse limits may be built from countable products and pullbacks we are only required to postulate the existence of the first two of these. This allows us to provide criteria for ascertaining that an ∞-category A is complete (or, dually, cocomplete):

18.4.2. THEOREM. *Suppose that κ is a regular cardinal and that A is an ∞-category that admits products of cardinality $< \kappa$ and pullbacks. If X is a κ-presentable simplicial set then A admits all limits of diagrams of shape X.*

To explain the proof strategy, consider a pushout diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ Z \arrow[r, hook] & P \end{tikzcd}$$

and suppose that an ∞-category A admits limits of shape X, Y, and Z and also pullbacks, $\lrcorner := \Lambda^2[2]$. A diagram d in A^P restricts to sub-diagrams d_X in A^X, d_Y in A^Y, and d_Z in A^X. By hypothesis, these each have limits ℓ_X, ℓ_Y, and ℓ_Z which can be seen to assemble into an internal diagram $d_\lrcorner := \ell_Y \to \ell_X \leftarrow \ell_Z$ in $A^\lrcorner$, which by hypothesis also has a limit $\ell_\lrcorner$. In Proposition 18.4.7, we argue that $\ell_\lrcorner$ defines a limit for the original P-shaped diagram d.

To explain why this is the case, we appeal to one of many equivalent definitions of a limit of a diagram valued in an ∞-category. Recall from Proposition 4.3.2 that an element ℓ in A defines a

limit for a diagram d in A^P if ℓ represents the ∞-category of cones $\mathsf{Hom}_{A^P}(\Delta, d)$ over d. This representability is encoded by an equivalence $\mathsf{Hom}_A(A, \ell) \simeq \mathsf{Hom}_{A^P}(\Delta, d)$ of *modules* from 1 to A. So our hypothesised limits for the sub-diagrams of d provides equivalences $\mathsf{Hom}_A(A, \ell_X) \simeq \mathsf{Hom}_{A^X}(\Delta, d_X)$, $\mathsf{Hom}_A(A, \ell_Y) \simeq \mathsf{Hom}_{A^Y}(\Delta, d_Y)$, and $\mathsf{Hom}_A(A, \ell_Z) \simeq \mathsf{Hom}_{A^Z}(\Delta, d_Z)$, and similarly, the universal property of $\ell_\lrcorner$ as the limit of the $\lrcorner$-shaped diagram $d_\lrcorner$ is encoded by an equivalence $\mathsf{Hom}_A(A, \ell_\lrcorner) \simeq \mathsf{Hom}_{A^\lrcorner}(\Delta, d_\lrcorner)$ of modules from 1 to A.

We must show that $\ell_\lrcorner$ has the stronger universal property of representing cones over the diagram d, i.e., that $\mathsf{Hom}(A, \ell_\lrcorner)$ is equivalent to $\mathsf{Hom}_{A^P}(\Delta, d)$. Since the diagram P is a pushout, we may show in Lemma 18.4.3 that the ∞-category of P-shaped cones $\mathsf{Hom}_{A^P}(\Delta, d)$ is isomorphic to the pullback:

$$\begin{tikzcd}
\mathsf{Hom}_{A^P}(\Delta, d) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Hom}_{A^Y}(\Delta, d_Y) \arrow[d, two heads] \\
\mathsf{Hom}_{A^Z}(\Delta, d_Z) \arrow[r] & \mathsf{Hom}_{A^X}(\Delta, d_X)
\end{tikzcd}$$

One of our first tasks will be to demonstrate, as an application of Theorem 7.4.2, that this limit in ${}_{1\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ is a "pseudo homotopy limit," defining a limit in the large quasi-category of modules from 1 to A.

With this result in hand, we may demonstrate the desired equivalence by arguing that $\mathsf{Hom}_A(A, \ell_\lrcorner)$ is the pullback of the equivalent cospan

$$\begin{tikzcd}
\mathsf{Hom}_A(A, \ell_Y) \arrow[r] \arrow[d, "\wr"'] & \mathsf{Hom}_A(A, \ell_X) \arrow[d, "\wr"] & \mathsf{Hom}_A(A, \ell_Z) \arrow[l] \arrow[d, "\wr"] \\
\mathsf{Hom}_{A^Y}(\Delta, d_Y) \arrow[r] & \mathsf{Hom}_{A^X}(\Delta, d_X) & \mathsf{Hom}_{A^Z}(\Delta, d_Z) \arrow[l]
\end{tikzcd}$$

in the large quasi-category of modules from 1 to A, a result which follows from Proposition 18.3.6. While the steps in the proof of Theorem 18.4.2 certainly contain more subtleties than in the classical case, the construction given here of a general limit out of iterations of simpler limits, is entirely analogous to the proof of the classical 1-category theoretic result presented, for instance, in [**80**, 3.4.12].

We now proceed to fill in the details of this outline, not just for a single diagram of shape X in A but for a family of diagrams $d\colon D \to A^X$. Our first result demonstrates that from any presentation of a simplicial set X as countable composite of pushouts of coproducts of monomorphisms of simplicial sets — for instance as displayed in (18.4.1) but not necessarily given by the skeletal filtration of X — the ∞-categories of cones over the corresponding restrictions of the diagram d assemble into into a dual limit diagram in the ∞-cosmos ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$. Since the original diagram is built as a cell complex from monomorphisms of simplicial sets, the dual diagram in ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ is of *cosmological limit type*, defined as an inverse sequence of pullbacks of products of isofibrations.

18.4.3. Lemma. *Consider a diagram $d\colon D \to A^X$ in an ∞-cosmos $\mathcal{K}$, and let X be a colimit of a diagram $F\colon J^{\mathrm{op}} \to s\mathcal{S}et$ whose colimit cone $\pi\colon F(-) \Rightarrow X$ presents X as a countable composite of pushouts of coproducts of monomorphisms of simplicial sets.*

(i) *The ∞-categories of cones over the restricted diagrams $d_j\colon D \xrightarrow{d} A^X \xrightarrow{\pi_j} A^{Fj}$ for each $j \in J$ assemble into a canonical cosmological limit type diagram*
$$\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})\colon J \to {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}.$$
(ii) *Moreover there is a canonical cosmological limit type cone over this diagram with summit $\mathsf{Hom}_{A^X}(\Delta, d)$.*

(iii) *This cone displays* $\mathsf{Hom}_{A^X}(\Delta, d)$ *as the limit of the diagram*

$$\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-}) \colon J \to {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$$

in the ∞*-cosmos of modules from* D *to* A *in* $\mathcal{K}$*.*

Proof. As the notation suggests, we define the value of the diagram of (i) at each $j \in J$ to be the ∞-category of cones $\mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj})$ over the restricted diagram

$$d|_{Fj} \colon D \xrightarrow{d} A^X \xrightarrow{A^{\pi_j}} A^{Fj}$$

Each morphism $f \colon j \to j'$ in J gives rise to a map of cospans,

$$\begin{tikzcd} D \arrow[r, "d|_{Fj}"] \arrow[d, equal] & A^{Fj} \arrow[d, "A^{Ff}"] & A \arrow[l, "\Delta"'] \arrow[d, equal] \\ D \arrow[r, "d|_{Fj'}"] & A^{Fj'} & A \arrow[l, "\Delta"'] \end{tikzcd}$$

which Proposition 3.4.5 converts into a map of comma ∞-categories

$$\begin{tikzcd} \mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj}) \arrow[rr, "Ff^*"] \arrow[dr, two heads, "(p_1,p_0)"'] & & \mathsf{Hom}_{A^{Fj'}}(\Delta, d|_{Fj'}) \arrow[dl, two heads, "(p_1,p_0)"] \\ & D \times A & \end{tikzcd}$$

Furthermore, if $Ff \colon Fj' \hookrightarrow Fj$ is an inclusion of simplicial sets, then A^{Fj} is an isofibration so Proposition 3.4.5 tells us that the map $Ff^* \colon \mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj}) \twoheadrightarrow \mathsf{Hom}_{A^{Fj'}}(\Delta, d|_{Fj'})$ is an isofibration as well. This tells us that the diagram $\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-}) \colon J \to {}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ is a cosmological limit type diagram, an inverse limit of pullbacks of products of isofibrations.

To verify (ii) observe equally that each transformation

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, equal] & A^{X} \arrow[d, "A^{\pi_j}"] & A \arrow[l, "\Delta"'] \arrow[d, equal] \\ D \arrow[r, "d|_{Fj}"] & A^{Fj} & A \arrow[l, "\Delta"'] \end{tikzcd}$$

induces a projection

$$\begin{tikzcd} \mathsf{Hom}_{A^{X}}(\Delta, d) \arrow[rr, "\pi_j^*"] \arrow[dr, two heads, "(p_1,p_0)"'] & & \mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj}) \arrow[dl, two heads, "(p_1,p_0)"] \\ & D \times A & \end{tikzcd}$$

defining the legs of the desired cone, and once again Proposition 3.4.5 tells us that if $\pi_j \colon Fj \hookrightarrow X$ is a monomorphism, then $\pi_j^* \colon \mathsf{Hom}_{A^X}(\Delta, d) \twoheadrightarrow \mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj})$ is an isofibration. This gives us a cosmological limit type cone with summit $\mathsf{Hom}_{A^X}(\Delta, d)$ over the diagram constructed in (i).

Finally, we verify that the cone just constructed defines a limit cone in the ∞-cosmos ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$. By the universal property of the simplicial cotensor, the functor $A^{(-)} \colon s\mathcal{S}et^{\mathrm{op}} \to \mathcal{K}$ carries the colimit cone $\pi \colon F(-) \Rightarrow X$ in $s\mathcal{S}et$ to a cosmological-type limit cone $\pi^* \colon A^X \Rightarrow A^{F-}$ in $\mathcal{K}$. This limit cone is then preserved by the cosmological functor $(-)^{\mathbb{2}} \colon \mathcal{K} \to \mathcal{K}^{\mathbb{2}}$, which carries the limit object A^X to the isofibration $(p_1, p_0) \colon A^{X \times \mathbb{2}} \twoheadrightarrow A^X \times A^X$ and carries A^{Fj} to $(p_1, p_0) \colon A^{Fj \times \mathbb{2}} \twoheadrightarrow A^{Fj} \times A^{Fj}$. We

abbreviate this diagram to $A^{F(-)\times 2}\colon J \to \mathcal{K}^2$, but recall that the objects in the image are isofibrations (p_1, p_0), not just the ∞-categories $A^{Fj\times 2}$.

Lemma 8.2.8 tells us how conical limits in the domain of a Grothendieck fibration $\mathsf{cod}\,\mathcal{K}^2 \twoheadrightarrow \mathcal{K}$ are constructed. Given a diagram $A^{F(-)\times 2}\colon J \to \mathcal{K}^2$, first we form the limit cone over the composite diagram $\mathsf{cod}\, A^{F(-)\times 2} \cong A^{F-} \times A^{F-}\colon J \to \mathcal{K}$, which in this case is given by $\pi^* \times \pi^*\colon A^X \times A^X \to A^{F-} \times A^{F-}$. Then form the diagram defined as the domain of a cod-cartesian lift χ of this limit cone, which in this case produces the diagram

$$\mathsf{Hom}_{A^{F-}}(\pi^*, \pi^*)\colon J \to \mathcal{K}_{/A^X\times A^X}$$

living in the fiber of $\mathsf{cod}\colon \mathcal{K}^2 \to \mathcal{K}$ over $A^X \times A^X$. Finally, we form a limit cone for this diagram in $\mathcal{K}_{/A^X\times A^X}$ and compose with χ to obtain a limit cone over the original diagram $A^{F(-)\times 2}\colon J \to \mathcal{K}^2$.

This analysis tells us that the given limit cone $\pi^*\colon A^{X\times 2} \Rightarrow A^{F(-)\times 2}$ in $\mathcal{K}^2$ factors through a limit cone $\pi^*\colon A^{X\times 2} \Rightarrow \mathsf{Hom}_{A^{F-}}(\pi^*, \pi^*)$ in $\mathcal{K}_{/A^X\times A^X}$. Applying the cosmological functor $(d \times \Delta)^*\colon \mathcal{K}_{/A^X\times A^X} \to \mathcal{K}_{/D\times A}$, this limit cone pulls back along $d \times \Delta\colon D \times A \to A^X \times A^X$ to a limit cone $\pi^*\colon \mathsf{Hom}_{A^X}(\Delta, d) \Rightarrow \mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})$ in $\mathcal{K}_{/D\times A}$. The objects in this limit cone lie in the full subcategory ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ and since the inclusion ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A} \hookrightarrow \mathcal{K}_{/D\times A}$ creates all cosmological limits, we conclude that the cone of (ii) defines a limit cone in there as well. This proves (iii). □

We now argue that the limit cone constructed in Lemma 18.4.3 gives rise to a corresponding limit cone in the quasi-category of modules ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$.

18.4.4. Corollary. *For any diagram $d\colon D \to A^X$ for which the indexing simplicial set X is presented as the colimit of a diagram $F\colon J^{\mathrm{op}} \to s\mathcal{S}et$ whose colimit cone $\pi\colon F(-) \Rightarrow X$ exhibits X as a sequential composite of pushouts of coproducts of monomorphisms, the canonical cone displays $\mathsf{Hom}_{A^X}(\Delta, d)$ as the limit of the diagram*

$$\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})\colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$$

in the quasi-category of modules from D to A in $\mathcal{K}$.

Proof. By Theorems 7.4.2 and 16.4.20, limits in the quasi-category of modules ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \coloneqq \mathfrak{N}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$ are constructed as pseudo homotopy limits in the Kan complex enriched category ${}_{D\backslash}\mathcal{M}od(\mathcal{K})_{/A}$. In the case of the diagrams under consideration—products, pullbacks, and limits of towers of isofibrations, Proposition 7.4.14 reveals that such limits are given by the "strict pseudo cones" of Definition 7.4.13 formed from the corresponding 1-categorical limit cones. These include the cone constructed in Lemma 18.4.3. □

Continuing in the context of Lemma 18.4.3, where we consider a diagram $d\colon D \to A^X$ in an ∞-cosmos $\mathcal{K}$, and let X be a colimit of a diagram $F\colon J^{\mathrm{op}} \to s\mathcal{S}et$ whose colimit cone $\pi\colon F(-) \Rightarrow X$ presents X as a countable composite of pushouts of coproducts of monomorphisms of simplicial sets, now assume for each $j \in J$ that the restricted diagram $d|_{Fj}\colon D \xrightarrow{d} A^X \xrightarrow{\pi_j} A^{Fj}$ admits a limit $\ell_j\colon D \to A$. The following result observes that these limits internalize to define a diagram $\ell_-\colon D \to A^J$.

18.4.5. Lemma. *Let X be a colimit of a diagram $F\colon J^{\mathrm{op}} \to s\mathcal{S}et$ whose colimit cone $\pi\colon F(-) \Rightarrow X$ presents X as a countable composite of pushouts of coproducts of monomorphisms of simplicial sets and suppose $d\colon D \to A^X$ is a diagram so that for each $j \in J$ the restricted diagram $d|_{Fj}\colon D \xrightarrow{d} A^X \xrightarrow{\pi_j} A^{Fj}$ admits a limit $\ell_j\colon D \to A$. Then there exists a diagram $\ell_-\colon D \to A^J$ whose components $D \xrightarrow{\ell_-} A^J \xrightarrow{\mathrm{ev}_j} A$ are given by the limit functors ℓ_j.*

Proof. The hypothesis that $\ell_j \colon D \to A$ defines a limit of the restricted diagram $d|_{Fj}$ tells us that there exists absolute right lifting diagram below-left, which induces the fibered equivalence of modules below-right

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell_j"] \arrow[r, "d|_{Fj}"'] \arrow[ur, phantom, "\Downarrow\rho_j", shift right=3] & A^{Fj} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathsf{Hom}_A(A, \ell_j) \arrow[rr, "\ulcorner\rho_j\urcorner", "\sim"'] \arrow[dr] & & \mathsf{Hom}_{A^{Fj}}(\Delta, d|_{Fj}) \arrow[dl] \\ & D \times A & \end{tikzcd}$$

This gives a family of isomorphisms in the quasi-category ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ between the vertices of the diagram $\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-}) \colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ considered in Corollary 18.4.4 and the modules $\mathsf{Hom}_A(A, \ell_j)$. By lifting against the isofibration

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})"] \arrow[d, hook] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}{}^{J} \arrow[d, two heads] \\ \mathbb{I} \arrow[ur, dashed] \arrow[r, "\ulcorner\rho_-\urcorner"'] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}{}^{\mathrm{ob}J} \end{tikzcd}$$

we may transfer the arrows in the J-shaped diagram constructed in Lemma 18.4.3 along these pointwise isomorphisms, to obtain a natural isomorphism whose domain is a functor $\mathsf{Hom}_A(A, \ell_-) \colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ extending our family of represented modules.

Note that the represented module $\mathsf{Hom}_A(A, \ell_j)$ under consideration is the image of the functor $\ell_j \colon D \to A$ under the generalized Yoneda embedding $よ \colon \mathsf{Fun}(D, A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$. Since Theorem 18.1.4 proves this map is fully faithful, it follows that we may lift the diagram $\mathsf{Hom}_A(A, \ell_-) \colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ to a diagram $\ell_- \colon J \to \mathsf{Fun}(D, A)$ up to natural isomorphism. We may summarizes these findings in an essentially commutative diagram

$$\begin{tikzcd} & \mathsf{Fun}(D, A) \arrow[dr, "よ"] \arrow[d, phantom, "\cong"] & \\ J \arrow[ur, "\ell_-"] \arrow[rr, "\mathsf{Hom}_A(A, \ell_-)"] \arrow[rr, bend right, "\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})"', "\wr\|"] & {} & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \end{tikzcd} \tag{18.4.6}$$

The functor $\ell_- \colon J \to \mathsf{Fun}(D, A)$ carries an arrow $f \colon j \to j'$ in $\mathcal{J}$ to a representative of the unique 2-cell ℓ_f induced by the lifting property of the right-hand triangle in the following diagram

$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d, "\Delta"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell_j"] \arrow[r, "d|_{Fj}"'] \arrow[ur, phantom, "\Downarrow\rho_j", shift right=3] & A^{Fj} \arrow[r, "A^{Ff}"'] & A^{Fj'} \end{tikzcd} = \begin{tikzcd} & A \arrow[r, equal] \arrow[d, phantom, "\Downarrow\ell_f"] & A \arrow[d, "\Delta"] \\ D \arrow[ur, "\ell_j"] \arrow[r, equal] & D \arrow[ur, "\ell_{j'}"] \arrow[r, "d|_{Fj'}"'] \arrow[ur, phantom, "\Downarrow\rho_{j'}", shift right=3] & A^{Fj'} \end{tikzcd}$$

The functor of quasi-categories $\ell_- \colon J \to \mathsf{Fun}(D, A)$ transposes to define a diagram $\ell_- \colon D \to A^J$ in the ∞-cosmos $\mathcal{K}$ of the statement. Since $\ell_- \colon J \to \mathsf{Fun}(D, A)$ maps a vertex $j \in J$ to the functor $\ell_j \colon D \to A$, the components of this internalized diagram are given by these maps as claimed. $\square$

We now prove that the limits of the original diagram $d\colon D \to A^X$ coincide with limits of the internalized diagram $\ell_{-}\colon D \to A^J$.

18.4.7. Proposition. *Let X be a simplicial set defined as a coproduct, pushout of a monomorphism, or countable composite of monomorphisms of simplicial sets, a colimit of a diagram $F\colon J^{\mathrm{op}} \to s\mathcal{S}et$ and consider a fixed diagram $d\colon D \to A^X$ in an ∞-category A with the property that the restricted diagrams $d|_{Fj}\colon D \xrightarrow{d} A^X \xrightarrow{\pi_j^*} A^{Fj}$ for each $j \in J$ have limits $\ell_j\colon D \to A$. Then a functor $\ell\colon D \to A$ is a limit of the diagram $d\colon D \to A^X$ if and only if it is a limit of the diagram $\ell_{-}\colon D \to A^J$ formed from the limit functors in Lemma 18.4.5.*

Proof. Suppose first that $\ell\colon D \to A$ is a limit of the diagram $\ell_{-}\colon D \to A^J$. By Lemma 18.3.5, this is the case if and only if $\ell\colon 1 \to \mathsf{Fun}(D,A)$ is the limit of the dual diagram $\ell_{-}\colon 1 \to \mathsf{Fun}(D,A)^J$ of quasi-categories and this limit is stable under precomposition. Proposition 18.3.6 then applies to tell us that this limit is preserved by the generalized Yoneda embedding $\text{よ}\colon \mathsf{Fun}(D,A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$. By construction, however, the diagram $\text{よ}\ell_{-}\colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is isomorphic to the diagram $\mathsf{Hom}_A(A,\ell_{-})\colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ which is in turn, by the construction given in the proof of Lemma 18.4.5, is isomorphic to the diagram $\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})\colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$, as displayed in (18.4.6). It follows that the limits of the diagrams $\text{よ}\ell_{-}$ and $\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})$ are isomorphic when they exist. The limit of the former diagram is $\text{よ}\ell \cong \mathsf{Hom}_A(A,\ell)$, since the generalized Yoneda embedding preserves the limit ℓ, while the limit of the latter is $\mathsf{Hom}_{A^X}(\Delta, d)$, as demonstrated in Corollary 18.4.4. Now the resulting isomorphism $A \downarrow \ell \cong \mathsf{Hom}_{A^X}(\Delta, d)$ in the quasi-category ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ provides an equivalence $A \downarrow \ell \simeq \mathsf{Hom}_{A^X}(\Delta, d)$ of modules over $D \times A$ which presents ℓ as a limit of $d\colon D \to A^X$ as claimed.

Conversely, suppose $\ell\colon D \to A$ is a limit of $d\colon D \to A^X$. Again by (18.4.6) the diagrams $\text{よ}\ell_{-}$ and $\mathsf{Hom}_{A^{F-}}(\Delta, d|_{F-})$ are isomorphic and so the limit $\mathsf{Hom}_{A^X}(\Delta, d)$ of the second of these diagrams, as supplied by Corollary 18.4.4, is also a limit of the first diagram. However our assumption that ℓ is a limit of the family of diagrams $d\colon D \to A^X$ may otherwise be read as saying that this limit module is covariantly represented by the functor ℓ. In other words, we have shown that the diagram $\text{よ}\ell_{-}\colon J \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ has the module $\text{よ}\ell$ as its limit in the quasi-category of modules ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$. Since the generalized Yoneda embedding $\text{よ}\colon \mathsf{Fun}(D,A) \to {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$ is fully faithful by Theorem 18.1.4, the cone presenting this limit factors through the generalized Yoneda embedding up to isomorphism to give a cone in $\mathsf{Fun}(D,A)$ that displays ℓ as a limit of $\ell_{-}\colon J \to \mathsf{Fun}(D,A)$. Now to conclude that this data defines a limit cone over the transposed diagram $\ell_{-}\colon D \to A^J$, we appeal to Lemma 18.3.5 which tells us that we must show that this limit is stable under precomposition, meaning that it is preserved by the precomposition functor $k^*\colon \mathsf{Fun}(D,A) \to \mathsf{Fun}(C,A)$ associated with any functor $k\colon C \to D$ in $\mathcal{K}$.

To see this, we use the essentially commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}(D,A) \arrow[r, "\text{よ}"] \arrow[d, "k\circ-"'] \arrow[dr, phantom, "\cong"] & {}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A} \arrow[d, "(k\times A)^*"] \\
\mathsf{Fun}(C,A) \arrow[r, "\text{よ}"'] & {}_{C\backslash}\mathsf{Mod}(\mathcal{K})_{/A}
\end{tikzcd}$$

of Corollary 18.1.8. The limit under consideration is preserved by the top Yoneda embedding, since it was reflected from ${}_{D\backslash}\mathsf{Mod}(\mathcal{K})_{/A}$, and preserved by the right-hand vertical functor by Proposition 16.4.9.

It is then reflected by the bottom Yoneda embedding by Proposition 4.3.12, since this functor is fully faithful. Thus, we conclude that the limit must be preserved by the left-hand vertical as required. $\square$

We are now able to prove Theorem 18.4.2, and show that if A is an ∞-category which admits products of cardinality bounded by a regular cardinal κ and pullbacks, then A admits all limits of diagrams indexed by κ-presentable simplicial sets.

18.4.8. Remark. Recall from Definition 2.3.2 that to say that an ∞-category A admits all limits of diagrams of shape X means that the diagonal functor $\Delta\colon A \to A^X$ admits a right adjoin, or equivalently as in Definition that every family of diagrams $d\colon D \to A^X$ admits a limit. In ∞-cosmoi of $(\infty,1)$-categories, Corollary 16.2.10 proves that this is equivalent to postulating that every diagram $d\colon 1 \to A^X$ has a limit, but this reduction to the case $D = 1$ is not possible in all ∞-cosmoi.

Proof of Theorem 18.4.2. Our proof proceed by induction on the skeleta of the simplicial set X, which is κ-presentable if and only if it has a set of non-degenerate simplices of cardinality $< \kappa$.

When X is 0-skeletal it comprises a disjoint set of vertices of cardinality $< \kappa$ and so it follows that the limit of any family of diagrams $d\colon D \to A^X$ exists by our assumption that A admits all products of cardinality $< \kappa$.

So fix a natural number n and adopt the inductive hypothesis that the result of the statement holds for all $(n-1)$-skeletal κ-presentable simplicial sets. If X is a n-skeletal κ-presentable simplicial set, then we may express X as a pushout

$$\begin{tikzcd} \coprod_{L_nX} \partial\Delta[n] \arrow[r, hook] \arrow[d] & \coprod_{L_nX} \Delta[n] \arrow[d] \\ \mathrm{sk}_{n-1} X \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & X \end{tikzcd} \tag{18.4.9}$$

in which we attach an n-simplex for each non-degenerate n-simplex of X. By hypothesis the set L_nX of non-degenerate n-simplices of X has cardinality $< \kappa$.

The inclusion $\{0\}\colon \Delta[0] \to \Delta[n]$ of the initial element is left adjoint to the constant functor $!\colon \Delta[n] \to \Delta[0]$, so the diagonal $A \to A^{\Delta[n]}$ always admits a right adjoint given by evaluation at the vertex 0. Hence, A admits all limits of diagrams of shape $\Delta[n]$, given by evaluating at 0; see Proposition 2.4.7.

For the same reason, the projection $\pi\colon \coprod_{L_nX} \Delta[n] \cong L_nX \times \Delta[n] \to L_nX$ also admits a left adjoint, and hence the functor $A^{L_nX} \to A^{\coprod_{L_nX}\Delta[n]}$ determined by precomposition with this projection also admits a right adjoint. In addition, the diagonal $A \to A^{L_nX}$ admits a right adjoint, by our hypothesis that A possesses all products of cardinality $< \kappa$, and composing these adjunctions

$$\begin{tikzcd} A \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & A^{L_nX} \arrow[l, bend left, "\lim"] \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & A^{\coprod_{L_nX}\Delta[n]} \arrow[l, bend left, "\lim"] \end{tikzcd}$$

we see that A admits all limits of shape $\coprod_{L_nX} \Delta[n]$.

Furthermore the simplicial sets $\mathrm{sk}_{n-1} X$ and $\coprod_{L_nX} \Delta[n]$ are both $(n-1)$-skeletal and κ-presentable, so the inductive hypothesis suffices to show that A admits all limits of diagrams of those shapes. In this way, we have established the hypothesis required to apply Proposition 18.4.7 to the diagram whose

pushout is depicted in (18.4.9) to infer that A since A admits pullbacks, then A admits all limits of diagrams of shape X as required.

It remains only to prove that this result also holds when X is a κ-presentable simplicial set which is not n-skeletal for any n. Since any finitely presentable simplicial set is always n-skeletal for some n, it suffices to assume that $\kappa > \omega$. Note also that what we have shown already suffices to demonstrate that if A admits countable products and pullbacks then A limits of countable sequences, by which we mean limits of diagrams of shape $\boldsymbol{\omega}^{\mathrm{op}}$. This is because the category $\boldsymbol{\omega}^{\mathrm{op}}$ is freely generated by the graph $\mathbb{N}^{\mathrm{op}}$, with objects $n \in \boldsymbol{\omega}$ and non-identity edges $n+1 \to n$, and this graph may be regarded as a 1-skeletal simplicial set. Since $\boldsymbol{\omega}^{\mathrm{op}}$ is a free category, the inclusion $\mathbb{N}^{\mathrm{op}} \hookrightarrow \boldsymbol{\omega}^{\mathrm{op}}$ is inner anodyne, so $A^{\boldsymbol{\omega}^{\mathrm{op}}} \simeq A^{\mathbb{N}^{\mathrm{op}}}$. Thus, by specializing the argument given above, a limit of a countable sequence may be computed as a pullback of a diagram involving countable products; see Exercise 18.4.i.

To finish suppose A admits pullbacks and products of size bounded by κ for some regular cardinal $\kappa > \omega$ and suppose X is a κ-presentable simplicial set. Now the simplicial set X may be expressed as the countable composite of its skeleta inclusions

$$\mathrm{sk}_0 X \hookrightarrow \mathrm{sk}_1 X \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_{n-1} X \hookrightarrow \mathrm{sk}_n X \hookrightarrow \cdots \tag{18.4.10}$$

and each of these skeleta is κ-presentable. Hence, by the inductive hypothesis, A admits all limits of diagrams of shape $\mathrm{sk}_n X$. Since $\kappa > \omega$, A also admits countable products and hence limits of inverse sequences. In this way we have established the hypotheses required to apply Proposition 18.4.7 to the diagram depicted in (18.4.10) to infer that A since A admits limits of inverse sequences, A admits all limits of diagrams of shape X as required. □

Dually:

18.4.11. Theorem (colimit constructions). *Suppose that κ is a regular cardinal and that A is an ∞-category in an ∞-cosmos $\mathcal{K}$ that admits coproducts of cardinality $< \kappa$ and pushouts. If X is a κ-presentable simplicial set then A admits all colimits of diagrams of shape X.*

Proof. Colimits of X-indexed diagrams valued in an ∞-category A in an ∞-cosmos $\mathcal{K}$ coincide with limits of X^{op}-indexed diagrams in A in the ∞-cosmos $\mathcal{K}^{\mathrm{co}}$ of Definition 1.2.23. Thus Theorem 18.4.2 applies. □

Exercises.

18.4.i. Exercise. Give a direct construction, along the lines described in the proof of Theorem 18.4.2, that demonstrates that any ∞-category with countable products and pullbacks admits inverse limits of countable sequences.

18.4.ii. Exercise. Give a direct construction that demonstrates that

(i) Any ∞-category with finite products and equalizers has pullbacks.
(ii) Any ∞-category with finite products and pullbacks has equalizers.

CHAPTER 19

Cartesian exponentiation and monadicity

Famously the category $\mathcal{C}at$ of small categories is not a topos because, among other things, it fails to be locally cartesian closed. A finitely complete category $\mathcal{E}$ is *locally cartesian closed* just when each slice category $\mathcal{E}_{/B}$ is cartesian closed, or equivalently, which the pullback functor associated to any morphism $f\colon A \to B$ admits a right adjoint (as well as a left adjoint given by composition with f):

$$\begin{tikzcd} \mathcal{E}_{/B} \arrow[r, "f^*" description] & \mathcal{E}_{/A} \arrow[l, bend right, "\Sigma_f"'] \arrow[l, bend left, "\Pi_f"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

In the case $\mathcal{E} = \mathcal{C}at$, those functors f for which the pullback functor f^* does admit a right adjoint Π_f are called *exponentiable* and have been characterized by Conduché [**25**]. Famously, all cartesian and cocartesian fibrations $p\colon E \to B$ of 1-categories are *exponentiable*, in the sense that the pullback along p functor p^* has a right adjoint called *pushforward* along p. Moreover, if $p\colon E \to B$ is a cocartesian fibration and $q\colon F \to E$ is a cartesian fibration then the pushforward $\Pi_p(q)$ is also a cartesian fibration. Our aim in this chapter is to prove the quasi-categorical analogues of these two key results, whose conclusions then apply to any ∞-cosmos of $(\infty, 1)$-categories.[1] We then present a number of applications of these exponentiability results.

Recall from Chapter 5 that a *cocartesian fibration* of quasi-categories is an isofibration $p\colon E \twoheadrightarrow B$ whose fibers depend covariantly functorially on B. In the simplest non-trivial case, when $B = \Delta[1]$, the data is given by a pair of quasi-categories E_0 and E_1 together with a functor $E_0 \to E_1$. In general, the *comprehension construction* of Chapter 17 "straightens" $p\colon E \twoheadrightarrow B$ into a simplicial functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$ that sends each vertex $b \in B$ to the fiber E_b. The domain category appearing here is the *homotopy coherent realization* of the quasi-category B, a cofibrant simplicial category that indexes B-shaped homotopy coherent diagrams. At the level of objects and 1-arrows $f\colon a \to b$ in B, the comprehension construction is defined by lifting the 1-arrow f to a *p-cocartesian 1-arrow* with codomain E:

$$\begin{tikzcd} E_a \arrow[rrr, bend left=15, "\ell^E_a"] \arrow[dd, "p_a"'] \arrow[dr, dotted, "E_f"'] & & \arrow[dl, phantom, "\Leftarrow \ell^E_f"] & E \arrow[dd, two heads, "p"] \\ & E_b \arrow[urr, "\ell^E_b"'] \arrow[dd, "p_b"'] & & \\ 1 \arrow[rrr] \arrow[dr, equals] & \arrow[d, phantom, "\Leftarrow f"] & & B \\ & 1 \arrow[urr] & & \end{tikzcd}$$

[1] We decline to give a general characterization of the exponentiable functors between quasi-categories, instead referring the interested reader to [**62**, §B.3] or [**2**].

As noted in Observation 17.4.4 and recalled in Example 19.1.5, together this data defines a *lax cocone* ℓ^E under the comprehension functor c_p with nadir E. In fact, as we shall discover in Corollary 19.1.7, this lax cocone ℓ^E is colimit cocone: in particular the domain E of the cocartesian fibration p can be recovered up to equivalence as the oplax colimit of a the comprehension functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$. Gepner, Haugseng, and Nikolaus, who obtain a similar result to Corollary 19.1.7 as one of the main theorems of [**41**], interpret this result as a proof that "Lurie's unstraightening functor is a model for the ∞-categorical analogue of the Grothendieck construction." [2]

The comprehension functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$ can be used to define a "straightening" of the pullback of p along any generalized element $b\colon X \to B$, even in the case where X is not a quasi-category, simply by restricting the comprehension functor (and its lax cocone) along b. In §19.1, we derive Corollary 19.1.7 as a special case of our first main theorem, which proves that the fiber E_b is equivalent to the lax colimit of this straightened diagram.

19.1.8. Theorem. *For any cocartesian fibration $p\colon E \twoheadrightarrow B$ of quasi-categories and any map $b\colon X \to B$ of simplicial sets, the comprehension cocone induces a canonical map over E from the oplax colimit of the diagram*

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} \mathcal{QC}at$$

to the fiber

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "b"'] & B \end{tikzcd}$$

and this map is a natural equivalence in the Joyal model structure.

In §19.2, we study the pushforward functor Π_p assocated to a cocartesian fibration p between quasi-categories. The canonical natural transformation of Theorem 19.1.8 defines a natural Joyal equivalence relating the pullback functor p^* to a functor $\tilde{p}^*$ defined by forming oplax colimits of restrictions of the comprehension cocone:

$$\begin{tikzcd} s\mathcal{S}et_{/B} \arrow[r, bend left, "\tilde{p}^*"] \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\Downarrow\gamma"] & s\mathcal{S}et_{/E} \end{tikzcd}$$

Both functors p^* and $\tilde{p}^*$ are left Quillen with respect to the sliced Joyal model structures, admitting right Quillen adjoints $p^* \dashv \Pi_p$ and $\tilde{p}^* \dashv \tilde{\Pi}_p$ described in Proposition 19.2.4. By taking mates, it follows that there is a canonical natural transformation $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$ whose component at any isofibration $q\colon F \twoheadrightarrow E$ is an equivalence. In this way we obtain an alternate model $\tilde{\Pi}_p$ for the pushforward functor Π_p, appearing as Proposition 19.2.2, that is more easily understood. At an isofibration $q\colon F \twoheadrightarrow E$ the right adjoint Π_p is equivalent to the pullback along the comprehension cocone of the induced map between lax slices induced by whiskering with q. To prove Proposition 19.2.4, we show that the "whiskering with q" map is an isofibration. This establishes the quasi-categorical analogue of our first desideratum above. We then show further that if $q\colon F \twoheadrightarrow E$ is a cartesian fibration, then the

[2]Unfortunately, the assignment of the terms "oplax colimit" and "lax colimit" given in [**41**, 2.8] is the opposite of the one used here. See Exercise 17.1.i for an explanation of our convention.

"whiskering with q" map has a certain right horn lifting property, thereby proving the quasi-categorical analogue of our second desideratum:

19.2.7. Corollary. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow E$ is a cartesian fibration between quasi-categories, then*

$$\Pi_p q\colon \Pi_p F \twoheadrightarrow B$$

is a cartesian fibration between quasi-categories.

We show also that the pullback and pushforward functors along a cocartesian fibration preserve the accompanying class of *cartesian functors* between cartesian fibrations. These results are summarized in the following theorem:

19.2.8. Theorem. *For a cocartesian fibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the pullback-pushforward adjunction restricts to define an adjunction*

$$\begin{tikzcd} \mathcal{QC}at_{/E} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{QC}at_{/B} \arrow[l, bend right, "p^*"'] \\ \mathcal{C}art(\mathcal{QC}at)_{/E} \arrow[u, hook] \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{C}art(\mathcal{QC}at)_{/B} \arrow[u, hook] \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

and moreover all of these functors are cosmological.

After dualizing in §19.3 to obtain corresponding results about cartesian fibrations between quasi-categories, we argue in §19.4 that the concluding Theorem 19.2.8 and its dual extend to arbitrary ∞-cosmoi of $(\infty, 1)$-categories, though in the absence of a point-set level analytic argument the pushfoward might only define a quasi-pseudofunctor. As an immediate corollary, we construct "exponentials" whose exponents are either cartesian or cocartesian, justifying the appelation "exponentiable" for these maps, and prove:

19.4.7. Proposition. *If If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow B$ is a cartesian fibration, then*

$$(q\colon F \twoheadrightarrow B)^{p\colon E \twoheadrightarrow B}$$

is a cartesian fibration. Dually, if $p\colon E \twoheadrightarrow B$ is a cartesian fibration and $q\colon F \twoheadrightarrow B$ is a cocartesian fibration, then the exponential is a cocartesian fibration.

The final sections of this chapter supply some first applications of these results.

19.1. Pullback along a cocartesian fibration as an oplax colimit

Recall from Definition 17.1.4 that a **lax cocone** indexed by a simplicial set X and valued in a simplicially enriched category $\mathcal{E}$ is given by a simplicial functor

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[dl, hook] \arrow[dr, "\langle B_\bullet, B\rangle"] & \\ \mathfrak{C}X^{\triangleright} \arrow[rr, "\ell"'] & & \mathcal{E} \end{tikzcd} \tag{19.1.1}$$

The **base** of a lax cocone refers to the restricted homotopy coherent diagram $B_\bullet\colon \mathfrak{C}X \to \mathcal{E}$, while the object $B\colon \mathbb{1} \to \mathcal{E}$ defines its **nadir**. The shape of a lax cocone is dictated by the weight $W_{X^{\mathrm{op}}}$

introduced in Definition 7.2.15, which in present context serves as the **weight for oplax colimits** of homotopy coherent diagrams of shape X. As explained in Exercise 17.1.i, oplax colimits represent lax cocones under a diagram, which explains our terminology.

For example:

19.1.2. Definition (canonical lax cocones). For any simplicial set X, the **canonical X-shaped lax cocone**, is a lax cocone

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle 1, X\rangle"] & \\ \mathfrak{C}X^{\triangleright} \arrow[rr, "k^X"'] & & s\mathcal{S}et \end{tikzcd}$$

whose base is constant at the terminal simplicial set 1 and whose nadir is X.

To define $k^X \colon \mathfrak{C}X^{\triangleright} \to s\mathcal{S}et$ it remains to define the images in $\mathsf{Fun}(1, X) \cong X$ of the atomic r-arrows in $\mathfrak{C}X^{\triangleright}$ whose domain is a vertex $x \in X$ and whose codomain is the cocone vertex $\top$. By Theorem 6.3.10, atomic r-arrows correspond to pairs comprised of a non-degenerate $n+1$-simplex in $X^{\triangleright}$ from x to $\top$ together with an r-dimensional bead shape in $\mathfrak{C}\Delta[n+1]$, as in Definition 6.3.8. By Definition D.2.2, non-degenerate $n+1$-simplices in $X^{\triangleright}$ from x to $\top$ correspond to non-degenerate n-simplices z in X whose initial vertex is x. Putting this together, each atomic r-arrow from x to $\top$ in $\mathfrak{C}X^{\triangleright}$ corresponds to a non-degenerate n-simplex z in X whose initial vertex is x together with a r-dimensional bead shape

$$\{0, n+1\} = T^0 \subset T^1 \subset \cdots \subset T^{r-1} \subset T^r = [0, n+1]$$

in $\mathfrak{C}\Delta[n]$. Define $\mu \colon \Delta[r] \to \Delta[n]$ by $\mu(i) = \max(T^i \backslash \{n+1\})$. Then we define the image of this atomic r-arrow to be the r-simplex $z \cdot \mu$ in the hom simplicial set $X \cong \mathsf{Fun}(1, X)$.

Recall from Lemma 17.1.5 that lax cocones can be whiskered in two directions. Given a lax cocone (19.1.1) and a map $f \colon B \to A$, there is a whiskered lax cocone of shape X

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle B_\bullet, A\rangle"] & \\ \mathfrak{C}(X^{\triangleright}) \arrow[rr, "f \cdot \ell"'] & & \mathcal{E} \end{tikzcd} \tag{19.1.3}$$

with the same base diagram and whose components from $x \in X$ to $\top$ are defined by whiskering with f. In the other direction we have

19.1.4. Lemma. *For any map of simplicial sets $f \colon Y \to X$, the canonical lax cocone of shape X restricts along $\mathfrak{C}f^{\triangleright} \colon \mathfrak{C}Y^{\triangleright} \to \mathfrak{C}X^{\triangleright}$ to the whiskered composite*

$$\begin{tikzcd} & \mathfrak{C}Y + \mathbb{1} \arrow[dl] \arrow[rr, "\mathfrak{C}f + \mathbb{1}"] & & \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle 1, X\rangle"] & \\ \mathfrak{C}Y^{\triangleright} \arrow[rr, "\mathfrak{C}f^{\triangleright}"'] & & \mathfrak{C}X^{\triangleright} \arrow[rr, "k^X"'] & & s\mathcal{S}et \end{tikzcd} = \begin{tikzcd} & \mathfrak{C}Y + \mathbb{1} \arrow[dl] \arrow[dr, "\langle 1, X\rangle"] & \\ \mathfrak{C}Y^{\triangleright} \arrow[rr, "f \cdot k^Y"'] & & s\mathcal{S}et \end{tikzcd}$$

of the canonical lax cocone of shape Y with $f \colon Y \to X$.

Proof. By a direct verification, left as Exercise 19.1.i, applying Lemma 17.1.5 to Definition 19.1.2. □

Of particular interest are the lax cocones produced by the comprehension construction in Theorem 17.4.2, mentioned briefly in Observation 17.4.4.

19.1.5. EXAMPLE. The cocartesian cocone constructed in (17.4.3) defines a lax cocone

$$\begin{tikzcd} & \mathfrak{C}B_0 + \mathbb{1} \arrow[dl, hook'] \arrow[dr, "\langle c_p, E\rangle"] & \\ \mathfrak{C}B_0^{\triangleright} \arrow[rr, "\ell^E"'] & & \mathcal{K} \end{tikzcd} \tag{19.1.6}$$

of shape B_0, the underlying quasi-category of the codomain of the cocartesian fibration $p\colon E \twoheadrightarrow B$, whose base diagram is the comprehension functor and whose nadir is the ∞-category E, the domain of the cocartesian fibration.

Throughout the remainder of this section, we fix a cocartesian fibration between quasi-categories $p\colon E \twoheadrightarrow B$. Our aim is to demonstrate that the lax cocone ℓ^E is colimit cocone.

19.1.7. COROLLARY. *The domain of a cocartesian fibration $p\colon E \twoheadrightarrow B$ of quasi-categories is equivalent to the oplax colimit of the associated comprehension functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$, with*

$$\begin{tikzcd} & \mathfrak{C}B + \mathbb{1} \arrow[dl, hook'] \arrow[dr, "\langle c_p, E\rangle"] & \\ \mathfrak{C}B^{\triangleright} \arrow[rr, "\ell^E"'] & & \mathcal{QC}at \end{tikzcd}$$

as the associated colimit cocone.

This result will be achieved as the $b = \mathrm{id}_B$ special case of our main theorem:

19.1.8. THEOREM. *For any cocartesian fibration $p\colon E \twoheadrightarrow B$ of quasi-categories and any map $b\colon X \to B$ of simplicial sets, the comprehension cocone induces a canonical map over E from the oplax colimit of the diagram*

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} \mathcal{QC}at$$

to the fiber

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "b"'] & B \end{tikzcd}$$

and this map is a natural equivalence in the Joyal model structure.

Before proving this result, we tighten up its statement. As we explain presently, there is a functor $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ that acts on objects by carrying a generalized element $b\colon X \to B$ to a canonical map $\mathrm{colim}_{\mathrm{oplax}}(c_p \circ \mathfrak{C}b) \to E$. After defining this more formally, we construct a comparison natural transformation

$$\begin{tikzcd} s\mathcal{S}et_{/B} \arrow[r, bend left, "\tilde{p}^*"] \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\Downarrow\gamma"] & s\mathcal{S}et_{/E} \end{tikzcd} \tag{19.1.9}$$

Theorem 19.1.8 asserts that this map is a componentwise Joyal equivalence. Our first task is to give a precise definition of the functor $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$. For ease of exposition, we first describe its action on objects before establishing the functoriality of this construction.

19.1.10. Definition. Given an generalized element $b\colon X \to B$ in $s\mathcal{S}et_{/B}$, define a simplicial set

$$\tilde{E}_b := \mathrm{colim}_{\mathrm{oplax}}\left(\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} s\mathcal{S}et \right).$$

The oplax colimit $\tilde{E}_b$ is the nadir of the universal lax cocone under the diagram $c_p \circ \mathfrak{C}X$. This is the weighted colimit weighted by the **weight for oplax colimits**

$$M_{X^{\mathrm{op}}}\colon \mathfrak{C}X^{\mathrm{op}} \to s\mathcal{S}et, \qquad \text{where} \qquad M_{X^{\mathrm{op}}}(x) := \mathfrak{C}(X^{\triangleright})(x, \top),$$

where x is a vertex in X and $\top$ is the unique object not in the image of the inclusion $\mathfrak{C}X \hookrightarrow \mathfrak{C}(X^{\triangleright})$.[3]

The simplicial functor

$$\begin{tikzcd} & \mathfrak{C}X + \mathbb{1} \arrow[rr, "\mathfrak{C}b+\mathbb{1}"] \arrow[dl] & & \mathfrak{C}B + \mathbb{1} \arrow[dl] \arrow[dr, "\langle c_p, E\rangle"] & \\ \mathfrak{C}X^{\triangleright} \arrow[rr, "\mathfrak{C}b^{\triangleright}"'] & & \mathfrak{C}B^{\triangleright} \arrow[rr, "\ell^E"'] & & s\mathcal{S}et \end{tikzcd}$$

defines a lax cocone $\ell^E|_b\colon \mathfrak{C}X^{\triangleright} \to s\mathcal{S}et$ under the diagram $c_p \circ \mathfrak{C}b$ with nadir E, inducing a unique simplicial map $\tilde{e}_b\colon \tilde{E}_b \to E$ from the oplax colimit. This constructs an object of $s\mathcal{S}et_{/E}$.

19.1.11. Remark. Recall the comprehension functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$ is a "straightening" of the cocartesian fibration $p\colon E \twoheadrightarrow B$. By Proposition 17.4.6, this construction is stable under pullback along any $b\colon X \to B$ in $\mathcal{QC}at$. If X is not a quasi-category, as may well be the case here, the pullback of p along b is no longer a cocartesian fibration between quasi-categories. While we can no longer construct the comprehension functor associated to this map directly, we may a posteriori regard the composite $c_p \circ \mathfrak{C}b\colon \mathfrak{C}X \to \mathcal{QC}at$ as defining the comprehension functor associated to this map, as it would be isomorphic to this if it happened to exist.

To establish the functoriality of the construction of Definition 19.1.10, it will be convenient to re-express the oplax colimits of Definition 19.1.10.

19.1.12. Lemma.

(i) For any simplicial map $b\colon X \to B$ define a weight $M_b\colon \mathfrak{C}B^{\mathrm{op}} \to s\mathcal{S}et$ by taking the left Kan extension along $\mathfrak{C}b\colon \mathfrak{C}X^{\mathrm{op}} \to \mathfrak{C}B^{\mathrm{op}}$ of the weight for oplax colimits. Then for any diagram $F\colon \mathfrak{C}B \to s\mathcal{S}et$, there is an isomorphism

$$\mathrm{colim}_{\mathrm{oplax}}(F \circ \mathfrak{C}b) := \mathrm{colim}_{M_{X^{\mathrm{op}}}}(F \circ \mathfrak{C}b) \cong \mathrm{colim}_{M_b} F.$$

(ii) The weight $M_b\colon \mathfrak{C}B^{\mathrm{op}} \to s\mathcal{S}et$ is flexible and its collage is given by the pushout

$$\begin{tikzcd} \mathfrak{C}X + \mathbb{1} \arrow[r, "\mathfrak{C}b+\mathbb{1}"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \mathfrak{C}B + \mathbb{1} \arrow[d, hook] \\ \mathfrak{C}X^{\triangleright} \arrow[r] & \mathfrak{C}(X^{\triangleright} \underset{X+\Delta[0]}{\cup} (B + \Delta[0])) \cong \mathrm{coll} M_b \end{tikzcd}$$

in the category $s\mathcal{S}et\text{-}\mathcal{C}ptd$.

[3]The weight $M_{X^{\mathrm{op}}}$ is naturally isomorphic to the weight $W_{X^{\mathrm{op}}}$ introduced in Definition 7.2.15.

Proof. By Lemma 7.1.20, the weighted colimit of a restricted diagram is isomorphic to the colimit of the original diagram weighted by the left Kan extension of the weight. This specializes to prove (i).

By Lemma 7.2.10 the collage of the left Kan extended weight M_b is computed by the pushout of (ii). Since $\mathrm{coll}(M_b)$ is the homotopy coherent realization of the pushout $X^{\triangleright} \cup_{X+\Delta[0} (B + \Delta[0])$ of simplicial sets, Theorem 6.3.10 tells us that $\mathfrak{C}B + \mathbb{1} \hookrightarrow \mathrm{coll}M_b$ is a relative simplicial computad, and hence Theorem 7.2.12 tells us that M_b is a flexible weight. □

19.1.13. Observation. The utility of Lemma 19.1.12 is as follows. Suppose now that we have a map

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "b"'] & & Y \arrow[dl, "c"] \\ & B & \end{tikzcd}$$

in the slice category $s\mathcal{S}et_{/B}$. This gives rise to a commutative diagram of simplicial computads

$$\begin{tikzcd} \mathfrak{C}X^{\triangleright} \arrow[d, "\mathfrak{C}u^{\triangleright}"'] & \mathfrak{C}X + \mathbb{1} \arrow[l, hook'] \arrow[r, "\mathfrak{C}b+\mathbb{1}"] \arrow[d, "\mathfrak{C}u+\mathbb{1}"'] & \mathfrak{C}B + \mathbb{1} \arrow[d, equal] \\ \mathfrak{C}Y^{\triangleright} & \mathfrak{C}Y + \mathbb{1} \arrow[l, hook'] \arrow[r, "\mathfrak{C}c+\mathbb{1}"'] & \mathfrak{C}B + \mathbb{1} \end{tikzcd} \tag{19.1.14}$$

inducing a simplicial computad morphism $\mathrm{coll}(M_b) \to \mathrm{coll}(M_c)$ in the category ${}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}ptd$.

This construction is functorial, defining the horizontal functor in the following square

$$\begin{tikzcd} s\mathcal{S}et_{/B} \arrow[r, "\mathrm{coll}M_\bullet"] \arrow[d, dashed, "M_\bullet"'] & {}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}ptd \arrow[d, hook] \\ s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}} \arrow[r, hook, "\mathrm{coll}"] & {}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at \end{tikzcd}$$

By the description of the essential image of the collage functor given in Proposition 7.2.8, we see that $\mathrm{coll}M_\bullet$ factors as indicated defining a functor $M_\bullet \colon s\mathcal{S}et_{/B} \to {}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$.

Finally note that the collage $\mathrm{coll}M_{B^{\mathrm{op}}} \cong \mathfrak{C}B^{\triangleright}$ for the weight for oplax colimits of shape B defines a cone under the pushout diagram of Lemma 19.1.12(ii). Thus the codomain of the functor $\mathrm{coll}(M_\bullet)$ lifts to the slice category

$$s\mathcal{S}et_{/B} \xrightarrow{\mathrm{coll}M_\bullet} \left({}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}ptd\right)_{/\mathfrak{C}B^{\triangleright}}$$

Correspondingly, by the fully faithfulness of the collage construction, we can equally regard $M_\bullet$ as a functor

$$s\mathcal{S}et_{/B} \xrightarrow{M_\bullet} \left(s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}}\right)_{/M_{B^{\mathrm{op}}}}$$

landing in the full subcategory spanned by the flexible weights.

Observation 19.1.13 allows us to extend Definition 19.1.10 to a functor.

19.1.15. Definition. Define $\tilde{p}^* \colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ to be the composite functor

$$\tilde{p}^* \coloneqq \; s\mathcal{S}et_{/B} \xrightarrow{M_\bullet} \left(s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}}\right)_{/M_{B^{\mathrm{op}}}} \xrightarrow{\mathrm{colim}_{-} c_p} s\mathcal{S}et_{/\tilde{E}_B} \xrightarrow{\tilde{e}_B} s\mathcal{S}et_{/E}$$

where $\tilde{E}_B \coloneqq \mathrm{colim}_{\mathrm{oplax}}\, c_p$ and $\tilde{e}_B \colon \tilde{E}_B \to E$ is the map induced by the lax cocone (19.1.6).

For later use, we record a few properties of the functor just constructed.

19.1.16. Lemma. *The functor* $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ *preserves colimits.*

Proof. In Definition 19.1.15 the functor under consideration is defined as a composite of three functors, the latter two of which manifestly preserve colimits. Since colimits in a slice category over an object are created by the forgetful functor, it remains only to prove that the functor $M_\bullet\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}}$ preserves colimits. Since Proposition 7.2.8 demonstrates that the inclusion $s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}} \hookrightarrow {}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$ is full and coreflective, to show that this functor preserves colimits, it suffices to show that

$$s\mathcal{S}et_{/B} \xrightarrow{\mathrm{coll}M_\bullet} {}^{\mathfrak{C}B+\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$$

preserves them.

By Observation 19.1.13, the action of this functor on objects and morphisms is defined by the pushout of Lemma 19.1.12(ii), which we regard as a diagram in ${}^{\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$. The functors

$$s\mathcal{S}et_{/B} \xrightarrow{\mathfrak{C}(-)+\mathbb{1}} {}^{\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at \qquad \text{and} \qquad s\mathcal{S}et_{/B} \xrightarrow{\mathfrak{C}(-)^{\triangleright}} {}^{\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$$

both preserve colimits. Thus, the functor from $s\mathcal{S}et_{/B}$ to the category of pushout diagrams in ${}^{\mathbb{1}/}s\mathcal{S}et\text{-}\mathcal{C}at$ with one vertex fixed at $\mathfrak{C}B + \mathbb{1}$ preserves colimits. The pushout preserves colimits as well so we conclude that $\mathrm{coll}(M_\bullet)$ and hence $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ preserves colimits, as desired. □

19.1.17. Lemma. *The functor* $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ *preserves monomorphisms.*

Proof. Consider a monomorphism

$$\begin{tikzcd} X \arrow[rr, hook, "u"] \arrow[dr, "b"'] & & Y \arrow[dl, "c"] \\ & B & \end{tikzcd}$$

over B. We'll prove that the functor $u_*\colon M_b \to M_c \in s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}}$ is a projective cell complex, as in Definition 7.2.1. As described in Remark 7.2.2 and Digression 7.2.6, the projective cell complexes are projective monomorphisms in $s\mathcal{S}et^{\mathfrak{C}B^{\mathrm{op}}}$. Hence, by Theorem C.3.12, the functor $\mathrm{colim}_{-}\, c_p$ carries projective cell complexes to monomorphisms. Thus, this argument will demonstrate that $\tilde{p}^*$ preserves monomorphisms as claimed.

By Theorem 7.2.12, $u_*\colon M_b \to M_c$ is a projective cell complex if and only if $\mathrm{coll}(u_*)\colon \mathrm{coll}M_b \to \mathrm{coll}M_c$ is a relative simplicial computad. These collages are identified in Lemma 19.1.12 and the map between them is constructed by the pushout of the displayed map in 19.1.14. Since u is a monomorphism, each of the vertical functors are simplicial subcomputad inclusions. Since the left-hand horizontal functors are also simplicial subcomputad inclusions, it follows by the dual of Proposition C.1.12 that the pushout $\mathrm{coll}(u_*)\colon \mathrm{coll}M_b \to \mathrm{coll}M_c$ is a relative simplicial computad, as required. □

Now that we've precisely defined a functor $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ that carries a generalized element to the oplax colimit of the restricted comprehension functor, our next task is to define the natural transformation (19.1.9) alluded to in the statement of Theorem 19.1.8. This is achieved by the following lemma:

19.1.18. LEMMA. *For any $b\colon X \to B$, the diagram $c_p \circ \mathfrak{C}b$ is the base of a lax cocone with nadir E_b*

$$\begin{tikzcd}
& \mathfrak{C}X + \mathbb{1} \arrow[dl] \arrow[dr, "\langle E_b, c_p\circ\mathfrak{C}b\rangle"] & \\
\mathfrak{C}X^{\triangleright} \arrow[rr, "\ell^E"'] & & s\mathcal{S}et
\end{tikzcd}$$

Hence, by the universal property of the oplax colimit, there is a natural map $\tilde{E}_b \to E_b$ over E.

PROOF. By Theorem 17.4.2, the comprehension construction defines a cocartesian cone whose domain is the lax cocone of Example 19.1.5 and whose codomain is the canonical lax cocone $k^B\colon \mathfrak{C}B^{\triangleright} \to s\mathcal{S}et$ whose definition is recalled in Definition 19.1.2. Lemma 19.1.4 observes that this canonical lax cocone restricts along $b\colon X \to B$ to the canonical lax cocone of shape X. Thus, Lemma 17.1.9 applies, and the domain component of the restricted "cocartesian cocone"[4] is the lax cocone of the statement. □

To prove Theorem 19.1.8 we must verify that (19.1.9) is a componentwise Joyal weak equivalence. We first demonstrate this from generalized elements $b\colon \Delta[n] \to B$ whose domains are simplices and then use the results of Lemmas 19.1.16 and 19.1.17 to extend these results to the general case.

19.1.19. EXAMPLE. By definition $\tilde{p}^*(b\colon \Delta[0] \to B)$ is the oplax colimit of the diagram

$$\mathfrak{C}\Delta[0] \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} \mathcal{QC}at$$

that sends the unique object to the fiber E_b of $p\colon E \twoheadrightarrow B$ over $b\colon \Delta[0] \to B$. The weight for lax cocones of shape $\Delta[0]$ is the terminal weight so the weighted colimit is just the ordinary colimit of this one object diagram. Thus $\tilde{p}^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$ sends $b\colon \Delta[b] \to B$ to $E_b \to E$, which is isomorphic to the strict pullback $p^*(b\colon \Delta[0] \to B)$.

For $b\colon \Delta[1] \to B$, $\tilde{p}^*(b\colon \Delta[1] \to B)$ is the oplax colimit of the diagram

$$\mathfrak{C}\Delta[1] \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} \mathcal{QC}at$$

whose image is diagram $e_b\colon E_{b_0} \to E_{b_1}$ of quasi-categories constructed in (19.1.21). In this case, the oplax colimit has a simple description: it is given by an "mapping cylinder" formed by attaching E_{b_1} along the codomain edge of the cylinder $E_{b_0} \times \Delta[1]$ via the map e_b. We now show this simplicial set is Joyal weak equivalence to the quasi-category E_b.

19.1.20. PROPOSITION. *The data formed by applying the comprehension construction*

$$\begin{tikzcd}
E_{b_0} \arrow[dr, dotted, "e_b"'] \arrow[rr, bend left, "\ell_0^E"] \arrow[dd, "p_0"'] & & E_b \arrow[r] \arrow[dd, two heads, "p_b"] \arrow[ddr, phantom, "\lrcorner", very near start] & E \arrow[dd, two heads, "p"] \\
& E_{b_1} \arrow[ur, "\ell_1^E"'] \arrow[dd, two heads, "p_1"] \arrow[ur, phantom, "\chi", near start, shift left=4] & & \\
1 \arrow[rr, "0"] \arrow[dr, equal] & & \Delta[1] \arrow[r, "b"] & B \\
& 1 \arrow[ur, "1"'] \arrow[u, phantom, "\Downarrow\kappa", shift right=3] & &
\end{tikzcd} \tag{19.1.21}$$

[4]Since X is not necessarily a quasi-category, the pullback of $p\colon E \twoheadrightarrow B$ along $b\colon X \to B$ might not be a cocartesian fibration and hence it doesn't entirely make sense to refer to the restricted data as a "cocartesian cocone." Nonetheless, the construction detailed in the proof of Lemma 17.1.9 constructs the lax cocone that we seek.

to the pullback $p_b\colon E_b \to \Delta[1]$ of $p\colon E \twoheadrightarrow B$ along $b\colon \Delta[1] \to B$ induces a Joyal weak equivalence

$$\begin{tikzcd}
E_{b_0} \arrow[r, "e_b"] \arrow[d, hook, "\mathrm{id}\times\delta^0"'] & E_{b_1} \arrow[d, hook] \arrow[ddr, bend left, "\ell_1^E"] & \\
E_{b_0}\times\Delta[1] \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] \arrow[drr, bend right, "\chi"'] & \tilde{E}_b \arrow[dr, dashed, "\gamma_b", "\sim"'] & \\
& & E_b
\end{tikzcd}$$

Proof. By Proposition F.5.5, the oplax colimit $\tilde{E}_b$ is weakly equivalent to the quasi-categorical collage $\mathrm{coll}(e_b, E_{b_1})$, the quasi-category defined by the formula

$$\mathrm{col}(e_b, E_{b_1})_n = \left\{ \left(\Delta[i] \xrightarrow{a} E_{b_0}, \Delta[n] \xrightarrow{c} E_{b_1}\right) \middle| c|_{\{0,\ldots,i\}} = e_b(a), -1 \le i \le n. \right\}$$

which is elaborated upon in Definition F.5.2. Note there is a canonical map $\rho\colon \mathrm{coll}(e_b, \tilde{E}_{b_1})$ defined so that E_{b_0} is the pullback over the endpoint 0.

Proposition F.5.5 demonstrates that the equivalence $k\colon \tilde{E}_b \xrightarrow{\sim} \mathrm{coll}(e_b, E_{b_1})$ is inner anodyne. In particular, there exists a lift

$$\begin{tikzcd}
\tilde{E}_b \arrow[r, "\gamma_b"] \arrow[d, tail, "k"', "\wr"] & E_b \arrow[d, two heads, "p_b"] \\
\mathrm{coll}(e_b, E_{b_1}) \arrow[r, "\rho"'] \arrow[ur, dashed, "\ell"'] & \Delta[1]
\end{tikzcd}$$

defining a direct comparison map $\ell\colon \mathrm{coll}(e_b, E_{b_1}) \to E_b$ over $\Delta[1]$.

To prove that ℓ is an equivalence, observe by Lemma F.5.4 and Proposition 5.1.20 that ρ and p_b are both cocartesian fibrations. Hence, Proposition 16.2.11 tells us that if ℓ is a cartesian functor, then to demonstrate that ℓ is an equivalence, we need only show that it restricts to an equivalence on the fibers over 0 and 1. Indeed, ℓ is an isomorphism on both fibers, so now our only remaining task is to demonstrate that it is a cartesian functor.

The ρ-cocartesian lift, constructed in the proof of Lemma F.5.4, of the non-trivial 1-arrow with domain $a\colon \Delta[0] \to E_{b_0}$ is

$$\chi_a := (a\colon \Delta[0] \to E_{b_0}, e_b(a)\cdot\sigma^0\colon \Delta[1] \to E_{b_1})$$

given by the copy degenerate edge at $e_b(a) \in E_{b_1}$ lying over the 1-simplex in $\Delta[1]$. Such edges lie in the image of the functor $\chi\colon E_{b_0}\times\Delta[1] \to E_b$ used to define the map $\gamma_b\colon \tilde{E}_b \to E_b$. This functor in turn is defined to be a representative for the p-cocartesian lift χ of the non-degenerate 1-simplex in $\Delta[1]$. In particular, by Definition F.4.7, the functor χ is a cocartesian cylinder, meaning that its components indexed by vertices a in E_{b_0} are p-cocartesian 1-arrows. This proves that ℓ carries ρ-cocartesian arrows to p-cocartesian arrows, completing the proof that ℓ is an equivalence. □

We argue inductively that $\gamma_b\colon \tilde{E}_b \to E_b$ is an equivalence for any n-simplex $b\colon \Delta[n] \to B$ under the assumption that this is true for simplices of lower dimension. Our strategy mirrors that adopted for the 1-simplex: we construct a quasi-categorical model for the oplax colimit of a homotopy coherent diagram $c_p \circ \mathfrak{C}b\colon \mathfrak{C}\Delta[n] \to \mathcal{QC}at$, i.e., a quasi-category equivalent to the simplicial set $\tilde{E}_b$ defined as

the oplax colimit of $c_p \circ \mathfrak{C}b$, and then show that this is equivalent to the strict pullback E_b. The inductive step makes use of the following weights.

19.1.22. Definition (weights for oplax colimits of homotopy coherent simplices). To streamline notation write

$$M_{\Delta[n]^{\mathrm{op}}} \coloneqq \mathfrak{C}\Delta[n]^{\triangleright}(-,\top) \colon \mathfrak{C}\Delta[n]^{\mathrm{op}} \to s\mathcal{S}et$$

for the weight for oplax colimits of homotopy coherent diagrams of shape $\Delta[n]$, which coincides with the weight $W_{\Delta[n]^{\mathrm{op}}}$ introduced in Definition 7.2.15. To compare the weights for oplax colimits of a homotopy coherent $n-1$-simplex an n-simplex, we left Kan extend the former along the inclusion $\mathfrak{C}(\delta^n)^{\mathrm{op}} \colon \mathfrak{C}\Delta[n-1]^{\mathrm{op}} \to \mathfrak{C}\Delta[n]^{\mathrm{op}}$, writing $M_{\Delta[n-1]^{\mathrm{op}}} \colon \mathfrak{C}\Delta[n]^{\mathrm{op}} \to s\mathcal{S}et$ for the result. Explicitly,

$$M_{\Delta[n-1]^{\mathrm{op}}}(i) = \begin{cases} \mathfrak{C}\Delta[n](i,n) & i < n \\ \varnothing & i = n. \end{cases}$$

There is a natural inclusion $M_{\Delta[n-1]^{\mathrm{op}}} \hookrightarrow Y^n$ in $s\mathcal{S}et^{\mathfrak{C}\Delta[n]^{\mathrm{op}}}$ whose codomain is the representable weight $Y^n \coloneqq \mathfrak{C}\Delta[n](-,n) \colon \mathfrak{C}\Delta[n]^{\mathrm{op}} \to s\mathcal{S}et$ that is the identity on all components except the one indexed by $n \in \mathfrak{C}\Delta[n]$.

19.1.23. Lemma.

(i) The following diagram defines a pushout of weights in $s\mathcal{S}et^{\mathfrak{C}\Delta[n]^{\mathrm{op}}}$:

$$\begin{tikzcd} M_{\Delta[n-1]^{\mathrm{op}}} \arrow[r, hook] \arrow[d, "\mathrm{id}\times\delta^0"'] & Y^n \arrow[d] \\ M_{\Delta[n-1]^{\mathrm{op}}} \times \Delta[1] \arrow[r] & M_{\Delta[n]^{\mathrm{op}}} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

(ii) Let $F \colon \mathfrak{C}\Delta[n] \to s\mathcal{S}et$ be a homotopy coherent diagram whose $M_{\Delta[n-1]^{\mathrm{op}}}$-weighted colimit is Joyal weakly equivalent to the simplicial set E_{n-1}. Then the oplax colimit of F is Joyal weakly equivalent to the pushout

$$\begin{tikzcd} E_{n-1} \arrow[r, "\iota_n"] \arrow[d, "\mathrm{id}\times\delta^0"'] & F_n \arrow[d] \\ E_{n-1} \times \Delta[1] \arrow[r] & E_n \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

along a canonical map ι_n induced by the diagram F.

Proof. The pushout in (i) can be verified componentwise at each $i \in \mathfrak{C}\Delta[n]$ at which point this relationship is evident from the definitions.

The pushout of (ii) follows. If E_{n-1} is isomorphic to the $M_{\Delta[n-1]^{\mathrm{op}}}$-weighted colimit of F, then the pushout diagram of (ii) is obtained by applying the cocontinuous functor $\mathrm{colim}_{\_} F$ to the pushout diagram of (i). In this case, the map ι_n has a natural explicit description. By Lemma 7.1.20, the $M_{\Delta[n-1]^{\mathrm{op}}}$-weighted colimit of F coincides with the oplax colimit of the restricted diagram $F \circ \mathfrak{C}(\delta^n)^{\mathrm{op}}$. The functor F itself defines a canonical lax cocone under this restricted diagram with nadir F_n. Hence there is a natural comparison ι_n from the $M_{\Delta[n-1]^{\mathrm{op}}}$-weighted colimit to F_n.

By Theorem 7.2.12 all of the weights appearing in (i) are flexible. Since $E_{n-1} \to E_{n-1} \times \Delta[1]$ is a monomorphism, the dual of Proposition 7.3.1 implies that the pushout being constructed is invariant under Joyal weak equivalence. □

This lemma provides the inductive step in the following computation:

19.1.24. Proposition. *For any simplex $b\colon \Delta[n] \to B$, the component $\gamma_b\colon \tilde{E}_b \to E_b$ from the oplax colimit to the strict pullback is a Joyal weak equivalence.*

Proof. The base cases for $n = 0$ and $n = 1$ appear as Example 19.1.19 and Proposition 19.1.20. For the induction step, suppose we have shown this componentwise weak equivalence for all $n-1$-simplices in B. By Lemma 7.1.20 and Definition 19.1.22, the $M_{\Delta[n-1]^{\mathrm{op}}}$-weighted colimit of the diagram $\mathfrak{C}\Delta[n] \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} s\mathcal{S}et$ is isomorphic to the oplax weighted colimit of the restricted diagram

$$\mathfrak{C}\Delta[n-1] \xrightarrow{\mathfrak{C}\delta^n} \mathfrak{C}\Delta[n] \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} s\mathcal{S}et.$$

By the inductive hypothesis, this weighted colimit $\tilde{E}_{b\cdot\delta^n}$ is weakly equivalent to the pullback $E_{b\cdot\delta^n}$. By Lemma 19.1.23, the diagram

$$\begin{tikzcd} E_{b\cdot\delta^n} \arrow[r, "\iota_n"] \arrow[d, "\mathrm{id}\times\delta^0"'] & E_{b_n} \arrow[d] \\ E_{b\cdot\delta^n}\times\Delta[1] \arrow[r] & \tilde{E}_b \end{tikzcd}$$

is then a pushout up to Joyal weak equivalence. So it follows from Proposition F.5.5 that $\tilde{E}_b$ is equivalent to the quasi-categorical collage $\mathrm{coll}(\iota_n, E_{b_n})$, and as in the proof of Proposition 19.1.20, the map γ_b factors to define a map

$$\begin{tikzcd} \mathrm{coll}(\iota_n, E_{b_n}) \arrow[rr, dashed, "\ell"] \arrow[dr, two heads, "\rho"'] & & E_b \arrow[dl, two heads, "\pi_\ell\cdot b"] \\ & \Delta[1] & \end{tikzcd}$$

in this case involving the map $\pi_\ell\colon \Delta[n] \to \Delta[1]$ that carries every element but the last one to 0. Observe that π_ℓ a cocartesian fibration, and indeed a bifibration, as it is covariantly represented by the functor $!\colon [n-1] \to [0]$, which admits both left and right adjoints; see Theorem F.5.6.

Our task, again, is to show that ℓ is an equivalence. By Lemma F.5.4 and the fact that cocartesian fibrations compose, it is a functor between cocartesian fibrations. Moreover, ℓ is bijective on the fibers over $0, 1 \in \Delta[1]$, the latter being E_{b_n} in both cases and the former being $E_{b\cdot\delta^n}$. As in the proof of Proposition 19.1.20, ℓ is a cartesian functor, so Proposition 16.2.11 implies that ℓ is an equivalence, as desired. ☐

Combining the work in this section, we can finally prove our main result.

Proof of Theorem 19.1.8. Our task is to demonstrate that the canonical natural transformation

$$\begin{tikzcd} s\mathcal{S}et_{/B} \arrow[r, bend left, "\tilde{p}^*"] \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\Downarrow\gamma"] & s\mathcal{S}et_{/E} \end{tikzcd}$$

is a componentwise Joyal weak equivalence using the result of Proposition 19.1.24, which demonstrates that this is the case for the simplices $b\colon \Delta[n] \to B$ of B.

The category $s\mathcal{S}et_{/B}$ is equivalent to the category $s\mathcal{S}et^{\mathrm{el}\,B^{\mathrm{op}}}$ of presheaves indexed by the category $\mathrm{el}\,B$ of simplices of B; its objects are simplices $b\colon \Delta[n] \to B$ and a morphism from b to $c\colon \Delta[m] \to B$

is a simplicial operator $\alpha\colon \Delta[n] \to \Delta[m]$ so that $c \cdot \alpha = b$. The representable presheaves generate $s\mathcal{S}et^{\mathrm{el}\,B^{\mathrm{op}}}$ under colimits, and such colimits are preserved by both of the functors p^* and $\tilde{p}^*$, the former case because of the right adjoint Π_p that exists in the locally cartesian closed category $s\mathcal{S}et$ and the latter case by Lemma 19.1.16. Under the equivalence $s\mathcal{S}et^{\mathrm{el}\,B^{\mathrm{op}}} \cong s\mathcal{S}et_{/B}$, these representables correspond to the objects $b\colon \Delta[n] \to B$ whose domain is a simplex. Proposition 19.1.24 verifies that the components of γ indexed by such objects are equivalences, which is the moral reason why γ is an equivalence at all objects.

To demonstrate this, note that $b\colon X \to B$ is a colimit indexed by the category $\mathrm{el}\,X$ of its simplices $\Delta[n] \xrightarrow{x} X \xrightarrow{b} B$, i.e.,

$$(X \xrightarrow{b} B) \cong \operatorname*{colim}_{\mathrm{el}\,X}(\Delta[n] \to B).$$

The map γ_b factors as

$$\gamma_b\colon \tilde{p}^*(\operatorname*{colim}_{\mathrm{el}\,X} \Delta[n] \to B) \cong \operatorname*{colim}_{\mathrm{el}\,X} \tilde{p}^*(\Delta[n] \to B) \longrightarrow \operatorname*{colim}_{\mathrm{el}\,X} p^*(\Delta[n] \to B) \cong p^*(\operatorname*{colim}_{\mathrm{el}\,X} \Delta[n] \to B),$$

so it remains only to show that this middle map, the colimit of the equivalences γ_{bx} indexed by the simplices of X, is itself an equivalence. The indexing category $\mathrm{el}\,X$ is a Reedy category, so by Corollary C.5.17 and Example C.5.18 if we can show that the two $\mathrm{el}\,X$-indexed diagrams are Reedy cofibrant and that the category $\mathrm{el}\,X$ has fibrant constants, then the pointwise equivalence between the diagrams will induce the desired equivalence between their colimits. As explained in Example C.5.18, to say that the Reedy category $\mathrm{el}\,X$ has fibrant constants, means that for each element $x\colon \Delta[n] \to X$, the category of elements of the covariant representable boundary functor $\partial\,\mathrm{el}\,X_x$ is either empty or connected; see §C.4–C.5 for more. This category is empty just when x is non-degenerate and has a terminal object, and is connected in particular, when x is degenerate. So $\mathrm{el}\,X$ has fibrant constants and the colimit functor $(s\mathcal{S}et_{/B})^{\mathrm{el}\,X} \to s\mathcal{S}et_{/B}$ carries pointwise weak equivalences between Reedy cofibrant diagrams to weak equivalences.

To verify this Reedy cofibrancy, it suffices to show

(i) the canonical diagram $\mathrm{el}\,X \to s\mathcal{S}et_{/B}$ is Reedy cofibrant
(ii) $\tilde{p}^*$ and p^* preserve Reedy cofibrant objects.

Since $\tilde{p}^*$ and p^* preserve colimits, they in particular preserve latching objects, so for this second item it suffices to show that both functors also preserve monomorphisms. Here, the fact that the pullback functor p^* preserves monomorphisms is standard, and the fact that its replacement $\tilde{p}^*$ preserves monomorphisms was proven in Lemma 19.1.17.

So it remains only to prove (i), that is, to argue that the functor

$$\begin{array}{ccc} \mathrm{el}\,X & \longrightarrow & s\mathcal{S}et_{/B} \\ x\colon \Delta[n] \to X & \longmapsto & bx\colon \Delta[n] \to X \end{array}$$

is Reedy cofibrant. The latching object associated to $x\colon \Delta[n] \to X$ is the composite $\partial\Delta[n] \hookrightarrow \Delta[n] \xrightarrow{x} X \xrightarrow{b} B$ and the latching map is the inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ over B, which is obviously a monomorphism. This completes the proof. $\square$

Exercises.

19.1.i. Exercise. Prove Lemma 19.1.4.

19.2. Pushforward along a cocartesian fibration

Theorem 19.1.8 demonstrates that the pullback $E_b \to E$ of a functor $b\colon X \to B$ along a cocartesian fibration $p\colon E \twoheadrightarrow B$ is computed, up to equivalence, as the oplax colimit of a particular diagram

$$\mathfrak{C}X \xrightarrow{\mathfrak{C}b} \mathfrak{C}B \xrightarrow{c_p} \mathcal{QC}at$$

When the oplax colimit is defined strictly as a simplicial set it enjoys the universal property of Definition 7.1.7: maps $\mathsf{colim}_{\mathsf{oplax}}(c_p \circ \mathfrak{C}b) \to F$ correspond to lax cocones under $c_p \circ \mathfrak{C}b$ with nadir F. This correspondence defines a right adjoint

$$\begin{tikzcd} s\mathcal{S}et_{/E} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B} \arrow[l, bend right, "\tilde{p}^*"'] \end{tikzcd}$$

characterized on an object $q\colon F \to E$ by the bijection

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr, "b"'] & & \tilde{\Pi}_p F \arrow[dl, "\tilde{\Pi}_p(q)"] \\ & B & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathsf{colim}_{\mathsf{oplax}}(c_p \circ \mathfrak{C}b) \arrow[rr] \arrow[dr, "\tilde{p}^*(b) := \ell^E|_b"'] & & F \arrow[dl, "q"] \\ & E & \end{tikzcd}$$

That is, n-simplices in $\tilde{\Pi}_p F$ over $b\colon \Delta[n] \to B$ correspond to lax cocones under the homotopy coherent n-simplex $c_p \circ \mathfrak{C}b$ with nadir F whose whiskered composite with q recovers the restriction $\ell^E|_b$ of the lax cocone produced by the comprehension construction.

To make this simplex level construction of $\tilde{\Pi}_p(q)$ precise, we require a simplicial set whose n-simplices correspond to lax cocones under a homotopy coherent n-simplex in $\mathcal{QC}at$ with nadir F. One might think that the slice quasi-category $\mathfrak{N}\mathsf{core}_*\mathcal{QC}at_{/F}$ provides just such a gadget but this isn't quite correct: since we've passed to the $(\infty,1)$-categorical core of $\mathcal{QC}at$ before taking the homotopy coherent nerve, simplices in $\mathfrak{N}\mathsf{core}_*\mathcal{QC}at_{/F}$ correspond to *pseudo* cocones rather than lax cocones. The solution is to drop the core functor, in which case the homotopy coherent nerve $\mathfrak{N}\mathcal{QC}at$ is not a quasi-category. Instead, when this simplicial set is equipped with the natural markings of Definition D.8.10 it becomes a *2-complicial set* in the sense of Definition D.7.11. Lemma D.2.20 introduces a slice construction for marked simplicial sets, which does *not* commute with the functor that forgets the markings, but this is a good thing. The marked slice $\mathfrak{N}\mathcal{QC}at_{/\!/F}$ has exactly the property we desire, in that its n-simplices correspond to lax cocones under a homotopy coherent n-simplex in $\mathcal{QC}at$ with nadir F.

At a high level, this can be explained in the following way. The oplax colimits defining the functor $\tilde{p}^*$ in §19.1 are properly understood as a variety of $(\infty,2)$-categorical colimits. Consequently, the description of the corresponding right adjoint $\tilde{\Pi}_p$ involves an $(\infty,2)$-categorical cocone construction, instantiated by forming the slice of 2-complicial set over a vertex. As we explain in Appendix D, 2-complicial sets are simplicial sets in which certain simplices are designated as equivalences. This notion is not as unfamiliar as it may seem at first: Example D.7.12 reveals that Kan complexes are precisely the 0-complicial sets while quasi-categories correspond to 1-complicial sets.

The formal definition of the right adjoint $\tilde{\Pi}_p$ in Proposition 19.2.2 below, makes use of the following lemma.

19.2.1. Lemma. *Let $q\colon F \to E$ be a 0-arrow in a quasi-categorically enriched category $\mathcal{K}$.*

(i) There is a functor of slice 2-complicial sets

$$\mathfrak{N}\mathcal{K}_{/\!\!/F} \xrightarrow{q\circ -} \mathfrak{N}\mathcal{K}_{/\!\!/E}$$

induced from the whiskering operation for lax cocones.

(ii) If $q\colon F \twoheadrightarrow E$ is a representably-defined isofibration, then $q \circ -\colon \mathfrak{N}\mathcal{K}_{/\!\!/F} \twoheadrightarrow \mathfrak{N}\mathcal{K}_{/\!\!/E}$ is a complicial isofibration.

Proof. We temporarily ignore the naturally defined markings (see Definition D.8.10) on the homotopy coherent nerve of a quasi-categorically enriched category, and define the whiskering map $q \circ -$ on underlying simplicial sets. By Lemma D.2.20 and Proposition 6.3.3, maps of simplicial sets $X \to \mathfrak{N}\mathcal{K}_{/\!\!/F}$ correspond to maps $X^{\triangleright} \to \mathfrak{N}\mathcal{K}$ that send the cone point to F, which correspond to homotopy coherent functors $\mathfrak{C}X^{\triangleright} \to \mathcal{K}$ which send the cone point to F. Thus to define the map in (i), by the Yoneda lemma it suffices to provide a natural operation that converts a lax cocone of shape X with nadir F into a lax cocone of shape X with nadir E. This is achieved by the whiskering construction of (19.1.3). Since whiskering preserves fibered equivalences and isomorphisms, which correspond to marked 1- and 2-simplices in $\mathfrak{N}\mathcal{K}_{/\!\!/F}$, this defines the desired map of 2-complicial sets.

By Definition D.1.15 and Corollary D.4.9, to prove (ii), we must verify the lifting property

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathfrak{N}\mathcal{K}_{/\!\!/F} \arrow[d, "q\circ -"] \\ \Delta^k[n] \arrow[r] \arrow[ur, dashed] & \mathfrak{N}\mathcal{K}_{/\!\!/E} \end{tikzcd}$$

for $n \geq 1$ and $0 < k \leq n$ under the additional hypothesis that $q\colon F \twoheadrightarrow E$ is a representably-defined isofibration in $\mathcal{K}$. The bottom horizontal functor is given by a homotopy coherent $n+1$-simplex $\mathfrak{C}\Delta[n+1] \to \mathcal{K}$ that sends the first $n+1$ objects to $E_0, \ldots E_n$ and the final object to E satisfying one additional condition forced by the markings on $\Delta^k[n]$ and $\mathfrak{N}\mathcal{K}_{/\!\!/E}$. If $k < n$ this condition says that the 1-arrow

$$\begin{tikzcd} E_{k-1} \arrow[rr, "f_{k-1,k+1}"] \arrow[dr, "f_{k-1,k}"'] & \arrow[d, phantom, "\Downarrow\alpha"] & E_{k+1} \\ & E_k \arrow[ur, "f_{k,k+1}"'] & \end{tikzcd}$$

in $\mathsf{Fun}(E_{k-1}, E_{k+1})$ is invertible. If $k = n$, then the 1-arrow

$$\begin{tikzcd} E_{n-2} \arrow[rr, "f_{n-2,n}"] \arrow[dr, "f_{n-2,n-1}"'] & \arrow[d, phantom, "\Downarrow\alpha"] & E_n \\ & E_{n-1} \arrow[ur, "\sim", "f_{n-1,n}"'] & \end{tikzcd}$$

in $\mathsf{Fun}(E_{n-2}, E_n)$ must be invertible and $f_{n-1,n}\colon E_{n-1} \to E_n$ must admit an equivalence inverse. The δ^{n+1}-face of this homotopy coherent simplex and the top horizontal functor together define a simplicial functor $\mathfrak{C}\Lambda^k[n+1] \to \mathcal{K}$ that caries the $n+2$ objects to $E_0, \ldots, E_n, F$, respectively, and has the

property that for each $0 \le j \le n$ the diagram of function complexes

$$\begin{tikzcd} \mathfrak{C}\Lambda^k[n+1](j,n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_j, E) \arrow[d, two heads, "q\circ-"] \\ \mathfrak{C}\Delta[n+1](j,n+1) \arrow[r] & \mathsf{Fun}(E_j, F) \end{tikzcd}$$

commutes.

Since $0 < k < n+1$ to solve the original problem we need only construct a single lift

$$\begin{tikzcd} \sqcap^{n,k}_1 \cong \mathfrak{C}\Lambda^k[n+1](0,n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_0, E) \arrow[d, two heads, "q\circ-"] \\ \square^n \cong \mathfrak{C}\Delta[n+1](0,n+1) \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E_0, F) \end{tikzcd}$$

Lemma 6.3.7 identifies the left-hand map with a cubical horn inclusion. By hypothesis, the right-hand map is an isofibration of quasi-categories. By Theorem D.4.18 isofibrations admit fillers against "special outer horns" $\Lambda^m[m] \to \Delta[m]$, those for which the image of the final edge is invertible, and such extensions solve this lifting problem. $\square$

19.2.2. Proposition. *There is a right adjoint*

$$\begin{tikzcd} s\mathcal{S}et_{/E} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B} \arrow[l, bend right, "\tilde{p}^*"'] \end{tikzcd}$$

to the oplax colimit functor of Definition 19.1.15 defined at $q\colon F \to E$ by the pullback

$$\begin{tikzcd} \tilde{\Pi}_p F \arrow[r] \arrow[d, two heads, "\tilde{\Pi}_p(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/F} \arrow[d, two heads, "q\circ-"] \\ B \arrow[r, "\ell^E"] & \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/E} \end{tikzcd}$$

Moreover, when $q\colon F \twoheadrightarrow E$ is an isofibration, $\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow B$ is an isofibration between quasi-categories.

Proof. Recall from Lemma 19.2.1 that n-simplices in $\mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/F}$ correspond to lax cocones under homotopy coherent simplices with nadir F, and observe that the whiskering functor $q\circ-\colon \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/F} \to \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/E}$ does not change the underlying homotopy coherent diagram. By the defining universal property, an n-simplex in the pullback over $b\colon \Delta[n] \to B$ corresponds to lax cocone under the homotopy coherent n-simplex $c_p \circ \mathfrak{C}b\colon \mathfrak{C}\Delta[n] \to \mathcal{Q}\mathcal{C}at$ with nadir F that whiskers with q to the lax cocone of Example 19.1.5. This recovers the characterization of the right adjoint $\tilde{\Pi}_p$ given above and Lemma 19.1.16 demonstrates that this adjoint correspondence extends to all elements of $s\mathcal{S}et_{/B}$.

The action of $\tilde{\Pi}_p$ on morphisms $u\colon G \to F$ over E is given similarly by the pullback

$$\begin{tikzcd} \tilde{\Pi}_p G \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{N}\mathcal{QC}at_{/\!/G} \arrow[d, "u\circ -"] \\ \tilde{\Pi}_p F \arrow[r] \arrow[d, two heads, "\tilde{\Pi}_p q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{N}\mathcal{QC}at_{/\!/F} \arrow[d, two heads, "q\circ -"] \\ B \arrow[r, "\ell^E"] & \mathfrak{N}\mathcal{QC}at_{/\!/E} \end{tikzcd}$$

Now the quasi-category B defines a 1-complicial set by Theorem D.4.13. Thus, by Lemma 19.2.1, when $q\colon F \twoheadrightarrow E$ is an isofibration, then $\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow B$ is a complicial isofibration, which implies that $\tilde{\Pi}_p F$ is a 2-complicial set. We argue that in fact all 1-simplices in $\tilde{\Pi}_p F$ are marked. By the defining universal property, a 2-simplex in $\tilde{\Pi}_p F$ corresponds to a pair comprised of a 2-simplex in B and a 2-simplex in $\mathfrak{N}\mathcal{QC}at_{/\!/F}$, the latter of which corresponds to a 3-simplex in $\mathfrak{N}\mathcal{QC}at$. Since all 2-simplices in B and all 3-simplices in $\mathfrak{N}\mathcal{QC}at$ are marked, we see that all 2-simplices in $\tilde{\Pi}_p F$ are marked. Thus $\tilde{\Pi}_p F$ is a 1-complicial set, and from Theorem D.4.13 we can conclude that the underlying simplicial set $\tilde{\Pi}_p F$ is a quasi-category, and now the complicial isofibration $\tilde{\Pi}_p q$ becomes an isofibration between quasi-categories. □

19.2.3. Corollary. *When $p\colon E \twoheadrightarrow B$ is a cocartesian fibration between quasi-categories, the right adjoint functor $\tilde{\Pi}_p\colon s\mathcal{S}et_{/E} \to s\mathcal{S}et_{/B}$ restricts to a functor $\tilde{\Pi}_p\colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$ that preserves isofibrations.*

Proof. Proposition 19.2.2 demonstrates that $\tilde{\Pi}_p$ carries isofibrations to isofibrations, restricting to define a functor $\tilde{\Pi}_p\colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$. Moreover this functor preserves isofibrations, now considered as morphisms in these slice categories, since the action of $\tilde{\Pi}_p$ on an isofibration $u\colon G \twoheadrightarrow F$ over E is defined by pulling back the complicial isofibration $u\circ -\colon \mathfrak{N}\mathcal{QC}at_{/\!/G} \twoheadrightarrow \mathfrak{N}\mathcal{QC}at_{/\!/F}$. □

It would be tempting to claim that the conical limit-preserving functor $\tilde{\Pi}_p\colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$ is cosmological, but this functor fails to be simplicially enriched. However, Theorem 19.2.8 mitigates this disappointment. As a first step towards that result, we transfer the properties of the functor $\tilde{\Pi}_p$ to the right adjoint $\Pi_p\colon s\mathcal{S}et_{/E} \to s\mathcal{S}et_{/B}$ to the strict pullback functor $p^*\colon s\mathcal{S}et_{/B} \to s\mathcal{S}et_{/E}$.

19.2.4. Proposition. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration, then the adjunctions*

$$\begin{tikzcd} s\mathcal{S}et_{/E} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B} \arrow[l, bend right, "p^*"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} s\mathcal{S}et_{/E} \arrow[r, bend right, "\tilde{\Pi}_p"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B} \arrow[l, bend right, "\tilde{p}^*"'] \end{tikzcd}$$

are Quillen with respect to the sliced Joyal model structure.

Consequently, the natural Joyal equivalence $\gamma\colon \tilde{p}^* \Rightarrow p$ of Theorem 19.1.8, which defines a natural isomorphism of total left derived functors, transposes to a natural equivalence $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$, which defines a natural isomorphism of total right derived functors.

Proof. By an observation of Joyal and Tierney [**50**, 7.15], to show that $\tilde{p}^* \dashv \tilde{\Pi}_p$ is Quillen it suffices to show that the left adjoint preserves cofibrations and the right adjoint preserves fibrations

between fibrant objects. Lemma 19.1.17 demonstrates the first of these and Corollary 19.2.3 proves the second.

To prove that $p^* \dashv \Pi_p$ is Quillen, we prove that p^* is left Quillen. Thus functor preserves cofibrations because pullbacks preserve monomorphisms. By Theorem 19.1.8, p^* is naturally weakly equivalent to the left Quillen functor $\tilde{p}^*$. Since all objects in $s\mathcal{S}et_{/E}$ are cofibrant, the left Quillen functor $\tilde{p}^*$ preserves all Joyal weak equivalences, and hence by the 2-of-3 property p^* does as well. $\square$

We now consider the actions of the pushforward functors $\tilde{\Pi}_p$ and Π_p along a cocartesian fibration $p\colon E \twoheadrightarrow B$ when applied to a cartesian fibration $q\colon F \twoheadrightarrow E$. As before, we demonstrate directly that $\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow B$ is a then a cartesian fibration and then use Theorem 19.1.8 to conclude the same for Π_p.

19.2.5. Lemma. *Let $q\colon F \twoheadrightarrow E$ be a cartesian fibration in an ∞-cosmos $\mathcal{K}$. Then the correspond map $q \circ -\colon \mathfrak{N}\mathcal{K}_{/\!/F} \twoheadrightarrow \mathfrak{N}\mathcal{K}_{/\!/E}$ of 2-complicial sets has the right lifting property with respect to any outer horn inclusion*

$$\begin{tikzcd} \Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d, hook] & \mathfrak{N}\mathcal{K}_{/\!/F} \arrow[d, two heads, "q\circ -"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathfrak{N}\mathcal{K}_{/\!/E} \end{tikzcd}$$

whose final edge defines a cartesian 1-arrow

$$\begin{tikzcd} E_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, "f_{n-1,n-2}"'] & \arrow[d, phantom, "\Downarrow\chi"] & F \\ & E_n \arrow[ur, "f_n"'] & \end{tikzcd}$$

for the cartesian fibration $q \circ -\colon \mathsf{Fun}(E_{n-1}, F) \to \mathsf{Fun}(E_{n-1}, E)$.

Proof. As in the proof of Lemma 19.2.1, the bottom horizontal functor is given by a homotopy coherent $n+1$-simplex $\mathfrak{C}\Delta[n+1] \to \mathcal{K}$ that sends the first $n+1$ objects to $E_0, \ldots E_n$ and the final object to E, while the δ^{n+1}-face of this homotopy coherent simplex and the top horizontal functor together define a simplicial functor $\mathfrak{C}\Lambda^n[n+1] \to \mathcal{K}$ that caries the $n+2$ objects to $E_0, \ldots, E_n, F$ and has the property that for each $0 \le j \le n$ the diagram of function complexes

$$\begin{tikzcd} \mathfrak{C}\Lambda^n[n+1](j, n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_j, E) \arrow[d, two heads, "q\circ -"] \\ \mathfrak{C}\Delta[n+1](j, n+1) \arrow[r] & \mathsf{Fun}(E_j, F) \end{tikzcd}$$

commutes.

By Lemma 6.3.7, to solve the original lifting problem we need only construct a single lift

$$\begin{tikzcd} \sqcap^{n,n}_1 \cong \mathfrak{C}\Lambda^n[n+1](0, n+1) \arrow[r] \arrow[d, hook] & \mathsf{Fun}(E_0, E) \arrow[d, two heads, "q\circ -"] \\ \square^n \cong \mathfrak{C}\Delta[n+1](0, n+1) \arrow[r] \arrow[ur, dashed] & \mathsf{Fun}(E_0, F) \end{tikzcd}$$

This extension problem can be solved by filling inner horns and "special outer horns" $\Lambda^m[m] \to \Delta[m]$, those whose final edges are complies of the 1-simplex $\chi \in \mathsf{Fun}(E_{n-1}, F)$ pre-composed with some

functor $E_0 \to E_{n-1}$. Such 1-simplices represent $(q \circ -)$-cartesian cells so these special outer horn lifting problems also admit solutions by Proposition F.4.6(ii). $\square$

19.2.6. PROPOSITION. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow E$ is a cartesian fibration between quasi-categories, then*

$$\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow B$$

is a cartesian fibration between quasi-categories. Moreover, $\tilde{\Pi}_p$ preserves cartesian functors, restricting to define a functor

$$\tilde{\Pi}_p\colon \mathcal{C}art(\mathcal{Q}\mathcal{C}at)_{/E} \to \mathcal{C}art(\mathcal{Q}\mathcal{C}at)_{/B}.$$

PROOF. On account of Proposition 19.2.2, we are free use the "analytic" characterization of cartesian fibrations between quasi-categories of Proposition F.4.1(v). By Lemma 19.2.5, it remains only to show that any 1-arrow $b\colon \Delta[1] \to B$ admits a lift $\chi\colon \Delta[1] \to \tilde{\Pi}_p F$ with specified codomain whose image in $\mathfrak{N}\mathcal{K}_{/\!/F}$ is a $(q \circ -)$-cartesian 1-arrow. By Lemma 5.3.5, this is the same as saying that χ is a q-cartesian 1-arrow.

Recall from the construction of Proposition 19.2.2, that a 1-simplex $\chi\colon \Delta[1] \to \tilde{\Pi}_p F$ in the fiber over $b\colon \Delta[1] \to B$ corresponds to a 1-arrow

$$\begin{tikzcd} E_0 \arrow[rr, "f_0"] \arrow[dr, "e_b"'] & \arrow[d, phantom, "\Downarrow\chi"] & F \\ & E_1 \arrow[ur, "f_1"'] & \end{tikzcd}$$

in $\mathsf{Fun}(E_0, F)$ that whiskers with $q\colon F \to E$ to define the lax cocone that restricts the lax cocone associated to the comprehension construction along b. As observed previously, this compatibility condition tells us that the quasi-categories E_0 and E_1 are the fibers of $p\colon E \twoheadrightarrow B$ over the vertices in b and the functor $e_b\colon E_0 \to E_1$ is the comprehension of b. To form such a lift with codomain $f_1\colon E_1 \to F$, start by lifting b to the lax cocone

$$\begin{tikzcd} E_0 \arrow[rr, "\ell_0^E"] \arrow[dr, "e_b"'] & \arrow[d, phantom, "\Downarrow\epsilon"] & E \\ & E_1 \arrow[ur, "\ell_1^E"'] & \end{tikzcd}$$

under $e_b\colon E_0 \to E_1$ with nadir E associated with the comprehension construction, as displayed in (19.1.21). Since f_1 is in the fiber of $\tilde{\Pi}_p q\colon \tilde{\Pi}_p F \twoheadrightarrow B$ over the codomain of b, we must have $qf_1 = \ell_1^E$. Now we can lift ϵ along the cartesian fibration q to a q-cartesian cell with codomain $f \circ e_b$. This defines the $(q \circ -)$-cartesian cell χ.

Now if $u\colon G \to F$ is a cartesian functor from $r\colon G \twoheadrightarrow E$ to $q\colon F \twoheadrightarrow E$, then u is representabily cartesian in the sense that $u \circ -\colon \mathsf{Fun}(X, G) \to \mathsf{Fun}(X, F)$ carries $(r \circ -)$-cartesian 1-arrows to $(q \circ -)$-cartesian 1-arrows. Since $u \circ -\colon \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/G} \to \mathfrak{N}\mathcal{Q}\mathcal{C}at_{/\!/F}$ preserves the cartesian 1-arrows just identified, proving that $\tilde{\Pi}_p$ carries this map to a cartesian functor between cartesian fibrations over B. $\square$

19.2.7. COROLLARY. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow E$ is a cartesian fibration between quasi-categories, then*

$$\Pi_p q\colon \Pi_p F \twoheadrightarrow B$$

is a cartesian fibration between quasi-categories. Moreover, Π_p preserves cartesian functors, restricting to define a functor

$$\mathcal{C}art(\mathcal{QC}at)_{/E} \xrightarrow{\Pi_p} \mathcal{C}art(\mathcal{QC}at)_{/B}.$$

Proof. By Proposition 19.2.4, the components

$$\begin{tikzcd} \Pi_p F \arrow[rr, "\hat{\gamma}_q", "\sim"'] \arrow[dr, "\Pi_p q"', two heads] & & \tilde{\Pi}_p F \arrow[dl, "\tilde{\Pi}_p q", two heads] \\ & B & \end{tikzcd}$$

at an isofibration $q\colon F \twoheadrightarrow E$ of the transpose $\hat{\gamma}\colon \Pi_p \Rightarrow \tilde{\Pi}_p$ of the natural weak equivalence of Theorem 19.1.8 are equivalences of isofibrations over B. If q is a cartesian fibration, then Proposition 19.2.6 proves that $\tilde{\Pi}_p q$ is a cartesian fibration, so by Corollary 5.1.17, $\Pi_p q$ must be as well. □

19.2.8. Theorem. *For a cocartesian fibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the pullback-pushforward adjunction restricts to define an adjunction*

$$\begin{tikzcd} \mathcal{QC}at_{/E} \arrow[rr, bend right, "\Pi_p"'] \arrow[rr, phantom, "\perp"] & & \mathcal{QC}at_{/B} \arrow[ll, bend right, "p^*"'] \\ \mathcal{C}art(\mathcal{QC}at)_{/E} \arrow[u, hook] \arrow[rr, bend right, "\Pi_p"'] \arrow[rr, phantom, "\perp"] & & \mathcal{C}art(\mathcal{QC}at)_{/B} \arrow[u, hook] \arrow[ll, bend right, "p^*"'] \end{tikzcd}$$

and moreover all of these functors are cosmological.

Proof. By Proposition 19.2.4, the adjoint functors $p^* \dashv \Pi_p$ define an adjunction

$$\begin{tikzcd} \mathcal{QC}at_{/E} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{QC}at_{/B} \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

By Proposition 5.1.20, the left adjoint restricts to a cosmological functor

$$p^*\colon \mathcal{C}art(\mathcal{QC}at)_{/B} \to \mathcal{C}at(\mathcal{QC}at)_{/E}.$$

By Corollary 19.2.7, the right adjoint also restricts to a functor $\Pi_p\colon \mathcal{C}art(\mathcal{QC}at)_{/E} \to \mathcal{C}art(\mathcal{QC}at)_{/B}$. Since the inclusion $\mathcal{C}art(\mathcal{QC}at)_{/B} \hookrightarrow \mathcal{QC}at_{/B}$ is not full, this isn't quite enough to demonstrate adjointness of the restricted adjunction: it remains to argue that the adjoint transpose of a cartesian functor is a cartesian functor.

To that end, let $q\colon F \twoheadrightarrow E$ and $r\colon G \twoheadrightarrow B$ be cartesian fibrations. A functor $f\colon G \to \tilde{\Pi}_p q$ over B is cartesian if and only if the square

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "r"', two heads] & \mathfrak{N}\mathcal{QC}at_{/\!/F} \arrow[d, "q\circ-", two heads] \\ B \arrow[r, "\ell^E"'] & \mathfrak{N}\mathcal{QC}at_{/\!/E} \end{tikzcd}$$

carries r-cartesian arrows to representably q-cartesian arrows in $\mathfrak{R}\mathcal{QC}at_{/\!/F}$, as described in Lemma 19.2.5. Fixing an 1-arrow arrow $\zeta\colon \Delta[1] \to G$ over $b\colon \Delta[1] \to B$ as below-left, the arrow $f\zeta$ transposes to the functor over E displayed below-right

$$\begin{tikzcd}
\Delta[1] \arrow[r, "f\zeta"] \arrow[d, "b"'] & \mathfrak{R}\mathcal{QC}at_{/\!/F} \arrow[d, two heads, "q\circ-"] \\
B \arrow[r, "\ell^E"'] & \mathfrak{R}\mathcal{QC}at_{/\!/E}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\operatorname{colim}_{\text{oplax}}(c_p \circ \mathfrak{C}b) \arrow[rr, "\widehat{f\zeta}"] \arrow[dr, "\ell^E|_b"'] & & F \arrow[dl, "q"] \\
& E &
\end{tikzcd}$$

By Proposition 19.1.20, the oplax colimit is equivalent to the fiber E_b and the functor $\widehat{f\zeta}$ represents the whiskered lax cocone

$$\begin{tikzcd}
E_{b_0} \arrow[drr, bend left, "\ell_0^E"] \arrow[ddd, "p_0"'] \arrow[ddr, dotted, "e_b"'] & & & F \arrow[dr, "q"] & \\
& & E_b \arrow[ur, "\widehat{f\zeta}"] \arrow[r] \arrow[ddd, two heads, "p_b"] & E_r \arrow[u, "\hat{f}"'] \arrow[r] \arrow[ddd, two heads] & E \arrow[ddd, two heads, "p"] \\
& E_{b_1} \arrow[ur, bend right, "\ell_1^E"'] \arrow[ddd, two heads, "p_1"'] \arrow[u, phantom, "\Leftarrow\chi"] & & & \\
1 \arrow[drr, bend left, "0"] \arrow[ddr, equal] & & & & \\
& & \Delta[1] \arrow[r, "\zeta"] \arrow[rr, bend right, "b"'] & G \arrow[r, two heads, "r"] & B \\
& 1 \arrow[ur, bend right, "1"'] \arrow[u, phantom, "\Leftarrow\kappa"] & & &
\end{tikzcd}$$

Now f is a cartesian functor if and only if the whiskered composite $\widehat{f\zeta}\chi$ is q-cartesian whenever ζ is r-cartesian. Since Proposition 5.1.20 demonstrates that cartesian arrows are created by pullbacks, this proves that f is a cartesian functor if and only if the transposed functor

$$\begin{tikzcd}
E_r \arrow[rr, "\hat{f}"] \arrow[dr, two heads, "p^*r"'] & & F \arrow[dl, two heads, "q"] \\
& E &
\end{tikzcd}$$

is cartesian.

The functor $\Pi_p\colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$ is a restriction of a Quillen right adjoint, so to prove that this functor is cosmological, it remains only to demonstrate that the adjoint

$$\begin{tikzcd}
s\mathcal{S}et_{/E} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B} \arrow[l, bend right, "p^*"']
\end{tikzcd}$$

is simplicially enriched, which we do by arguing that the left adjoint preserves tensors with simplicial sets U. The tensor of $b\colon X \to B$ with U in $s\mathcal{S}et_{/B}$ is the right-hand vertical composite which pulls

back to the left-hand vertical composite

$$\begin{tikzcd} F \times U \arrow[r, two heads] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times U \arrow[d, "\pi"] \\ F \arrow[r, two heads] \arrow[d, "p^*(b)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "b"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

to define the tensor of $p^*(b)\colon F \to E$ with U in $s\mathcal{S}et_{/E}$. By Proposition 8.2.12, the further restriction $\Pi_p\colon \mathcal{C}art(\mathcal{QC}at)_{/E} \to \mathcal{C}art(\mathcal{QC}at)_{/B}$ is also cosmological. □

We can reinterpret the argument just given to obtain a characterization of the $\Pi_p q$-cartesian 1-arrows in the cartesian fibration constructed from a cocartesian fibration $p\colon E \twoheadrightarrow B$ and a cartesian fibration $q\colon F \twoheadrightarrow E$ between quasi-categories that lift a specified arrow $\beta\colon \mathbb{2} \to B$.

19.2.9. Lemma. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow B$ is a cartesian fibration between quasi-categories, then the cartesian 1-arrows χ in $\Pi_p q\colon \Pi_p F \twoheadrightarrow B$ are those maps that transpose to define functors*

$$\begin{tikzcd} \mathbb{2} \arrow[rr, "\chi"] \arrow[dr, "\beta"'] & & \Pi_p F \arrow[dl, two heads, "\Pi_p q"] \\ & B & \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} & F \arrow[d, two heads, "q"] \\ E_\beta \arrow[ur, "\hat{\chi}"] \arrow[r, "\ell_\beta"] \arrow[d, two heads, "p_\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ \mathbb{2} \arrow[r, "\beta"'] & B \end{tikzcd}$$

that carry p-cocartesian lifts of β to q-cartesian lifts of β.

Proof. By Theorem 19.1.8, E_β may be identified with the oplax colimit of the canonical lax cocone formed by taking a p-cocartesian lift of β. From this perspective, the transposed functor $\hat{\chi}\colon E_\beta \to F$ acts by whiskering this p-cocartesian arrow. By the construction in the proof of Theorem 19.2.8, the $\Pi_p q$-cartesian lifts of β are those arrows for which this whiskered composite is q-cartesian, as claimed. □

19.3. Pullback and pushforward along a cocartesian fibration

In this section, we explain how the results of the previous two sections dualize to the case of a *cartesian* fibration $p\colon E \twoheadrightarrow B$ between quasi-categories. There are two approaches to defining the canonical cocone under comprehension functor $c_p\colon \mathfrak{C}B^{\mathrm{op}} \to \mathcal{QC}at^{\mathrm{co}}$ whose results are equivalent. The first, adopted in, the proof of Theorem 17.4.7 is to regard p as a cocartesian fibration in the dual ∞-cosmos $\mathcal{QC}at^{\mathrm{op}}$. Under that interpretation, the diagram (17.4.3) has the form

$$\begin{tikzcd} & \mathfrak{C}\emptyset^{\triangleright} \arrow[r, "p"] \arrow[d, hook] & co\mathcal{C}art(\mathcal{QC}at^{\mathrm{co}}) \arrow[r, "\mathrm{dom}"] \arrow[d, "\mathrm{cod}"] & \mathcal{QC}at^{\mathrm{co}} \\ \mathfrak{C}B^{\mathrm{op}} \arrow[r, hook] & \mathfrak{C}(B^{\mathrm{op}})^{\triangleright} \arrow[r, "k^B"'] \arrow[ur, dashed] & \mathcal{QC}at^{\mathrm{co}} & \end{tikzcd}$$

Note in particular that the "underlying quasi-category" of maps from $\mathbb{1}$ to B in $\mathcal{QC}at^{\mathrm{co}}$ is B^{op}. We introduce the following terminology to describe the data of the comprehension cocone $\ell^E \colon \mathfrak{C}(B^{\mathrm{op}})^{\triangleright} \to \mathcal{QC}at^{\mathrm{co}}$.

19.3.1. Definition (oplax cocones). For a simplicial set X and a simplicially enriched category $\mathcal{E}$, an **oplax cocone** of shape X is given by a simplicial functor

$$\begin{tikzcd} & \mathfrak{C}X^{\mathrm{co}} + \mathbb{1} \arrow[dl] \arrow[dr, "{\langle F,B\rangle}"] & \\ \mathfrak{C}(X^{\triangleright})^{\mathrm{co}} \arrow[rr, "\ell"'] & & \mathcal{E} \end{tikzcd}$$

The **base** of a lax cocone refers to the restricted homotopy coherent diagram $F \colon \mathfrak{C}X^{\mathrm{co}} \to \mathcal{E}$, while the object $B \colon \mathbb{1} \to \mathcal{E}$ defines its **nadir**.

19.3.2. Remark. It's necessary that the base homotopy coherent diagram of an oplax cocone has the form $F \colon \mathfrak{C}X^{\mathrm{co}} \to \mathcal{E}$. By the conventions of §6.3, the non-degenerate 1-arrow in $\mathfrak{C}\Delta[1]^{\triangleright}(0,\top) \cong \mathfrak{C}\Delta[2](0,2)$ defines an lax cocone under the 0-arrow from 0 to 1. In $(\mathfrak{C}\Delta[1]^{\triangleright})^{\mathrm{co}} \cong \mathfrak{C}\Delta[2]^{\mathrm{co}}$ the source and target of this 1-arrow are exchanged, so that it now defines an oplax cocone under the 0-arrow from 0 to 1.

19.3.3. Example. When $p \colon E \twoheadrightarrow B$ is a cartesian fibration, the cocartesian cocone constructed in (17.4.3) defines an oplax cocone

$$\begin{tikzcd} & \mathfrak{C}(B_0)^{\mathrm{op}\,\mathrm{co}} + \mathbb{1} \arrow[dl] \arrow[dr, "{\langle c_p,E\rangle}"] & \\ \mathfrak{C}((B_0)^{\mathrm{op}})^{\triangleright\,\mathrm{co}} \arrow[rr, "\ell^E"'] & & \mathcal{K} \end{tikzcd}$$

of shape $(B_0)^{\mathrm{op}}$, the opposite of the underlying quasi-category of the ∞-category $B \in \mathcal{K}$, whose base diagram is the comprehension functor and whose nadir is the ∞-category E, the domain of the cocartesian fibration.

In the ∞-cosmos of quasi-categories, the oplax cocone $\ell^E \colon \mathfrak{C}(B^{\mathrm{op}})^{\triangleright\,\mathrm{co}} \to \mathcal{QC}at$ can be arrived at in another way. By Proposition 16.1.8, if $p \colon E \twoheadrightarrow B$ is a cartesian fibration of quasi-categories, then $p^{\mathrm{op}} \colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ is a cocartesian fibration of quasi-categories. However, the comprehension functor

$$c_{p^{\mathrm{op}}} \colon \mathfrak{C}B^{\mathrm{op}} \to \mathcal{QC}at$$

does not agree with the comprehension functor c_p obtained from $\mathcal{QC}at^{\mathrm{co}}$. The functor $c_{p^{\mathrm{op}}}$ acts on objects by sending an element $b \colon \mathbb{1} \to B^{\mathrm{op}}$ to the fiber E_b^{op}, while c_p sends b to E_b. This deficiency is easily corrected by post-composing with the functor $(-)^{\mathrm{op}} \colon \mathcal{QC}at \to \mathcal{QC}at^{\mathrm{co}}$ that sends each quasi-category to its op-dual; see Remark 16.1.9. The commutative diagram

$$\begin{tikzcd} \mathfrak{C}B^{\mathrm{op}} \arrow[rr, "c_p"] \arrow[dr, "c_{p^{\mathrm{op}}}"'] & & \mathcal{QC}at^{\mathrm{co}} \\ & \mathcal{QC}at \arrow[ur, "(-)^{\mathrm{op}}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}(B^{\mathrm{op}})^{\triangleright} \arrow[rr, "\ell^E"] \arrow[dr, "\ell^{E^{\mathrm{op}}}"'] & & \mathcal{QC}at^{\mathrm{co}} \\ & \mathcal{QC}at \arrow[ur, "(-)^{\mathrm{op}}"'] & \end{tikzcd}$$

extends to oplax cocones.

We similarly define the co-dual $s\mathcal{S}et^{\mathrm{co}}$ of the simplicially enriched category $s\mathcal{S}et$ by defining the simplicial set of maps from X to Y to be $(Y^X)^{\mathrm{op}}$. The functor $(-)^{\mathrm{op}} \colon s\mathcal{S}et \to s\mathcal{S}et^{\mathrm{co}}$ defines an isomorphism of simplicially enriched categories. Recalling from Exercise 17.1.i that the universal *oplax*

cocone under a diagram defines that *lax* colimit, these observations allow us to obtain the following duals of the results of §19.1.

19.3.4. Theorem. *For any cartesian fibration $p\colon E \twoheadrightarrow B$ of quasi-categories and any map $b\colon X \to B$ of simplicial sets, the comprehension cocone induces a canonical map over E from the lax colimit of the diagram*

$$\mathfrak{C}(X^{\mathrm{op}})^{\mathrm{co}} \xrightarrow{\mathfrak{C}(b^{\mathrm{op}})^{\mathrm{co}}} \mathfrak{C}(B^{\mathrm{op}})^{\mathrm{co}} \xrightarrow{c_p} \mathcal{QC}at$$

to the fiber

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "b"'] & B \end{tikzcd}$$

and this map is a natural equivalence in the Joyal model structure. ☐

Specializing to $b = \mathrm{id}_B$:

19.3.5. Corollary. *The domain of a cartesian fibration $p\colon E \twoheadrightarrow B$ of quasi-categories is equivalent to the lax colimit of the associated comprehension functor $c_p\colon \mathfrak{C}B \to \mathcal{QC}at$, with*

$$\begin{tikzcd} & \mathfrak{C}(B^{\mathrm{op}})^{\mathrm{co}} + \mathbb{1} \arrow[dl] \arrow[dr, "\langle c_p, E\rangle"] & \\ \mathfrak{C}(B^{\mathrm{op}})^{\triangleright\,\mathrm{co}} \arrow[rr, "\ell^E"'] & & \mathcal{QC}at \end{tikzcd}$$

as the associated colimit cocone. ☐

The results of §19.2 dualize similarly, for instance by defining the functors $\Pi_{p^{\mathrm{op}}}$ and $\tilde{\Pi}_{p^{\mathrm{op}}}$ for the cocartesian fibration $p^{\mathrm{op}}\colon E^{\mathrm{op}} \twoheadrightarrow B^{\mathrm{op}}$ and then pre- and post-composing with the involution $s\mathcal{S}et \cong s\mathcal{S}et^{\mathrm{co}}$, for instance as displayed by the adjunction:

$$\begin{tikzcd} s\mathcal{S}et_{/E} \arrow[r, leftrightarrow, "(-)^{\mathrm{op}}", "\cong"'] & (s\mathcal{S}et_{/E^{\mathrm{op}}})^{\mathrm{co}} \arrow[r, bend right, "\Pi_{p^{\mathrm{op}}}"'] \arrow[r, phantom, "\perp"] & (s\mathcal{S}et_{/B^{\mathrm{op}}})^{\mathrm{co}} \arrow[l, bend right, "(p^{\mathrm{op}})^*"'] \arrow[r, leftrightarrow, "(-)^{\mathrm{op}}", "\cong"'] & s\mathcal{S}et_{/B} \end{tikzcd}$$

Note that the composite right adjoint $\Pi_p\colon s\mathcal{S}et_{/E} \to s\mathcal{S}et_{/B}$ carries a cocartesian fibration $q\colon F \twoheadrightarrow E$ over E to a cartesian fibration over E^{op}, to a cartesian fibration over B^{op}, by Corollary 19.2.7, and then finally to a cocartesian fibration over B. This proves:

19.3.6. Corollary. *If $p\colon E \twoheadrightarrow B$ is a cartesian fibration and $q\colon F \twoheadrightarrow E$ is a cocartesian fibration between quasi-categories, then*

$$\Pi_p q\colon \Pi_p F \twoheadrightarrow B$$

is a cocartesian fibration between quasi-categories. Moreover, Π_p preserves cartesian functors, restricting to define a functor

$$co\mathcal{C}art(\mathcal{QC}at)_{/E} \xrightarrow{\Pi_p} co\mathcal{C}art(\mathcal{QC}at)_{/B}. \quad ☐$$

From this, we again conclude:

19.3.7. THEOREM. *For a cartesian fibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the pullback-pushforward adjunction restricts to define an adjunction*

$$\begin{tikzcd} \mathcal{QC}at_{/E} \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{QC}at_{/B} \arrow[l, bend right, "p^*"'] \\ co\mathcal{C}art(\mathcal{QC}at)_{/E} \arrow[u, hook] \arrow[r, bend right, "\Pi_p"'] \arrow[r, phantom, "\perp"] & co\mathcal{C}art(\mathcal{QC}at)_{/B} \arrow[u, hook] \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

and moreover all of these functors are cosmological. □

19.4. Exponentiation

For efficiency's sake, before deducing some consequences of our work, we extend it to arbitrary ∞-cosmoi of $(\infty, 1)$-categories, using the techniques developed in Chapter 16.

19.4.1. THEOREM. *Let $p\colon E \twoheadrightarrow B$ be a cartesian or cocartesian fibration in an ∞-cosmos of $(\infty, 1)$-categories $\mathcal{K}$. Then the pullback functor admits a quasi-pseudofunctorial right adjoint Π_p, defining the* ***pushforward*** *of an isofibration along p.*

$$\begin{tikzcd} \mathcal{K}_{/E} \arrow[r, bend right, squiggly, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{K}_{/B} \arrow[l, bend right, "p^*"'] \end{tikzcd} \qquad\qquad \mathsf{Fun}_E(p^*(r), q) \simeq \mathsf{Fun}_B(r, \Pi_p q)$$

PROOF. The pushforward functor $\Pi_p\colon \mathcal{K}_{/E} \rightsquigarrow \mathcal{K}_{/B}$ is defined as the composite of the pushforward associated to the underlying co/cartesian fibration $p_0\colon E_0 \twoheadrightarrow B_0$ between quasi-categories with the zig-zag of cosmological biequivalences

$$\begin{tikzcd} \mathcal{K}_{/E} \arrow[r, dashed, "\Pi_p"] \arrow[d, "(-)_0"', "\wr"] & \mathcal{K}_{/B} \arrow[d, "(-)_0", "\wr"'] \\ \mathcal{QC}at_{/E_0} \arrow[r, "\Pi_{p_0}"'] & \mathcal{QC}at_{/B_0} \end{tikzcd} \tag{19.4.2}$$

Note that both pullback and the pushforward commute up to equivalence with the sliced underlying quasi-category functors $(-)_0\colon \mathcal{K}_{/B} \to \mathcal{QC}at_{/B_0}$.

The composite quasi-pseudofunctor can be understood as a pointwise right biadjoint, satisfying the quasi-categorical analog of Definition 16.2.19, on account of the composite equivalences

$$\mathsf{Fun}_E(p^*(r), q) \xrightarrow[\sim]{(-)_0} \mathsf{Fun}_{E_0}((p_0)^*(r_0), q_0) \cong \mathsf{Fun}_{B_0}(r_0, \Pi_{p_0}(q_0)) \xleftarrow[\sim]{(-)_0} \mathsf{Fun}_B(r, \Pi_p q)$$

that are quasi-pseudonaturally defined in $q\colon F \twoheadrightarrow E$ and $r\colon G \twoheadrightarrow B$. □

19.4.3. THEOREM. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration in an ∞-cosmos of $(\infty,1)$-categories, then Π_p preserves cartesian fibrations and cartesian functors and restricts to define a quasi-pseudofunctorial right adjoint:*

$$\begin{tikzcd} \mathcal{K}_{/E} \arrow[r, squiggly, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{K}_{/B} \arrow[l, bend right, "p^*"'] \\ \mathcal{C}art(\mathcal{K})_{/E} \arrow[u, hook] \arrow[r, squiggly, "\Pi_p"'] \arrow[r, phantom, "\perp"] & \mathcal{C}art(\mathcal{K})_{/B} \arrow[u, hook] \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

Dually, if $p\colon E \twoheadrightarrow B$ is a cartesian fibration, then Π_p preserves cocartesian fibrations and cartesian functors and restricts to a quasi-pseudofunctorial right adjoint to $p^*\colon co\mathcal{C}art(\mathcal{K})_{/B} \to co\mathcal{C}art(\mathcal{K})_{/E}$.

PROOF. Let $\mathcal{K}$ be an ∞-cosmos of $(\infty,1)$-categories and let $p\colon E \twoheadrightarrow B$ be a cocartesian fibration and let $q\colon F \twoheadrightarrow E$ be a cartesian fibration in $\mathcal{K}$. Then by Proposition 14.3.5, $\Pi_p q$ is a cartesian fibration in $\mathcal{K}$ if and only if its underlying functor is a cartesian fibration on quasi-categories. By the essential commutativity of (19.4.2), this functor is equivalent to $\Pi_{p_0}(q_0)$, which is a cartesian fibration of quasi-categories by Corollary 19.2.7. Since cartesian fibrations are replete up to equivalence, the preservation result follows. Finally, Theorem 19.2.8 proves that cartesian functors between cartesian fibrations of quasi-categories transpose to cartesian functors under the $p^* \dashv \Pi_p$. Since cartesian functors are preserved and reflected by cosmological biequivalences, the same holds for any ∞-cosmos of $(\infty,1)$-categories. The proof that pushforward preserves cartesian functors is similar. $\square$

19.4.4. LEMMA (Beck-Chevalley). *Let*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

be a pullback diagram in an ∞-cosmos of $(\infty,1)$-categories, and suppose that p is a cocartesian fibration, or cartesian fibration. Then for all $r\colon G \twoheadrightarrow E$, the maps $f^*\Pi_p r \simeq \Pi_q g^* r$ are equivalent over A.

PROOF. Applying the cosmological biequivalence $(-)_0\colon \mathcal{K} \rightsquigarrow \mathcal{QC}at$ it suffices to prove this for quasi-categories, in which case the functors in question have left adjoints:

$$\begin{tikzcd} \mathcal{K}_{/F} \arrow[d, squiggly, "\Pi_q"'] & \mathcal{K}_{/E} \arrow[l, "g^*"'] \arrow[d, squiggly, "\Pi_p"] \\ \mathcal{K}_{/A} & \mathcal{K}_{/B} \arrow[l, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{QC}at_{/F_0} \arrow[d, "\Pi_{q_0}"'] & \mathcal{QC}at_{/E_0} \arrow[l, "g_0^*"'] \arrow[d, "\Pi_{p_0}"] \\ \mathcal{QC}at_{/A_0} & \mathcal{QC}at_{/B_0} \arrow[l, "f_0^*"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et_{/F_0} \arrow[r, bend right, "\Sigma_{g_0}"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\Pi_{q_0}"'] \arrow[d, phantom, "\vdash"] & s\mathcal{S}et_{/E_0} \arrow[l, bend right, "g_0^*"'] \arrow[d, bend left, "\Pi_{p_0}"] \arrow[d, phantom, "\dashv"] \\ s\mathcal{S}et_{/A_0} \arrow[u, bend right, "q_0^*"'] \arrow[r, bend left, "\Sigma_{f_0}"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_{/B_0} \arrow[l, bend left, "f_0^*"] \arrow[u, bend left, "p_0^*"] \end{tikzcd}$$

For any map of simplicial sets $a\colon X \to A_0$, by examining the pullback rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & F_0 \arrow[r, "g_0"] \arrow[d, two heads, "q_0"'] \arrow[dr, phantom, "\lrcorner" very near start] & E_0 \arrow[d, two heads, "p_0"] \\ X \arrow[r, "a"'] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

it is clear that $\Sigma_{g_0} q_0^* a \cong p_0^* \Sigma_{f_0} a$. Thus, the right adjoints $f_0^*\Pi_{p_0} \cong \Pi_{q_0} g_0^*$ also commute up to isomorphism, and conservativity provides the claimed equivalence in $\mathcal{K}$. □

Proof. Applying the cosmological biequivalence $(-)_0\colon \mathcal{K} \rightleftharpoons \mathcal{Q}Cat$ it suffices to prove this for quasi-categories, in which case the functors in question have left adjoints:

$$\begin{tikzcd} \mathcal{K}_{/F} \arrow[d, squiggly, "\Pi_q"'] & \mathcal{K}_{/E} \arrow[l, "g^*"'] \arrow[d, squiggly, "\Pi_p"] \\ \mathcal{K}_{/A} & \mathcal{K}_{/B} \arrow[l, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Q}Cat_{/F_0} \arrow[d, "\Pi_{q_0}"'] & \mathcal{Q}Cat_{/E_0} \arrow[l, "g_0^*"'] \arrow[d, "\Pi_{p_0}"] \\ \mathcal{Q}Cat_{/A_0} & \mathcal{Q}Cat_{/B_0} \arrow[l, "f_0^*"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et_{/F_0} \arrow[r, bend right, "\Sigma_{g_0}"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\Pi_{q_0}"'] \arrow[d, phantom, "\vdash"] & s\mathcal{S}et_{/E_0} \arrow[l, bend right, "g_0^*"'] \arrow[d, bend right, "\Pi_{p_0}"'] \arrow[d, phantom, "\dashv"] \\ s\mathcal{S}et_{/A_0} \arrow[u, bend right, "q_0^*"'] \arrow[r, bend left, "\Sigma_{f_0}"] \arrow[r, phantom, "\bot"] & s\mathcal{S}et_{/B_0} \arrow[u, bend right, "p_0^*"'] \arrow[l, bend left, "f_0^*"] \end{tikzcd}$$

For any map of simplicial sets $a\colon X \to A_0$, by examining the pullback rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & F_0 \arrow[r, "g_0"] \arrow[d, two heads, "q_0"'] \arrow[dr, phantom, "\lrcorner" very near start] & E_0 \arrow[d, two heads, "p_0"] \\ X \arrow[r, "a"'] & A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

it is clear that $\Sigma_{g_0} q_0^* a \cong p_0^* \Sigma_{f_0} a$. Thus, the right adjoints $f_0^*\Pi_{p_0} \cong \Pi_{q_0} g_0^*$ also commute up to isomorphism, and conservativity provides the claimed equivalence in $\mathcal{K}$. □

Recall that an isofibration $p\colon E \twoheadrightarrow B$ is called *exponentiable* if the pullback functor $p^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/E}$ admits a right adjoint Π_p. Theorem 19.4.1 proves that cartesian and cocartesian fibrations between $(\infty,1)$-categories are exponentiable in this sense. The following construction justifies this terminology.

19.4.5. Definition. Let $p\colon E \twoheadrightarrow B$ be either a cartesian or a cocartesian fibration in an ∞-cosmos $\mathcal{K}$ of $(\infty,1)$-categories and let $q\colon F \twoheadrightarrow B$ be any isofibration. Then define the exponential

$$(F \overset{q}{\twoheadrightarrow} B)^{(E \overset{p}{\twoheadrightarrow} B)} \quad \in \quad \mathcal{K}_{/B}$$

to be the image of q under the composite functor

$$\mathcal{K}_{/B} \xrightarrow{p^*} \mathcal{K}_{/E} \overset{\Pi_p}{\rightsquigarrow} \mathcal{K}_{/B}$$

19.4.6. Lemma. *Let $p\colon E \twoheadrightarrow B$, $q\colon F \twoheadrightarrow B$, and $r\colon G \twoheadrightarrow B$ be isofibrations in an ∞-cosmos of $(\infty,1)$-categories and suppose p and r are either cocartesian fibrations or cartesian fibrations or one of each. Then there exist quasi-pseudonatural equivalences*

$$\mathsf{Fun}_B(G \twoheadrightarrow B, (F \twoheadrightarrow B)^{E \twoheadrightarrow B}) \simeq \mathsf{Fun}_B(E \times_B G \twoheadrightarrow B, F \twoheadrightarrow B) \simeq \mathsf{Fun}_B(E \twoheadrightarrow B, (F \twoheadrightarrow B)^{G \twoheadrightarrow B})$$

the left- and right-hand of which do not require r or p, respectively, to be a co/cartesian fibration.

PROOF. Starting from the left under the hypothesis that p is a cocartesian or cartesian fibration, by Theorem 19.4.1 we have

$$\mathsf{Fun}_B(G \twoheadrightarrow B, (F \twoheadrightarrow B)^{E \twoheadrightarrow B}) := \mathsf{Fun}_B(r, \Pi_p p^* q) \simeq \mathsf{Fun}_E(p^* r, p^* q) \cong \mathsf{Fun}_B(E \times_B G \twoheadrightarrow B, F \twoheadrightarrow B),$$

this last isomorphism a consequence of the simplicial universal property of the pullback in $\mathcal{K}$. □

19.4.7. PROPOSITION. *If $p\colon E \twoheadrightarrow B$ is a cocartesian fibration and $q\colon F \twoheadrightarrow B$ is a cartesian fibration in an ∞-cosmos of $(\infty,1)$-categories, then the exponential q^p is a cartesian fibration whose cartesian 1-arrows are those maps*

$$\begin{tikzcd} \mathbb{2} \arrow[rr, "\chi"] \arrow[dr, "\beta"'] & & (F \overset{q}{\twoheadrightarrow} B)^{(E \overset{p}{\twoheadrightarrow} B)} \arrow[dl] \\ & B & \end{tikzcd}$$

that transpose to define functors

$$\begin{tikzcd} E_\beta \arrow[r] \arrow[d, two heads, "p_\beta"'] \arrow[rr, bend left, "\hat{\chi}"] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] & F \arrow[dl, two heads, "q"] \\ \mathbb{2} \arrow[r, "\beta"'] & B & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} E_\beta \arrow[rr, "\hat{\chi}"] \arrow[dr, two heads, "p_\beta"'] & & F_\beta \arrow[dl, two heads, "q_\beta"] \\ & \mathbb{2} & \end{tikzcd}$$

that carry p-cocartesian lifts of β to q-cartesian lifts of β.

PROOF. By Proposition 5.1.20 and Theorem 19.4.3, the functor $(-)^p\colon \mathcal{K}_{/B} \to \mathcal{K}_{/B}$ preserves cartesian fibrations. In the case of quasi-categories, the characterization of the q^p-cartesian arrows is a special case of Lemma 19.2.9. The general case follows by model-independence, which also provides an interpretation of the ∞-category $\mathbb{2}$. □

For later use, we note the following lemma, which makes use of the product-projection bifibration of Example 5.2.4.

19.4.8. LEMMA. *Let $q\colon F \twoheadrightarrow B$ be a cartesian fibration and let $p\colon E \twoheadrightarrow B$ be a cocartesian fibration in an ∞-cosmos of $(\infty,1)$-categories, and consider the product projection bifibration $\pi\colon A \times B \twoheadrightarrow B$. Then the equivalence*

$$\mathsf{Fun}_B(F \overset{q}{\twoheadrightarrow} B, (A \times B \twoheadrightarrow B)^{E \overset{p}{\twoheadrightarrow} B}) \simeq \mathsf{Fun}_B(E \overset{p}{\twoheadrightarrow} B, (A \times B \twoheadrightarrow B)^{F \overset{q}{\twoheadrightarrow} B})$$

restricts to an equivalence

$$\mathsf{Fun}^{\mathrm{cart}}_B(F \overset{q}{\twoheadrightarrow} B, (A \times B \twoheadrightarrow B)^{E \overset{p}{\twoheadrightarrow} B}) \simeq \mathsf{Fun}^{\mathrm{cart}}_B(E \overset{p}{\twoheadrightarrow} B, (A \times B \twoheadrightarrow B)^{F \overset{q}{\twoheadrightarrow} B})$$

between the functor complexes in $\mathcal{C}art(\mathcal{K})_{/B}$ and $co\mathcal{C}art(\mathcal{K})_{/B}$, which is to say that cartesian functors between teh cartesian fibrations on the left transpose to define cartesian functors between the cocartesian fibrations on the right.

PROOF. By adjunction, the data of a map over B

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr, "q"'] & & (A \times B \twoheadrightarrow B)^{E \overset{p}{\twoheadrightarrow} B} \arrow[dl, "\pi^p"] \\ & B & \end{tikzcd}$$

is given by a single functor $\hat{f}\colon F \times_B E \to A$. By Proposition 19.4.7, f is a cartesian functor if and only if for each q-cocartesian cell $\chi\colon \mathbb{2} \to F$ over $\beta\colon \mathbb{2} \to B$, the induced functor

$$\begin{tikzcd} E_\beta \arrow[r, "\hat{\chi}"] \arrow[d, two heads, "p_\beta"'] & A \times B \arrow[d, two heads, "\pi"] \\ \Delta[1] \arrow[r, "\beta"'] & B \end{tikzcd}$$

carroes p-cartesian cell $\gamma\colon \mathbb{2} \to E$ over β to π-cocartesian ones, these being those maps $\mathbb{2} \to A \times B$ whose component along the other projection $A \times B \to A$ is invertible. In summary, the functor f is cartesian if and only if for every cocartesian lift γ of β the composite morphism

$$\mathbb{2} \xrightarrow{\langle \chi, \gamma \rangle} F_\beta \times_{\mathbb{2}} E_\beta \hookrightarrow F \times_B E \xrightarrow{\hat{f}} A$$

represents an invertible arrow in A. The symmetry of this condition in $q\colon F \twoheadrightarrow B$ and $p\colon E \twoheadrightarrow B$ proves the claim. ☐

Exercises.

19.4.i. EXERCISE. Give a second proof of Lemma 19.4.4 that argues directly in the ∞-cosmos $\mathcal{K}$ by replacing f with an isofibration so that $f^*\colon \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ has a left adjoint.

19.4.ii. EXERCISE. Consider a cocartesian or cartesian fibration $p\colon E \twoheadrightarrow B$, an isofibration $r\colon G \twoheadrightarrow B$, and a functor $f\colon A \to B$ in an ∞-cosmos of $(\infty,1)$-categories. Show that

$$f^*((G \overset{r}{\twoheadrightarrow} B)^{E \overset{p}{\twoheadrightarrow} B}) \qquad \text{and} \qquad (f^*G \overset{f^*r}{\twoheadrightarrow} A)^{f^*E \overset{f^*p}{\twoheadrightarrow} A}$$

define equivalent isofibrations over A.

19.5. Co/monadicity of cartesian fibrations

The 0-skeleton of a quasi-category B defines the "underlying set of objects" $\mathrm{ob}\, B$, together with a canonical inclusion $\mathrm{ob}\, B \hookrightarrow B$. By Proposition 5.1.20, pulling back along the inclusion $\mathrm{ob}\, B \hookrightarrow B$ induces a forgetful functor

$$\begin{array}{ccc} \mathcal{C}art(\mathcal{QC}at)_{/B} & \longrightarrow & \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\, B} \cong \prod_{\mathrm{ob}\, B} \mathcal{QC}at \\ (E \overset{p}{\twoheadrightarrow} B) & \longmapsto & (E_b)_{b \in \mathrm{ob}\, B} \end{array}$$

whose codomain is isomorphic to the product of the quasi-categorically enriched categories of quasi-categories, a cartesian fibration over a set being simply an indexed family of quasi-categories. Our

aim in this section is to construct left and right adjoints and prove that this functor is monadic and comonadic in a suitable sense.

The adjoint functors are constructed as what we refer to as *biadjoint* functors of quasi-categorically enriched categories: that is, we construct quasi-categorically enriched functors

$$L, R\colon \mathcal{C}art(\mathcal{QC}at)_{/\,\mathrm{ob}\,B} \longrightarrow \mathcal{C}art(\mathcal{QC}at)_{/B}$$

together with natural equivalences of function complexes that encode the adjoint transpose correspondence. The right adjoint makes use of the exponentiation construction of §19.4 and the Yoneda lemma is used to prove biadjointness.

Recall from Definition 16.1.12 that any quasi-categorically enriched category $\mathcal{K}$ has an $(\infty,1)$-categorical core that presents the **underlying quasi-category** of the quasi-categorically enriched category, defined by applying the homotopy coherent nerve. For example, by Lemma 16.4.8 we may defined the quasi-category $\mathsf{Cat}_{(\infty,1)}$ of ∞-categories and functors as the quasi-categorical core of $\mathcal{QC}at$. The biadjoint functors of quasi-categorically enriched categories pass to biadjoint functors on the $(\infty,1)$-categorical cores, satisfying the condition of Definition 16.2.19, so by Proposition 16.2.20 these descend to adjoint functors between their quasi-categorical cores.

In particular, this implies that the map of large quasi-categories of cartesian fibrations and cartesian functors

$$\mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B} \to \mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/\,\mathrm{ob}\,B} \cong \prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)}$$

admits both left and right adjoints. We use the dual of Theorem 10.4.12 to prove first that this forgetful functor is comonadic and then appeal to Proposition 10.4.13. The monadicity of this forgetful functor will be used in §?? to construct a "discrete reflection" functor for cartesian fibrations.

19.5.1. Proposition. *For any quasi-category B, the functor $\mathcal{C}art(\mathcal{QC}at)_{/B} \to \mathcal{C}art(\mathcal{QC}at)_{/\,\mathrm{ob}\,B}$ admits a quasi-categorically enriched right biadjoint defined by*

$$\begin{array}{ccc} \mathcal{C}art(\mathcal{QC}at)_{/\,\mathrm{ob}\,B} & \longrightarrow & \mathcal{C}art(\mathcal{QC}at)_{/B} \\ (E_b)_{b\in\mathrm{ob}\,B} & \longmapsto & \displaystyle\prod_{b\in\mathrm{ob}\,B} (E_b\times B \xrightarrow{\pi} B)^{\mathrm{Hom}_B(b,B)\xrightarrow{p_1} B} \end{array}$$

sending an $\mathrm{ob}\,B$-indexed family of quasi-categories $(E_b)_{b\in\mathrm{ob}\,B}$ to the $\mathrm{ob}\,B$-indexed product of the cartesian fibrations $(E_b\times B \xrightarrow{\pi} B)^{\mathrm{Hom}_B(b,B)\xrightarrow{p_1} B}$.

Proof. By Example 5.2.4 the product projection $\pi\colon E_b\times B \twoheadrightarrow B$ is a bifibration, while Proposition 5.1.23 demonstrates that the codomain projection $p_1\colon \mathrm{Hom}_B(b,B)\twoheadrightarrow B$ is a cocartesian fibration. Thus, Proposition 19.4.7 implies that the exponential π^{p_1} is a cartesian fibration. The enriched functoriality of this construction can be established in a number of steps, by considering the composite functor

$$\begin{array}{ccccccc} \mathcal{QC}at & \xrightarrow{-\times B\xrightarrow{\pi} B} & \mathcal{C}art(\mathcal{QC}at)_{/B} & \xrightarrow{p_1^*} & \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{Hom}_B(b,B)} & \xrightarrow{\Pi_{p_1}} & \mathcal{C}art(\mathcal{QC}at)_{/B} \\ E_b & & & \longmapsto & & & (E_b\times B \xrightarrow{\pi} B)^{\mathrm{Hom}_B(b,B)\xrightarrow{p_1} B} \end{array}$$

given by Corollary 19.2.7 and appealing to the enriched universal property of the conical product.

To establish the biadjointness, consider another cartesian fibration $q\colon F \twoheadrightarrow B$ with fibers $(F_b)_{b\in \mathrm{ob}\, B}$. By Lemma 19.4.8 and the universal property of the product

$$\begin{aligned}
&\mathsf{Fun}^{\mathsf{cart}}_B\Big(F \overset{q}{\twoheadrightarrow} B, \prod_{b\in\mathrm{ob}\,B} (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{\mathsf{Hom}_B(b,B)\overset{p_1}{\twoheadrightarrow} B}\Big)\\
&\qquad\simeq \prod_{b\in\mathrm{ob}\,B} \mathsf{Fun}^{\mathsf{cart}}_B\big(F \overset{q}{\twoheadrightarrow} B, (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{\mathsf{Hom}_B(b,B)\overset{p_1}{\twoheadrightarrow} B}\big)\\
&\qquad\simeq \prod_{b\in\mathrm{ob}\,B} \mathsf{Fun}^{\mathsf{cart}}_B\big(\mathsf{Hom}_B(b,B) \overset{p_1}{\twoheadrightarrow} B, (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{F\overset{q}{\twoheadrightarrow} B}\big).
\end{aligned}$$

By the dual of the Yoneda lemma, proven as Theorem 5.5.4, restricting along the element $\ulcorner \mathrm{id}_b \urcorner\colon 1 \to \mathsf{Hom}_B(b,B)$ defines an equivalence

$$\mathsf{Fun}^{\mathsf{cart}}_B\big(\mathsf{Hom}_B(b,B) \overset{p_1}{\twoheadrightarrow} B, (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{F\overset{q}{\twoheadrightarrow} B}\big) \simeq \mathsf{Fun}_B\big(1 \overset{b}{\to} B, (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{F\overset{q}{\twoheadrightarrow} B}\big).$$

By adjointness

$$\mathsf{Fun}_B\big(1 \overset{b}{\to} B, (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{F\overset{q}{\twoheadrightarrow} B}\big) \simeq \mathsf{Fun}_B(F_b \overset{!}{\to} 1 \overset{b}{\to} B, E_b \times B \overset{\pi}{\twoheadrightarrow} B) \simeq \mathsf{Fun}(F_b, E_b).$$

These equivalences compose to define the required natural equivalence

$$\mathsf{Fun}^{\mathsf{cart}}_B\Big(F \overset{q}{\twoheadrightarrow} B, \prod_{b\in\mathrm{ob}\,B} (E_b \times B \overset{\pi}{\twoheadrightarrow} B)^{\mathsf{Hom}_B(b,B)\overset{p_1}{\twoheadrightarrow} B}\Big) \simeq \prod_{b\in\mathrm{ob}\,B} \mathsf{Fun}(F_b, E_b). \qquad \square$$

The construction of the left biadjoint requires a lemma. Recall from the proof of Theorem 19.2.8 that the tensor of an isofibration $p\colon E \twoheadrightarrow B$ with a quasi-category U in $\mathcal{QC}at_{/B}$ is the composite

$$U \times E \overset{\pi}{\twoheadrightarrow} E \overset{p}{\twoheadrightarrow} B$$

For another isofibration $q\colon F \twoheadrightarrow B$, we have a pullback square

$$\begin{tikzcd}
\mathsf{Fun}_B(U \times E \overset{p\pi}{\twoheadrightarrow} B, F \overset{q}{\twoheadrightarrow} B) \cong \mathsf{Fun}_B(E \overset{p}{\twoheadrightarrow} B, F \overset{q}{\twoheadrightarrow} B)^U \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Fun}(E,F)^U \arrow[d, two heads, "(q\circ -)^U"] \\
1 \arrow[r, "p\pi"] & \mathsf{Fun}(U \times E, B) \cong \mathsf{Fun}(E,B)^U
\end{tikzcd}$$

establishing the simplicially enriched universal property of the tensor.

19.5.2. Lemma. *For any quasi-category U and cartesian fibrations $p\colon E \twoheadrightarrow B$ and $q\colon F \twoheadrightarrow B$, the isomorphism* $\mathsf{Fun}_B(U \times E, F) \cong \mathsf{Fun}_B(E,F)^U$ *restricts to an isomorphism* $\mathsf{Fun}^{\mathsf{cart}}_B(U \times E, F) \cong \mathsf{Fun}^{\mathsf{cart}}_B(E,F)^U$.

Proof. Any functor $f\colon E \to F$ over B induces a commutative square over B

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "i"', shift right] \arrow[d, phantom, "\vdash"] & F \arrow[d, "i", shift left] \arrow[d, phantom, "\dashv"] \\
\mathsf{Hom}_B(B,p) \arrow[r, "\mathsf{Hom}_B(B,f)"'] \arrow[u, bend left, dashed, "r"] & \mathsf{Hom}_B(B,q) \arrow[u, bend right, dashed, "r"']
\end{tikzcd}$$

whose vertical functors are the canonical ones induced by p and q. Because p and q are cartesian, Theorem 5.1.11 proves that these functors admit right adjoints over B. Theorem 5.1.19 then establishes that f is cartesian if and only if the mate of this canonical isomorphism is an isomorphism.

The mate that detects whether f is a cartesian functor lives as a 1-simplex in the quasi-category

$$\mathsf{Sq}_B(\mathsf{Hom}_B(B,p) \to E, \mathsf{Hom}_B(B,q) \to F) \coloneqq \mathsf{Fun}_B(E,F) \underset{\mathsf{Fun}_B(\mathsf{Hom}_B(B,p),F)}{\times} \mathsf{Fun}_B(\mathsf{Hom}_B(B,p), \mathsf{Hom}_B(B,q))$$

of commutative squares from $r\colon \mathsf{Hom}_B(B,p) \to E$ to $r\colon \mathsf{Hom}_B(B,q) \to F$.

Now the adjunction over B associated to the composite cartesian fibration $U \times E \xrightarrow{\pi} E \xrightarrow{p} B$ is

$$\begin{tikzcd} U \times E \arrow[rr, "U\times i"] \arrow[dr, two heads, "p\pi"'] & \arrow[d, phantom, "\perp"] & U \times \mathsf{Hom}_B(B,p) \arrow[ll, dashed, bend left, "U\times r"] \arrow[dl, two heads, "p_0\pi"] \\ & B & \end{tikzcd}$$

the product of the adjunction for p with U. In particular,

$$\mathsf{Sq}_B(U \times \mathsf{Hom}_B(B,p) \to U \times E, \mathsf{Hom}_B(B,q) \to F) \cong \mathsf{Sq}_B(\mathsf{Hom}_B(B,p) \to E, \mathsf{Hom}_B(B,q) \to F)^U.$$

Now a 1-simplex in $\mathsf{Sq}_B(r,r)^U$ is an isomorphism if and only if it is a pointwise isomorphism, which proves that $\mathsf{Fun}^{\mathsf{cart}}_B(p\pi,q) \cong \mathsf{Fun}^{\mathsf{cart}}_B(p,q)^U$ as claimed. □

19.5.3. Proposition. *For any quasi-category B, the functor $\mathcal{C}art(\mathcal{QC}at)_{/B} \to \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\,B}$ admits a quasi-categorically enriched left biadjoint defined by*

$$\begin{array}{ccc} \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\,B} & \longrightarrow & \mathcal{C}art(\mathcal{QC}at)_{/B} \\ (E_b)_{b\in\mathrm{ob}\,B} & \longmapsto & \coprod\limits_{b\in\mathrm{ob}\,B} E_b \times \mathsf{Hom}_B(B,b) \twoheadrightarrow B \end{array}$$

sending an $\mathrm{ob}\,B$*-indexed family of quasi-categories* $(E_b)_{b\in\mathrm{ob}\,B}$ *to the* $\mathrm{ob}\,B$*-indexed coproduct of the cartesian fibrations* $E_b \times \mathsf{Hom}_B(B,b) \xrightarrow{\pi} \mathsf{Hom}_B(B,b) \xrightarrow{p_0} B$.

It follows easily from the connectedness of the simplicial sets appearing on the left-hand side of the lifting problem in (F.4.3) of Proposition F.4.1(v) that the coproduct of cartesian fibrations of quasi-categories is a cartesian fibration. The construction implicit in the statement of Proposition 19.5.3 is more subtle, as this coproduct is taken in $\mathcal{QC}at_{/B}$ rather than in $\mathcal{QC}at^{\mathbb{2}}$ but it holds as well. See Exercise 19.5.i.

Proof. Simplicial functoriality of the construction $(E_b)_{b\in\mathrm{ob}\,B} \mapsto \coprod_{b\in\mathrm{ob}\,B} E_b \times \mathsf{Hom}_B(B,p) \twoheadrightarrow B$ follows from the enriched universal properties of tensors and coproducts established by Lemma 19.5.2 and Exercise 19.5.i.

To establish the biadjointness, consider another cartesian fibration $q\colon F \twoheadrightarrow B$ with fibers $(F_b)_{b\in\mathrm{ob}\,B}$. Appealing again to the universal properties of the tensor and coproduct we have natural isomorphisms

$$\begin{aligned} \mathsf{Fun}^{\mathsf{cart}}_B\Big(\coprod_{b\in\mathrm{ob}\,B} E_b \times \mathsf{Hom}_B(B,b) \twoheadrightarrow B, F \overset{q}{\twoheadrightarrow} B\Big) &\cong \prod_{b\in\mathrm{ob}\,B} \mathsf{Fun}^{\mathsf{cart}}_B(E_b \times \mathsf{Hom}_B(B,b) \twoheadrightarrow B, F \overset{q}{\twoheadrightarrow} B) \\ &\cong \prod_{b\in\mathrm{ob}\,B} \mathsf{Fun}^{\mathsf{cart}}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, F \overset{q}{\twoheadrightarrow} B)^{E_b}. \end{aligned}$$

By the Yoneda lemma, proven as Theorem 5.5.4, restricting along the element $\ulcorner \mathrm{id}_b \urcorner\colon 1 \to \mathsf{Hom}_B(B,b)$ defines an equivalence

$$\mathsf{Fun}^{\mathsf{cart}}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, F \overset{q}{\twoheadrightarrow} B) \simeq \mathsf{Fun}_B(1 \overset{b}{\to} B, F \overset{q}{\twoheadrightarrow} B) \cong F_b.$$

This equivalence of quasi-categories is preserved by the cotensor and product yielding an equivalence

$$\prod_{b\in \mathrm{ob}\,B} \mathsf{Fun}^{\mathrm{cart}}_B(\mathsf{Hom}_B(B,b) \overset{p_0}{\twoheadrightarrow} B, F \overset{q}{\twoheadrightarrow} B)^{E_b} \simeq \prod_{b\in \mathrm{ob}\,B} \mathsf{Fun}(E_b, F_b),$$

which composes to define the sought-for natural equivalence

$$\mathsf{Fun}^{\mathrm{cart}}_B(\coprod_{b\in \mathrm{ob}\,B} E_b \times \mathsf{Hom}_B(B,b) \twoheadrightarrow B, F \overset{q}{\twoheadrightarrow} B) \simeq \prod_{b\in \mathrm{ob}\,B} \mathsf{Fun}(E_b, F_b). \qquad \square$$

Upon passing to the $(\infty,1)$-cores of the quasi-categorically enriched categories $\mathcal{C}art(\mathcal{QC}at)_{/B}$ and $\prod_{\mathrm{ob}\,B} \mathcal{QC}at$, the biadjoints of Proposition 19.5.1 and 19.5.3 satisfy the premises of Proposition 16.2.20 and hence descend to an adjoint triple between the quasi-categories defined as the homotopy coherent nerves of these Kan-complex enriched categories:

$$\begin{tikzcd} \mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B} \arrow[r, "u" description] & \prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)} \arrow[l, bend right, "\ell"', "\perp"] \arrow[l, bend left, "r", "\perp"'] \end{tikzcd}$$

We now argue that this adjunction is monadic and comonadic. We have nearly all of the hypotheses of the ∞-categorical comonadicity theorem in hand. The final one is aided by the following lemma, whose proof follows directly from an elaboration of the statement.

19.5.4. LEMMA. *Let $U\colon \mathcal{K} \to \mathcal{L}$ be a functor of quasi-categorically enriched categories that reflects equivalences. Then the corresponding functor $U\colon \mathfrak{N}\mathsf{core}_*\mathcal{K} \to \mathfrak{N}\mathsf{core}_*\mathcal{L}$ between quasi-categorical cores is conservative.*

PROOF. Exercise 19.5.ii. $\square$

19.5.5. THEOREM. *The forgetful functor $u\colon \mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B} \to \prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)}$ is monadic and comonadic.*

PROOF. We've established the adjunctions $\ell \dashv u \dashv r$ between the quasi-categories of cartesian fibrations over B and $\mathrm{ob}\,B$-indexed families. We use the dual of Theorem 10.4.12 to argue that $u \dashv r$ is comonadic and then appeal to Proposition 10.4.13 to conclude that $\ell \dashv u$ is then monadic.

By the dual of Theorem 10.4.12, to prove that $u \dashv r$ is comonadic we must demonstrate that $\mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B}$ has and $u\colon \mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B} \to \prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)}$ preserves limits of u-split simplicial objects. Indeed, by construction $\mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B}$ and $\prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)}$ are the quasi-categories of ∞-categories in the ∞-cosmoi $\mathcal{C}art(\mathcal{QC}at)_{/B}$ and $\mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\,B}$ and u arises from the cosmological functor

$$u\colon \mathcal{C}art(\mathcal{QC}at)_{/B} \to \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\,B}$$

defined by pulling back along the inclusion $\mathrm{ob}\,B \hookrightarrow B$. Hence Proposition 16.4.9 applies to prove that these quasi-categories admit and u preserves all limits.

The remaining hypothesis of Theorem 10.4.12 asks that we show that u is conservative. By Proposition 16.2.11 a cartesian functor between cartesian fibrations of quasi-categories is an equivalence if and only if it is a fiberwise equivalence. This says exactly that the quasi-categorically enriched functor $u\colon \mathcal{C}art(\mathcal{QC}at)_{/B} \to \mathcal{C}art(\mathcal{QC}at)_{/\mathrm{ob}\,B}$ reflects equivalences, so Lemma 19.5.4 allows us to conclude that $u\colon \mathsf{Cart}(\mathsf{Cat}_{(\infty,1)})_{/B} \to \prod_{\mathrm{ob}\,B} \mathsf{Cat}_{(\infty,1)}$ is conservative. Hence, by the dual of Theorem 10.4.12, $u \dashv r$ is comonadic and hence $\ell \dashv u$ is monadic, as claimed. $\square$

Exercises.

19.5.i. Exercise.

(i) Show that the ∞-cosmoi $\mathcal{QC}at$ and $\mathcal{QC}at^{\mathbb{2}}$ have coproducts.

(ii) Argue further that the slice ∞-cosmos has $\mathcal{QC}at_{/B}$ has coproducts, created by the forgetful functor $\mathcal{QC}at_{/B} \to \mathcal{QC}at$.

(iii) Conclude that the ∞-cosomoi $\mathcal{C}art(\mathcal{QC}at)$, $co\mathcal{C}art(\mathcal{QC}at)$, $\mathcal{C}art(\mathcal{QC}at)_{/B}$, and $co\mathcal{C}art(\mathcal{QC}at)_{/B}$ have coproducts, created by the inclusions into $\mathcal{QC}at^{\mathbb{2}}$ and $\mathcal{QC}at_{/B}$.

19.5.ii. Exercise. Prove Lemma 19.5.4.

19.6. Homs for modules between quasi-categories

Recall from Definition 13.1.2 that a **right extension** of a module $A \overset{G}{\nrightarrow} C$ along a module $A \overset{E}{\nrightarrow} B$ consists of a pair given by a module $B \overset{\hom_A(E,G)}{\nrightarrow} C$ together with a binary cell

$$\begin{tikzcd} A \arrow[r, "E", "|"] \arrow[d, equal] & B \arrow[r, "\hom_A(E,F)", "|"] \arrow[d, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\ A \arrow[rr, "F"', "|"] & & C \end{tikzcd}$$

that is universal among the cells in the virtual equipment of modules in a sense detailed there. Dually, a **right lifting** of $A \overset{G}{\nrightarrow} C$ through $B \overset{F}{\nrightarrow} C$ consists of a pair given by a module $A \overset{\hom(F,G)_C}{\nrightarrow} B$ together with a universal binary cell

$$\begin{tikzcd} A \arrow[r, "\hom(F,G)_C", "|"] \arrow[d, equal] & B \arrow[r, "F", "|"] \arrow[d, phantom, "\Downarrow\epsilon"] & C \arrow[d, equal] \\ A \arrow[rr, "G"', "|"] & & C \end{tikzcd}$$

Here we've deployed different notation for these notions with the aim of rebranding them. We refer to the module $\hom_A(E,G)$ has the **left hom** from E to F and refer to $\hom(F,G)_C$ as the **right hom** from F and G. Extending the "module" metaphor, we think of the left hom as the object of homomorphisms from E to G that respect the left A-actions, while the right hom is the object of homomorphisms from F to G that respect the right C-actions. Our aim in this section is to demonstrate that the virtual equipment of modules between quasi-categories has left and right homs for all suitable pairs of modules. By model independence, this result extends to any ∞-cosmos of $(\infty,1)$-categories.

19.6.1. Theorem. *The virtual equipment of modules between quasi-categories has left and right homs.*

Proof. The cases of left and right homs are dual so we focus on the former. Fixing a module $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, define $\hom_A(E,G)$ to be the image of the module $A \overset{G}{\nrightarrow} C$ under the composite right adjoint functor

$$\begin{tikzcd} \mathcal{QC}at_{/A\times C} \arrow[r, "\perp", "(q\times C)^*"'] & \mathcal{QC}at_{/E\times C} \arrow[l, bend right, dashed, "(q\times C)\circ-"'] \arrow[r, "\perp", "\Pi_{p\times C}"'] & \mathcal{QC}at_{/B\times C} \arrow[l, bend right, dashed, "(p\times C)^*"'] \end{tikzcd}$$

Note that the left adjoint carries a span of isofibrations $B \overset{s}{\twoheadleftarrow} F \overset{r}{\twoheadrightarrow} C$ to the composite two-sided isofibration:

$$\begin{tikzcd} & & E \times_B F \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & E \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & & F \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "r"] & \\ A & & B & & C \end{tikzcd}$$

and adjointness provides natural equivalences (or in this case isomorphisms) of quasi-categories

$$\mathsf{Fun}_{A\times C}(E \times_B F, G) \cong \mathsf{Fun}_{B\times C}(F, \hom_A(E, G)).$$

These equivalences induce bijections on the sets of isomorphism classes of objects in these quasi-categories. By Proposition 12.1.7 this will establish the universal property of required by the right hom in the virtual equipment of modules as spelled out in Definition 13.1.2 — at least once we prove that $\hom_A(E, G)$ defines a module under the hypothesis that E and G do.

Thus it remains only to show that when $A \overset{E}{\nrightarrow} B$ is a module, then the functor

$$\hom_A(E, -) \colon \mathcal{QC}at_{/A\times C} \xrightarrow{(q\times C)^*} \mathcal{QC}at_{/E\times C} \xrightarrow{\Pi_{p\times C}} \mathcal{QC}at_{/B\times C}$$

preserves modules. Proposition 11.4.4 tells us that modules are stable under pullback, so we need only demonstrate that $\Pi_{p\times C}$ carries a module $E \overset{v}{\twoheadleftarrow} M \overset{u}{\twoheadrightarrow} C$ to a module from B to C. Lemma 11.4.2 characterizes modules from E to C as those two-sided isofibrations $(v, u) \colon M \twoheadrightarrow E \times C$ so that

(i) $(v, u) \colon M \twoheadrightarrow E \times C$ is a cocartesian fibration in $\mathcal{QC}at_{/C}$,
(ii) $(v, u) \colon M \twoheadrightarrow E \times C$ is a cartesian fibration in $\mathcal{QC}at_{/E}$,
(iii) $(v, u) \colon M \twoheadrightarrow E \times C$ is discrete as an object in $\mathcal{QC}at_{/E\times C}$.

We will show that $\Pi_{p\times C}$ preserves each of these properties, proving that $\Pi_{p\times C}M$ defines a module from B to C. We start with (iii). Theorem 19.2.8 demonstrates that the pushforward is a cosmological functor and Corollary 14.1.6 observes that cosmological functors preserve discrete objects. So (iii) is automatic.

We next turn our attention to (ii), arguing that $\Pi_{p\times C}M \twoheadrightarrow B \times C$ is a cartesian fibration in $\mathcal{QC}at_{/B}$. First note, by Proposition 14.1.9, that the cosmological functor $\Pi_p \colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$ preserves cartesian fibrations[5], so we conclude that $\Pi_p M \twoheadrightarrow \Pi_p(B \times C)$ is a cartesian fibration in $\mathcal{QC}at_{/B}$. However, this functor does not coincide with $\Pi_{p\times C}M \twoheadrightarrow B \times C$. Rather, as demonstrated by Lemma 19.6.2 below, $\Pi_{p\times C}M \twoheadrightarrow B \times C$ is a pullback of $\Pi_p M \twoheadrightarrow \Pi_p(B \times C)$ in $\mathcal{QC}at_{/B}$, so by pullback stability of cartesian fibrations, this proves (ii).

For the final property (i), recall from Lemma 11.1.1 that to say that $(v, u) \colon M \twoheadrightarrow E \times C$ defines a cocartesian fibration in $\mathcal{QC}at_{/C}$ is equivalently to say that $(v, u) \colon M \twoheadrightarrow E \times C$ defines a morphism in the sub ∞-cosmos $co\mathcal{C}art(\mathcal{QC}at)_{/E} \subset \mathcal{QC}at_{/E}$. By Theorem 19.3.7, Π_p restricts to define a cosmological functor $co\mathcal{C}art(\mathcal{QC}at)_{/E} \to co\mathcal{C}art(\mathcal{QC}at)_{/B} \subset \mathcal{QC}at_{/B}$. In particular, by Lemma 19.6.2 the pullback $\Pi_{p\times C}M \twoheadrightarrow B \times C$ of $\Pi_p M \twoheadrightarrow \Pi_p(B \times C)$ lies in the sub cosmos $co\mathcal{C}art(\mathcal{QC}at)_{/B}$. By Lemma 11.1.1 again this tells us that $\Pi_{p\times C}M \twoheadrightarrow B \times C$ is a cocartesian fibration in $\mathcal{QC}at_{/C}$ as required. $\square$

[5]Note this is not an application of Theorem 19.4.3, which in this context would tell us that Π_p carries a *cocartesian* fibration over E to a cocartesian fibration over B. Rather, this says that Π_p, simply by virtue of being cosmological, carries cartesian fibrations between isofibration with codomain E to cartesian fibrations between isofibrations with codomain B.

19.6.2. LEMMA. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration or cocartesian fibration between quasi-categories. Then the image of an isofibration $M \twoheadrightarrow E \times C$ under $\Pi_{p\times C}\colon \mathcal{QC}at_{/E\times C} \to \mathcal{QC}at_{/B\times C}$ is the left vertical isofibration defined by the pullback*

$$\begin{tikzcd} \Pi_{p\times C}M \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \Pi_p M \arrow[d, two heads] \\ B\times C \arrow[r, "\eta"'] & \Pi_p(E\times C) \end{tikzcd}$$

whose right-hand vertical morphism is the image of the map $M \twoheadrightarrow E\times C$ under $\Pi_p\colon \mathcal{QC}at_{/E} \to \mathcal{QC}at_{/B}$.

The conclusion of Lemma 19.6.2 holds, up to equivalence, in any ∞-cosmos of $(\infty,1)$-categories $\mathcal{K}$. However, we find it easier to work with the strict adjunction of Theorem 19.2.8 rather than the weaker adjunctions of Theorem 19.4.1 and this is the only case we really need.

PROOF. To make sense of the statement note that p^* carries $\pi\colon B\times C \twoheadrightarrow B$ to $\pi\colon E\times C \twoheadrightarrow E$. Hence the map $\eta\colon B\times C \to \Pi_p(E\times C)$ is a component of the unit of the adjunction $p^* \dashv \Pi_p$. Now the claim follows directly by verifying that the displayed pullback has the universal property that defines $\Pi_{p\times C}M \in \mathcal{QC}at_{/B\times C}$. A cone over the pullback diagram may be interpreted as a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Pi_p M \arrow[d, two heads] \\ B\times C \arrow[r, "\eta"'] & \Pi_p(E\times C) \end{tikzcd}$$

in $\mathcal{QC}at_{/B}$, which then transposes to define a commutative square

$$\begin{tikzcd} X\times_B E \arrow[r] \arrow[d] & M \arrow[d, two heads] \\ E\times C \arrow[r, equal] & E\times C \end{tikzcd}$$

in $\mathcal{QC}at_{/E}$. In fact, this diagram lies in $\mathcal{QC}at_{/E\times C}$ and the left-hand vertical map $X\times_B E \to E\times C$ is isomorphic to $(p\times C)^*(X \to B\times C)$. Hence, this square transposes along $(p\times C)^* \dashv \Pi_{p\times C}$ to a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Pi_{p\times C}M \arrow[d, two heads] \\ B\times C \arrow[r, equal] & B\times C \end{tikzcd}$$

which proves the claim. $\square$

For later use, we observe a second way to compute the pushforwards appearing in the construction of the right or left homs between modules in a special case, which is easily proven by directly comparing the defining universal properties.

19.6.3. Lemma. *The image of an isofibration $M \twoheadrightarrow E \times C$ under $\Pi_{!\times C} \colon \mathcal{QC}at_{/E\times C} \to \mathcal{QC}at_{/1\times C}$ is the left vertical isofibration defined by the pullback*

$$\begin{tikzcd} \Pi_{!\times C}M \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & M^E \arrow[d, two heads] \\ C \arrow[r, "\eta"'] & (E\times C)^E \end{tikzcd}$$

Proof. Exercise 19.6.iii. □

The result of Theorem 19.6.1 extends to biequivalent ∞-cosmoi. By this point, the routine argument is safely left to the reader.

19.6.4. Corollary. *In any ∞-cosmos of $(\infty,1)$-categories, the virtual equipment of modules has left and right homs.*

Proof. Exercise 19.6.ii. □

Exercises.

19.6.i. Exercise. Adapt Lemma 19.6.2 to a model-independent statement that applies in any ∞-cosmos of $(\infty,1)$-categories and then prove this result.

19.6.ii. Exercise. Use Theorems 19.6.1 and 15.3.3 to prove Corollary 19.6.4.

19.6.iii. Exercise. Prove Lemma 19.6.3.

19.7. Existence of pointwise Kan extensions

19.7.1. Theorem. *Let $k \colon A \to B$ be any map between small simplicial sets and let E be a complete quasi-category, admitting limits indexed by any small simplicial set. Then E admits pointwise right Kan extensions along k.*

Proof. It will be convenient to replace the map $k \colon A \to B$ by an equivalent functor $k \colon A \to B$ between small categories, constructed as a fibrant replacement in the Joyal model structure. Consider the module $\mathrm{Hom}_B(B,k)$ from A to B covariantly represented by this functor.

Theorem 19.6.1 proves that the virtual equipment of modules between quasi-categories admits left and right homs, defining right extensions and right liftings in the virtual equipment of modules between quasi-categories. In particular given any diagram $f \colon A \to E$, there exists a right extension in the virtual equipment of modules

$$\begin{tikzcd} A \arrow[r, "\mathrm{Hom}_B(B,k)", "|"{marking}] \arrow[d, equal] & B \arrow[r, dashed, "\mathrm{hom}_A(\mathrm{Hom}_B(B,k)\text{,}\mathrm{Hom}_E(E,f))", "|"{marking}] \arrow[d, phantom, "\Downarrow"] & E \arrow[d, equal] \\ A \arrow[rr, "\mathrm{Hom}_E(E,f)"', "|"{marking}] & & E \end{tikzcd}$$

Now the quasi-category E will admit a pointwise right Kan extension of $f \colon A \to E$ along $k \colon A \to B$ just when the module $\mathrm{hom}_A(\mathrm{Hom}_B(B,k),\mathrm{Hom}_E(E,f))$ is covariantly represented by a functor $r \colon B \to E$. By Corollary **??**, this module is covariantly represented if and only if its pullbacks along each vertex $b \colon 1 \to B$ are covariantly represented, which is the case just when the fiber of the module $\mathrm{hom}_A(\mathrm{Hom}_B(B,k),\mathrm{Hom}_E(E,f)) \twoheadrightarrow B \times E$ over b has a terminal element.

By Lemma 13.1.4 the fiber $\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))(1,b)$ arises as the the right extension

$$\begin{tikzcd}
B \arrow[r, "\mathsf{Hom}_B(b,B)"] \arrow[d, equal] & 1 \arrow[rr, dashed, "\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))(1,b)"] \arrow[d, phantom, "\Downarrow\rho"] & & E \arrow[d, equal] \\
B \arrow[rrr, "\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))"'] & {} & & E
\end{tikzcd}$$

Since $\mathsf{Hom}_B(B,k) \otimes \mathsf{Hom}_B(b,B) \cong \mathsf{Hom}_B(b,k)$ by Proposition 12.4.7, Proposition 13.1.5 allows us to combine these two right extensions into a single one

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_B(b,k)"] \arrow[d, equal] & 1 \arrow[rr, dashed, "\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))(1,b)"] \arrow[d, phantom, "\Downarrow"] & & E \arrow[d, equal] \\
A \arrow[rrr, "\mathsf{Hom}_E(E,f)"'] & {} & & E
\end{tikzcd}$$

By exactness of the comma square

$$\begin{tikzcd}
& \mathsf{Hom}_B(b,k) \arrow[dl, two heads, "p_1"'] \arrow[dr, two heads, "!"] & \\
A \arrow[dr, "k"'] & \overset{\phi}{\Leftarrow} & 1 \arrow[dl, "b"] \\
& B &
\end{tikzcd}$$

we also have $\mathsf{Hom}_B(b,k) \cong \mathsf{Hom}_A(p_1,A) \otimes \mathsf{Hom}_1(1,!)$. By Lemma 13.1.4, there is a right extension diagram

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Hom}_A(p_1,A)"] \arrow[d, equal] & \mathsf{Hom}_B(b,k) \arrow[rr, dashed, "\mathsf{Hom}_E(E,fp_1)"] & {} \arrow[d, phantom, "\Downarrow"] & E \arrow[d, equal] \\
A \arrow[rrr, "\mathsf{Hom}_E(E,f)"'] & & {} & E
\end{tikzcd}$$

So by Proposition 13.1.5, since $\mathsf{Hom}_E(E,f)$ admits a right extension along $\mathsf{Hom}_A(p_1,A) \otimes \mathsf{Hom}_1(1,!)$, the right extension of $\mathsf{Hom}_E(E,fp_1)$ along $\mathsf{Hom}_1(1,!)$ exists and is given by

$$\begin{tikzcd}
\mathsf{Hom}_B(b,k) \arrow[r, "\mathsf{Hom}_1(1,!)"] \arrow[d, equal] & 1 \arrow[rr, dashed, "\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))(1,b)"] \arrow[d, phantom, "\Downarrow"] & & E \arrow[d, equal] \\
\mathsf{Hom}_B(b,k) \arrow[rrr, "\mathsf{Hom}_E(E,fp_1)"'] & {} & & E
\end{tikzcd}$$

In summary, the fiber of the module $\mathrm{hom}_A(\mathsf{Hom}_B(B,k),\mathsf{Hom}_E(E,f))$ over $b\colon 1 \to B$ is the module defined by the right extension of $\mathsf{Hom}_E(E,fp_1)$ along $\mathsf{Hom}_1(1,!)$. Thus, we see that if this module is represented by an element $e\colon 1 \to E$, that element defines a pointwise right extension of the diagram

$$\begin{tikzcd}
\mathsf{Hom}_B(b,k) \arrow[r, two heads, "p_1"] \arrow[d, "!"'] & A \arrow[r, "f"] \arrow[dl, phantom, "\Uparrow"] & E \\
1 \arrow[urr, dashed, "e"'] & &
\end{tikzcd}$$

By Proposition 13.5.2, this exists if and only if the diagram $fp_1 \colon \mathsf{Hom}_B(b,k) \twoheadrightarrow E$ has a limit in E. If E is complete, this limit exists and consequently so does the pointwise right extension of f along k, as claimed. □

19.7.2. Remark. The proof of Theorem 19.7.1 demonstrates that the fiber over $b \colon 1 \to B$ of the module $\hom_A(\mathsf{Hom}_B(B,k), \mathsf{Hom}_E(E,f))$ is equivalently the module $\hom_{\mathsf{Hom}_B(b,k)}(\mathsf{Hom}_1(1,!), \mathsf{Hom}_E(E,fp_1))$ defined as the right extension of $\mathsf{Hom}_E(E,fp_1)$ along $\mathsf{Hom}_1(1,!)$. In Theorem 19.6.1, this right extension of modules is defined by pulling back the isofibration $\mathsf{Hom}_E(E,fp_1) \twoheadrightarrow \mathsf{Hom}_B(b,k) \times E$ along the left leg of the span

$$\begin{tikzcd} & \mathsf{Hom}_B(b,k) \times E \arrow[dl, equal] \arrow[dr, "! \times E"] & \\ \mathsf{Hom}_B(b,k) \times E & & E \end{tikzcd}$$

and then pushing it forward along the right leg of the span. In this case, the pullback along the identity does nothing and Lemma 19.6.3 allows us to compute the pushforward along the projection as left-hand vertical morphism in the left-hand pullback square

$$\begin{tikzcd} \hom_A(\mathsf{Hom}_B(B,k), \mathsf{Hom}_E(E,f))(1,b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Hom}_E(E,fp_1)^{\mathsf{Hom}_B(b,k)} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E^{\mathsf{Hom}_B(b,k) \times 2} \arrow[d, two heads, "{(p_1,p_0)}"] \\ 1 \times E \arrow[r, "\mathrm{id} \times \Delta"'] & \mathsf{Hom}_B(b,k)^{\mathsf{Hom}_B(b,k)} \times E^{\mathsf{Hom}_B(b,k)} \arrow[r, "(fp_1)^{\mathsf{Hom}_B(b,k)} \times \mathrm{id}"'] & E^{\mathsf{Hom}_B(b,k)} \times E^{\mathsf{Hom}_B(B,k)} \end{tikzcd}$$

Upon pasting on the right-hand pullback square, obtained by exponentiating the defining pullback for $\mathsf{Hom}_E(E,fp_1)$ by $\mathsf{Hom}_B(b,k)$, we see that the fiber is equivalent to the ∞-category of cones over fp_1 constructed in Definition 4.2.1, i.e.,

$$\hom_A(\mathsf{Hom}_B(B,k), \mathsf{Hom}_E(E,f))(1,b) \simeq_E \mathsf{Hom}_{E^{\mathsf{Hom}_B(b,k)}}(\Delta, fp_1). \tag{19.7.3}$$

Thus, by Corollary **??**, to define the value of the right extension $r \colon B \to E$ of f along k at $b \colon 1 \to B$ is define a terminal element in the quasi-category (19.7.3), which by Proposition 4.3.2 amounts to defining a limit for the diagram $fp_1 \colon \mathsf{Hom}_B(b,k) \twoheadrightarrow E$. This recovers the classical limit formula for pointwise Kan extensions as can be found for instance in [**64**, X.5.3].

Appendix of Abstract Nonsense

APPENDIX A

Basic concepts of enriched category theory

Enriched category theory exists because enriched categories exist in nature. To explain, consider the data of a (small) 1-category C, given by:

- a set of objects
- for each pair of objects $x, y \in C$, a set $C(x,y)$ of arrows in C from x to y
- for each $x \in C$, a specified identity element $\mathrm{id}_x \colon 1 \to C(x,x)$, and for each $x, y, z \in C$, a specified composition map $\circ \colon C(y,z) \times C(x,y) \to C(x,z)$ satisfying the associativity and unit conditions:

$$\begin{tikzcd} C(y,z)\times C(x,y)\times C(w,x) \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\circ"'] & C(x,z)\times C(w,x) \arrow[d, "\circ"] \\ C(y,z)\times C(w,y) \arrow[r, "\circ"'] & C(w,z) \end{tikzcd} \qquad \begin{tikzcd} C(x,y) \arrow[r, "\mathrm{id}\times\mathrm{id}_x"] \arrow[d, "\mathrm{id}_y\times\mathrm{id}"'] \arrow[dr, equal] & C(x,y)\times C(x,x) \arrow[d, "\circ"] \\ C(y,y)\times C(x,y) \arrow[r, "\circ"'] & C(x,y) \end{tikzcd} \tag{A.0.1}$$

In many mathematical examples of interest, the set $C(x,y)$ can be given additional structure, in which case it would be strange not to take it account when performing further categorical constructions.

Perhaps there exists a specified zero arrow $0_{x,y} \in C(x,y)$, the set of which defines a two-sided ideal for composition: $g \circ 0 \circ f = 0$. Or extending this, perhaps $C(x,y)$ admits an abelian group structure, for which composition is $\mathbb{Z}$-bilinear. Or in another direction, perhaps the set of arrows from x to y in C form the objects of a 1-category. In this setting, we regard the objects of $C(x,y)$ as "1-dimensional" morphisms from x to y and the arrows of $C(x,y)$ as "2-dimensional" morphisms from x to y in C; now, it's natural to require the composition map to define a functor. Or perhaps the set of arrows from x to y in C form the vertices of a simplicial set, whose higher simplices now provides arrows in each positive dimension; in this setting, it is natural to ask composition to define a simplicial map. In such settings, one says that the 1-category C can be enriched over the category $\mathcal{V}$ in which the objects $C(x,y)$ and diagrams (A.0.1) live[1] — with $\mathcal{V}$ equal to the category of pointed sets, abelian groups, categories, or simplicial sets in the examples just described.

An alternate point of view of enriched category theory is often emphasized — adopted for instance in the classic textbook **[56]** from which we stole the title of this chapter. To borrow a distinction used by Peter May, the term "enriched" can be used as a compound noun — *enriched categories* — or as an adjective — *enriched* categories. In the noun form, an enriched category C has no pre-existing underlying ordinary category, although we shall see below that the underlying unenriched 1-category can always be identified a posteriori. When used as an adjective, an enriched category C is perhaps

[1] To interpret the diagrams (A.0.1) in $\mathcal{V}$ one needs to specify an interpretation for the monoidal product "$\times$" and its unit object 1 (which is not displayed in the diagram). In the examples we will consider, this product is the cartesian product and this unit is the terminal object.

most naturally an ordinary category, whose hom-sets can be given additional structure.[2] While the noun perspective is arguably more elegant when discussing the general theory of enriched categories, the adjective perspective dominates when discussing examples, so we choose to emphasize the adjective form and focus on enriching unenriched categories here.

Before giving a precise definition of enriched category and the enriched functors between them in §A.2, in §A.1 we study that category $\mathcal{V}$ that defines the base for enrichment in which the hom-objects will ultimately live. Here we are interested primarily in two examples $\mathcal{V} = \mathcal{C}at$ and $\mathcal{V} = s\mathcal{S}et$, as well as the unenriched case $\mathcal{V} = \mathcal{S}et$, each of which has the special property of being cartesian closed categories. Since there are some simplifications in enriching over a cartesian closed category, we grant ourselves the luxury of working explicitly with this notion; see [**56**] or [**79**, Chapter 3] for an introduction to categories enriched over a more general symmetric monoidal category.

We continue in §A.3 with an introduction to enriched natural transformations and the enriched Yoneda lemma. These notions allow us to correctly state the universal properties that characterize *tensors* and *cotensors* in §A.4 and *conical limits* and *colimits* in §A.5, both of which are characterized by universal properties involving an enriched natural isomorphism. We conclude in §A.6 with a general theory of change of base — the one part of the theory of enriched categories that is not covered in encyclopedic detail in [**56**], the original reference instead being [**37**] — which allows us to be more precise about the procedure by which a 2-category may be regarded as a simplicial category (as used to great effect in Chapters 9 and 10) or by which a simplicial category may be quotiented to define a 2-category (see e.g., Definition 1.4.1) as alluded to in Digression 1.4.2.

A.1. Cartesian closed categories

Throughout this text, the base category for enrichment will always be taken to be a cartesian closed category:

A.1.1. Definition. A category $\mathcal{V}$ is **cartesian closed** when it

- admits finite products, or equivalently, a terminal object $1 \in \mathcal{V}$ and binary products and
- for each $A \in \mathcal{V}$, the functor $A \times - \colon \mathcal{V} \to \mathcal{V}$ admits a right adjoint $(-)^A \colon \mathcal{V} \to \mathcal{V}$.

A.1.2. Lemma. *In a cartesian closed category $\mathcal{V}$, the product bifunctor is the left adjoint of a two-variable adjunction, this being captured by a commutative triangle of natural isomorphisms*

$$\begin{tikzcd} & \mathcal{V}(A \times B, C) \arrow[dl, phantom, "\cong" sloped] \arrow[dr, phantom, "\cong" sloped] & \\ \mathcal{V}(A, C^B) \arrow[rr, phantom, "\cong"] & & \mathcal{V}(B, C^A) \end{tikzcd} \tag{A.1.3}$$

Proof. The family of functors $(-)^A \colon \mathcal{V} \to \mathcal{V}$ extend to bifunctors

$$(-)^- \colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

in a unique way so that the isomorphism defining each adjunction $A \times - \dashv (-)^A$

$$\mathcal{V}(A \times B, C) \cong \mathcal{V}(B, C^A)$$

becomes natural in A (as well as B and C). The details are left as Exercise A.1.i or to [**80**, 4.3.6]. This defines the natural isomorphism on the right-hand side of (A.1.3). The natural isomorphism on the

[2]To quote [**66**] "Thinking from the two points of view simultaneously, it is essential that the constructed ordinary category be isomorphic to the ordinary category that one started out with. Either way, there is a conflict of notation between that preferred by category theorists and that in common use by 'working mathematicians' (to whom [**64**] is addressed).

left-hand is defined by composing with the symmetry isomorphism $A \times B \cong B \times A$. The third natural isomorphism is taken to be the composite of these two. □

A.1.4. Example (cartesian closed categories).

(i) The category of sets is cartesian closed, with B^A defined to be the set of functions from A to B. Transposition across the natural isomorphism (A.1.3) is referred to as "currying."

(ii) The category $\mathcal{C}at$ of small categories is cartesian closed, with B^A defined to be the category of functors and natural transformations from A to B.[3] The natural isomorphism

$$\mathcal{C}at(A, C^B) \cong \mathcal{C}at(A \times B, C) \cong \mathcal{C}at(B, C^A)$$

identifies natural transformations, which are arrows $\mathbb{2} \to C^A$ in the category of functors, with "homotopies" $A \times \mathbb{2} \to C$.

(iii) For any small category C, the category $\mathcal{S}et^{C^{\mathrm{op}}}$ is cartesian closed. By the Yoneda lemma for $F, G \in \mathcal{S}et^{C^{\mathrm{op}}}$, the value of G^F at $c \in C$ must be defined by

$$G^F(c) \cong \mathcal{S}et^{C^{\mathrm{op}}}(C(-,c), G^F) \cong \mathcal{S}et^{C^{\mathrm{op}}}(F \times C(-,c), G).$$

By the proof of Lemma A.1.2, the action of G^F on a morphism $f \colon c \to c' \in C$ is defined by precomposition with the corresponding natural transformation $f_* \colon C(-,c) \to C(-,c')$. This defines the functor G^F. Since any functor $H \in \mathcal{S}et^{C^{\mathrm{op}}}$ is canonically a colimit of representables, this definition extends to the required natural isomorphism $\mathcal{S}et^{C^{\mathrm{op}}}(H, G^F) \cong \mathcal{S}et^{C^{\mathrm{op}}}(F \times H, G)$.

(iv) In particular taking $C = \mathbf{\Delta}^{\mathrm{op}}$, the category of simplicial sets $s\mathcal{S}et := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ is cartesian closed.

The exponential B^A is frequently referred to as an *internal hom*. As this name suggests, the internal hom B^A can be viewed as a lifting of the hom-set $\mathcal{V}(A, B)$ along a functor that we now introduce.

A.1.5. Definition. For any cartesian closed category $\mathcal{V}$ the **underlying set functor** is the functor

$$\mathcal{V} \xrightarrow{(-)_0 := \mathcal{V}(1,-)} \mathcal{S}et$$

represented by the terminal object $1 \in \mathcal{V}$.

A.1.6. Lemma. *For any pair of objects $A, B \in \mathcal{V}$ in a cartesian closed category, the underlying set of the internal hom B^A is $\mathcal{V}(A, B)$, i.e.:*

$$(B^A)_0 \cong \mathcal{V}(A, B).$$

Proof. Combining Definition A.1.5 with (A.1.3):

$$(B^A)_0 := \mathcal{V}(1, B^A) \cong \mathcal{V}(1 \times A, B) \cong \mathcal{V}(A, B)$$

since there is a natural isomorphism $1 \times A \cong A$. □

It makes sense to ask whether an isomorphism of underlying sets can be "enriched" to lie in $\mathcal{V}$, that is, lifted along the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$.

[3] Size matters here: the category of large but locally small categories is not cartesian closed, though it is still possible to define B^A in the case where A is a small category.

A.1.7. Lemma. *The natural isomorphisms* (A.1.3) *characterizing the defining two-variable adjunction of a cartesian closed category lift to $\mathcal{V}$: for any $A, B, C \in \mathcal{V}$*

$$\begin{tikzcd}[column sep=small] & C^{A\times B} \arrow[dl, phantom, "\rotatebox{45}{$\cong$}"] \arrow[dr, phantom, "\rotatebox{-45}{$\cong$}"] & \\ (C^B)^A & \cong & (C^A)^B \end{tikzcd} \tag{A.1.8}$$

Proof. This follows from the associativity of finite products and the Yoneda lemma. To prove (A.1.8), it suffices to show that $C^{A\times B}$, $(C^B)^A$, and $(C^A)^B$ represent the same functor. We have a sequence of natural isomorphisms:

$$\begin{aligned}\mathcal{V}(X,(C^B)^A) &\cong \mathcal{V}(X\times A, C^B)\\ &\cong \mathcal{V}((X\times A)\times B, C) \cong \mathcal{V}(X\times(A\times B), C)\\ &\cong \mathcal{V}(X, C^{A\times B})\\ &\cong \mathcal{V}((A\times B)\times X, C) \cong \mathcal{V}(A\times(B\times X), C)\\ &\cong \mathcal{V}(B\times X, C^A)\\ &\cong \mathcal{V}(X,(C^A)^B),\end{aligned}$$

and hence

$$\mathcal{V}(X,(C^B)^A) \cong \mathcal{V}(X, C^{A\times B}) \cong \mathcal{V}(X,(C^A)^B). \qquad \square$$

A.1.9. Remark. Note $(-)^1 \colon \mathcal{V} \to \mathcal{V}$ is naturally isomorphic to the identity functor — i.e., $B^1 \cong B$, — since it is right adjoint to a functor $-\times 1 \colon \mathcal{V} \to \mathcal{V}$ that is naturally isomorphic to the identity.

A bicomplete cartesian closed category is a special case of a complete and cocomplete closed symmetric monoidal category, this being deemed a **cosmos** by Bénabou, to signify that such bases are an ideal setting for enriched category theory. For obvious reasons, we won't use this term here. There is a competing 2-categorical notion of (fibrational) "cosmos" due to Street [**93**] that is more similar to the notion we consider here, which was the direct inspiration for the terminology we introduce in Definition 1.2.1.

Exercises.

A.1.i. Exercise. Prove that in a cartesian closed category $\mathcal{V}$, the family of functors $(-)^A \colon \mathcal{V} \to \mathcal{V}$ extend to bifunctors

$$(-)^- \colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

in a unique way so that the isomorphism defining each adjunction $A \times - \dashv (-)^A$

$$\mathcal{V}(A\times B, C) \cong \mathcal{V}(B, C^A)$$

becomes natural in A (as well as B and C).

A.1.ii. Exercise. The data of a **closed symmetric monoidal category** generalizes Definition A.1.1 by replacing finite products by an arbitrary bifunctor $-\otimes- \colon \mathcal{V}\times\mathcal{V} \to \mathcal{V}$, replacing the terminal object by an object $I \in \mathcal{V}$, and requiring the additional specification of natural isomorphisms

$$A\otimes(B\otimes C) \cong (A\otimes B)\otimes C \qquad I\otimes A \cong A \cong A\otimes I \qquad A\otimes B \cong B\otimes A$$

satisfying various coherence axioms [**37, 52**]; see also [**53**]. Explain why these isomorphisms do not need to specified in the special case of a cartesian closed category and why the coherence conditions are automatic.

A.2. Enriched categories

We now briefly switch perspectives and explain the meaning of the noun phrase "enriched category" before discussing what is required to "enrich" an ordinary 1-category. Throughout we fix a cartesian closed category $(\mathcal{V}, \times, 1)$ to serve as the base for enrichment.

A.2.1. Definition. A **$\mathcal{V}$-enriched category** C is given by:

- a set of objects
- for each pair of objects $x, y \in C$, an **object of arrows** $C(x,y) \in \mathcal{V}$
- for each $x \in C$, a specified identity element encoded by a map $\mathrm{id}_x \colon 1 \to C(x,x) \in \mathcal{V}$, and for each $x,y,z \in C$, a specified composition map $\circ \colon C(y,z) \times C(x,y) \to C(x,z) \in \mathcal{V}$ satisfying the associativity and unit conditions, both commutative diagrams lying in $\mathcal{V}$:[4]

$$\begin{tikzcd} C(y,z) \times C(x,y) \times C(w,x) \arrow[r, "\circ \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \circ"'] & C(x,z) \times C(w,x) \arrow[d, "\circ"] \\ C(y,z) \times C(w,y) \arrow[r, "\circ"'] & C(w,z) \end{tikzcd} \qquad \begin{tikzcd} C(x,y) \arrow[r, "\mathrm{id} \times \mathrm{id}_x"] \arrow[d, "\mathrm{id}_y \times \mathrm{id}"'] \arrow[dr, equal] & C(x,y) \times C(x,x) \arrow[d, "\circ"] \\ C(y,y) \times C(x,y) \arrow[r, "\circ"'] & C(x,y) \end{tikzcd}$$

Evidently from the diagrams of (A.0.1), a locally-small 1-category defines a category enriched in $\mathcal{S}et$. The underlying set functor of Definition A.1.5 can be used to define the "underlying category" of an enriched category.

A.2.2. Definition. If C is a $\mathcal{V}$-category, its **underlying category** is the 1-category with the same set of objects and with hom-sets $C(x,y)_0$ defined by applying the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$ to the hom-objects $C(x,y) \in \mathcal{V}$.

Note the identity arrow $\mathrm{id}_x \colon 1 \to C(x,x)$ of the $\mathcal{V}$-category is by definition an element of the hom-set $C(x,x)_0 := \mathcal{V}(1, C(x,x))$. The composite of two arrows $f \colon 1 \to C(x,y)$ and $g \colon 1 \to C(y,z)$ in the underlying category is defined to be the arrow constructed as the composite

$$1 \xrightarrow{g \times f} C(y,z) \times C(x,y) \xrightarrow{\circ} C(x,z)$$

In analogy with the discussion around Definition A.1.5, when one speaks of "enriching" an a priori unenriched category C over $\mathcal{V}$, the task is to define a $\mathcal{V}$-enriched category as in Definition A.2.1 whose underlying category recovers C. When $\mathcal{V} = \mathcal{C}at$, the task is to define a 2-category whose underlying 1-category is the one given. When $\mathcal{V} = s\mathcal{S}et$, the task is define simplicial hom-sets of n-arrows so that the 0-arrows are the ones given. When a simplicially enriched category C is encoded as a simplicial object $C_\bullet$ in $\mathcal{C}at$ as explained in Digression 1.2.3, its underlying category is the category C_0, further justifying the notion introduced in Definition A.2.2.

For example, a cartesian closed category $\mathcal{V}$ as in Definition A.1.1 can be enriched to define a $\mathcal{V}$-category.

A.2.3. Lemma. *A cartesian closed category $\mathcal{V}$ defines a $\mathcal{V}$-category whose:*

- *objects are the objects of $\mathcal{V}$,*
- *hom-object in $\mathcal{V}$ from A to B is given by the internal hom B^A, and*

[4] These diagrams suppose the associativity and unit natural isomorphisms involving the product bifunctors $\times$ and its unit object 1. In a cartesian closed category these are *canonical* — rather than given by extra data, as is the case in the more general *closed symmetric monoidal category*; see Exercise A.1.ii.

- *the identity map* $\mathrm{id}_A \colon 1 \to A^A$ *and composition map* $\circ \colon C^B \times B^A \to C^A$ *are defined to be the transposes of*
$$1 \times A \xrightarrow{\cong} A \qquad \text{and} \qquad C^B \times B^A \times A \xrightarrow{\mathrm{id} \times \epsilon} C^B \times B \xrightarrow{\epsilon} C$$
the latter defined using the counit of the cartesian closure adjunction.

Proof. The task is to verify the commutative diagrams of (A.2.1) in $\mathcal{V}$ and then observe that Lemma A.1.6 reveals that the underlying category of the $\mathcal{V}$-category defined by the statement is the 1-category $\mathcal{V}$. We leave the identity conditions to the reader and verify associativity.

The definition of the composition map as an adjoint transpose implies that *its* adjoint transpose, the top-right composite below, is given by the left-bottom composite:

$$\begin{tikzcd} C^B \times B^A \times A \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\epsilon"'] & C^A \times A \arrow[d, "\epsilon"] \\ C^B \times B \arrow[r, "\epsilon"'] & C \end{tikzcd}$$

The associativity diagram below-left commutes if and only if the transposed diagram appearing as the outer boundary composite below-right commutes:

$$\begin{tikzcd} D^C \times C^B \times B^A \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\circ"'] & D^B \times B^A \arrow[d, "\circ"] \\ D^C \times C^A \arrow[r, "\circ"'] & D^A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} D^C \times C^B \times B^A \times A \arrow[rr, "\circ\times\mathrm{id}\times\mathrm{id}"] \arrow[dd, "\mathrm{id}\times\circ\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times\epsilon"] & & D^B \times B^A \times A \arrow[d, "\mathrm{id}\times\epsilon"] \\ & D^C \times C^B \times B \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\epsilon"'] & D^B \times B \arrow[d, "\epsilon"] \\ D^C \times C^A \times A \arrow[r, "\mathrm{id}\times\epsilon"'] & D^C \times C \arrow[r, "\epsilon"'] & D \end{tikzcd}$$

which follows from bifunctoriality of $\times$ and two instances of the commutative square above. $\square$

A.2.4. Definition. The **free $\mathcal{V}$-category** on a 1-category C has

- the same objects as C
- the hom-objects defined to be coproducts $\coprod_{C(x,y)} 1$ of the terminal object 1 indexed by the hom-set $C(x,y)$
- the identity map $\mathrm{id}_x \colon 1 \to \coprod_{C(x,x)} 1$ given by the inclusion of the component indexed by the identity arrow
- the composition map given by acting by the composition function on the indexing sets:

$$\coprod_{C(y,z)} 1 \times \coprod_{C(x,y)} 1 \cong \coprod_{C(y,z)\times C(x,y)} 1 \xrightarrow{\coprod_\circ 1} \coprod_{C(x,z)} 1$$

For example, free $\mathcal{C}at$-enriched categories are those with no non-identity 2-cells and free $s\mathcal{S}et$-enriched categories are those with no non-degenerate arrows in positive dimensions. When context allows, we use the same name for the 1-category C and the free $\mathcal{V}$-category it generates, using language to disambiguate.

A.2.5. Definition. A **$\mathcal{V}$-enriched functor** $F \colon C \to \mathcal{D}$ is given by

- a mapping on objects that carries each $x \in C$ to some $Fx \in \mathcal{D}$

- for each pair of objects $x, y \in C$, an internal action on the $\mathcal{V}$-objects of arrows given by a morphism $F_{x,y}\colon C(x,y) \to \mathcal{D}(Fx, Fy) \in \mathcal{V}$ so that the $\mathcal{V}$-functoriality diagrams commute:

$$\begin{tikzcd} C(y,z) \times C(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z}\times F_{x,y}"'] & C(x,z) \arrow[d, "F_{x,z}"] \\ \mathcal{D}(Fy,Fz) \times \mathcal{D}(Fx,Fy) \arrow[r, "\circ"] & \mathcal{D}(Fx,Fz) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{Fx}"'] & C(x,x) \arrow[d, "F_{x,x}"] \\ & \mathcal{D}(Fx,Fx) \end{tikzcd}$$

A prototypical example is given by the representable functors:

A.2.6. Example. For any $\mathcal{V}$-category C and object $c \in C$, the **enriched representable $\mathcal{V}$-functor** $C(c,-)\colon C \to \mathcal{V}$ is defined on objects by the assignment $x \in C \mapsto C(c,x) \in \mathcal{V}$ and whose internal action on arrows

$$C(x,y) \xrightarrow{C(c,-)_{x,y}} C(c,y)^{C(c,x)} \qquad \leftrightsquigarrow \qquad C(x,y) \times C(c,x) \xrightarrow{\circ} C(c,y)$$

is given by the adjoint transpose of the internal composition map for C. The $\mathcal{V}$-functoriality diagrams are transposes of associativity and identity diagrams in C.

The contravariant enriched representable functors are defined similarly; see Exercise A.2.ii.

As the next result explains, an enriched representable functor can be thought of as a "two-step" enrichment of the corresponding unenriched representable functor: the first step enriches the hom-sets to hom-objects in $\mathcal{V}$ and the second step enriches the composition function to an internal composition map in $\mathcal{V}$.

A.2.7. Proposition. *To enrich a 1-category C over $\mathcal{V}$ one must:*

(i) First lift each of the unenriched representable functors through the underlying set functor to define an unenriched functor that encodes the data of the objects of arrows in $\mathcal{V}$:

$$\begin{tikzcd} C \arrow[rr, dashed, "C(c,-)"] \arrow[dr, "C(c,-)"'] & & \mathcal{V} \arrow[dl, "(-)_0"] \\ & \mathcal{S}et & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "C(c,-)"] & \mathcal{V} \\ x \arrow[r, mapsto] & C(c,x) \end{tikzcd}$$

(ii) Then enriched each of the unenriched $\mathcal{V}$-valued representable functors defined in (i) to a $\mathcal{V}$-functor $C(c,-)\colon C \to \mathcal{V}$ whose internal action on arrows encodes the data of the internal composition map for C.

Proof. The correspondence between the data in (i) and (ii) and the data of a $\mathcal{V}$-enriched category whose underlying category is C is clear. As discussed in Definition A.2.2, there is no additional data required by the identity arrows in an enriched category. Now the $\mathcal{V}$-functoriality of the enriched representable functor $C(c,-)\colon C \to \mathcal{V}$ is expressed by commutative diagrams

$$\begin{tikzcd} C(y,z) \times C(x,y) \arrow[r, dashed, "\circ"] \arrow[d, "C(c,-)_{y,z}\times C(c,-)_{x,y}"'] & C(x,z) \arrow[d, "C(c,-)_{x,z}"] \\ C(c,z)^{C(c,y)} \times C(c,y)^{C(c,x)} \arrow[r, "\circ"] & C(c,z)^{C(c,x)} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{C(c,x)}"'] & C(x,x) \arrow[d, "C(c,-)_{x,x}"] \\ & C(c,x)^{C(c,x)} \end{tikzcd}$$

— where the composition map in C is the one that has just been defined — and these transpose to the required associativity and identity axioms for the $\mathcal{V}$-category C. ☐

Both of the constructions of underlying unenriched categories and free categories extend functorially to functors; see Exercise A.2.iii. The relationship between these constructions is summarized by the following proposition, which we also decline to prove, because it will be generalized by a result that we do provide a proof for in §A.6.

A.2.8. PROPOSITION. *The free $\mathcal{V}$-category functor defines a fully faithful left adjoint to the underlying category functor. Consequently, a $\mathcal{V}$-category is free just when it is isomorphic to the free category on its underlying category via the counit of this adjunction.*

PROOF. Exercise A.2.iv or see §A.6. □

Exercises.

A.2.i. EXERCISE. Verify the unit condition left to the reader in the proof of Lemma A.2.3.

A.2.ii. EXERCISE. Define the **opposite** of a $\mathcal{V}$-category and dualize Example A.2.6 to define contravariant enriched representable functors.

A.2.iii. EXERCISE.

(i) Define the underlying functor of an enriched functor.
(ii) Prove that the passage from enriched functors to underlying unenriched functors is functorial.
(iii) Define the free enriched functor on an unenriched functor.
(iv) Prove the passage from unenriched functors to free enriched functors is functorial.

A.2.iv. EXERCISE. Prove A.2.8.

A.3. Enriched natural transformations and the enriched Yoneda lemma

Recall an (unenriched) natural transformation $\alpha\colon F \Rightarrow G$ between parallel functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ is given by:

- the data of an arrow $\alpha_x \in \mathcal{D}(Fx, Gx)$ for each $x \in \mathcal{C}$
- subject to the condition that for each morphism $f \in \mathcal{C}(x, y)$, the diagram

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\alpha_y"'] & Gy \end{tikzcd} \tag{A.3.1}$$

  commutes in $\mathcal{D}$.

This enriches to the notion of a $\mathcal{V}$*-natural transformation* whose data is exactly the same — a family of arrows in the underlying category of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ — but with a stronger $\mathcal{V}$-naturality condition expressed by internalizing the naturality condition (A.3.1) given above.

A.3.2. DEFINITION. A **$\mathcal{V}$-enriched natural transformation** $\alpha\colon F \Rightarrow G$ between $\mathcal{V}$-enriched functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ is given by:

- an arrow $\alpha_x\colon 1 \to \mathcal{D}(Fx, Gx)$ for each $x \in \mathcal{C}$

- so that for each pair of objects $x, y \in C$, the follow $\mathcal{V}$-naturality square commutes in $\mathcal{V}$:

$$\begin{tikzcd} C(x,y) \arrow[r, "\alpha_y \times F_{x,y}"] \arrow[d, "G_{x,y}\times\alpha_x"'] & \mathcal{D}(Fy,Gy)\times\mathcal{D}(Fx,Fy) \arrow[d, "\circ"] \\ \mathcal{D}(Gx,Gy)\times\mathcal{D}(Fx,Gx) \arrow[r, "\circ"'] & \mathcal{D}(Fx,Gy) \end{tikzcd} \tag{A.3.3}$$

A.3.4. Example. An arrow $f\colon 1 \to C(x,y)$ in the underlying category of a $\mathcal{V}$-category C defines a $\mathcal{V}$-natural transformation $f^*\colon C(y,-) \Rightarrow C(x,-)$ between the enriched representable functors whose component at $z \in C$ is defined by evaluating the adjoint transpose of the composition map at f:

$$1 \xrightarrow{f} C(x,y) \xrightarrow{C(-,z)_{x,y}} C(y,z)^{C(x,y)}$$

Evaluating one component of the associative diagram for C at f provides the required $\mathcal{V}$-naturality square.

A.3.5. Lemma.

(i) The ***vertical composite*** *of $\mathcal{V}$-enriched natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, both from C to $\mathcal{D}$ has component $(\beta\cdot\alpha)_x$ at $x \in C$ defined by the composite*

$$1 \xrightarrow{\beta_x\times\alpha_x} \mathcal{D}(Gx,Hx)\times\mathcal{D}(Fx,Gx) \xrightarrow{\circ} \mathcal{D}(Fx,Hx)$$

(ii) The ***horizontal composite*** *of $\alpha\colon F \Rightarrow G$ from C to $\mathcal{D}$ and $\gamma\colon H \Rightarrow K$ from $\mathcal{D}$ to $\mathcal{E}$ has component $(\gamma * \alpha)_x$ at $x \in C$ defined by the composite*

$$\begin{tikzcd} 1 \arrow[r, "\alpha_x"] & \mathcal{D}(Fx,Gx) \arrow[r, "\gamma_{Gx}\times H_{Fx,Gx}"] \arrow[d, "K_{Fx,Gx}\times\gamma_{Fx}"'] & \mathcal{E}(KGx,HGx)\times\mathcal{E}(HFx,HGx) \arrow[d, "\circ"] \\ & \mathcal{E}(KFx,KGx)\times\mathcal{E}(HFx,KFx) \arrow[r, "\circ"'] & \mathcal{D}(HFx,KGx) \end{tikzcd}$$

which is well-defined by $\mathcal{V}$-naturality of γ.

Proof. Exercise A.3.i. ☐

The data of the **underlying natural transformation** of a $\mathcal{V}$-naturality transformation is given by the same family of arrows. The unenriched naturality condition (A.3.1) is proven by evaluating the enriched naturality condition (A.3.3) at an underlying arrow $f\colon 1 \to C(x,y)$. In particular, the middle four interchange rule for horizontal and vertical composition of $\mathcal{V}$-natural transformations follows from the middle four interchange rule for horizontal and vertical composition of unenriched natural transformations for the data of the latter determines the data of the former. Consequently, Lemma A.3.5 implies that:

A.3.6. Corollary. *For any cartesian closed category $\mathcal{V}$, there is a 2-category $\mathcal{V}$-$\mathcal{C}at$ of $\mathcal{V}$-enriched categories, $\mathcal{V}$-enriched functors, and $\mathcal{V}$-enriched natural transformations. Moreoever, the underlying category functor $\mathcal{V}$-$\mathcal{C}at \to \mathcal{C}at$ and the free $\mathcal{V}$-category functor $\mathcal{C}at \to \mathcal{V}$-$\mathcal{C}at$ are both 2-functors.* ☐

We now turn our attention to the $\mathcal{V}$-enriched Yoneda lemma, which we present in several forms. One role of the Yoneda lemma is to give a representable characterization of isomorphic objects in C. When C is a $\mathcal{V}$-category, this has several forms involving each of the three notions of "representable"

functor appearing in Proposition A.2.7. The notion of $\mathcal{V}$-**natural isomorphism** referred to in the following result is defined to be a $\mathcal{V}$-natural transformation with an inverse for vertical composition.

A.3.7. LEMMA. *For objects x, y in a $\mathcal{V}$-category C the following are equivalent:*

(i) x and y are isomorphic as objects of the underlying category of C.
(ii) The $\mathcal{S}et$-valued unenriched representable functors $C(x,-)_0, C(y,-)_0 \colon C \rightrightarrows \mathcal{S}et$ are naturally isomorphic.
(iii) The $\mathcal{V}$-valued unenriched representable functors $C(x,-), C(y,-) \colon C \rightrightarrows \mathcal{V}$ are naturally isomorphic.
(iv) The $\mathcal{V}$-valued $\mathcal{V}$-functors $C(x,-), C(y,-) \colon C \rightrightarrows \mathcal{V}$ are naturally $\mathcal{V}$-isomorphic.

PROOF. Applying the underlying category functor $(-)_0 \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at$, the fourth statement implies the third. The third statement implies the second by composing the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$. The second statement implies the first by the unenriched Yoneda lemma; this is still the main point. Finally, the first statement implies the last by a direct construction: if $f \colon 1 \to C(x,y)$ and $g \colon 1 \to C(y,x)$ define an isomorphism in the underlying category of C, the corresponding representable $\mathcal{V}$-natural transformations of Example A.3.4 define a $\mathcal{V}$-natural isomorphism. □

Lemma A.3.7 defines a common notion of **isomorphism** between two objects of an enriched category, which turns out to be no different the usual unenriched notion of isomorphism. This can be thought of as defining a "cheap" form of the enriched Yoneda lemma. The full form of the $\mathcal{V}$-Yoneda lemma enriches the usual statement — a natural isomorphism between the set of natural transformations whose domain is a representable functor to the set defined by evaluating the codomain at the representing object — to an isomorphism in $\mathcal{V}$. The first step to make this precise is to enrich the set of $\mathcal{V}$-enriched natural transformations between a parallel pair of $\mathcal{V}$-functors can to define an object of $\mathcal{V}$.

A.3.8. DEFINITION. Let $\mathcal{V}$ be a complete cartesian closed category and consider a parallel pair of $\mathcal{V}$-functors $F, G \colon C \rightrightarrows \mathcal{D}$, with C a small $\mathcal{V}$-category. Then the **$\mathcal{V}$-object of $\mathcal{V}$-natural transformations** is defined by the equalizer diagram

$$\mathcal{V}^C(F,G) \rightarrowtail \prod_{z \in C} \mathcal{D}(Fz, Gz) \rightrightarrows \prod_{x,y \in C} \mathcal{D}(Fx, Gy)^{C(x,y)}$$

where one map in the equalizer diagram is defined by projecting to $\mathcal{D}(Fx, Gx)$, applying the internal action of G on arrows, and then composing, while the other is defined by projection to $\mathcal{D}(Fy, Gy)$, applying the internal action of F on arrows, and then composing.

A.3.9. LEMMA. *The underlying set of the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations $\mathcal{V}^C(F,G)$ is the set of $\mathcal{V}$-natural transformations from F to G.*

PROOF. By its defining universal property, elements of the underlying set of $\mathcal{V}^C(F,G)$ correspond to maps $\alpha \colon 1 \to \prod_{z \in C} \mathcal{D}(Fz, Gz)$ that equalize the parallel pair of maps described in Definition A.3.8. The map α defines the components of a $\mathcal{V}$-natural transformation $\alpha \colon F \Rightarrow G$ and the commutativity condition transposes to (A.3.3). □

The Yoneda lemma is usually expressed by the slogan "evaluation at the identity is an isomorphism," but since in the enriched context the enriched object of natural transformations is defined via a limit, it is easier to define the map that induces a natural transformation instead. Given an object $a \in \mathcal{A}$

in a small $\mathcal{V}$-category and a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, the internal action of F on arrows transposes to define a map that equalizes the parallel pair

$$Fa \overset{F_{a,-}}{\dashrightarrow} \prod_{z\in\mathcal{A}} Fz^{\mathcal{A}(a,z)} \rightrightarrows \prod_{x,y\in\mathcal{A}} Fy^{Fx\times\mathcal{A}(x,y)} \tag{A.3.10}$$

and thus induces a map $Fa \to \mathcal{V}^{\mathcal{A}}(\mathcal{A}(a,-),F)$ in $\mathcal{V}$.

A.3.11. Theorem (enriched Yoneda lemma). *For any small $\mathcal{V}$-category $\mathcal{A}$, object $a \in \mathcal{A}$, and $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, the canonical map defines an isomorphism in $\mathcal{V}$*

$$Fa \xrightarrow{\cong} \mathcal{V}^{\mathcal{A}}(\mathcal{A}(a,-),F),$$

that is $\mathcal{V}$-natural in both a and F.

Proof. To prove the isomorphism, it suffices to verify that (A.3.10) is a limit cone. To that end consider another cone over the parallel pair

$$V \xrightarrow{\lambda} \prod_{z\in\mathcal{A}} Fz^{\mathcal{A}(a,z)} \rightrightarrows \prod_{x,y\in\mathcal{A}} Fy^{Fx\times\mathcal{A}(x,y)}$$

and define a candidate factorization by evaluating the component λ_a at id_a:

$$\lambda_a(\mathrm{id}_a) \coloneqq V \xrightarrow{\mathrm{id}_a\times\mathrm{id}} \mathcal{A}(a,a)\times V \xrightarrow{\lambda_a} Fa$$

To see that $\lambda_a(\mathrm{id}_a)\colon V \to Fa$ indeed defines a factorization of λ through the limit cone, it suffices to show commutativity at each component $Fz^{\mathcal{A}}(a,z)$ of the product, which we verify in transposed form:

$$\begin{tikzcd}
\mathcal{A}(a,z)\times V \arrow[d, "\mathrm{id}\times\mathrm{id}_a\times\mathrm{id}"'] \arrow[dr, equal] & \\
\mathcal{A}(a,z)\times\mathcal{A}(a,a)\times V \arrow[r, "\circ\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\lambda_a"'] & \mathcal{A}(a,z)\times V \arrow[d, "\lambda_z"] \\
\mathcal{A}(a,z)\times Fa \arrow[r, "F_{a,z}"] & Fz
\end{tikzcd}$$

The upper triangle commutes by the identity law for $\mathcal{A}$ while the bottom square commutes because λ defines a cone over the parallel pair. Uniqueness of the factorization $\lambda_a(\mathrm{id}_a)$ follows from the same diagram by taking $z = a$ and evaluating at id_a.

We leave the verification of $\mathcal{V}$-naturality to the reader or to [**56**, §2.4]. □

Passing to underlying sets:

A.3.12. Corollary. *For any small $\mathcal{V}$-category $\mathcal{A}$, object $a \in \mathcal{A}$, and $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$, there is a bijection between $\mathcal{V}$-natural transformations $\alpha\colon \mathcal{A}(a,-) \Rightarrow F$ and elements $1 \to Fa$ in the underlying set of Fa implemented by evaluating the component at $a \in \mathcal{A}$ at the identity id_a.* □

This gives a criterion for establishing the representability of a $\mathcal{V}$-functor by presenting the minimal data required to establish the defining $\mathcal{V}$-natural isomorphism.

A.3.13. Corollary. *For a $\mathcal{V}$-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ and an object $r \in \mathcal{A}$ the following are equivalent and define what it means for r to* ***represent*** *$\mathcal{V}$:*

(i) There exists an isomorphism

$$\mathcal{A}(a,r) \cong Fa$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in $a \in \mathcal{A}$.

(ii) There exists an element $u\colon 1 \to Fr$ in the underlying set of Fr so that the composite map

$$\mathcal{A}(a,r) \xrightarrow{F_{a,r}} \mathcal{V}(Fr,Fa) \xrightarrow{-\circ u} \mathcal{V}(1,Fa) \cong Fa$$

defines an isomorphism in $\mathcal{V}$ for all $a \in \mathcal{A}$.

Proof. By Corollary A.3.12 the element $u\colon 1 \to Fr$ in the underlying set of Fr determines a unique $\mathcal{V}$-natural transformation $\mathcal{A}(-,r) \Rightarrow F$ whose component at $a \in \mathcal{A}$ is the map of the statement. Thus, the universal element u defines a $\mathcal{V}$-natural isomorphism and not just a $\mathcal{V}$-natural transformation just when the map of the statement is an isomorphism. □

Since we have assumed our bases for enrichment to be cartesian closed, the 2-category $\mathcal{V}$-$\mathcal{C}at$ admits finite products, allowing us to define multivariable $\mathcal{V}$-functors. The following result will imply that the structures characterized by $\mathcal{V}$-natural isomorphisms in §A.4 and §A.5 assemble into $\mathcal{V}$-functors.

A.3.14. Proposition. *Let $M\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ be a $\mathcal{V}$-functor so that for each $a \in \mathcal{A}$, the $\mathcal{V}$-functor $M(-,a)\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ is represented by some $Fa \in \mathcal{B}$, meaning there exists a $\mathcal{V}$-natural isomorphism*

$$\mathcal{B}(b,Fa) \cong M(b,a).$$

Then there is a unique way of extending the mapping $a \in \mathcal{A} \mapsto Fa \in \mathcal{B}$ to a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ so that the isomorphisms are $\mathcal{V}$-natural in $a \in \mathcal{A}$ as well as $b \in \mathcal{B}$.

Proof. By the Yoneda lemma in the form of Corollary A.3.12, to define a family of isomorphisms $\alpha_{b,a}\colon \mathcal{B}(b,Fa) \cong M(b,a)$ for each $a \in \mathcal{A}$ that are $\mathcal{V}$-natural in $b \in \mathcal{B}$ is to define a family of elements $\eta_a\colon 1 \to M(Fa,a)$ for each $a \in \mathcal{A}$. By the $\mathcal{V}$-naturality statement in the Yoneda lemma, for the former isomorphism to be $\mathcal{V}$-natural in a is equivalent to the family of elements $\eta_a\colon 1 \to M(Fa,a)$ being "extraordinarily" $\mathcal{V}$-natural in a. What this means is that for any pair of objects $a,a' \in \mathcal{A}$, the outer square commutes:

$$\begin{tikzcd}
\mathcal{A}(a',a) \arrow[rr, "M(F-{,}a)_{a,a'}"] \arrow[dr, dashed, "F"] \arrow[dd, "M(Fa'{,}-)_{a',a}"'] & & M(Fa',a)^{M(Fa,a)} \arrow[dd, "M(Fa'{,}a)^{\eta_a}"] \\
& \mathcal{B}(Fa',Fa) \arrow[ur, "M(-{,}a)"] \arrow[dr, "\alpha_{Fa',a}", "\cong"'] & \\
M(Fa',a)^{M(Fa',a')} \arrow[rr, "M(Fa'{,}a)^{\eta_{a'}}"'] & & M(Fa',a)
\end{tikzcd}$$

The definition of the top-horizontal map $M(F-,a)_{a',a}$ depends on the internal action of F on arrows, which we seek to define, but note that the composite of the other factor with the right vertical map is the natural isomorphism $\alpha_{Fa',a}$. Thus, there is a unique way to define $F_{a',a}$ making the extraordinary $\mathcal{V}$-naturality square commute, which is exactly the claim. $\mathcal{V}$-functoriality of these internal action maps for F follows from $\mathcal{V}$-functoriality of M in the $\mathcal{A}$ variable. □

We close this section with some applications. The correct notions of $\mathcal{V}$-enriched equivalence or $\mathcal{V}$-enriched adjunction are given by interpreting the standard 2-categorical notions of equivalence and adjunction in the 2-category $\mathcal{V}$. For sake of contrast, we present both notions in an alternate form here and leave it to the reader to apply Theorem A.3.11 to relate these to the 2-categorical notions.

A.3.15. Definition. A pair of $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ are **$\mathcal{V}$-equivalent** if there exists a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ that is

- **$\mathcal{V}$-fully faithful**: i.e., each $F_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is an isomorphism in $\mathcal{V}$ and
- essentially surjective on objects: i.e., each $d \in \mathcal{D}$ is isomorphic to Fc for some $c \in \mathcal{C}$.

A.3.16. Definition. A **$\mathcal{V}$-enriched adjunction** is given by a pair of $\mathcal{V}$-functors $F\colon \mathcal{B} \to \mathcal{A}$ and $U\colon \mathcal{A} \to \mathcal{B}$ together with isomorphisms

$$\mathcal{A}(Fb, a) \cong \mathcal{B}(b, Ua)$$

that are $\mathcal{V}$-natural in both $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

A.3.17. Remark. By Proposition A.3.14, a $\mathcal{V}$-functor $U\colon \mathcal{A} \to \mathcal{B}$ admits a $\mathcal{V}$-left adjoint if and only if each $\mathcal{B}(b, U-)\colon \mathcal{A} \to \mathcal{V}$ is represented by some object $Fb \in \mathcal{A}$, in which case the data of the $\mathcal{V}$-natural isomorphism $\mathcal{A}(Fb, -) \cong \mathcal{B}(b, U-)$ equips $b \in \mathcal{B} \mapsto Fb \in \mathcal{A}$ with the structure of a $\mathcal{V}$-functor. Dual remarks construct enriched right adjoints.

Exercises.

A.3.i. Exercise. Prove Lemma A.3.5.

A.3.ii. Exercise. Use Corollary A.3.12 to show that the notions of $\mathcal{V}$-equivalence and $\mathcal{V}$-adjunction given in Definitions A.3.15 and A.3.16 are equivalent to the 2-categorical notions in $\mathcal{V}\text{-}\mathcal{C}at$.

A.4. Tensors and cotensors

A $\mathcal{V}$-category $\mathcal{C}$ admits *tensors* just when for all $C \in \mathcal{C}$, the covariant representable functor $\mathcal{C}(C, -)\colon \mathcal{C} \to \mathcal{V}$ admits a left $\mathcal{V}$-adjoint $- \otimes C\colon \mathcal{V} \to \mathcal{C}$. Dually, a $\mathcal{V}$-category $\mathcal{C}$ admits *cotensors* just when the contravariant representable functor $\mathcal{C}(-, C)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ admits a mutual right adjoint $C^{-}\colon \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$. The aim in this section is to introduce both constructions formally. In the next section, we establish some useful formal properties of enriched categories that are either *tensored* or *cotensored*.

A.4.1. Definition. A $\mathcal{V}$-category $\mathcal{C}$ is **cotensored**, if for all $V \in \mathcal{V}$ and $C \in \mathcal{C}$, the $\mathcal{V}$-functor $\mathcal{C}(-, C)^V\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is represented by an object $C^V \in \mathcal{C}$, i.e., there exists an isomorphism

$$\mathcal{C}(X, C^V) \cong \mathcal{C}(X, C)^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By Proposition A.3.14, the cotensor product defines a unique $\mathcal{V}$-functor

$$\mathcal{C} \times \mathcal{V}^{\mathrm{op}} \xrightarrow{(-)^{-}} \mathcal{C}$$

making the defining isomorphism $\mathcal{V}$-natural in all three variables.

A.4.2. Definition. Dually, a $\mathcal{V}$-category $\mathcal{C}$ is **tensored** if for all $V \in \mathcal{V}$ and $\mathcal{C}$ the $\mathcal{V}$-functor $\mathcal{C}(C, -)^V\colon \mathcal{C} \to \mathcal{V}$ is represented by an object $V \otimes C \in \mathcal{C}$, i.e., there exists an isomorphism

$$\mathcal{C}(V \otimes C, X) \cong \mathcal{C}(C, X)^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By Proposition A.3.14, the tensor product defines a unique $\mathcal{V}$-functor

$$\mathcal{V}\times\mathcal{C} \xrightarrow{-\otimes-} \mathcal{C}$$

making the defining isomorphism $\mathcal{V}$-natural in all three variables.

Immediately from these definitions

A.4.3. LEMMA. *A $\mathcal{V}$-category $\mathcal{C}$ is tensored and cotensored if and only if the $\mathcal{V}$-functor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{V}$ is part of a two-variable $\mathcal{V}$-adjunction*

$$\begin{tikzcd}[column sep=small] & \mathcal{C}(V\times A,B) \arrow[dl, phantom, sloped, "\cong"] \arrow[dr, phantom, sloped, "\cong"] & \\ \mathcal{C}(A,B^V) \arrow[rr, phantom, "\cong"] & & \mathcal{C}(A,B)^V \end{tikzcd}$$

as expressed by the commutative triangle of $\mathcal{V}$-natural isomorphisms. ☐

A.4.4. LEMMA. *In any category $\mathcal{C}$ that is enriched and cotensored over $\mathcal{V}$, there are $\mathcal{V}$-natural isomorphisms*

$$C^1\cong C \qquad \textit{and} \qquad \begin{tikzcd}[column sep=small] & C^{U\times V} \arrow[dl, phantom, sloped, "\cong"] \arrow[dr, phantom, sloped, "\cong"] & \\ (C^V)^U \arrow[rr, phantom, "\cong"] & & (C^U)^V \end{tikzcd}$$

for $U,V\in\mathcal{V}$ and $C\in\mathcal{C}$.

PROOF. By Lemma A.3.7, to define the displayed $\mathcal{V}$-natural isomorphisms, it suffices to prove that these objects represent the same $\mathcal{V}$-functors $\mathcal{C}^{\mathrm{op}}\to\mathcal{V}$. By the defining universal property of the cotensor, for any $X\in\mathcal{C}$,

$$\mathcal{C}(X,C^1)\cong\mathcal{C}(X,C)^1\cong\mathcal{C}(X,C)$$

by Remark A.1.9. Similarly, by the defining universal property of the cotensor, we have $\mathcal{V}$-natural isomorphisms in each of the three vertices of the triangle below

$$\begin{tikzcd}[column sep=small] & \mathcal{C}(X,C^{U\times V})\cong\mathcal{C}(X,C)^{U\times V} \arrow[dl, phantom, sloped, "\cong"] \arrow[dr, phantom, sloped, "\cong"] & \\ \mathcal{C}(X,(C^V)^U)\cong(\mathcal{C}(X,C)^V)^U \arrow[rr, phantom, "\cong"] & & \mathcal{C}(X,(C^U)^V)\cong(\mathcal{C}(X,C)^U)^V \end{tikzcd}$$

whence the connecting $\mathcal{V}$-natural isomorphisms are given by Lemma A.1.7. ☐

Extending Lemma A.2.3:

A.4.5. LEMMA. *A cartesian closed category $(\mathcal{V},\times,1)$ is enriched, tensored, and cotensored over itself, with tensors defined by the cartesian product and cotensors defined by the internal hom*[5]*:*

$$V\otimes W := V\times W \qquad \textit{and} \qquad W^V := W^V.$$

PROOF. Lemma A.1.7 establishes the required isomorphisms (A.1.8) in $\mathcal{V}$. The proof of their $\mathcal{V}$-naturality is left to the reader or [**56**, §1.8]. ☐

The presence of tensors and cotensors provides a convenient mechanism for testing whether an a priori unenriched adjunction may be enriched to a $\mathcal{V}$-adjunction.

A.4.6. PROPOSITION. *Let $U\colon\mathcal{A}\to\mathcal{B}$ and $F\colon\mathcal{B}\to\mathcal{A}$ be unenriched adjoint functors between categories $\mathcal{A}$ and $\mathcal{B}$ that are both enriched over $\mathcal{V}$.*

[5]This justifies the abuse of exponential notation for both internal homs and cotensors.

(i) If $\mathcal{A}$ and $\mathcal{B}$ are cotensored over $\mathcal{V}$ and $U\colon \mathcal{A} \to \mathcal{B}$ is a cotensor-preserving $\mathcal{V}$-functor, then $F \dashv U$ may be enriched to a $\mathcal{V}$-adjunction.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are tensored over $\mathcal{V}$ and $F\colon \mathcal{B} \to \mathcal{A}$ is a tensor-preserving $\mathcal{V}$-functor, then $F \dashv U$ may be enriched to a $\mathcal{V}$-adjunction.

Conversely, if $F \dashv U$ is a $\mathcal{V}$-enriched adjunction then certainly F and U are $\mathcal{V}$-enriched and respectively tensor- and cotensor-preserving.

Proof. Exercise A.4.i. □

There is a convenient device for turning an a priori unenriched category $\mathcal{C}$ into a category that is both enriched and cotensored over simplicial sets that is of utility in constructing ∞-cosmoi. This is a specialization of the notion of *closed $\mathcal{V}$-module* of [**88**, 14.3] to the case $\mathcal{V} = s\mathcal{S}et$, in which case less data is required to prove Proposition A.4.8.

A.4.7. Definition. An unenriched category $\mathcal{C}$ is a **right $s\mathcal{S}et$-module** if there exists a bifunctor

$$\mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \xrightarrow{(-)^{-}} \mathcal{C}$$

equipped with natural isomorphisms

$$\begin{tikzcd} \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \times s\mathcal{S}et^{\mathrm{op}} \arrow[r, "(-)^{-}\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times(-\times-)"'] \arrow[dr, phantom, "\cong"] & \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \arrow[d, "(-)^{-}"] \\ \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \arrow[r, "(-)^{-}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "(-{,}1)"] \arrow[dr, "\mathrm{id}"'] & \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \arrow[d, "(-)^{-}"] \arrow[dl, phantom, "\cong"] \\ & \mathcal{C} \end{tikzcd}$$

asserting that for $U, V \in s\mathcal{S}et$ and $A \in \mathcal{C}$

$$A^{U\times V} \cong (A^U)^V \qquad \text{and} \qquad A \cong A^1$$

with these natural isomorphisms satisfying the manifest pseudo-algebra laws. A right $s\mathcal{S}et$-module is **continuous** if the bifunctor $\mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \to \mathcal{C}$ carries limits in its first variable to limits in $\mathcal{C}$ and carries colimits in its second variable to limits in $\mathcal{C}$.

A.4.8. Proposition. *A continuous right $s\mathcal{S}et$-module $\mathcal{C}$ can be enriched to a $s\mathcal{S}et$-category in a unique way so that there exist isomorphisms in $s\mathcal{S}et$*

$$\mathcal{C}(A, B^V) \cong \mathcal{C}(A, B)^V$$

for all $A, B \in \mathcal{C}$ and $V \in s\mathcal{S}et$ natural[6] in all three variables.

Proof. We must construct an enrichment from the supplied right pseudo-action. This is done straightforwardly by defining the n-arrow in $\mathcal{C}(A, B)$ to be arrows $A \to B^{\Delta[n]}$ in $\mathcal{C}$ and using the pseudo-action laws and the canonical diagonal co-algebra structure on $\Delta[n]$ to define an associative and unital composition of such. Note that the identity axiom of the right pseudo-action ensures that arrows $A \to B^{\Delta[0]}$ are in bijective correspondence with arrows $A \to B$ and thus that this construction really does give an enrichment of $\mathcal{C}$.

Furthermore, we must show that the pseudo-action supplies cotensors for this derived enrichment: that is that

$$\mathcal{C}(A, B^V) \cong \mathcal{C}(A, B)^V$$

[6] In fact these isomorphisms are $s\mathcal{S}et$-natural.

for all $A, B \in \mathcal{C}$ and $V \in s\mathcal{S}et$ natural in all three-variables, which fact follows from one of the assumed continuity properties of $(-)^{-} : \mathcal{C} \times s\mathcal{S}et^{\mathrm{op}} \to \mathcal{C}$ using the fact that every simplicial set can be expressed as a colimit of representables. $\square$

We have not yet made use of continuity in of the right action bifunctor in its first variable. The reason for this hypothesis will explained in §A.5.

Exercises.

A.4.i. Exercise. Prove Proposition A.4.6.

A.4.ii. Exercise. Let $\mathcal{J}$ be a small unenriched category and let $\mathcal{C}$ be a $\mathcal{V}$-category. Prove that if $\mathcal{C}$ is tensored or cotensored then so is $\mathcal{C}^{\mathcal{J}}$.

A.5. Conical limits

Let $\mathcal{V}$ be a cartesian closed category. The **conical limit** of a $\mathcal{V}$-enriched functor is the limit weighted by the terminal weight, the $\mathcal{V}$-functor that is constant at $1 \in \mathcal{V}$. In what follows, we focus on the slightly less general case of conical limits of unenriched diagrams — those indexed by 1-categories instead of $\mathcal{V}$-categories — as they will be of greatest interest here. Conical limits necessarily define 1-categorical limits, so we pay particular attention to what is required to enriched a 1-categorical limit to a conical limit.

To motivate the universal property that characterizes conical limits, consider a diagram $F : \mathcal{J} \to \mathcal{V}$ indexed by a 1-category $\mathcal{J}$ and valued in $\mathcal{V}$ itself. A cone λ over the diagram F with summit suggestively denoted $\lim F \in \mathcal{V}$ is a limit cone if it satisfies the universal property

$$\mathcal{V}(A, \lim F) \cong \lim_{j \in \mathcal{J}} \mathcal{V}(A, Fj),$$

an isomorphism of hom-sets. The next lemma reveals that this isomorphism can always be enriched to lie in $\mathcal{V}$, i.e., lifted along the underlying set functor of Definition A.1.5.

A.5.1. Lemma. *If $\mathcal{V}$ is a cartesian closed category, then any 1-categorical limit cone $\lambda : \Delta \lim F \Rightarrow F$ over a diagram $F : \mathcal{J} \to \mathcal{V}$ gives rise to an isomorphism in $\mathcal{V}$*

$$(\lim F)^A \cong \lim_{j \in \mathcal{J}} (Fj^A)$$

that is $\mathcal{V}$-natural in $A \in \mathcal{V}$.

Proof. For any $A \in \mathcal{V}$, the exponential $(-)^A : \mathcal{V} \to \mathcal{V}$ is right adjoint to the product functor $- \times A : \mathcal{V} \to \mathcal{V}$; as such it preserves limits, giving rise to the isomorphism of the statement. $\square$

In terminology we now introduce, Lemma A.5.1 asserts that all 1-categorical limits in a cartesian closed category $\mathcal{V}$ automatically enriched to define *conical limits*.

A.5.2. Definition. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $\mathcal{J}$ be a 1-category. The **conical limit** of an unenriched diagram $F : \mathcal{J} \to \mathcal{C}$ is given by an object $L \in \mathcal{C}$ and a cone $\lambda : \Delta L \Rightarrow F$ inducing a $\mathcal{V}$-natural isomorphism of hom-objects in $\mathcal{V}$

$$\mathcal{C}(X, L) \xrightarrow[\cong]{\lambda_*} \lim_{j \in \mathcal{J}} \mathcal{C}(X, Fj) \quad \in \quad \mathcal{V} \tag{A.5.3}$$

for all $X \in \mathcal{C}$.

The isomorphism (A.5.3) looks identical to the usual isomorphism that characterizes 1-categorical limits except for one very important difference: it postulates an isomorphism in $\mathcal{V}$ rather than an isomorphism in $\mathcal{S}et$. In the case where $\mathcal{V} = s\mathcal{S}et$, the isomorphism of vertices that underlies this isomorphism of simplicial sets describes the usual 1-categorical universal property. To say that the limit is *conical* and not merely *1-categorical* is to assert that this universal property extends to all positive dimensions.

Inspecting Definition A.5.2, we see immediately that:

A.5.4. PROPOSITION. *A 1-categorical limit cone is conical just when it is preserved by all $\mathcal{V}$-valued representable functors $C(X,-): C \to \mathcal{V}$.* □

The proof of Lemma A.5.1 generalizes to show:

A.5.5. PROPOSITION. *If C has tensors, then all 1-categorical limits in C are conical.*

PROOF. By Proposition A.5.4, to show that a 1-categorical limit is conical, it suffices to show that it is preserved by the $\mathcal{V}$-valued representable functors $C(X,-): C \to \mathcal{V}$ for all $X \in \mathcal{V}$. If C admits tensors, then each of these functors admits a left adjoint; as right adjoints, they necessarily preserve the 1-categorical limits of the statement. □

A.5.6. PROPOSITION. *Let C be enriched and cotensored over $\mathcal{V}$. A limit of an unenriched diagram $F: \mathcal{J} \to C$ is a conical limit if and only if it is preserved by cotensors with all objects $V \in \mathcal{V}$.*

PROOF. Cotensors are $\mathcal{V}$-enriched right adjoints, which preserve conical limits. The content is that preservation by cotensors suffices to enriched a 1-categorical limit to a $\mathcal{V}$-categorical one.

Recall that a 1-categorical limit cone $(\lambda_j: L \to Fj)_{j\in\mathcal{J}}$ is conical just when it is preserved by all $\mathcal{V}$-valued representable functors $C(X,-): C \to \mathcal{V}$. To see that the natural map

$$C(X,L) \to \lim_{j\in\mathcal{J}} C(X,Fj)$$

is an isomorphism in $\mathcal{V}$ we appeal to the (unenriched) Yoneda lemma and prove that for all $V \in \mathcal{V}$, the map of hom-sets

$$\mathcal{V}(V, C(X,L))_0 \to \mathcal{V}(V, \lim_{j\in\mathcal{J}} C(X,Fj))_0 \tag{A.5.7}$$

is an isomorphism. Since unenriched representables preserve 1-categorical limits,

$$\mathcal{V}(V, \lim_{j\in\mathcal{J}} C(X,Fj))_0 \cong \lim_{j\in\mathcal{J}} \mathcal{V}(V, C(X,F_j))_0.$$

By the unenriched universal property of cotensors, maps $V \to C(X,L)$ in $\mathcal{V}$ correspond bijectively to maps $X \to L^V$ in C. So the map (A.5.7) is isomorphic to the map of hom-sets

$$C(X,L^V)_0 \to \lim_{j\in\mathcal{J}} C(X,Fj^V)_0 \cong C(X, \lim_{j\in\mathcal{J}}(Fj^V))_0,$$

where again the unenriched representable commutes with the 1-categorical limit. To say that cotensors preserve the limit $L \cong \lim_{j\in\mathcal{J}} Fj$ means that $\lim_{j\in\mathcal{J}}(Fj^V) \cong (\lim_{j\in\mathcal{J}} Fj)^V \cong L^V$, so by the Yoneda lemma, the map of hom-sets

$$C(X,L^V)_0 \to C(X, \lim_{j\in\mathcal{J}}(Fj^V))_0,$$

is an isomorphism, and thus (A.5.7) and hence $C(X,L) \cong \lim_{j\in\mathcal{J}} C(X,Fj)$ is an isomorphism as desired. □

A.5.8. Proposition. *Suppose C is a $\mathcal{V}$-category that admits conical limits indexed by a small category $\mathcal{J}$. Then the limit functor* $\lim\colon C^{\mathcal{J}} \to C$ *is a $\mathcal{V}$-functor.*

Proof. We first prove that the conical limit functor $\lim\colon \mathcal{V}^{\mathcal{J}} \to \mathcal{V}$ is a $\mathcal{V}$-functor. The result for a general $\mathcal{V}$-category C then follows from the defining $\mathcal{V}$-natural isomorphism (A.5.3) via Proposition A.3.14.

The advantage of this reduction is that we can make use of the fact that $\mathcal{V}$ and $\mathcal{V}^{\mathcal{J}}$ are tensored, the latter following from the former by Exercise A.4.ii. Now by Proposition A.4.6, to show that the conical limit is a $\mathcal{V}$-functor it suffices to show that it's unenriched left adjoint $\Delta\colon \mathcal{V} \to \mathcal{V}^{\mathcal{J}}$ is a $\mathcal{V}$-functor that preserves tensors, and this is straightforward. ☐

Exercises.

A.5.i. Exercise. Specialize the result of Proposition A.5.5 to prove the following: in any 2-category C that admits tensors with the walking-arrow category $\mathbb{2}$, any 1-categorical limits that C admits are automatically conical.[7]

A.6. Change of base

Change of base for enriched categories was first considered in [**37**]. Their main result, appearing as Proposition A.6.4 below, is that a *lax monoidal functor* $T\colon \mathcal{V} \to \mathcal{W}$ induces a change-of-base 2-functor $T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$.

A.6.1. Definition. A **(lax) monoidal functor** between cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$ is a functor $T\colon \mathcal{V} \to \mathcal{W}$ equipped with natural transformations

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, "T\times T"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\ \mathcal{V} \arrow[r, "T"'] & \mathcal{W} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[dr, "1"'] & \mathcal{V} \arrow[d, "T"] \arrow[dl, phantom, "\Uparrow\phi_0"] \\ & \mathcal{W} \end{tikzcd}$$

so that the evident associativity and unit diagrams commute.

A.6.2. Definition. A **(lax) monoidal natural transformation** between monoidal functors between cartesian closed categories $T, U\colon \mathcal{V} \rightrightarrows \mathcal{W}$ is given by a natural transformation $\theta\colon T \Rightarrow U$ so that the pasting diagrams commute

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, bend left, "T\times T"] \arrow[r, bend right, "U\times U"'] \arrow[r, phantom, "\Downarrow\theta\times\theta"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\ \mathcal{V} \arrow[r, "U"'] & \mathcal{W} \end{tikzcd} = \begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, "T\times T"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathcal{W}\times\mathcal{W} \arrow[d, "\times"] \\ \mathcal{V} \arrow[r, bend left, "T"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{W} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[dr, bend right, "1"'] & \mathcal{V} \arrow[d, bend right, "T"'] \arrow[d, bend left, "U"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \arrow[dl, phantom, "\Uparrow\phi_0"] \\ & \mathcal{W} \end{tikzcd} = \begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[dr, "1"'] & \mathcal{V} \arrow[d, "U"] \arrow[dl, phantom, "\Uparrow\phi_0"] \\ & \mathcal{W} \end{tikzcd}$$

[7] The statement asserts that the presence of tensors with $\mathbb{2}$ implies that the universal property of 1-dimensional limits automatically has an additional 2-dimensional aspect. See the discussion around Proposition 1.4.5 for an illustration of this and see [**55**, pp. 306] for a proof.

A.6.3. Example. A functor $T\colon \mathcal{V} \to \mathcal{W}$ between cartesian closed categories **preserves finite products** just when the natural maps defined for any $A, B \in \mathcal{V}$

$$T(A \times B) \xrightarrow{\cong} TA \times TB \qquad \text{and} \qquad T1 \xrightarrow{\cong} 1$$

are isomorphisms. The inverse isomorphisms then make T into a **strong monoidal functor** between the cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$, with the structure maps of Definition A.6.1 given by natural isomorphisms. Moreover, any natural transformation between product-preserving functors is automatically a monoidal natural transformation.

The following result was first stated as [**37**, II.6.3], with the proof left to the reader. We adopt the same tactic and leave the diagram chases to the reader or to [**27**, 4.2.4] and instead just give the definition.

A.6.4. Proposition. *A monoidal functor $T\colon \mathcal{V} \to \mathcal{W}$ induces a change-of-base 2-functor*

$$T_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at.$$

Proof. The construction of the change-of-base 2-functor is the important thing. Let C be a $\mathcal{V}$-category and define a $\mathcal{W}$-category T_*C to have the same objects and to have mapping objects $T_*C(x,y) := TC(x,y)$. The composition and identity maps are given by the composites

$$TC(y,z) \times TC(x,y) \xrightarrow{\phi} T(C(y,z) \times C(x,y)) \xrightarrow{T\circ} TC(x,z) \qquad 1 \xrightarrow{\phi_0} T1 \xrightarrow{T\mathrm{id}_x} TC(x,x)$$

which make use of the structure maps of the monoidal functor. A straightforward diagram chases verifies that T_*C is a $\mathcal{W}$-category.

If $F\colon C \to \mathcal{D}$ is a $\mathcal{V}$-functor, then we define a $\mathcal{W}$-functor $T_*F\colon T_*C \to T_*\mathcal{D}$ to act on objects by $c \in C \mapsto Fc \in \mathcal{D}$ and with internal action on arrows defined by

$$TC(x,y) \xrightarrow{TF} \mathcal{D}(Tx, Ty)$$

Again, a straightforward diagram chase verifies that T_*F is $\mathcal{V}$-functorial. It is evident from this definition that $T_*(GF) = T_*G \cdot T_*F$.

Finally, let $\alpha\colon F \Rightarrow G$ be a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors $F, G\colon C \rightrightarrows \mathcal{D}$ and define a $\mathcal{W}$-natural transformation $T_*\alpha\colon T_*F \Rightarrow T_*G$ to have components

$$1 \xrightarrow{\phi_0} T1 \xrightarrow{T\alpha_c} T\mathcal{D}(Fc, Gc)$$

Another straightforward diagram chase verifies that $T_*\alpha$ is $\mathcal{W}$-natural.

It remains to verify this assignment is functorial for both horizontal and vertical composition of enriched natural transformations. Consulting Lemma A.3.5, we see that the component of $T_*(\beta \cdot \alpha)$ is defined by the top-horizontal composite while the component of the vertical composite of $T_*\alpha$ with $T_*\beta\colon T_*G \Rightarrow T_*H$ is defined by the bottom composite:

$$\begin{tikzcd}
1 \arrow[r, "\phi_0"] \arrow[dr, "\phi_0 \times \phi_0"'] & T1 \arrow[r, "T(\beta_c \times \alpha_c)"] & T(\mathcal{D}(Gc,Hc) \times \mathcal{D}(Fc,Gc)) \arrow[r, "T\circ"] & T\mathcal{D}(Fc,Hc) \\
& T1 \times T1 \arrow[u, "\phi"'] \arrow[r, "T\beta_c \times T\alpha_c"'] & T\mathcal{D}(Gc,Hc) \times T\mathcal{D}(Fc,Gc) \arrow[u, "\phi"'] &
\end{tikzcd}$$

By the unit axiom for the monoidal functor and naturality of ϕ, these composites agree. The argument for functoriality of horizontal composites is similar. □

For any cartesian closed category $\mathcal{V}$, the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$ is monoidal (Exercise A.6.i). Consequently:

A.6.5. COROLLARY. *For any cartesian closed category $\mathcal{V}$, the underlying category construction defines a 2-functor*

$$(-)_0 \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at. \quad \square$$

A.6.6. REMARK. In fact, the change of base procedure is *itself* a 2-functor from the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations to the 2-category of 2-categories, 2-functors, and 2-natural transformations. See [**27**, §4.3] for a discussion and proof.

If the lax monoidal functor $T \colon \mathcal{V} \to \mathcal{W}$ does not commute with the underlying set functors for $\mathcal{V}$ and $\mathcal{W}$ up to natural isomorphism, the change-of-base 2-functor $T_* \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$ will not preserve underlying categories, even up to natural isomorphism. Consulting Definition A.2.2, the following condition on T is required to ensure that underlying categories are preserved.

A.6.7. LEMMA. *The change-of-base 2-functor induced by a monoidal functor $(T, \phi, \phi_0) \colon \mathcal{V} \to \mathcal{W}$ preserves underlying categories, if and only if, for each $V \in \mathcal{V}$ the composite function on hom-sets*

$$\mathcal{V}(1, V)_0 \xrightarrow{T} \mathcal{W}(T1, TV)_0 \xrightarrow{-\circ\phi_0} \mathcal{W}(1, TV)_0$$

is a bijection.

PROOF. The displayed function defines the component at $V \in \mathcal{V}$ of the unique monoidal natural transformation from the underlying set-functor for $\mathcal{V}$ with the composite of T with the underlying set functor for $\mathcal{W}$. By Remark A.6.6, if it defines a monoidal natural isomorphism, then it induces a 2-natural isomorphism between the underlying category 2-functor $(-)_0 \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at$ and the composite of the change of base 2-functor $T_* \colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$ with the underlying category 2-functor $(-)_0 \colon \mathcal{W}\text{-}\mathcal{C}at \to \mathcal{C}at$.

Conversely, this condition is necessary for the underlying category of the $\mathcal{W}$-category $T_*\mathcal{V}$ to coincide with the underlying category of the cartesian closed category $\mathcal{V}$. $\square$

One situation in which the condition of Lemma A.6.7 is automatic is when the lax monoidal functor is the right adjoint of a monoidal adjunction, which in this context might be thought of as a *change-of-base adjunction*. The proof, originally given in [**54**], is by a short diagram chase left to Exercise A.6.ii.

Another immediate consequence of Remark A.6.6 is that the monoidal adjunctions of Exercise A.6.ii induce an adjunction between the corresponding change-of-base 2-functors. Between cartesian closed categories $\mathcal{V}$ and $\mathcal{W}$, the statement of this more general result simplifies by applying the ideas of Example A.6.3. The right adjoint functor preserves finite products and so is automatically strong monoidal, and by Exercise A.6.ii(i) the left adjoint is as well. Any natural transformation between product-preserving functors automatically defines a monoidal natural transformation. Consequently:

A.6.8. PROPOSITION. *Any adjunction comprised of finite-product preserving functors between cartesian closed categories induces a change-of-base 2-adjunction*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{V}\text{-}\mathcal{C}at \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{W}\text{-}\mathcal{C}at \arrow[l, bend left, "U_*"] \end{tikzcd} \quad \square$$

A.6.9. Lemma. *Any adjunction comprised of finite-product preserving functors between cartesian closed categories*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

defines a $\mathcal{V}$-enriched adjunction between F and $U_*\mathcal{W}$; i.e., there exists a $\mathcal{V}$-natural isomorphism

$$U\mathcal{W}(Fv, w) \cong \mathcal{V}(v, Uw).$$

Proof. It's easy to verify the $\mathcal{V}$-functoriality of $U\colon U_*\mathcal{W} \to \mathcal{V}$, which implies that for all $v \in \mathcal{V}$, the map

$$U\mathcal{W}(Fv, w) \xrightarrow{U_{Fv,w}} \mathcal{V}(UFv, Uw) \xrightarrow{-\circ\eta_v} \mathcal{V}(v, Uw)$$

is $\mathcal{V}$-natural in $w \in U_*\mathcal{W}$. By Remark A.3.17, to construct a compatible $\mathcal{V}$-enrichment of F, we need only demonstrate that this map in an isomorphism in $\mathcal{V}$.

We do this by constructing an explicit inverse, namely

$$\mathcal{V}(v, Uw) \xrightarrow{\eta} UF\mathcal{V}(v, Uw) \xrightarrow{U(F_{v,Uw})} U\mathcal{W}(Fv, FUw) \xrightarrow{\epsilon_w\circ-} U\mathcal{W}(Fv, w)$$

where the middle map is defined by applying the unenriched functor U to the canonical action map $F_{v,Uw}\colon F\mathcal{V}(v, Uw) \to \mathcal{W}(Fv, FUw)$ from the $\mathcal{W}$-functor $F\colon F_*\mathcal{V} \to \mathcal{W}$.

The proof that this maps are inverses involves a pair of diagram chases, the first of which demonstrates that the top-right composite reduces to the left-bottom composite, which is the identity:

$$\begin{tikzcd} \mathcal{V}(v, Uw) \arrow[r, "\eta"] \arrow[ddrr, "\eta_{Uw}\circ-"'] & UF\mathcal{V}(v, Uw) \arrow[r, "U(F_{v,Uw})"] \arrow[dr, "UF_{v,Uw}"] & U\mathcal{W}(Fv, FUw) \arrow[r, "\epsilon_w\circ-"] \arrow[d, "U_{Fv,FUw}"] & U\mathcal{W}(Fv, w) \arrow[d, "U_{Fv,w}"] \\ & & \mathcal{V}(UFv, UFUw) \arrow[r, "U\epsilon_w\circ-"] \arrow[d, "-\circ\eta_v"] & \mathcal{V}(UFv, Uw) \arrow[d, "-\circ\eta_v"] \\ & & \mathcal{V}(v, UFUw) \arrow[r, "U\epsilon_w\circ-"] & \mathcal{V}(v, Uw) \end{tikzcd}$$

The only subtle point is the commutativity of the trapezoidal region, which expresses the fact that $\eta\colon \mathrm{id}_{\mathcal{V}} \Rightarrow UF$ is a *closed natural transformation* between product-preserving functors between cartesian closed categories. This region commutes because the transposed diagram does:

$$\begin{tikzcd} v \times \mathcal{V}(v, Uw) \arrow[r, equal] \arrow[d, "\eta_v\times\eta"'] & v \times \mathcal{V}(v, Uw) \arrow[r, "\mathrm{ev}"] \arrow[d, "\eta"'] & Uw \arrow[d, "\eta_{Uw}"] \\ UFv \times UF\mathcal{V}(v, Uw) \arrow[r, "\cong"] & UF(v \times \mathcal{V}(v, UW)) \arrow[r, "UF\mathrm{ev}"] & UFUw \end{tikzcd}$$

the right-hand square by naturality, and the left-hand square because any naturally transformation between product-preserving functors is automatically a (cartesian) *monoidal* natural transformation. The other diagram chase is similar. □

A.6.10. PROPOSITION. *Given a change of base adjunction comprised of finite-product preserving functors between cartesian closed categories*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

then if $\mathcal{M}$ is tensored or cotensored as a $\mathcal{W}$-category, then $U_*\mathcal{M}$ is tensored or cotensored as $\mathcal{V}$-category with the tensors or cotensor of $M \in \mathcal{M}$ by $V \in \mathcal{V}$ defined by

$$V \otimes M := FV \otimes M \qquad \textit{and} \qquad M^V := M^{FV}.$$

PROOF. The statements are dual. Suppose $\mathcal{M}$ admits cotensors as a $\mathcal{W}$-category. To verify that $U_*\mathcal{M}$ admits cotensors as a $\mathcal{V}$-category we must supply an isomorphism

$$U\mathcal{M}(X, M^{FV}) \cong (U\mathcal{M}(X, M))^V$$

in $\mathcal{V}$ that is $\mathcal{V}$-natural in X. By the enriched Yoneda lemma, we can extract this isomorphism from an

$$\mathcal{V}(Z, U\mathcal{M}(X, M^{FV})) \cong \mathcal{V}(Z, (U\mathcal{M}(X, M))^V)$$

natural in $Z \in \mathcal{V}$. To that end, we have

$$\begin{aligned} \mathcal{V}(Z, U\mathcal{M}(X, M^{FV})) &\cong U\mathcal{W}(FZ, \mathcal{M}(X, M^{FV})) \cong U\mathcal{W}(FZ, \mathcal{M}(X, M)^{FV}) \\ &\cong U\mathcal{W}(FZ \times FV, \mathcal{M}(X, M)) \cong U\mathcal{W}(F(Z \times V), \mathcal{M}(X, M)) \\ &\cong \mathcal{V}(Z \times V, U\mathcal{M}(X, M)) \cong \mathcal{V}(Z, (U\mathcal{M}(X, M))^V), \end{aligned}$$

by composing the $\mathcal{V}$-natural isomorphisms of Lemma A.6.9 arising from the adjunction $F \dashv U$, the enriched natural isomorphisms arising from the cartesian closed structure on $\mathcal{V}$ and on $U_*\mathcal{W}$, and the fact that F preserves binary products. $\square$

A.6.11. EXAMPLE. Both adjoints of the adjunction

$$\begin{tikzcd} \mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathsf{h}"'] \end{tikzcd}$$

of Proposition 1.1.11 preserve finite products. Hence, by Proposition A.6.8 induces a change-of-base adjunction defined by the 2-functors

$$\begin{tikzcd} 2\text{-}\mathcal{C}at \arrow[r, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend right, "\mathsf{h}_*"'] \end{tikzcd}$$

that act identically on objects and act by applying the homotopy category functor or nerve functor, respectively, on homs. Note h also satisfies the condition of Lemma A.6.7 so both adjoints preserve underlying categories, as is evident from direct computation.

Exercises.

A.6.i. EXERCISE. For any cartesian closed category $\mathcal{V}$, prove that the underlying set functor $(-)_0 \colon \mathcal{V} \to \mathcal{S}et$ is lax monoidal

A.6.ii. EXERCISE. Consider an adjunction in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations.

(i) Prove that the left adjoint is strong monoidal.[8]
(ii) Prove that the right adjoint is "pro-normal," satisfying the condition of Lemma A.6.7.

[8]Hint: take the mates of the structure maps of the monoidal right adjoint.

APPENDIX B

An introduction to 2-category theory

2-categories — categories *enriched* in $\mathcal{C}at$ — and double categories — categories defined *internally* to $\mathcal{C}at$ — were first introduced by Charles Ehresmann. A notable early expository account appeared in [**57**]. The basic definitions are given in §B.1, which pays particular attention to the composition of 2-cells in a 2-category by means of *pasting*.

In §B.2, we briefly answer the question: what do 2-categories form? We define three dimensions of morphisms between 2-categories — the 2-functors, 2-natural transformations, and modifications — and observe that these collectively assemble into a *3-category*, this being a category enriched over 2-categories.

In §B.3 and §B.4 we develop the calculus of adjunctions and mates in any 2-category, complementing the results of §2.1, and the special case of right adjoint right inverse adjunctions. Miscellaneous 2-categorical lemmas needed elsewhere appear in §B.5. Finally, in §B.6 we consider the representability of various 2-categorical structures and comment briefly on the bicategorical Yoneda lemma.

B.1. 2-categories and the calculus of pasting diagrams

The category $\mathcal{C}at$ of small categories is cartesian closed with exponentials B^A defined to be the category of functors and natural transformations from A to B and terminal object $\mathbb{1}$. Exploiting the work in Appendix A, we can concisely define a 2-category to be a category enriched over this cartesian closed category. Unpacking this definition, we see it contains a considerable amount of structure:

B.1.1. Definition (2-category). A **2-category** $\mathcal{C}$ is a category enriched in $\mathcal{C}at$. Explicitly it has:

- a collection of objects A, B;
- a collection of arrows $f\colon A \to B$, frequently called **1-cells**, these being the objects of the hom-categories $\mathcal{C}(A, B)$; and
- a collection of arrows between arrows $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$, called **2-cells**,[1] these being the morphisms of the hom-categories $\mathcal{C}(A, B)$ from f to g

so that:

(i) For each fixed pair of objects $A, B \in \mathcal{C}$, the 1-cells and 2-cells form a category. In particular:

[1]Implicitly in this graphical representation, is the requirement that a 2-cell α has a domain 1-cell f and a codomain 1-cell g, and these 1-cells have a common domain object A and codomain object B. The objects A and B may be referred to as the 0-cell source and 0-cell target of α.

- A pair of 2-cells as below-left admits a **vertical composite** as below-right:

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \end{tikzcd} \qquad =: \qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & B \end{tikzcd}$$

- Each 1-cell $f\colon A \to B$ has an identity 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & B \end{tikzcd}$.

(ii) The objects and 1-cells define a category in the ordinary sense; in particular, each object has an identity arrow $\mathrm{id}_A\colon A \to A$.

(iii) The objects and 2-cells form a category. In particular:

- A pair of 2-cells as below-left admits a **horizontal composite** as below-right:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd} \qquad =: \qquad \begin{tikzcd} A \arrow[r, bend left, "jf"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow\gamma*\alpha"] & C \end{tikzcd}$$

- The identity 2-cells on identity 1-cells

$$\begin{tikzcd} A \arrow[r, bend left, "\mathrm{id}_A"] \arrow[r, bend right, "\mathrm{id}_A"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{\mathrm{id}_A}"] & A \end{tikzcd}$$

define identities for horizontal composition.

(iv) Finally, the horizontal composition is functorial with respect to the vertical composition:

- The horizontal composite of the identity 2-cells is an identity 2-cell:

$$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_g"] & B \arrow[r, bend left, "k"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_k"] & C \end{tikzcd} \qquad = \qquad \begin{tikzcd} A \arrow[r, bend left, "kg"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{kg}"] & C \end{tikzcd}$$

- In the situation below, the horizontal composite of the vertical composites coincides with the vertical composite of the horizontal composites, a property referred to as **middle-four interchange**

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[r, bend left=50, "j"] \arrow[r, "k" description] \arrow[r, bend right=50, "\ell"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & C \end{tikzcd}$$

A degenerate special case of horizontal composition, in which all but one of the 2-cells is an identity id_f on its boundary 1-cell f, is called "whiskering."

B.1.2. Definition (whiskering). The **whiskered composite** of a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$ with a pair of 1-cells $a\colon X \to A$ and $b\colon B \to Y$ is defined by the horizontal composite:

$$\begin{tikzcd} X \arrow[r, bend left, "bfa"] \arrow[r, bend right, "bga"'] \arrow[r, phantom, "\Downarrow b\cdot\alpha\cdot a"] & Y \end{tikzcd} \quad := \quad \begin{tikzcd} X \arrow[r, "a"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "b"] & Y \end{tikzcd} \quad := \quad \begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_a"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_b"] & Y \end{tikzcd}$$

As the following lemma reveals, horizontal composition can be recovered from vertical composition and whiskering. Our primary interest in this result, however, has to with a rather prosaic consequence appearing as the final part of the statement, which will be applied shockingly often.

B.1.3. Lemma (naturality of whiskering). *Any horizontally-composable pair of 2-cells in a 2-category gives rise to a commutative square in the hom-category from the domain object to the codomain object whose diagonal defines the horizontal composite:*

$$\mathcal{C} \ni \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} hf \arrow[r, Rightarrow, "\beta f"] \arrow[d, Rightarrow, "h\alpha"'] \arrow[dr, Rightarrow, "\beta*\alpha"] & kf \arrow[d, Rightarrow, "k\alpha"] \\ hg \arrow[r, Rightarrow, "\beta g"'] & kg \end{tikzcd} \quad \in \mathcal{C}(A,C)$$

In particular, if any three of the four whiskered 2-cells $h\alpha$, $k\alpha$, βf, and βg are invertible, so is the fourth.

Proof. By middle-four interchange:

$$\beta g \cdot h\alpha = (\beta * \mathrm{id}_g)\cdot(\mathrm{id}_h * \alpha) = (\beta\cdot\mathrm{id}_h)*(\mathrm{id}_g\cdot\alpha) = \beta*\alpha = (\mathrm{id}_k\cdot\beta)*(\alpha\cdot\mathrm{id}_f) = (\mathrm{id}_k*\alpha)\cdot(\beta*\mathrm{id}_f) = k\alpha\cdot\beta f.$$

□

The operations of horizontal and vertical composition are special cases of composition by *pasting*, an operation first introduced by Bénabou [**6**]. The main result, proven in the 2-categorical context by Power [**70**] is that a well-formed pasting diagram such as

$$\begin{tikzcd} & B \arrow[rrr, "b"] \arrow[dr, "d"] \arrow[d, phantom, "\Downarrow\gamma"] & & \arrow[d, phantom, "\Downarrow\alpha"] & F \arrow[dr, "g"] \arrow[drr, bend left, "c"] & \arrow[d, phantom, "\Downarrow\beta"] & \\ A \arrow[ur, "a"] \arrow[rr, "i"] \arrow[drr, "j"'] & {} & C \arrow[r, "e"] \arrow[d, phantom, "\Downarrow\delta"] & E \arrow[ur, "f"] & \Downarrow\epsilon & H \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\zeta"] & Z \\ & & D \arrow[ur, "k"] \arrow[rr, "\ell"'] & & G \arrow[ur, "m"] \arrow[urr, bend right, "n"'] & {} & \end{tikzcd} \tag{B.1.4}$$

has a unique 2-cell composite.[2] We leave the formal statement and proof of this result to the literature and instead describe informally how such pasting composites should be interpreted.

B.1.5. Digression (how to read a pasting diagram). A pasting diagram in a 2-category $\mathcal{C}$ represents a unique composite 2-cell, defining a morphism in one of the hom-categories between a pair of objects. To identify these objects, look at the underlying directed graph of objects and 1-cells in the pasting diagram. If the pasting diagram is well-formed, that graph should have a unique source object A and

[2]This result was generalized to the bicategorical context by Verity [**103**], in which case the composite 2-cell is well-defined once its source and target 1-cells are specified.

a unique target object Z. This indicates that the pasting diagram defines a 2-cell in the hom-category $C(A, Z)$. The object A is its **source 0-cell** and the object Z is its **target 0-cell**.

The next step is to identify the source 1-cell and the target 1-cell of the pasting diagram. These should both be objects of $C(A, Z)$, i.e., 1-cells in the 2-category from A to Z. Again if the pasting diagram is well-formed, the **source 1-cell** should be the unique composable path of 1-cells none of which occur as codomains of any 2-cells in the pasting diagram. In the diagram (B.1.4), these are the 1-cells whose labels appear above the arrow, and their composite is cba. Dually, the **target 1-cell** should be the unique composable path of 1-cells, none of which occur as domains of any 2-cells in the pasting diagram. In (B.1.4), these are the 1-cells whose labels appear below the diagram, and their composite is $n\ell j$.

The final step is to represent the pasting diagram as a vertical composite of 2-cells, each of which is a map between a pair of composite 1-cells from A to Z that trace a composable path through the directed graph of the pasting diagram. Each 2-cell in the pasting diagram will label precisely one of the 2-cells of this composite. The expressions of these vertical 2-cell composites are not necessarily unique and may not necessarily pass through every possible composable path of 1-cells (though there will be some vertical composite of 2-cells that does pass through each path of 1-cells).

To start, pick any 2-cell in the pasting diagram whose 1-cell source can be found as a subsequence of the source 1-cell; in the (B.1.4), either α or β can be chosen first. Whisker it so that it defines a 2-cell from the source 1-cell cba to another path of composable 1-cells from A to Z through the pasting diagram. Then this whiskered composite forms the first step in the sequence of composable 2-cells. Remove this part of the pasting diagram and repeat until you arrive at the target 1-cell. In the example above, there are three possible ways to express the composite pasted cell (B.1.4) as vertical composites of whiskered 2-cell, represented by the three paths through the following commutative diagram in the category $C(A, Z)$:

$$\begin{tikzcd}
cba \arrow[r, Rightarrow, "c\alpha a"] \arrow[d, Rightarrow, "\beta ba"'] & cfeda \arrow[r, Rightarrow, "cfe\gamma"] \arrow[d, Rightarrow, "\beta feda"'] & cfei \arrow[d, Rightarrow, "\beta fei"] \\
hgba \arrow[r, Rightarrow, "hg\alpha a"'] & hgfeda \arrow[r, Rightarrow, "hgfe\gamma"'] & hgfei \arrow[r, Rightarrow, "hgf\delta"] & hgfkj \arrow[r, Rightarrow, "h\epsilon j"] & hm\ell j \arrow[r, Rightarrow, "\zeta\ell j"] & h\ell j
\end{tikzcd} \quad \in \; C(A, Z)$$

B.1.6. Digression (notions of sameness inside a 2-category). From the point of view of 2-category theory, the most natural notion of "sameness" for two objects of a 2-category is *equivalence*: A and B are to be regarded as the same, if there exist 1-cells $f\colon A \to B$ and $g\colon B \to A$ together with invertible 2-cells $\alpha\colon \mathrm{id}_A \cong gf$ and $\beta\colon fg \cong \mathrm{id}_B$.

From the point of view of 2-category theory, the most natural notion of "sameness" for a parallel pair of morphisms in a 2-category is *isomorphism*: $h, k\colon A \rightrightarrows B$ are to be regarded as the same if there exists an invertible 2-cell $\gamma\colon h \cong k$.

From the point of view of 2-category theory, the most natural notion of "sameness" for a pair of 2-cells with common boundary is *equality*. Because a 2-category lacks any higher dimensional morphisms to mediate between 2-cells, there is no weaker notion of sameness available.

A 2-category has four duals, including itself, each of which have the same objects, 1-cells, and 2-cells, but with some domains and codomains swapped.

B.1.7. Definition (op and co duals). Let C be a 2-category.

- Its **op-dual** C^{op} is the 2-category with $C^{\mathrm{op}}(A, B) \coloneqq C(B, A)$. This reverses the direction of the 1-cells but not the 2-cells.

- Its **co-dual** C^{co} is the 2-category with $C^{\mathrm{co}}(A,B) \coloneqq C(A,B)^{\mathrm{op}}$. This reverses the direction of the 2-cells but not the 1-cells.
- Its **coop-dual** C^{coop} is the 2-category with $C^{\mathrm{coop}}(A,B) \coloneqq C(B,A)^{\mathrm{op}}$. This reverses the direction of both the 2-cells and the 1-cells.

Further details about the specification of C^{op} C^{co}, and C^{coop} as $\mathcal{C}at$-enriched categories are left to Exercise B.1.iii.

The data of an equivalence in C also defines an equivalence in each of its duals.

In Digression 1.2.3, we saw that the data of a simplicial category could be expressed as a diagram of a particular type valued in $\mathcal{C}at$. A small 2-category can be similarly encoded, in fact in two different ways, as a category defined *internally* to the category of categories.

B.1.8. Definition (internal category). Let $\mathcal{E}$ be any category with pullbacks. An **internal category** in $\mathcal{E}$ is given by the data

$$\begin{tikzcd} & C_1 \times_{C_0} C_1 \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_r"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ C_1 \arrow[dr, "d"'] & & C_1 \arrow[dl, "c"] \\ & C_0 & \end{tikzcd} \qquad C_1 \underset{C_0}{\times} C_1 \xrightarrow{\ \circ\ } C_1 \underset{c}{\overset{d}{\underset{\longleftarrow i \longleftarrow}{\rightrightarrows}}} C_0$$

subject to commutative diagrams that define the domains and codomains of composites and identities and encode the fact that composition is associative and unital. The details are left to Exercise B.1.i or the literature.

For example, a **double category** is a category internal to $\mathcal{C}at$. A 2-category can be realized as a special case of this construction in the following two ways.

B.1.9. Digression (2-categories as category objects). A 2-category may be defined to be an internal category in $\mathcal{C}at$

$$\begin{tikzcd} C_{1,2} \underset{C_0}{\times} C_{1,2} \arrow[r, "\circ"] & C_{1,2} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

in which the category C_0 is a set, namely the set of objects of the 2-category. The 1- and 2-cells occur as the objects and arrows of the category $C_{1,2}$. The functors $d, c\colon C_{1,2} \to C_0$ send 1- and 2-cells to their domain and codomain 0-cells. The functor $i\colon C_0 \to C_{1,2}$ sends each object to its identity 1-cell. The action of the functor $\circ\colon C_{1,2} \times_{C_0} C_{1,2} \to C_{1,2}$ on objects defines composition of 1-cells and the action on morphisms defines the horizontal composition on 2-cells. Vertical composition on 2-cells is the composition inside the category $C_{1,2}$. Functoriality of this map encodes middle-four interchange. This definition motivates the Segal category model of $(\infty,1)$-categories described in Appendix E.

Dually, a 2-category may be defined to be an internal category in $\mathcal{C}at$

$$\begin{tikzcd} C_{0,2} \underset{C_{0,1}}{\times} C_{0,2} \arrow[r, "\circ"] & C_{0,2} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_{0,1} \arrow[l, "i" description] \end{tikzcd}$$

in which the categories $C_{0,1}$ and $C_{0,2}$ have the same set of objects and all four functors are identity-on-objects. Here the common set of objects defines the objects of the 2-category and the arrows of $C_{0,1}$ and $C_{0,2}$ define the 1- and 2-cells, respectively. The functors $d, c\colon C_{1,2} \rightrightarrows C_{0,2}$ define the

domain and codomain 1-cells for a 2-cell, which the functor $\circ\colon C_{0,2} \times_{C_{0,1}} C_{0,2} \to C_{0,2}$ encodes vertical composition of 2-cells. The composition inside the category $C_{0,2}$ defines horizontal composition of 2-cells. Functoriality of this map encodes middle-four interchange.

Exercises.

B.1.i. EXERCISE. Complete the definition of an internal category sketched in Definition B.1.8.

B.1.ii. EXERCISE. Define a duality involution on double categories that exchanges the two expressions of a 2-category as an internal category appearing in Digression B.1.9.

B.1.iii. EXERCISE. For any 2-category $\mathcal{C}$, define $\mathcal{C}at$-enriched categories $\mathcal{C}^{\mathrm{op}}$, $\mathcal{C}^{\mathrm{co}}$, and $\mathcal{C}^{\mathrm{coop}}$ along the lines specified by Definition B.1.7.

B.2. The 3-category of 2-categories

Ordinary 1-categories form the objects of a 2-category of categories, functors, and natural transformations. Similarly, 2-categories form the objects of a 3-category of 2-categories, 2-functors, 2-natural transformations, and modifications. In this section, we briefly introduce all of these notions.

Recall from Definition B.1.1, that a 2-category is a category enriched in $\mathcal{C}at$. Similarly, 2-functors and 2-natural transformations are precisely the $\mathcal{C}at$-enriched functors and $\mathcal{C}at$-enriched natural transformations of Appendix A. By Corollary A.3.6, 2-categories, 2-functors, and 2-natural transformations assemble into a 2-category. The 3-dimensional cells between 2-categories — the modifications — are defined using the 2-cells of a 2-category, like the 2-dimensional cells between 1-categories — the natural transformations — are defined using the 1-cells in a 1-category.

B.2.1. DEFINITION. A **2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ between 2-categories is given by:

- a mapping on objects $\mathcal{C} \ni x \mapsto Fx \in \mathcal{D}$;
- a functorial mapping on 1-cells $\mathcal{C} \ni f\colon x \to y \mapsto Ff\colon Fx \to Fy \in \mathcal{D}$ respecting domains and codomains; and
- a mapping on 2-cells

$$\begin{tikzcd} \mathcal{C} \ni & x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y & \mapsto & Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & Fy & \in \mathcal{D} \end{tikzcd}$$

that respects 0- and 1-cell sources and targets that is functorial for both horizontal and vertical composition and horizontal and vertical identities.

B.2.2. DEFINITION. A **2-natural transformation** $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$ between a parallel pair of 2-functors F and G is given by a family of 1-cells $(\phi_c\colon Fc \to Gc)_{c\in\mathcal{C}}$ in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ that are natural with respect to the 1-cells $f\colon x \to y$ in $\mathcal{C}$, in the sense that the square

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\phi_x"'] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

commutes in $\mathcal{D}$, and also natural with respect to the 2-cells $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$ in $\mathcal{C}$, in the sense that the top-right and bottom-left whiskered composites

$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] \arrow[d, "\phi_x"'] & Fy \arrow[d, "\phi_y"] \\ Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & Gy \end{tikzcd}$$

are equal: i.e., $\phi_y \cdot F\alpha = G\alpha \cdot \phi_x$.

B.2.3. Definition. A **modification** $\Xi \colon \phi \Rrightarrow \psi$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] & \mathcal{D} \end{tikzcd} \quad \phi \overset{\Xi}{\Rrightarrow} \psi$$

between parallel 2-natural transformations is given by a family of 2-cells in $\mathcal{D}$

$$\begin{tikzcd} Fc \arrow[r, bend left, "\phi_c"] \arrow[r, bend right, "\psi_c"'] \arrow[r, phantom, "\Downarrow\Xi_c"] & Gc \end{tikzcd}$$

indexed by the objects $c \in \mathcal{C}$ with the property that for any 1-cell $f \colon x \to y$ in $\mathcal{C}$, the whiskered composites $\Xi_y \cdot Ff = Gf \cdot \Xi_x$ are equal in $\mathcal{D}$ and for any 2-cell $\alpha \colon f \Rightarrow g$ in $\mathcal{D}$, the horizontal composites in $\mathcal{D}$

$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & Fy \arrow[r, bend left, "\phi_y"] \arrow[r, bend right, "\psi_y"'] \arrow[r, phantom, "\Downarrow\Xi_y"] & Gy \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, bend left, "\phi_x"] \arrow[r, bend right, "\psi_x"'] \arrow[r, phantom, "\Downarrow\Xi_x"] & Gx \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & Gy \end{tikzcd}$$

are equal.

Finally, the category of 2-categories is cartesian closed, with internal hom $\mathcal{B}^{\mathcal{A}}$ defined to be the 2-category of 2-functors, 2-natural transformations, and modifications. So now we can define a **3-category** to be a category enriched in 2-categories.

Exercises.

B.2.i. Exercise. For the reader who has a lot of blank paper, unpack the definition of a 3-category just given.

B.3. Adjunctions and mates

As discussed in Chapter 2, any 2-category has an internally-defined notion of adjunction:

B.3.1. Definition (adjunction). An **adjunction** in a 2-category C is comprised of:

- a pair of objects A and B,
- a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$,
- and a pair of 2-cells $\eta\colon 1_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow 1_A$, called the **unit** and **counit** respectively,

so that the triangle equalities hold:

$$\begin{tikzcd} & B \arrow[rr, equals] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[rr, equals] & {} & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equals] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equals] & {} & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

The 1-cell f is called the **left adjoint** and u is called the **right adjoint**, a relationship that is denoted symbolically in text by writing $f \dashv u$ or in a displayed diagram such as

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd} \tag{B.3.2}$$

In the presence of an adjunction as in (B.3.2), certain 2-cells with codomain A "transpose" into 2-cells with codomain B; op-dually, certain 2-cells with domain A "transpose" into 2-cells with domain B:

$$fb \xRightarrow{\alpha} a \quad \leftrightsquigarrow \quad b \xRightarrow{\beta} ua \qquad\qquad df \xRightarrow{\gamma} c \quad \leftrightsquigarrow \quad d \xRightarrow{\delta} cu \tag{B.3.3}$$

Both of these transposition operations admit a common generalization due to **[57]** referred to as the "mates correspondence" which describes a duality between 2-cells induced by a pair of adjunctions.

B.3.4. Definition (mates). Given any pair of adjunctions and functors

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, bend right, "f"'] \arrow[d, phantom, "\dashv"] & B' \arrow[d, bend right, "f'"'] \arrow[d, phantom, "\dashv"] \\ A \arrow[r, "a"'] \arrow[u, bend right, "u"'] & A' \arrow[u, bend right, "u'"'] \end{tikzcd}$$

there is exists a bijection between 2-cells as below-left and 2-cells as below-right

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\swArrow\alpha"] & B' \arrow[d, "f'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} B \arrow[r, "b"] \arrow[dr, phantom, "\seArrow\beta"] & B' \\ A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"'] \end{tikzcd} \tag{B.3.5}$$

implemented by pasting with the units and counits of the adjunctions:

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\swArrow\alpha"] & B' \arrow[d, "f'"] \\ A \arrow[r, "a"'] & A' \end{tikzcd} := \begin{tikzcd} B \arrow[r, equals] \arrow[dr, "f"'] & B \arrow[r, "b"] \arrow[d, phantom, "\seArrow\eta"] \arrow[dr, phantom, "\seArrow\beta"] & B' \arrow[dr, "f'"] \arrow[d, phantom, "\seArrow\epsilon'"] & \\ & A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"] \arrow[r, equals] & A' \end{tikzcd}$$

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[dr, phantom, "\Searrow\beta"] & B' \\
A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"']
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
 & B \arrow[r, "b"] \arrow[d, dashed, "f"] \arrow[dr, phantom, "\Swarrow\alpha"] & B' \arrow[r, equal] \arrow[d, dashed, "f'"] & B' \\
A \arrow[ur, "u"] \arrow[r, equal] \arrow[ur, phantom, "\Swarrow\epsilon", near end, shift right=3] & A \arrow[r, "a"'] & A' \arrow[ur, "u'"'] \arrow[ur, phantom, "\Swarrow\eta'", near start, shift left=3] &
\end{tikzcd}
$$

Pairs of corresponding 2-cells (B.3.5) under this bijection are referred to as **mates**.

The mates correspondence is respected by horizontal and vertical composition of squares (B.3.5) in the sense made precise by the following result:

B.3.6. Theorem (double-functoriality of the mates correspondence). *For any 2-category $\mathcal{C}$, there is an identity on objects, vertical morphisms, and horizontal morphisms, double isomorphism $\mathbb{L}\mathsf{adj}(\mathcal{C}) \cong \mathbb{R}\mathsf{adj}(\mathcal{C})$ between the double categories whose*

- *objects and horizontal morphisms are the objects and 1-cells of $\mathcal{C}$*
- *vertical morphisms are fully-specified adjunctions (f, u, η, ϵ) pointing in the direction of the left adjoint*
- *cells in $\mathbb{L}\mathsf{adj}$ are 2-cells in $\mathcal{C}$ of the form displayed below-left, while cells in $\mathbb{R}\mathsf{adj}$ are 2-cells in $\mathcal{C}$ of the form displayed below-right:*

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\alpha"] & B' \arrow[d, "f'"] \\
A \arrow[r, "a"'] & A'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "b"] \arrow[dr, phantom, "\Searrow\beta"] & B' \\
A \arrow[u, "u"] \arrow[r, "a"'] & A' \arrow[u, "u'"']
\end{tikzcd}
$$

that acts on cells by taking mates.

Note that the composition of vertical morphisms in the double categories $\mathbb{L}\mathsf{adj}$ and $\mathbb{R}\mathsf{adj}$ makes use of Proposition 2.1.9.

Proof. The horizontal and vertical functoriality of the mates correspondence of Definition B.3.4 can be verified by an easy diagram chase, or see [**57**, 2.2]. ☐

B.3.7. Warning (mates of isomorphisms need not be isomorphisms). In general it is possible for one of the 2-cells in a mate pair (B.3.5) to be invertible without the other being so, for instance because the adjoint transpose of an isomorphism need not be invertible. However if both horizontal 1-cells a and b are equivalences, or if both adjunctions $f \dashv u$ are adjoint equivalences $f' \dashv u'$, then $\alpha\colon f'b \Rightarrow af$ is invertible if and only if its mate $\beta\colon bu \Rightarrow u'a$ is invertible.

Exercise B.3.ii suggests a new proof that any pair of left adjoints $f' \dashv u$ and $f \dashv u$ to a common 1-cell are isomorphic by applying the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$. A more complicated argument along the same lines can be used to prove:

B.3.8. Lemma. *Suppose given a triple of adjoint functors $\ell \dashv i \dashv r$. Then the counit of $\ell \dashv i$ is invertible if and only if the unit of $i \dashv r$ is invertible.*

Proof. Let $i\colon A \to B$ and write $\epsilon\colon \ell i \Rightarrow \mathrm{id}_A$ for the counit of $\ell \dashv i$ and $\eta\colon \mathrm{id}_A \Rightarrow ri$ for the unit of $i \dashv r$. If ϵ admits an inverse isomorphism $\epsilon^{-1}\colon \ell i \Rightarrow \mathrm{id}_A$, then the vertical composite in $\mathbb{L}\mathsf{adj}$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\swarrow \mathrm{id}_i"] & A \arrow[d, "i"] \\ A \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow \epsilon"] & B \arrow[d, "\ell"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

admits an inverse cell for horizontal composition in $\mathbb{L}\mathsf{adj}$:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\ell i"'] \arrow[dr, phantom, "\swarrow \epsilon^{-1}"] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd}$$

Applying the horizontal functoriality of the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$, the mates of these cells must also compose horizontally in $\mathbb{R}\mathsf{adj}$ to identities.[3] Applying the vertical functoriality of the double isomorphism $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$, the mate of the vertical composite equals the composite

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\searrow \eta"] & A \\ A \arrow[r, "i"'] \arrow[d, equal] \arrow[dr, phantom, "\searrow \mathrm{id}_i"] & B \arrow[u, "r"'] \\ A \arrow[r, equal] & A \arrow[u, "i"'] \end{tikzcd}$$

in $\mathbb{R}\mathsf{adj}$. In summary, we conclude that the counit of $\ell \dashv i$ is an isomorphism if and only if the unit of $i \dashv r$ is an isomorphism, in which case its inverse isomorphism $\eta^{-1}\colon ri \Rightarrow \mathrm{id}_A$ is the mate of $\epsilon^{-1}\colon \mathrm{id}_A \Rightarrow \ell i$ via the composite adjunction $\ell i \dashv ri$:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\ell i"'] \arrow[dr, phantom, "\swarrow \epsilon^{-1}"] & A \arrow[d, equal] \\ A \arrow[r, equal] & A \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, equal] \arrow[dr, phantom, "\searrow \eta^{-1}"] & A \arrow[d, equal] \\ A \arrow[r, equal] \arrow[u, "ri"] & A \end{tikzcd}$$ □

Elaborating upon Warning B.3.7 we have:

B.3.9. Proposition (equivalence invariance of adjointness). *Suppose given an essentially commutative square whose horizontal arrows are equivalences:*

$$\begin{tikzcd} B \arrow[r, "b", "\sim"'] \arrow[d, "f"] \arrow[dr, phantom, "\cong \swarrow \alpha"] & B' \arrow[d, "f'"'] \\ A \arrow[r, "\sim", "a"'] \arrow[u, bend left, dashed, "u"] \arrow[u, phantom, "\vdash"] & A' \arrow[u, bend right, dashed, "u'"'] \arrow[u, phantom, "\dashv"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} B \arrow[r, "b", "\sim"'] \arrow[dr, phantom, "\cong \searrow u'a\epsilon \cdot u'\alpha u \cdot \eta'"] & B' \\ A \arrow[r, "\sim", "a"'] \arrow[u, bend left, dashed, "u"] & A' \arrow[u, bend right, dashed, "u'"'] \end{tikzcd}$$

[3] Since the horizontal morphisms in the cells in question are all id_A, the concern raised in Warning B.3.7 does not apply.

Then f admits a right adjoint u if and only f' admits a right adjoint u', in which case the **mate** *of the isomorphism α is an isomorphism.*

Proof. Proposition 2.1.11 may be used to to choose inverse adjoint equivalences $b' \dashv b$ and $a \dashv a$. If f is a left adjoint, then by Proposition 2.1.9, $f' \cong afb'$ is isomorphic to a left adjoint and so by Proposition 2.1.10 f' is left adjoint to bua'. If $f' \dashv u'$ is defined to be the composite adjunction as in the previous paragraph, the mate of α works out to the whiskered composite of $a'a \cong \mathrm{id}_A$ with bu. By Proposition 2.1.10, any other choice of right adjoint to f' is isomorphic to this one, so the mate of α is still an isomoprhism. ☐

Exercises.

B.3.i. Exercise. Explain how the bijections (B.3.3) may be realized as special cases of the mates correspondence.

B.3.ii. Exercise. Consider two adjunctions $f \dashv u$ and $f' \dashv u$ as vertical morphisms in $\mathbb{L}\mathsf{adj} \cong \mathbb{R}\mathsf{adj}$ and apply the double functoriality of the mates correspondence to prove that $f \cong f'$.

B.4. Right adjoint right inverse adjunctions

An important class of adjunctions is those whose counit 2-cells are isomorphisms.

B.4.1. Definition. A 1-cell $f\colon B \to A$ in a 2-category admits a **right adjoint right inverse**, abbreviated **RARI**, if it admits a right adjoint $u\colon A \to B$ so that the counit of the adjunction $f \dashv u$ is an isomorphism.

In the situation of Definition B.4.1, f defines a **left adjoint left inverse**, abbreviated **LALI**, to u. The co-dual defines the class **left adjoint right inverse** or **right adjoint left inverse** adjunctions with invertible unit. We leave the dualizations of the results that follow to the reader.

When the counit of $f \dashv u$ is an isomorphism, the whiskered composites $f\eta$ and ηu of the unit must also be isomorphisms. Indeed, to construct an adjunction of this form it suffices to give 2-cells with these properties, as demonstrated by the following 2-categorical lemma.

B.4.2. Lemma. *Suppose we are given a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$ and a 2-isomorphism $fu \cong \mathrm{id}_A$ in a 2-category. If there exists a 2-cell $\eta'\colon \mathrm{id}_B \Rightarrow uf$ with the property that $f\eta'$ and $\eta' u$ are 2-isomorphisms, then f is left adjoint to u. Furthermore, in the special case where u is a section of f, then f is left adjoint to u with the counit of the adjunction an identity.*

Proof. Let $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ be the isomorphism, taken to be the identity in the case where u is a section of f. We will define an adjunction $f \dashv u$ with counit ϵ by modifying $\eta'\colon \mathrm{id}_B \Rightarrow uf$. The "triangle identity composite" $\theta \coloneqq u\epsilon \cdot \eta' u\colon u \Rightarrow u$ defines an automorphism of u. Define

$$\eta \coloneqq \mathrm{id}_B \xRightarrow{\eta'} uf \xRightarrow{\theta^{-1}f} uf \coloneqq \mathrm{id}_B \xRightarrow{\eta'} uf \xRightarrow{(u\epsilon f)^{-1}} ufuf \xRightarrow{(\eta' uf)^{-1}} uf.$$

Immediately, $u\epsilon \cdot \eta u = \mathrm{id}_u$, as is verified by the calculation:

$$\begin{tikzcd} u \arrow[r, Rightarrow, "\eta' u"'] \arrow[rr, Rightarrow, bend left, "\eta u"] \arrow[dr, Rightarrow, bend right, "\theta"'] & ufu \arrow[r, Rightarrow, "\theta^{-1} fu"'] \arrow[d, Rightarrow, "u\epsilon"] & ufu \arrow[d, Rightarrow, "u\epsilon"] \\ & u \arrow[r, Rightarrow, "\theta^{-1}"'] & u \end{tikzcd}$$

The other triangle identity composite $\phi \coloneqq \epsilon f \cdot f\eta$ is an isomorphism, as a composite of isomorphisms, and also an idempotent:

$$\begin{tikzcd}
f \arrow[r, Rightarrow, "f\eta"] \arrow[d, Rightarrow, "f\eta"'] & fuf \arrow[r, Rightarrow, "\epsilon f"] \arrow[d, Rightarrow, "fuf\eta"] & f \arrow[d, Rightarrow, "f\eta"] \\
fuf \arrow[r, Rightarrow, "f\eta uf"] \arrow[dr, equal] & fufuf \arrow[r, Rightarrow, "\epsilon fuf"] \arrow[d, Rightarrow, "fu\epsilon f"] & fuf \arrow[d, Rightarrow, "\epsilon f"] \\
& fuf \arrow[r, Rightarrow, "\epsilon f"'] & f
\end{tikzcd}$$

By cancelation, any idempotent isomorphism is the identity, proving that $\epsilon f \cdot f\eta = \mathrm{id}_f$. $\square$

B.4.3. Remark. If desired, the unit of the RARI constructed in the proof of Lemma B.4.2 can be taken to be η', with the counit ϵ modified to absorb the isomorphism θ^{-1}. The details are left as Exercise B.4.i.

By Proposition 13.4.5 to say that an adjunction

$$\begin{tikzcd}
A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"']
\end{tikzcd}$$

is a RARI is equivalently to say that the right adjoint $u\colon A \to B$ is fully faithful. The presence of a left adjoint to a fully faithful functor provides a convenient characterization of its essential image.

B.4.4. Definition. A generalized element $b\colon X \to B$ is in the **essential image** of the right adjoint $u\colon A \to B$ of a RARI if and only if the unit component $\eta b\colon b \Rightarrow ufb$ is an isomorphism.

B.4.5. Lemma. *A generalized element $b\colon X \to B$ is in the essential image of the right adjoint $u\colon A \to B$ of a RARI if and only if there exists a generalized element $a\colon X \to A$ and invertible 2-cell $\alpha\colon b \cong ua$.*

Proof. The statement generalizes Definition B.4.4, so it remains to argue that if given an invertible 2-cell $\alpha\colon b \cong ua$, the unit component ηb is also an isomorphism. This follows immediately from the banal final statement of Lemma B.1.3. From the horizontally-composable pair

$$\begin{tikzcd}
X \arrow[rr, "b"] \arrow[dr, "a"'] & \arrow[d, phantom, "\cong\Downarrow\alpha"] & B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \\
& A \arrow[ur, "u"'] & & A \arrow[ur, "u"'] &
\end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd}
b \arrow[r, Rightarrow, "\eta b"] \arrow[d, Rightarrow, "\alpha"', "\cong"] & ufb \arrow[d, Rightarrow, "\alpha", "\cong"'] \\
ua \arrow[r, Rightarrow, "\cong", "\eta ua"'] & ufua
\end{tikzcd}$$

and $f \dashv u$ is a RARI, ηu is invertible. Thus, if α is invertible so is ηb. $\square$

Sometimes it is more convenient to make use of a stricter notion of RARI, in which the counit ϵ is required to be the identity $fu = \mathrm{id}_A$. In this case it follows from the triangle equalities that $f\eta = \mathrm{id}_f$, so that the unit is fibered over A.

When the left adjoint has a 2-categorical property that we now introduce, we shall see that a right adjoint right inverse up to isomorphism can always be replaced by a right adjoint right inverse up to identity.

B.4.6. Definition. A 1-cell $f\colon B \to A$ in a 2-category defines an **isofibration** — in which case the arrow is typically denoted by "$\twoheadrightarrow$" — if given any invertible 2-cell as displayed below left abutting to A with a specified lift of one of its boundary 1-cells through f, there exists an invertible 2-cell abutting to B with this boundary 1-cell as displayed below right that whiskers with f to the original 2-cell:

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[dr, bend right, "a"'] \arrow[dr, phantom, "\cong\Downarrow\alpha" near end, shift left=2] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "\bar{b}"'] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd}$$

B.4.7. Lemma. *Let $f\colon B \twoheadrightarrow A$ be any isofibration that admits a right adjoint $u'\colon A \to B$ with counit $\epsilon\colon fu' \cong \mathrm{id}_A$ an isomorphism. Then u' is isomorphic to a functor u that lies strictly over A and defines a strict right adjoint right inverse to f. Thus any such f defines a fibered adjunction*

$$\begin{tikzcd} B \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\perp"] \arrow[dr, two heads, "f"'] & & A \arrow[ll, bend left, dashed, "u"] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

Proof. Define the 1-cell $u\colon A \to B$ by lifting the counit isomorphism through the isofibration $f\colon B \twoheadrightarrow A$

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[dr, bend right, equal] \arrow[dr, phantom, "\cong\Downarrow\epsilon" near end, shift left=2] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "u'"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\cong\Downarrow\beta"] & B \arrow[d, two heads, "f"] \\ & A \end{tikzcd}$$

Note by construction that $fu = \mathrm{id}_A$. By the triangle equalities for the adjunction $f \dashv u'$, the unit defines a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow u'f$ with $\eta u'$ and $f\eta$ both invertible. The composite 2-cell

$$\eta' \coloneqq \mathrm{id}_B \xRightarrow{\eta} u'f \underset{\cong}{\xRightarrow{\beta f}} uf$$

then has the properties that $\eta' u$ and $f\eta$ are both invertible. Applying Lemma B.4.2, this 2-cell may then be modified to define the unit of an adjunction $f \dashv u$ with counit $fu = \mathrm{id}_A$. $\square$

Exercises.

B.4.i. Exercise. Suppose we are given a pair of 1-cells $u\colon A \to B$ and $f\colon B \to A$ and a 2-isomorphism $fu \cong \mathrm{id}_A$ in a 2-category. Modify the proof of Lemma B.4.2 to show that if there exists a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ with the property that $f\eta'$ and $\eta' u$ are 2-isomorphisms, then f is left adjoint to u with unit η and counit invertible.

B.5. A bestiary of 2-categorical lemmas

B.5.1. LEMMA. *Suppose $f \dashv u$ is an adjunction under B in the sense that*

- *the solid-arrow triangles involving both adjoints commute*

$$\begin{tikzcd} & B \arrow[dl, "c"'] \arrow[dr, "a"] & \\ C \arrow[ur, dashed, bend left, "r"] \arrow[ur, phantom, "\dashv"] \arrow[rr, "u", ""'{name=U}] & & A \arrow[ll, bend left, "f", ""{name=F}] \arrow[from=U, to=F, phantom, "\top"] \end{tikzcd}$$

- *and $\eta a = \mathrm{id}_a$ and $\epsilon c = \mathrm{id}_c$.*

Then if c admits a right adjoint r with unit $\iota\colon \mathrm{id}_B \Rightarrow rc$ and counit $\nu\colon cr \Rightarrow \mathrm{id}_C$, then $u\nu$ exhibits r as an absolute right lifting of u through a.

$$\begin{tikzcd} & B \arrow[d, "a"] \\ C \arrow[ur, dashed, bend left, "r"] \arrow[r, "u"'] \arrow[ur, phantom, "\Downarrow u\nu"] & A \end{tikzcd}$$

PROOF. The argument is purely diagrammatic. Any 2-cell as below-left factors through $u\nu$ as below-right:

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \\ C \arrow[r, "u"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] & \\ C \arrow[r, "u"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\Downarrow\epsilon"] & A \arrow[d, "f"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\Downarrow\eta"] & \\ & C \arrow[r, "u"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \arrow[dd, bend left=60, "c"] & \\ C \arrow[r, "u"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\Downarrow\epsilon"] & A \arrow[d, "f"] & \\ & C \arrow[r, "u"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \arrow[dr, "c"] \arrow[rr, equal] \arrow[rr, phantom, shift right=3, "\Downarrow\iota"] & & B \arrow[d, "c"] \arrow[dd, bend left=60, "a"] \\ C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, shift right=3, "\Downarrow\epsilon"] & A \arrow[r, "f"'] & C \arrow[ur, "r"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\nu"] & C \arrow[d, "u"] \\ & & & A \end{tikzcd}$$

Conversely if ζ defines a factorization of χ through $u\nu$, then

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[r, phantom, shift right=4, "\Downarrow\zeta"] & B \\ C \arrow[ur, "r"'] & \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[r, phantom, shift right=4, "\Downarrow\zeta"] & B \arrow[d, "c"] \arrow[r, equal] \arrow[r, phantom, shift right=4, "\Downarrow\iota"] & B \\ C \arrow[ur, "r"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\nu"] & C \arrow[ur, "r"'] & \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[r, phantom, shift right=4, "\Downarrow\zeta"] & B \arrow[d, "c"] \arrow[dr, "a"] \arrow[drr, "c"] \arrow[rr, equal] \arrow[rr, phantom, shift right=4, "\Downarrow\iota"] & & B \\ C \arrow[ur, "r"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\nu"] & C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, shift right=3, "\Downarrow\epsilon"] & A \arrow[r, "f"] & C \arrow[u, "r"'] \end{tikzcd}$$

$$= \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[r, phantom, shift right=4, "\Downarrow\zeta"] & B \arrow[d, "a"] \arrow[dr, "c"] \arrow[r, equal] \arrow[r, phantom, shift right=4, "\Downarrow\iota"] & B \\ C \arrow[ur, "r"] \arrow[r, "u"] \arrow[ur, phantom, shift right=3, "\Downarrow u\nu"] \arrow[rr, equal, bend right] \arrow[rr, phantom, shift right=3, "\Downarrow\epsilon"] & A \arrow[r, "f"] & C \arrow[u, "r"'] \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "a"] \arrow[dr, "c"] \arrow[r, equal] \arrow[r, phantom, shift right=4, "\Downarrow\iota"] & B \\ C \arrow[r, "u"] \arrow[rr, equal, bend right] \arrow[rr, phantom, shift right=3, "\Downarrow\epsilon"] & A \arrow[r, "f"] & C \arrow[u, "r"'] \end{tikzcd}$$

proving that the factorization constructed above is unique. □

B.5.2. Example. For example, there is a diagram of adjoint functors

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook] \arrow[dr, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] \arrow[d, phantom, "\top"] & \mathbf{\Delta}_\perp \arrow[ll, dashed, bend right, "\top"'] \arrow[dl, "!"] \\ & \mathbb{1} \arrow[u, dashed, "{[-1]}"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

involving the categories introduced in Definition 2.3.9. The inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\perp$ freely adjoins a bottom element and its right adjoint is the forgetful functor, also an inclusion. The adjunction $[-1] \dashv !$ witnesses the fact that $[-1] \in \mathbf{\Delta}_\perp$ defines an initial object with the counit ν defining the canonical natural transformation from the initial object to the identity functor.

This diagram satisfies the premises of Lemma B.5.1 in $\mathcal{C}at^{\mathrm{op}}$. Let A be an object of any 2-category $\mathcal{C}$ that is cotensored over $\mathcal{C}at$. Then the 2-functor $A^{(-)} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}$ carries the given data to a diagram of adjoint functors in $\mathcal{C}$ as below-left and hence the triangle below-right is absolute right lifting:

$$\begin{tikzcd} & A \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] \arrow[dl, phantom, "\perp", bend right] & \\ A^{\mathbf{\Delta}_\perp} \arrow[ur, dashed, bend left, "\mathrm{ev}_{-1}"] \arrow[rr, "\mathrm{res}", "\top"'] & & A^{\mathbf{\Delta}} \arrow[ll, bend left, "f"] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^{\mathbf{\Delta}_\perp} \arrow[ur, dashed, bend left, "\mathrm{ev}_{-1}"] \arrow[r, "\mathrm{res}"'] \arrow[ur, phantom, "\Downarrow A^\nu"] & A^{\mathbf{\Delta}} \end{tikzcd}$$

This proves Proposition 2.3.11.

B.5.3. Example. There is a similar diagram of adjoint functors

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook] \arrow[dr, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] \arrow[d, phantom, "\top"] & \mathbf{\Delta}_\top \arrow[ll, dashed, bend right, "\top"'] \arrow[dl, "!"] \\ & \mathbb{1} \arrow[u, dashed, "{[-1]}"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

there is a diagram of adjoint functors

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook] \arrow[dr, "!"'] & \mathbf{\Delta}_+ \arrow[r, hook] \arrow[d, phantom, "\top"] & \mathbf{\Delta}_\perp \arrow[ll, dashed, bend right, "\top"'] \arrow[dl, "!"] \\ & \mathbb{1} \arrow[u, dashed, "{[-1]}"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

involving the categories introduced in Definition 2.3.9. The inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\top$ freely adjoins a top element and its right adjoint is the forgetful functor, also an inclusion. The adjunction $[-1] \dashv !$ witnesses the fact that $[-1] \in \mathbf{\Delta}_\top$ defines an initial object with the counit ν defining the canonical natural transformation from the initial object to the identity functor. This proves another version of Proposition 2.3.11 where the "splittings" occur on the other side of the co/simplicial objects.

Exercises.

B.5.i. Exercise. Search for other examples in $\mathcal{C}at$ satisfying the premises of Lemma B.5.1.

B.6. Representable characterizations of 2-categorical notions

Recall from Definition 1.4.6 that an **equivalence** in a 2-category is given by

- a pair of objects A and B
- a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$
- a pair of invertible 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, equal, ""{name=U}] \arrow[r, bend right, "gf"', ""{name=D}] & A \arrow[from=U, to=D, phantom, "\cong\Downarrow\alpha"] \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, bend left, "fg", ""{name=U}] \arrow[r, bend right, equal, ""{name=D}] & B \arrow[from=U, to=D, phantom, "\cong\Downarrow\beta"] \end{tikzcd}$$

In analogy with Theorem 1.4.7, we have the following result which tells us that equivalences in a 2-category represent equivalences of categories.

B.6.1. Proposition (equivalences are representably defined). *A 1-cell $f\colon A \to B$ in a 2-category C defines an equivalence if and only if for all $X \in C$ the induced functor*

$$C(X,A) \xrightarrow{f_*} C(X,B)$$

defines an equivalence of categories.

Proof. Each $X \in C$ defines a 2-functor $C(X,-)\colon C \to Cat$, so if $f\colon A \xrightarrow{\sim} B$ is an equivalence in C, then $f_*\colon C(X,A) \xrightarrow{\sim} C(X,B)$ is an equivalence in Cat.

Conversely, by essential surjectivity of the equivalence $f_*\colon C(B,A) \to C(B,B)$ there exists some $g\colon B \to A$ and isomorphism $\beta\colon fg \cong \mathrm{id}_B$. By fully-faithfulness of $f_*\colon C(A,A) \to C(A,B)$ the isomorphism $\beta^{-1}f\colon f \cong fgf$ lifts to an isomorphism $\alpha\colon \mathrm{id}_A \cong gf$. □

Similarly, an adjoint functor in a 2-category induces pointwise-defined adjunctions between the hom-categories, but in this case, a further "exactness" condition is required to convert a representably-defined adjunction into an adjunction in the 2-category.

B.6.2. Proposition (adjunctions are representably defined). *A 1-cell $u\colon A \to B$ in a 2-category C admits a left adjoint if and only if:*

(i) For all $X \in C$, the induced functor admits a left adjoint

$$\begin{tikzcd} C(X,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & C(X,B) \arrow[l, bend right, dashed, "f^X"'] \end{tikzcd}$$

(ii) For all $k\colon Y \to X \in C$, the mate of the identity 2-cell is an isomorphism:

$$\begin{tikzcd} C(X,A) \arrow[r, "k^*"] \arrow[d, "u_*"'] \arrow[dr, phantom, "\Nearrow\mathrm{id}"] & C(Y,A) \arrow[d, "u_*"] \\ C(X,B) \arrow[r, "k^*"'] & C(Y,B) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C(X,A) \arrow[r, "k^*"] \arrow[dr, phantom, "\cong\Nwarrow\epsilon^Y k^* f^X \cdot f^Y k^* \eta^X"] & C(Y,A) \\ C(X,B) \arrow[r, "k^*"'] \arrow[u, "f^X"] & C(Y,B) \arrow[u, "f^Y"'] \end{tikzcd}$$

Proof. In the cartesian closed category of 2-categories, the hom 2-functor $C(-,-)\colon C^{\mathrm{op}} \times C \to Cat$ transposes to define a Yoneda embedding 2-functor $C(-,-)\colon C \to Cat^{C^{\mathrm{op}}}$ whose codomain is the

2-category of 2-functors, 2-natural transformations, and modifications. This 2-functor preserves adjunctions, carrying an adjoint pair $f \dashv u$ in C to an adjunction between the representable 2-functors $C(-,A)$ and $C(-,B)$

$$\begin{tikzcd} C(-,A) \arrow[r, bend right, "u_*"'] \arrow[r, phantom, "\perp"] & C(-,B) \arrow[l, bend right, "f_*"'] \end{tikzcd}$$

whose left and right adjoints are the 2-natural transformations $f_* \dashv u_*$ and whose unit and counit are modifications. Evaluating at $X \in C$, this defines a family of adjunction as in (i) and **strict adjunction morphisms**, i.e., so that any $k\colon Y \to X$ induces a strictly commutative square with respect to the left and right adjoints inhabited by a mate pair of identity 2-cells.

The real content is in the converse. Assuming (i), define a candidate left inverse by $f := f^B(\mathrm{id}_B)$. By construction $uf := u_* f_B(\mathrm{id}_B)$ so we may define a candidate unit to be the component of the unit η^B of $f^B \dashv u_*$ at id_B:

$$\eta := \mathrm{id}_B \xrightarrow{\eta^B_{\mathrm{id}_B}} uf \quad \in C(B,B).$$

Note that these definitions do not a priori give any information about the other composite $fu \in C(A,A)$, but condition (ii) defines a natural isomorphism $\alpha\colon f^A u^* \cong u^* f^B$

$$\begin{tikzcd} C(B,A) \arrow[r, "u^*"] & C(A,A) \\ C(B,B) \arrow[u, "f^B"] \arrow[r, "u^*"'] \arrow[ur, phantom, "\cong\Nwarrow \epsilon^A u^* f^B \cdot f^A u^* \eta^B"] & C(A,B) \arrow[u, "f^A"'] \end{tikzcd}$$

whose component at id_B defines an isomorphism

$$\alpha_{\mathrm{id}_B} := f^A(u) \xrightarrow{f^A(\eta u)} f^A(ufu) = f^A u_*(fu) \xrightarrow{\epsilon^A(fu)} fu \quad \in C(A,A).$$

Using this, we define the counit to be the composite of the inverse of this isomorphism with the component of the counit ϵ^A of $f^A \dashv u_*$ at id_A:

$$\epsilon := fu \xrightarrow[\cong]{\alpha^{-1}_{\mathrm{id}_B}} f^A(u) \xrightarrow{\epsilon^A_{\mathrm{id}_A}} \mathrm{id}_A \quad \in C(A,A).$$

The commutative diagram

$$\begin{tikzcd} u \arrow[r, "\eta u"] \arrow[d, "\eta^A(u)"'] & ufu \arrow[d, "\eta^A(ufu)"] \arrow[dr, equal, bend left] & \\ uf^A(u) \arrow[r, "uf^A(\eta u)"] \arrow[rr, bend right, "u\alpha_{\mathrm{id}_B}"'] & uf^A(ufu) \arrow[r, "u\epsilon^A(fu)"] & ufu \end{tikzcd}$$

reveals that $u\alpha_{\mathrm{id}_B} \cdot \eta^A u = \eta u$, so

$$u\epsilon \cdot \eta u = (u\epsilon^A_{\mathrm{id}_A} \cdot u\alpha^{-1}_{\mathrm{id}_B}) \cdot (u\alpha_{\mathrm{id}_B} \cdot \eta^A u) = u\epsilon^A_{\mathrm{id}_A} \cdot u\alpha_{\mathrm{id}_B} = \mathrm{id}_u,$$

which verifies one of the two triangle equalities.

It is somewhat delicate to prove that the other triangle equality composite

$$f \xrightarrow{f\eta} fuf \xrightarrow{\epsilon f} f \in \mathcal{C}(B,A)$$

is the identity because we don't have any way to understand the arrow $f\eta$. Note, however, that this arrow defines an endomorphism of the object $f^B(\mathrm{id}_B) \in \mathcal{C}(B,A)$, so if we verify that its transpose under the adjunction $f^B \dashv u_*$ is the unit component $\eta^B_{\mathrm{id}_B}$, then by uniqueness of adjoint transposition, we must have $\epsilon f \cdot f\eta = \mathrm{id}_f$ as desired. This can be verified by direct calculation: the adjoint transpose is computed by applying the functor u_* and then precomposing with $\eta^B_{\mathrm{id}_B} = \eta$, which yields the left-bottom composite below.

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta"'] & uf \arrow[d, "\eta uf"] \arrow[dr, equal, bend left] & \\
uf \arrow[r, "uf\eta"'] & ufuf \arrow[r, "u\epsilon f"'] & uf
\end{tikzcd}$$

An easy diagram chase making use of the previously-verified triangle equality completes the proof. □

Condition (ii) of Proposition B.6.2 is referred to as a "Beck-Chevalley" or *exactness* condition. Another exactness condition appears in a representable characterization of absolute lifting diagrams.

B.6.3. Definition. A trio of functors (u, v, w) between a pair of absolute left lifting diagrams (ℓ, λ) and (ℓ', λ') as below defines a **left exact transformation** if and only if the 2-cell τ induced by the universal property of the absolute left lifting is invertible:

$$\begin{tikzcd}
& B \arrow[d, "f"] \arrow[dr, "v"] & \\
C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\lambda"] \arrow[dr, "w"'] & A \arrow[dr, "u"] & B' \arrow[d, "f'"] \\
& C' \arrow[r, "g'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[dr, "v"] & \\
C \arrow[ur, "\ell"] \arrow[dr, "w"'] \arrow[rr, phantom, "\exists!\Uparrow\tau"] & & B' \arrow[d, "f'"] \\
& C' \arrow[ur, "\ell'"] \arrow[r, "g'"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\lambda'"] & A'
\end{tikzcd} \tag{B.6.4}$$

This left exactness condition holds if and only if, in the diagram on the left of (B.6.4), the whiskered 2-cell $u\lambda$ displays $v\ell$ as the absolute left lifting of $g'w$ through f', which is to say that the left exact transformations are those that preserve absolute left lifting diagrams.

B.6.5. Lemma. *The mate of a commutative square between right adjoints as below*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"] \arrow[d, phantom, shift right=3ex, "\dashv"] & A' \arrow[d, "u'"'] \arrow[d, phantom, shift left=3ex, "\vdash"] \\
B \arrow[r, "b"'] \arrow[u, bend left=60, "f"] & B' \arrow[u, bend right=60, "f'"']
\end{tikzcd}$$

is invertible if and only if (b, a, b) defines a left exact transformation between the absolute left lifting diagrams (f, η) and (f', η') of id_B through u and $\mathrm{id}_{B'}$ through u'.

Proof. The unique 2-cell τ satisfying the pasting diagram below is the mate of $\mathrm{id}\colon bu \Rightarrow u'a$.

$$\begin{tikzcd} & A \arrow[d, "u"] \arrow[drr, "a"] & & \\ B \arrow[ur, "f"] \arrow[r, equal] \arrow[dr, "b"'] & B \arrow[drr, "b"] \arrow[u, phantom, "\Uparrow\eta"] & & A' \arrow[d, "u'"] \\ & B' \arrow[rr, equal] & & B' \end{tikzcd} = \begin{tikzcd} & A \arrow[dr, "a"] & \\ B \arrow[ur, "f"] \arrow[dr, "b"'] \arrow[rr, phantom, "\exists!\Uparrow\tau"] & & A' \arrow[d, "u'"] \\ & B' \arrow[ur, "f'"] \arrow[r, equal] & B' \end{tikzcd}$$ □

B.6.6. Proposition. *Consider a 2-cell in a 2-category $\mathcal{C}$*

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, "\Uparrow\lambda", shift right=3] & A \end{tikzcd}$$

(i) If (ℓ, λ) defines an absolute left lifting diagram in $\mathcal{C}$, then

(a) For all $X \in \mathcal{C}$,

$$\begin{tikzcd} & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[ur, "\ell_*"] \arrow[r, "g_*"'] \arrow[ur, phantom, "\Uparrow\lambda_*", shift right=3] & \mathcal{C}(X,A) \end{tikzcd}$$

defines an absolute left lifting diagram in $\mathcal{C}at$.

(b) For all $k\colon Y \to X \in \mathcal{C}$, the induced transformation

$$\begin{tikzcd} & \mathcal{C}(X,B) \arrow[d, "f_*"] \arrow[dr, "k^*"] & \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] \arrow[dr, "k^*"'] & \mathcal{C}(X,A) \arrow[dr, "k^*"] & \mathcal{C}(Y,B) \arrow[d, "f_*"] \\ & \mathcal{C}(Y,C) \arrow[r, "g_*"'] & \mathcal{C}(Y,A) \end{tikzcd}$$

is left exact.

(ii) Conversely if ((i)a) holds for each $X \in \mathcal{C}$, then (ℓ, λ) defines an absolute left lifting diagram in $\mathcal{C}$.

(iii) Moreover, if $g\colon C \to A$ and $f\colon B \to A$ are so that for all $X \in \mathcal{C}$, the functor $g_*\colon \mathcal{C}(X,C) \to \mathcal{C}(X,A)$ admits an absolute left lifting through $f_*\colon \mathcal{C}(X,B) \to \mathcal{C}(X,A)$ for which condition ((i)b) holds, then g admits an absolute left lifting through f in $\mathcal{C}$.

Proof. We won't make use of the first statement so we leave the details to the reader with the hint that to verify the universal property of an absolute lifting diagram in $\mathcal{C}at$, it suffices to consider cones over the cospan (g_*, f_*) whose summit is the terminal category $\mathbb{1}$.

For the second assertion, consider a cone

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

over the cospan (g, f) in $\mathcal{C}$. This data defines a diagram of categories as below-left, which factors uniquely as below-right:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] & \mathcal{C}(X,A) \end{tikzcd} = \begin{tikzcd} \mathbb{1} \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\exists!\Uparrow\zeta", near start] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ \mathcal{C}(X,C) \arrow[r, "g_*"'] \arrow[ur, "\ell_*"] \arrow[ur, phantom, "\Uparrow\lambda_*", near end, shift right=3] & \mathcal{C}(X,A) \end{tikzcd}$$

defining the desired unique factorization

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\chi"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow\zeta", near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] \arrow[ur, "\ell"] \arrow[ur, phantom, "\Uparrow\lambda", near end, shift right=3] & A \end{tikzcd}$$

For the final statement, we define the pair (ℓ, λ) by evaluating the functor and natural transformation of the postulated absolute left lifting (ℓ^C, λ^C) in the case $X = C$ at $\mathrm{id}_C \in \mathcal{C}(C, C)$. To verify that $\lambda \colon f\ell \Rightarrow g$ defines an absolute left lifting of g through f, consider a functor $c \colon X \to C$. The hypothesis of left-exactness tells us that the composite transformation

$$\begin{tikzcd} & & \mathcal{C}(C,B) \arrow[d, "f_*"] \arrow[dr, "c^*"] & \\ \mathbb{1} \arrow[r, "\mathrm{id}_C"] & \mathcal{C}(C,C) \arrow[ur, "\ell^C"] \arrow[r, "g_*"'] \arrow[dr, "c^*"'] \arrow[ur, phantom, "\Uparrow\lambda^c", shift right=3] & \mathcal{C}(C,A) \arrow[dr, "c^*"] & \mathcal{C}(X,B) \arrow[d, "f_*"] \\ & & \mathcal{C}(X,C) \arrow[r, "g_*"'] & \mathcal{C}(X,A) \end{tikzcd}$$

is absolute left lifting. By the proof of the second statement above, this tells us that $(\ell c, \lambda c)$ is an absolute left lifting of gc through f, which proves that (ℓ, λ) is an absolute left lifting as required. □

B.6.7. Remark. The results of Proposition B.6.2 and Proposition B.6.6 can be viewed as applications of the bicategorical Yoneda lemma, which defines a 2-fully faithful embedding of a bicategory $\mathcal{C}$ into the 2-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ of pseudofunctors, pseudonatural transformations, and modifications. If a 1-cell $u \colon A \to B$ in $\mathcal{C}$ satisfies condition (i) of Proposition B.6.2, then by Theorem B.3.6, the left adjoints $f^X \colon \mathcal{C}(X, B) \to \mathcal{C}(X, A)$ define the components of a lax natural transformation. Condition (ii) demands that this lax natural transformation is a pseudo natural transformation. Now 2-fully faithfulness allows us to lift this to an arrow $f \colon B \to A$ in $\mathcal{C}$, which is left adjoint to u.

In the case of Proposition B.6.1, where $u \colon A \to B$ induces equivalences $\mathcal{C}(X, A) \xrightarrow{\sim} \mathcal{C}(X, B)$, the inverse equivalences automatically assemble into a pseudonatural transformation, which is why no additional hypothesis was required.

Exercises.

B.6.i. Exercise. Confirm the assertion made in the proof of Lemma B.6.5.

B.6.ii. Exercise. Verify the forwards implication in Proposition B.6.6.

APPENDIX C

Abstract homotopy theory

The underlying 1-category of an ∞-cosmos, together with its classes of isofibrations, equivalences, and trivial fibrations, defines a *category of fibrant objects*, a classical context for abstract homotopy theory first studied by Brown [**20**]. In §C.1, we develop some of the general theory of categories of fibrant objects with the aim of presenting some classical proofs that were omitted in the main text.

The remainder of this chapter develops material that will be applied in later appendices. In Appendix E we will discover that examples of ∞-cosmoi can be found "in the wild" as model categories that are enriched as such over Joyal's model structure on the category of simplicial sets. For this reason, model categories, enriched model categories, and the functors between them are introduced in §C.3. In the introduction to Chapter I. Axiomatic Homotopy Theory [**71**] where Quillen first introduces the definition of a model category, he highlights the factorization and lifting axioms as being the most important. These are most clearly encapsulated in the categorical notion of a weak factorization system discussed in §C.2, the axioms for which were enumerated later.

Finally, some of the technical combinatorial proofs of Appendix D involve inductive arguments involving the Reedy category $\mathbf{\Delta}$. Thus, we conclude in §C.4 with a brief introduction to Reedy category theory following the presentation of [**82**].

C.1. Abstract homotopy theory in a category of fibrant objects

In this section we work in an (unenriched) category of fibrant objects, a notion first introduced by Brown [**20**]. Examples include the underlying category of an ∞-cosmos or the full subcategory of fibrant objects in a Quillen model category (hence the name).

C.1.1. Definition (category of fibrant objects). A **category of fibrant objects** consists of a category $\mathcal{M}$ together with two subcategories of morphisms $\mathcal{W}$ and $\mathcal{F}$ satisfying the following axioms:

(i) $\mathcal{M}$ has products and in particular a terminal object 1. Moreover, the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are each closed under products.
(ii) $\mathcal{W}$ has the **2-of-3 property**: for any composable pair of morphisms, if any two of f, g, and gf is in $\mathcal{W}$ then so is the third.
(iii) Pullbacks of maps in $\mathcal{F}$ exist and lie in $\mathcal{F}$, and pullback also preserves the class $\mathcal{F} \cap \mathcal{W}$.
(iv) Limits of towers of maps in $\mathcal{F}$ exist and also lie in $\mathcal{F}$, and the class $\mathcal{F} \cap \mathcal{W}$ is also closed under forming limits of towers.
(v) For every object B, there exists a **path object** PB together with a factorization of the diagonal into a map in $\mathcal{W}$ followed by a map in $\mathcal{F}$:

$$\begin{tikzcd} & PB \arrow[dr, two heads] & \\ B \arrow[ur, "\sim"] \arrow[rr, "\Delta"'] & & B \times B \end{tikzcd}$$

(vi) All objects are **fibrant**: for every $B \in \mathcal{M}$, the map $B \to 1$ lies in $\mathcal{F}$.

C.1.2. REMARK. The original definition only asked for finite products in axiom (i) and omitted axiom (iv). The fact that the classes $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$ are closed under finite products can be proven as a consequence of axiom (iii); see Corollary C.1.14. Here, we ask for these infinite limits to parallel the limit axiom 1.2.1(i) in our definition of an ∞-cosmos. In practice, the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are frequently characterized by a right lifting property, in which case all of these closure axioms are automatic; see Lemma C.2.3.

Our primary interest in this notion is on account of the following two examples.

C.1.3. LEMMA. *The underlying category of an ∞-cosmos defines a category of fibrant objects with $\mathcal{W}$ the class of equivalences and $\mathcal{F}$ the class of isofibrations.*

PROOF. Most of the axioms of Definition C.1.1 are subsumed by the limit and isofibration axioms of Definition 1.2.1. The remaining pieces are established in Lemma 1.2.11, Lemma 1.2.13, and Lemma 1.2.17. □

C.1.4. LEMMA. *The full subcategory of fibrant objects in a model category defines a category of fibrant objects with $\mathcal{W}$ the class of weak equivalences and $\mathcal{F}$ the class of fibrations between fibrant objects.*

PROOF. Exercise C.3.i. □

In general, it is customary is to refer to the maps in $\mathcal{W}$ as "weak equivalences", the maps in $\mathcal{F}$ as "fibrations," and the maps in $\mathcal{F} \cap \mathcal{W}$ as "trivial fibrations" — unless the specific context dictates alternate names — and depict these classes by the decorated arrows, $\xrightarrow{\sim}$, $\twoheadrightarrow$, and $\overset{\sim}{\twoheadrightarrow}$, respectively.

C.1.5. REMARK. Both of the examples just discussed have the additional property of being **right proper**, satisfying an additional axiom:

(vii) Pullbacks of maps in $\mathcal{W}$ along maps in $\mathcal{F}$ define maps in $\mathcal{W}$:

$$\begin{tikzcd} F \arrow[r, "g", "\sim"'] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

For ∞-cosmoi this is proven in Lemma 3.3.2 and for model categories, this was first observed by Reedy in [**74**, Theorem B] (see also [**66**, 15.4.2]).

The factorization axiom in a category of fibrant objects can be generalized to construct factorizations of any map; cf. Lemma 1.2.13

C.1.6. LEMMA (Brown factorization lemma). *Any map $f\colon A \to B$ in a category of fibrant objects may be factored as a weak equivalence followed by an fibration, where the weak equivalence is constructed as a section of a trivial fibration.*

$$\begin{tikzcd} & Pf \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "s", "\sim"'] \arrow[rr, "f"'] & & B \end{tikzcd}$$

Proof. The displayed factorization is constructed by the pullback of the path object fibration $PB \twoheadrightarrow B \times B$ of (v):

$$\begin{tikzcd} & B \arrow[drr, "\sim"] \arrow[ddrr, bend left, "\Delta"'] & & \\ A \arrow[ur, "f"] \arrow[r, dashed, "\sim", "s"'] \arrow[dr, "(A{,}f)"'] & Pf \arrow[rr] \arrow[d, two heads, "(q{,}p)"'] \arrow[drr, phantom, "\lrcorner" very near start] & & PB \arrow[d, two heads] \\ & A \times B \arrow[rr, "f \times B"'] & & B \times B \end{tikzcd}$$

By the 2-of-3 property for the weak equivalences, both projections $PB \overset{\sim}{\twoheadrightarrow} B$ are trivial fibrations. Since the map q is a pullback of one of these projections along $f \colon A \to B$, it follows from axiom (iii) that q is a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence. □

C.1.7. Corollary. *If $\mathcal{M}$ is a category of fibrant objects and $B \in \mathcal{M}$, then the category $\mathcal{M}_{/B}$ of fibrations in $\mathcal{M}$ with codomain B and maps over B becomes a category of fibrant objects with weak equivalences and fibrations created by the forgetful functor $\mathcal{M}_{/B} \to \mathcal{M}$.*

Proof. The construction of limits in the slice category $\mathcal{M}_{/B}$ is described in the proof of Proposition 1.2.19; note in particular, that id_B is the terminal object, so all objects in $\mathcal{M}_{/B}$, being fibrations in $\mathcal{M}$, are fibrant. Path objects for a fibration $f \colon A \twoheadrightarrow B$ are constructed by applying Lemma C.1.6 to the "diagonal" map $(f, f) \colon A \to A \times_B A$ from A to the pullback of f along itself. □

The dual of a result of Blumberg and Mandell [**15**, 6.4] demonstrates that the equivalences in any ∞-cosmos satisfy the 2-of-6 property. The proof reveals that this holds in any category of fibrant objects in which the class $\mathcal{W}$ is closed under retracts.

C.1.8. Proposition. *Let $\mathcal{M}$ be a category with classes of maps $\mathcal{W}$ and $\mathcal{F}$ so that:*

- *$\mathcal{W}$ satisfies the 2-of-3 property, and is closed under retracts.*
- *The pullback of a map in $\mathcal{F} \cap \mathcal{W}$ is in $\mathcal{F} \cap \mathcal{W}$ and these pullbacks always exist.*
- *Every map in $\mathcal{W}$ factors as a section of a map in $\mathcal{F} \cap \mathcal{W}$ followed by a map in $\mathcal{F}$.*

Then the class $\mathcal{W}$ satisfies the ***2-of-6 property****: for any composable triple of morphisms*

$$\begin{tikzcd} & B \arrow[dr, "hg", "\sim"'] \arrow[dd, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "hgf"] \arrow[dr, "\sim", "gf"'] & & D \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

if gf and hg are in a class $\mathcal{W}$ then f, g, h, and hgf are too.

Proof. Form the factorization of the weak equivalence hg as displayed below

$$\begin{tikzcd} & D' \arrow[dl, bend right, "r"', "\sim"] \arrow[dr, two heads, "p", "\sim"'] & \\ B \arrow[ur, "j"', "\sim"] \arrow[rr, "\sim", "hg"'] & & D \end{tikzcd}$$

and form the pullback of p along h and the induced map t:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[ddrr, bend right, "gf"', "\sim"] & B \arrow[dr, dashed, "t"] \arrow[ddr, bend right, "g"'] \arrow[drr, bend left, "j", "\sim"'] \arrow[drrr, bend left, equals] & & & \\ & & C' \arrow[r, "u"] \arrow[d, two heads, "q"', "\wr"] \arrow[dr, phantom, "\lrcorner" very near start] & D' \arrow[r, two heads, "\sim", "r"'] \arrow[d, two heads, "\wr"', "p"] & B \\ & & C \arrow[r, "h"'] & D & \end{tikzcd}$$

By pullback stability of the trivial fibrations, the map q is a weak equivalence, so by the 2-of-3 property and the assumption that gf is a weak equivalence, the composite $tf : A \to C'$ must be a weak equivalence. Since the map f is a retract of this composite

$$\begin{tikzcd} A \arrow[r, equals] \arrow[d, "f"'] & A \arrow[r, equals] \arrow[d, "\wr"', "tf"] & A \arrow[d, "f"] \\ B \arrow[r, "t"'] & C' \arrow[r, "ru"'] & B \end{tikzcd}$$

so by retract closure of the weak equivalences, $f \in \mathcal{W}$. Now it follows from the 2-of-3 property that g, h, and hgf lie in $\mathcal{W}$ as well. ☐

C.1.9. Corollary. *The equivalences in an ∞-cosmos satisfy the 2-of-6 property.*

Proof. It remains only to argue that the premises of Proposition C.1.8 hold for the classes of equivalences and trivial fibrations in any ∞-cosmos.

Lemma 1.2.17 proves that the equivalences in $\mathcal{K}$ are also closed under retracts and have the 2-of-3 property. Lemma 1.2.11 proves that the class of trivial fibrations is stable under pullbacks, which exist in any ∞-cosmos. Lemma 1.2.13 constructed the desired factorization, which by the 2-of-3 property factors an equivalence as a section of a trivial fibration followed by a trivial fibration. Now Proposition C.1.8 applies to prove that the equivalences in any ∞-cosmos satisfy the stronger 2-of-6 property. ☐

The following consequence of Lemma C.1.6, traditionally referred to as "Ken Brown's lemma," is the key to proving the equivalence invariance of many constructions in a category of fibrant objects.

C.1.10. Lemma (Ken Brown's lemma). *Consider a functor $F : \mathcal{M} \to \mathcal{K}$ whose domain is a category of fibrant objects and whose codomain is a category with a subcategory of "weak equivalences" satisfying the 2-of-3 property. If F carries trivial fibrations to weak equivalences, then F carries weak equivalences in $\mathcal{M}$ to weak equivalences in $\mathcal{K}$.*

Proof. By the 2-of-3 property of the weak equivalences in $\mathcal{M}$, any weak equivalence in a category of fibrant objects may be factored as a section of a trivial fibration followed by a trivial fibration.

$$\begin{tikzcd} & Pf \arrow[dl, two heads, bend right, "q"', "\sim"] \arrow[dr, two heads, "p", "\sim"'] & \\ A \arrow[ur, "s"', "\sim"] \arrow[rr, "\sim", "f"'] & & B \end{tikzcd}$$

By hypothesis, the images of the maps q and p under F are weak equivalences. By the 2-of-3 property of the weak equivalences in $\mathcal{K}$, it follows that the image of s and thus also the image of f are weak equivalences. ☐

The rest of this section is devoted to applications of Lemma C.1.10 to establish the weak equivalence invariance of limits in a category of fibrant objects. To warm up, as a very easy consequence:

C.1.11. LEMMA. *In a category of fibrant objects, a weak equivalence between fibrations pulls back to a weak equivalence between fibrations:*

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, "u", "\sim"'] \arrow[dd, "r"', two heads] & & E \arrow[dd, "p"', two heads] \arrow[dr, "e", "\sim"'] & \\
& Q \arrow[rr] \arrow[dl, "s", two heads] & & F \arrow[dl, "q", two heads] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

PROOF. By Corollary C.1.7, slices of a category $\mathcal{M}$ of fibrant objects define categories of fibrant objects and pullback along f defines a functor $f^*\colon \mathcal{M}_{/B} \to \mathcal{M}_{/A}$. Note that the map u in the displayed diagram is the pullback of the map e, so it follows directly from axiom (iii) of Definition C.1.1 that pullback preserves trivial fibrations. Now Lemma C.1.10 implies that it also preserves equivalences. □

Other results in a similar vein require somewhat more delicate arguments. The proofs appearing below are originally due to Reedy in an unpublished manuscript **[74]** that implicitly gave birth to the notion of a "Reedy category" that we introduce in §C.4.

C.1.12. PROPOSITION. *Consider a diagram in a category of fibrant objects:*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "r"'] & A \arrow[d, "p"] & B \arrow[l, "f"', two heads] \arrow[d, "q"] \\
\bar{C} \arrow[r, "\bar{g}"] & \bar{A} & \bar{B} \arrow[l, "\bar{f}"', two heads]
\end{tikzcd}$$

If the map r and the map $z\colon B \to A \times_{\bar{A}} \bar{B}$ are both

(i) fibrations, or

(ii) trivial fibrations

then the induced map from the pullback of f along g to the pullback of $\bar{f}$ along $\bar{g}$ again a fibration or trivial fibration, respectively.

PROOF. By considering the commutative diagram and repeatedly applying the pullback composition and cancelation lemma, one discovers that the induced map t factors as a pullback of z followed by a pullback of r as displayed below

$$\begin{tikzcd}
C \underset{A}{\times} B \arrow[rrr] \arrow[drr] \arrow[ddd, dashed, "t"'] \arrow[ddr, dotted, two heads] & & & B \arrow[drr] \arrow[ddr, "z"] \arrow[ddd, "q"', two heads] & & \\
& & C \arrow[rrr] \arrow[ddd, "r"', two heads] & & & A \arrow[ddd, "p", two heads] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[rrr, dotted] & & & \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] & \\
\bar{C} \underset{\bar{A}}{\times} \bar{B} \arrow[rrr] \arrow[drr] & & & \bar{B} \arrow[drr] & & \\
& & \bar{C} \arrow[rrr] & & & \bar{A}
\end{tikzcd}$$

and is thus an fibration or trivial fibration if both of these maps are. $\square$

The hypothesis of right properness allows us to prove the following result whose dual form is sometimes called the "gluing lemma."

C.1.13. PROPOSITION. *In a right proper category of fibrant objects, the induced map between the pullbacks of the horizontal rows of a diagram*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "r"', "\wr"] & A \arrow[d, "\wr"', "p"] & B \arrow[l, two heads, "f"'] \arrow[d, "\wr"', "q"] \\ \bar{C} \arrow[r, "\bar{g}"'] & \bar{A} & \bar{B} \arrow[l, two heads, "\bar{f}"] \end{tikzcd}$$

is again a weak equivalence.

PROOF. Using Lemma C.1.6, the 2-of-3 property from Definition C.1.1, and the right properness of Remark C.1.5, the proof of Proposition 3.3.3 works equally in any right proper category of fibrant objects. $\square$

C.1.14. COROLLARY. *In a category of fibrant objects, finite products of fibrations, trivial fibrations, or weak equivalences are again finite products, trivial fibrations, or weak equivalences.*

PROOF. This can be proven inductively from the case of binary products, which can be constructed as pullbacks over the terminal object. Lemma C.1.10 applies to deduce the result for weak equivalences from the result for trivial fibrations. $\square$

The category of diagrams valued in a category of fibrant objects may itself be equipped with the structure of a category of fibrant objects, at least for certain diagram shapes. A particularly useful family of diagrams includes those indexed by inverse categories.

C.1.15. DEFINITION. A category $\mathcal{I}$ is a **inverse category** if there exists a functor $\deg\colon \mathcal{I} \to \omega^{\mathrm{op}}$ that reflects identities.

The degree functor assigns a natural number degree to each object of $\mathcal{I}$ in such a way that all non-identity morphisms "lower degree," in the sense that the degree of their domain object is strictly greater than the degree of their codomain object. The utility of this axiomatization is it allows us to define the data of an $\mathcal{I}$-indexed diagram or natural transformation by inductively specifying diagrams indexed by the full subcategories

$$\mathcal{I}_{\leq 0} \hookrightarrow \cdots \hookrightarrow \mathcal{I}_{\leq n-1} \hookrightarrow \mathcal{I}_{\leq n} \hookrightarrow \cdots \hookrightarrow \mathcal{I}$$

of objects with bounded degree. To extend $X \in \mathcal{M}^{\mathcal{I}_{\leq n-1}}$ to $\mathcal{M}^{\mathcal{I}_{\leq n}}$ requires the specification, for each object i with degree n of an object $X_i \in \mathcal{M}$ together with a map

$$X_i \to \partial_i X := \lim\left({}^{i/}\mathcal{I}_{\leq n-1} \longrightarrow \mathcal{I}_{\leq n-1} \xrightarrow{X} \mathcal{M} \right) \tag{C.1.16}$$

The mathematics does not change in any substantial way if ω is replaced by the category of ordinals. The reason we restrict to finite degrees is because we've only asked for limits of ω^{op}-indexed towers in Definition C.1.1.

For reasons that will become clear momentarily we define:

C.1.17. DEFINITION. Let $\mathcal{M}$ be a category of fibrant objects and let $\mathcal{I}$ be an inverse category.

- If $X \in \mathcal{M}^{\mathcal{I}}$ is a diagram with the property that for each $i \in \mathcal{I}$, the map $X_i \twoheadrightarrow \partial_i X$ defined by (C.1.16) is a fibration, then call X a **fibrant diagram**.
- If $\alpha \colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ is a natural transformation between fibrant diagrams so that for each $i \in \mathcal{I}$ the map $\hat{m}_i$ defined by

$$\begin{tikzcd}
X_i \arrow[rr, "\alpha_i"] \arrow[dr, dashed, "\hat{m}_i"] \arrow[dd, two heads] & & Y_i \arrow[dd, two heads] \\
& \bullet \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\partial_i X \arrow[rr, "\partial_i \alpha"'] & & \partial_i Y
\end{tikzcd} \tag{C.1.18}$$

is a fibration, then call $\alpha \colon X \to Y$ a **fibrant natural transformation**.

C.1.19. Proposition.

(i) A category of fibrant objects $\mathcal{M}$ admits limits of any fibrant diagram indexed by an inverse category $X \in \mathcal{M}^{\mathcal{I}}$, with $\lim_{\mathcal{I}} F \in \mathcal{M}$ constructed as the limit of a tower

$$\lim_{\mathcal{I}} X \coloneqq \lim_{\omega^{\mathrm{op}}} \left(\cdots \twoheadrightarrow \lim_{\mathcal{I}_{\leq n}} X \twoheadrightarrow \lim_{\mathcal{I}_{\leq n-1}} X \twoheadrightarrow \cdots \twoheadrightarrow \lim_{\mathcal{I}_{\leq 0}} X \right)$$

each layer of which is a pullback

$$\begin{tikzcd}
\lim_{\mathcal{I}_{\leq n}} X \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_{\mathcal{I}_{\leq n-1}} X \arrow[d] \\
\prod\limits_{\deg(i)=n} X_i \arrow[r, two heads] & \prod\limits_{\deg(i)=n} \partial_i X
\end{tikzcd}$$

In particular, each leg of the limit cone $\lim_{\mathcal{I}} X \twoheadrightarrow X_i$ is a fibration as is each map in the image of the fibrant diagram X.

(ii) For any fibrant natural transformation $\alpha \colon X \to Y \in \mathcal{M}^{\mathcal{I}}$, the induced map $\lim_{\mathcal{I}} X \to \lim_{\mathcal{I}} Y$ is the limit composite of a tower whose n-th layer is the map p_n constructed as a pullback in the diagram below:

$$\begin{tikzcd}
\lim_{\mathcal{I}_{\leq n}} X_i \arrow[rrr] \arrow[drr, two heads] \arrow[ddd, two heads] \arrow[ddr, dotted, "p_n"] & & & \prod\limits_{\deg(i)=n} X_i \arrow[drr, two heads] \arrow[ddd, two heads] \arrow[ddr] & & \\
& & \lim_{\mathcal{I}_{\leq n-1}} X_i \arrow[rrr] \arrow[ddd, two heads] & & & \prod\limits_{\deg(i)=n} \partial_i X \arrow[ddd, two heads] \\
& \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[rrr, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & & & \bullet \arrow[ur, dotted, two heads] \arrow[dl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\lim_{\mathcal{I}_{\leq n}} Y_i \arrow[rrr] \arrow[drr, two heads] & & & \prod\limits_{\deg(i)=n} Y_i \arrow[drr, two heads] & & \\
& & \lim_{\mathcal{I}_{\leq n-1}} Y_i \arrow[rrr] & & & \prod\limits_{\deg(i)=n} \partial_i Y
\end{tikzcd} \tag{C.1.20}$$

Moreover, each component map $\alpha_i \colon X_i \twoheadrightarrow Y_i$ is a fibration.

Proof. Note that the slice category ${}^{i/}\mathcal{I}_{n-1}$ is again an inverse category in which every object has degree at most $n-1$. In the case where i has degree 1, this category is discrete, so by induction we may

assume that the limit $\partial_i X$ defined by (C.1.16) exists. Now the result of (i) follows by direct inspection of the universal property of this construction, or from a more conceptual argument that will prove a generalization of this result in §C.4. The final assertion follows from this construction and is left to Exercise C.1.i.

By (i), it follows that the induced map between the inverse limits is defined as as the limit of an ω^{op}-indexed diagram in the category $\mathcal{M}^{\mathbf{2}}$:

$$\begin{tikzcd} \lim_{\mathcal{I}} X \coloneqq \lim_{n\in\omega^{\mathrm{op}}} \lim_{\mathcal{I}_{\leq n}} X \arrow[r, dotted] \arrow[d, dashed, "\lim_{\mathcal{I}}\alpha"'] & \lim_{\mathcal{I}_{\leq n}} X \arrow[r] \arrow[d] & \lim_{\mathcal{I}_{\leq n-1}} X \arrow[r] \arrow[d] & \cdots \arrow[r] & \lim_{\mathcal{I}_{\leq 1}} X \arrow[r] \arrow[d] & \lim_{\mathcal{I}_{\leq 0}} X \arrow[d, "p_0"] \\ \lim_{\mathcal{I}} Y \coloneqq \lim_{n\in\omega^{\mathrm{op}}} \lim_{\mathcal{I}_{\leq n}} Y \arrow[r, dotted] & \lim_{\mathcal{I}_{\leq n}} Y \arrow[r] & \lim_{\mathcal{I}_{\leq n-1}} Y \arrow[r] & \cdots \arrow[r] & \lim_{\mathcal{I}_{\leq 1}} Y \arrow[r] & \lim_{\mathcal{I}_{\leq 0}} Y \end{tikzcd}$$

From this description it is clear that the map $\lim_{\mathcal{I}} \alpha$ factors as the limit composite of a tower whose bottom layer is the pullback of the map p_0 along the lower-horizontal composite above, whose next layer is the pullback of the map p_1 appearing in the right-most square, whose next layer is the pullback of the map p_2 appearing in the second right-most square, and so on, where in each square p_n is the map from the upper left-hand corner to the pullback of the lower-right cospan. Finally, by applying pullback composition and cancelation in the cube (C.1.20), it follows from the fact proven in (i) that the top and bottom faces are pullbacks that the map p_n from the initial vertex to the pullback in the left face is a pullback of the corresponding map from $\prod_{\deg(i)=n} X_i$ to the pullback in the right face. This proves all but the final clause of (ii), which is also left to Exercise C.1.i. ☐

C.1.21. Proposition. *Let $\mathcal{M}$ be a category of fibrant objects with fibrations $\mathcal{F}$ and weak equivalences $\mathcal{W}$ and let $\mathcal{I}$ be an inverse category. The category $\mathcal{M}^{\mathcal{I}}$ of fibrant diagrams and all natural transformations between them inherits the structure of a category of fibrant objects in which:*

- *the class of weak equivalences is the class of natural transformations whose components lie in $\mathcal{W}$*
- *the class of fibrations is the class of fibrant natural transformations, those $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ so that for each $i \in \mathcal{I}$ the map*
$$X_i \twoheadrightarrow Y_i \underset{\partial_i Y}{\times} \partial_i X$$
is in $\mathcal{F}$.
- *the class of trivial fibrations is the class of natural transformations $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ so that for each $i \in \mathcal{I}$ the map*
$$X_i \overset{\sim}{\twoheadrightarrow} Y_i \underset{\partial_i Y}{\times} \partial_i X$$
is in $\mathcal{F} \cap \mathcal{W}$.

Proof. The proof is a very lengthy exercise for the reader, which only entails specializing the corresponding arguments from §C.4 to this "one-sided" case. ☐

The payoff for all this work is that it is now easy to verify the following result.

C.1.22. Proposition. *Let $\mathcal{M}$ be a category of fibrant objects and let $\mathcal{I}$ be an inverse category. Then with the structure of C.1.21 the limit functor $\lim\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ preserves the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ and hence also $\mathcal{W}$.*

PROOF. Consider a map $\alpha\colon X \to Y \in \mathcal{M}^{\mathcal{I}}$ in $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$. By Proposition C.1.19(ii), this map is the limit composite of a tower of maps, each layer of which is the pullback of a product of the maps that we have assumed lies in $\mathcal{F}$ or $\mathcal{F} \cap \mathcal{W}$. Since the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under product, pullback, and limits of towers, it is now clear that the limit functor preserves these classes. The fact that it also proves the class $\mathcal{W}$ follows from Lemma C.1.10. □

Consequently:

C.1.23. COROLLARY. *In a category of fibrant objects:*

(i) A pointwise weak equivalence between cospans of fibrations induces a weak equivalence between their pullbacks.
(ii) A pointwise weak equivalence between towers of fibrations induces a weak equivalence between their inverse limits. □

Exercises.

C.1.i. EXERCISE.

(i) Verify that each leg of the limit cone constructed in Proposition C.1.19(i) is a fibration.
(ii) Conclude that each morphism in the image of a fibrant diagram is a fibration.
(iii) Arguing along the same lines, verify that each component of a fibrant natural transformation is a fibration.

C.2. Lifting properties, weak factorization systems, and Leibniz closure

Fixing two arrows j and p in a category $\mathcal{M}$, we regard any commutative square of the form

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "j"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dashed, "\ell"] & \bullet \end{tikzcd}$$

as presenting a **lifting problem** between j and p, which is solved by constructing a **lift**: a diagonal morphism ℓ making both triangles commute. If every lifting problem between j and p has a solution, we say that j has the **left lifting property** with respect to p and, equivalently, that p has the **right lifting property** with respect to j. When this is the case, we use the suggestive symbol $j \boxslash p$ to assert this lifting property.

Frequently in abstract homotopy theory a class of maps of interest is characterized by a left or right lifting property with respect to another class or set of maps.

C.2.1. DEFINITION. Let $\mathcal{J}$ be a class of maps in a category $\mathcal{M}$.

- Write $\mathcal{J}^{\boxslash}$ for the class of maps that have the **right lifting property** with respect to every morphism in $\mathcal{J}$.
- Write ${}^{\boxslash}\mathcal{J}$ for the class of maps that have the **left lifting property** with respect to every morphism in $\mathcal{J}$.

C.2.2. EXAMPLE. Definitions 1.1.17 and 1.1.24 charaterize the isofibrations and trivial fibrations between quasi-categories by right lifting properties against the sets of maps

$$\{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n} \cup \{\mathbb{1} \hookrightarrow \mathbb{I}\} \qquad \text{and} \qquad \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n\geq 0}$$

respectively.

Maps characterized by a right lifting property automatically satisfy various closure properties that may now be familiar.

C.2.3. Lemma. *Any class of maps $\mathcal{J}^{\boxslash}$ characterized by a right lifting property contains the isomorphisms and is closed under composition, product, pullback, retract, and limits of towers.*

In the statement, "products" and "retracts" refer to limits formed in the category of arrows, while the "pullbacks" are of a map in $\mathcal{J}^{\boxslash}$ along an arbitrary map. A "tower" refers to a diagram of shape α^{op}, where α is a limit ordinal (most likely ω). Closure under limits of towers asserts that if the images of each of the atomic arrows in the indexing category lie in $\mathcal{J}^{\boxslash}$, then the map from the limit object to the terminal object in the diagram is also in $\mathcal{J}^{\boxslash}$.

Proof. All of the arguments are similar. For instance, suppose q is a pullback of $p \in \mathcal{J}^{\boxslash}$. By juxtaposing a lifting problem as below-left with the pullback square as below-right, we may solve the composite lifting problem of j against p to obtain the dashed diagonal morphism ℓ, and then induce a solution s to the lifting problem of j against q via the cone formed by (v, ℓ) over the pullback diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "\mathcal{J}\ni j"'] & \bullet \arrow[r, "a"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dotted, "s"] \arrow[urr, dashed, "\ell"'] & \bullet \arrow[r, "b"'] & \bullet \end{tikzcd}$$

So q lifts against $\mathcal{J}$ and is therefore in $\mathcal{J}^{\boxslash}$. $\square$

On account of the dual of Lemma C.2.3, any set of maps in a cocomplete category "cellularly generates" a larger class of maps with the same left lifting property.

C.2.4. Definition. Let $\mathcal{J}$ be a class of maps that we think of as "basic cells." A **$\mathcal{J}$-cell complex** is a map built as a transfinite composite of pushouts of coproducts of maps in $\mathcal{J}$:

$$\begin{tikzcd} \bullet \arrow[r, "\coprod j\in\mathcal{J}"] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[r, "\coprod j\in\mathcal{J}"] \arrow[d] & \bullet \arrow[d] & \\ \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[rrrr, dashed, bend right, "\in\mathcal{J}\text{-cell}"'] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet \arrow[r, dotted] & \bullet \\ & \bullet \arrow[u] \arrow[r, "\coprod j\in\mathcal{J}"'] & \bullet \arrow[u] \arrow[ul, phantom, "\llcorner", very near end] & & \end{tikzcd}$$

The class of $\mathcal{J}$-cell complexes $\mathcal{J}$-cell is said to be **cellularly generated** by a set of maps $\mathcal{J}$. The class of maps $\mathcal{J}$-cof **cofibrantly generated** by a set of maps $\mathcal{J}$ is comprised of those maps obtained as retracts of sequential composites of pushouts of coproducts of those maps.

C.2.5. Definition. A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

(i) Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$.

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

(ii) The classes $\mathcal{L}$ and $\mathcal{R}$ respectively have the left and right lifting properties $\mathcal{L} \boxslash \mathcal{R}$ with respect to each other : that is, any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

admits a diagonal filler as indicated.

(iii) Moreover $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

In the presence of a pair of adjoint functors, lifting properties transpose.

C.2.6. Lemma. *In the presence of any adjunction*

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$$

(i) The lifting problem displayed below left in $\mathcal{N}$ admits a solution if and only if the transposed lifting problem displayed below right admits a solution in $\mathcal{M}$.

$$\begin{tikzcd} FA \arrow[r, "f^\sharp"] \arrow[d, "F\ell"'] & X \arrow[d, "r"] \\ FB \arrow[r, "g^\sharp"'] \arrow[ur, dashed, "k^\sharp"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "\ell"'] & UX \arrow[d, "Ur"] \\ B \arrow[r, "g^\flat"'] \arrow[ur, dashed, "k^\flat"] & UY \end{tikzcd}$$

(ii) If $\mathcal{M}$ has a weak factorization system $(\mathcal{L}, \mathcal{R})$ while $\mathcal{N}$ has a weak factorization system $(\mathcal{L}', \mathcal{R}')$ then F preserves the left classes if and only if U preserves the right classes:

$$F\mathcal{L} \subset \mathcal{L}' \qquad \leftrightsquigarrow \qquad \mathcal{R} \supset U\mathcal{R}'.$$

The factorizations of Definition C.2.5 are completely irrelevant to the statement of (ii) but we have stated this result for weak factorization systems because that is the context in which it will typically be applied.

Proof. Exercise C.2.iv. □

Lemma C.2.6(ii) defines the notion of **adjunction of weak factorization systems**, this being an adjoint pair of functors between categories equipped with weak factorization systems so that the left adjoint preserves the left classes and the right adjoint preserves the right classes. Our aim is now to extend this notion to two-variable adjunctions,[1] which are given by a triple of bifunctors, which we write using notation that will suggest the most common examples

$$\mathcal{K} \times \mathcal{L} \xrightarrow{\otimes} \mathcal{M}, \;\; \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\{,\}} \mathcal{L}, \;\; \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\hom} \mathcal{K} \tag{C.2.7}$$

equipped with a natural isomorphism

$$\mathcal{M}(K \otimes L, M) \cong \mathcal{L}(L, \{K, M\}) \cong \mathcal{K}(K, \hom(L, M)).$$

[1]There is an analogous generalization to n-variable adjunctions that can be found in [**22**, §4].

The "pushout-product" of a bifunctor $-\otimes-\colon \mathcal{K}\times\mathcal{L}\to\mathcal{M}$ defines a bifunctor $-\widehat{\otimes}-\colon \mathcal{K}^{2}\times\mathcal{L}^{2}\to\mathcal{M}^{2}$ that we refer to as the "Leibniz tensor" (when the bifunctor $\otimes$ is called a "tensor"). The "Leibniz cotensor" and "Leibniz hom"

$$\widehat{\{-,-\}}\colon (\mathcal{K}^{2})^{\mathrm{op}}\times\mathcal{M}^{2}\to\mathcal{L}^{2} \qquad \text{and} \qquad \widehat{\hom}(-,-)\colon (\mathcal{L}^{2})^{\mathrm{op}}\times\mathcal{M}^{2}\to\mathcal{K}^{2}$$

are defined dually, using pullbacks in $\mathcal{L}$ and $\mathcal{K}$ respectively.

C.2.8. Definition (Leibniz tensors and cotensors). Given a bifunctor $-\otimes-\colon \mathcal{K}\times\mathcal{L}\to\mathcal{M}$ valued in a category with pushouts, the **Leibniz tensor** of a map $k\colon I\to J$ in $\mathcal{K}$ and a map $\ell\colon A\to B$ in $\mathcal{L}$ is the map $k\mathbin{\widehat{\otimes}}\ell$ in $\mathcal{M}$ induced by the pushout diagram below-left:

$$\begin{tikzcd}
I\otimes A \arrow[r, "I\otimes\ell"] \arrow[d, "k\otimes A"'] & I\otimes B \arrow[d] \arrow[ddr, bend left, "k\otimes B"] & \\
J\otimes A \arrow[r] \arrow[drr, bend right, "J\otimes\ell"'] & \bullet \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "k\widehat{\otimes}\ell"] & \\
 & & J\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
\{J,X\} \arrow[drr, bend left, "\{k,X\}"] \arrow[dr, dashed, "\widehat{\{k,m\}}"] \arrow[ddr, bend right, "\{J,m\}"'] & & \\
 & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \{I,X\} \arrow[d, "\{I,m\}"] \\
 & \{J,Y\} \arrow[r, "\{k,Y\}"'] & \{I,Y\}
\end{tikzcd}$$

In the case of a bifunctor $\{-,-\}\colon \mathcal{K}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{L}$ contravariant in one of its variables valued in a category with pullbacks, the **Leibniz cotensor** of a map $k\colon I\to J$ in $\mathcal{K}$ and a map $m\colon X\to Y$ in $\mathcal{M}$ is the map $\widehat{\{k,m\}}$ induced by the pullback diagram above right.

C.2.9. Proposition. *The Leibniz construction preserves:*

(i) structural isomorphisms: a natural isomorphism

$$X*(Y\otimes Z)\cong(X\times Y)\square Z$$

between suitably composable bifunctors extends to a natural isomorphism

$$f\mathbin{\widehat{*}}(g\mathbin{\widehat{\otimes}}h)\cong(f\mathbin{\widehat{\times}}g)\mathbin{\widehat{\square}}h$$

between the corresponding Leibniz products;

(ii) adjointness: if $(\otimes,\{,\},\hom)$ define a two-variable adjunction, then the Leibniz functors $(\widehat{\otimes},\widehat{\{,\}},\widehat{\hom})$ define a two-variable adjunction between the corresponding arrow categories;

(iii) colimits in the arrow category: if $\otimes\colon\mathcal{K}\times\mathcal{L}\to\mathcal{M}$ is cocontinuous in either variable, then so is $\widehat{\otimes}\colon\mathcal{K}^{2}\times\mathcal{L}^{2}\to\mathcal{M}^{2}$;

(iv) pushouts: if $\otimes\colon\mathcal{K}\times\mathcal{L}\to\mathcal{M}$ is cocontinuous in its second variable, and if g' is a pushout of g, then $f\mathbin{\widehat{\otimes}}g'$ is a pushout of $f\mathbin{\widehat{\otimes}}g$;

(v) *composition, in a sense: the Leibniz tensor $f \mathbin{\widehat{\otimes}} (h \cdot g)$ factors as a composite of a pushout of $f \mathbin{\widehat{\otimes}} g$ followed by $f \mathbin{\widehat{\otimes}} h$*

$$\begin{tikzcd}
I \otimes A \arrow[r, "I\otimes g"] \arrow[d, "f\otimes A"'] & I \otimes B \arrow[r, "I\otimes h"] \arrow[d] & I \otimes C \arrow[d] \arrow[ddrr, bend left, "f\otimes C"] & & \\
J \otimes A \arrow[r] \arrow[drr, "J\otimes g"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] \arrow[dr, dashed, "f\widehat{\otimes}g"] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] \arrow[ddrr, bend left, "f\widehat{\otimes}(h\cdot g)"] & & \\
& & J \otimes B \arrow[r] \arrow[drr, "J\otimes h"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f\widehat{\otimes}h"] & \\
& & & & J \otimes C
\end{tikzcd}$$

(vi) *cell complex structures: if f and g may be presented as cell complexes with cells f_α and g_β, respectively, and if $\otimes$ is cocontinuous in both variables, then $f \mathbin{\widehat{\otimes}} g$ may be presented as a cell complex with cells $f_\alpha \mathbin{\widehat{\otimes}} g_\beta$.*

Proof. The components of the induced structural isomorphism between Leibniz products are instances of the given structure isomorphism and hence invertible, proving (i). For (ii), by naturality of the isomorphisms defining a two-variable adjunction $(\otimes, \{,\}, \mathrm{hom})$, each of the squares below commutes if and only if the other two do, under the hypothesis that the horizontal arrows given the same names in each diagram are transposes:

$$\begin{tikzcd}
J \otimes A \underset{I\otimes A}{\cup} I \otimes B \arrow[r, "(u,v)"] \arrow[d, "k\widehat{\otimes}\ell"'] & X \arrow[d, "m"] \\
J \otimes B \arrow[r, "w"'] \arrow[ur, dashed, "x"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\ell"'] & \{J, X\} \arrow[d, "\widehat{\{k,m\}}"] \\
B \arrow[r, "(w,v)"'] \arrow[ur, dashed, "x"] & \{J, Y\} \underset{\{I,Y\}}{\times} \{I, X\}
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "v"] \arrow[d, "k"'] & \mathrm{hom}(B, X) \arrow[d, "\widehat{\mathrm{hom}}(\ell,m)"] \\
J \arrow[r, "(u,w)"'] \arrow[ur, dashed, "x"] & \mathrm{hom}(A, X) \underset{\mathrm{hom}(A,Y)}{\times} \mathrm{hom}(B, Y)
\end{tikzcd}$$

This transposition correspondence extends to solutions to the lifting problems presented by these squares; see Exercise C.2.v.

Property (iii) is immediately since limits or colimits in arrow categories are computed pointwise. For (iv), consider the commutative cube:

$$\begin{tikzcd}
I \otimes A \arrow[rrr] \arrow[drr, "I\otimes g"] \arrow[ddd, "f\otimes A"'] & & & I \otimes C \arrow[drr, "I\otimes g'"] \arrow[ddd] & & \\
& & I \otimes B \arrow[rrr] \arrow[ddd] \arrow[dl, dotted] & & & I \otimes D \arrow[ddd] \arrow[dl, dotted] \\
& \bullet \arrow[rrr, dashed] \arrow[ddr, dashed, "f\widehat{\otimes}g"] & & & \bullet \arrow[ddr, dashed, "f\widehat{\otimes}g'"] & \\
J \otimes A \arrow[ur, dotted] \arrow[rrr] \arrow[drr, "J\otimes g"'] & & & J \otimes C \arrow[ur, dotted] \arrow[drr] & & \\
& & J \otimes B \arrow[rrr] & & & J \otimes D
\end{tikzcd}$$

Since $I \otimes -$ and $J \otimes -$ preserve the pushout defining g' as a pushout of g, the top and bottom faces of the cube are pushouts. The squares defining the domains of the Leibniz tensors define pushouts inside the left and right-hand faces. It follows by pushout composition and cancelation that $f \mathbin{\widehat{\otimes}} g'$ is a pushout of $f \mathbin{\widehat{\otimes}} g$ as claimed.

The displayed diagram in (v) proves the assertion made there, so it remains only to prove (vi). First note that pushouts of trasnfinite composites of pushouts are again transfinite composites of pushouts and transfinite composites of transfinite composites are transfinite composites, so it suffices to work one variable at a time and prove that $f \widehat{\otimes} -$ preserves cell complex presentations for g. To that end, suppose g is a α-composite of maps g_i each of which are pushouts of a coproduct of maps $g'_i = \coprod_j g'_{i,j}$. We may promote this colimit to the arrow category to regard $g = g_{0,\alpha}$ as the colimit of the diagram $\alpha \to \mathcal{L}^2$ with one-step maps

$$\begin{tikzcd}
\bullet \arrow[r, "g_i"] \arrow[d, "g_{i,\alpha}"'] & \bullet \arrow[d, "g_{i+1,\alpha}"] \\
\bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

Similarly, the pushout square defining g_i from g'_i can similarly be promoted to a pushout square in the arrow category:

$$\begin{tikzcd}
\bullet \arrow[rr, "g'_i"] \arrow[dd, "g'_i"'] \arrow[dr, "u_i"] & & \bullet \arrow[dd, equal] \arrow[dr, "v_i"] & \\
& \bullet \arrow[rr, "g_i", near start] \arrow[dd, "g_{i,\alpha}"', near start] & & \bullet \arrow[dd, "g_{i+1,\alpha}"] \\
\bullet \arrow[rr, equal] \arrow[dr, dashed, "g_{i+1,\alpha}v_i"'] & & \bullet \arrow[dr, dashed, "g_{i+1,\alpha}v_i"] & \\
& \bullet \arrow[rr, equal] & & \bullet
\end{tikzcd}$$

We interpret this cube as presenting the square in the front face as a pushout of the square in the back face, which decomposes as a coproduct of similar squares, one for each component of $g'_i = \coprod_j g'_{i,j}$. In this way we see that $g = g_{0,\alpha}$ is the domain component of the colimit of a diagram $\alpha \to \mathcal{L}^2$, each step of which is a pushout of a coproduct of maps in the arrow category. Now $f \widehat{\otimes} - \colon \mathcal{L}^2 \to \mathcal{M}^2$ preserves colimits in the arrow category, and the domain functor $\mathrm{dom}\colon \mathcal{M}^2 \to \mathcal{M}$ preserves colimit as well. Thus, $f \widehat{\otimes} g$ is a colimit of an α-sequence of pushouts of coproducts of the maps $f \widehat{\otimes} g_{i,j}$. $\square$

More details establishing these assertions are given in [**82**, §§4-5].

C.2.10. Definition. Let $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$ be cocomplete categories each equipped with weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathcal{R})$, respectively. A **left Leibniz bifunctor** is a bifunctor

$$\otimes\colon \mathcal{K} \times \mathcal{L} \to \mathcal{M}$$

that is

(i) cocontinuous in each variable separately, and
(ii) has the **Leibniz property**: $\otimes$-pushout products of a map in $\mathcal{M}$ with a map in $\mathcal{C}$ are in $\mathcal{L}$.

Dually, a bifunctor between complete categories equipped with weak factorization systems is a **right Leibniz bifunctor** if it is continuous in each variable separately and if pullback cotensors of maps in the right classes land in the right class. We most frequently apply this definition in the case of a bifunctor

$$\{-,-\}\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{L}$$

that is contravariant in one of its variables, in which we case the relevant hypothesis is that $\mathcal{K}$ is cocomplete and colimits in the first variable are carried to limits in $\mathcal{L}$. The nature of the duality

between left and right Leibniz bifunctors is somewhat subtle to articulate. We leave this as a puzzle for the reader, with the hint to see [**22**].

C.2.11. Lemma. *If the bifunctors*

$$\mathcal{K} \times \mathcal{L} \xrightarrow{\otimes} \mathcal{M}, \quad \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\{-,-\}} \mathcal{L}, \quad \text{and} \quad \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\hom} \mathcal{K}$$

define a two-variable adjunction, and $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, *and* $(\mathcal{L}, \mathscr{R})$ *are three weak factorization systems on* $\mathcal{K}$, $\mathcal{L}$, *and* $\mathcal{M}$ *respectively, then the following are equivalent*

(i) $\otimes \colon \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ *defines a left Leibniz bifunctor.*
(ii) $\{-,-\} \colon \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{L}$ *defines a right Leibniz bifunctor.*
(iii) $\hom \colon \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{K}$ *defines a right Leibniz bifunctor.*

When these conditions are satisfied, we say that $(\otimes, \{,\}, \hom)$ *defines a* ***Leibniz two-variable adjunction***.

Proof. The presence of the adjoints ensures that each of the bifunctors satisfy the required (co)continuity hypotheses. Note that, for instance, $\mathcal{M} \widehat{\otimes} \mathcal{C} \subset \mathcal{L}$ if and only if $\mathcal{M} \widehat{\otimes} \mathcal{C} \boxslash \mathscr{R}$. Now the equivalence of (i), (ii), and (iii) follows from the equivalence of the following three lifting properties:

$$\mathcal{M} \widehat{\otimes} \mathcal{C} \boxslash \mathscr{R} \quad \leftrightsquigarrow \quad \mathcal{C} \boxslash \widehat{\{\mathcal{M}, \mathscr{R}\}} \quad \leftrightsquigarrow \quad \mathcal{M} \boxslash \widehat{\hom}(\mathcal{C}, \mathscr{R}),$$

the proof of which is left to Exercise C.2.v. ☐

C.2.12. Lemma. *For any category* $\mathcal{M}$ *with products and coproducts that is equipped with a weak factorization system* $(\mathcal{L}, \mathscr{R})$ *the set-tensor, set-cotensor, and hom*

$$* \colon \mathcal{S}et \times \mathcal{M} \to \mathcal{M}, \quad \{-,-\} \colon \mathcal{S}et^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, \quad \text{and} \quad \hom \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et$$

respectively define a Leibniz two-variable adjunction relative to the mono-epi weak factorization system $(\mathcal{M}, \mathcal{E})$ *on* $\mathcal{S}et$.

Proof. By Lemma C.2.11, it suffices to prove any one of these bifunctors is Leibniz. When $A \hookrightarrow B$ is a monomorphism in $\mathcal{S}et$, the Leibniz tensor with $f \colon X \to Y$ decomposes as a coproduct of maps that are manifestly in $\mathcal{L}$.

$$\begin{tikzcd}
A * X \arrow[r, hook] \arrow[d, "A*f"'] & B * X \cong A * X \coprod B \backslash A * X \arrow[d, "A*f \coprod \mathrm{id}"] \arrow[ddr, bend left, "B*f"] & \\
A * Y \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right] & A * Y \coprod B \backslash A * X \arrow[dr, "\mathrm{id} \coprod B\backslash A*f"] & \\
& & B * Y \cong A * Y \coprod B \backslash A * Y
\end{tikzcd} \tag{C.2.13}$$

A slicker proof is also possible. Because every monomorphism may be presented as a cell complex built from a single cell $\varnothing \hookrightarrow 1$, by Proposition C.2.9(vi), it suffices to consider Leibniz tensor with the generating monomorphism $\varnothing \hookrightarrow 1$. But note that the functor

$$\mathcal{M}^{\mathbf{2}} \xrightarrow{(\varnothing \hookrightarrow 1) \hat{*} -} \mathcal{M}^{\mathbf{2}}$$

is naturally isomorphic to the identity, which certainly preserves the left class $\mathcal{L}$. ☐

C.2.14. Remark. To prove that $\hom\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et$ is right Leibniz is to show that for any $\ell \in \mathcal{L}$ and $r \in \mathscr{R}$, the morphism

$$\mathcal{M}(\operatorname{cod}\ell, \operatorname{dom} r) \xrightarrow{r\circ-\circ\ell} \mathcal{M}(\operatorname{dom}\ell, \operatorname{dom} r) \underset{\mathcal{M}(\operatorname{dom}\ell, \operatorname{cod} r)}{\times} \mathcal{M}(\operatorname{cod}\ell, \operatorname{cod} r)$$

is an epimorphism. The target of this map is the set of commutative squares in $\mathcal{M}$ from ℓ to r, while the fiber over any element is the set of solutions to the lifting problem so-presented. The fact that this is an epimorphism follows from the lifting property $\mathcal{L} \boxslash \mathscr{R}$.

Exercises.

C.2.i. Exercise. Finish the proof of Lemma C.2.3.

C.2.ii. Exercise.

(i) Prove the "retract argument": Suppose $f = r \circ \ell$ and f has the left lifting property with respect to its right factor r. Then f is a retract of its left factor ℓ.

(ii) Conclude that in the presence of axioms (i) and (ii) of Definition C.2.5, that axiom (iii) may be replaced by the hypothesis that the classes $\mathcal{L}$ and $\mathscr{R}$ are closed under retracts.

C.2.iii. Exercise.

(i) Suppose $\mathcal{M}$ is a category with products, pullbacks, and limits of towers equipped with a weak factorization system $(\mathcal{L}, \mathscr{R})$, and let $\mathcal{I}$ be an inverse category. Prove that the category of diagrams $\mathcal{M}^{\mathcal{I}}$ has a weak factorization system whose left class is comprised of those maps whose components are in $\mathcal{L}$ and whose right class is comprised of those maps $\alpha\colon X \to Y$ so that for each $i \in I$, the map $\hat{m}_i\colon X_i \to \partial_i X \times_{\partial_i Y} Y_i$ over (C.1.18) lies in $\mathscr{R}$.

(ii) Give a new proof of Proposition C.1.22 under the additional hypothesis that $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are the right classes of weak factorization systems.

C.2.iv. Exercise. Prove Lemma C.2.6

C.2.v. Exercise. Given a two variable adjunction (C.2.7) and classes of maps $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathcal{K}, \mathcal{L}, \mathcal{M}$, respectively, prove that the following lifting properties are equivalent

$$\mathcal{A} \hat{\otimes} \mathcal{B} \boxslash \mathcal{C} \quad \Leftrightarrow \quad \mathcal{B} \boxslash \widehat{\{\mathcal{A}, \mathcal{C}\}} \quad \Leftrightarrow \quad \mathcal{A} \boxslash \widehat{\hom}(\mathcal{B}, \mathcal{C}).$$

C.3. Model categories and Quillen functors

The following reformulation of Quillen's definition of "closed model categories" [**71**, I.5.1] was given by Joyal and Tierney [**50**, 7.7], who prove that a category $(\mathcal{M}, \mathcal{W})$ with weak equivalences satisfying the two-of-three property admits a model structure just when there exist classes $\mathcal{C}$ and $\mathcal{F}$ that define a pair of weak factorization systems as follows:

C.3.1. Definition (model category). A **model structure** on a category $\mathcal{M}$ with all small limits and colimits consists of three classes of maps — the **weak equivalences** $\mathcal{W}$ denoted "$\xrightarrow{\sim}$" which must satisfy the two-of-three property, the **cofibrations** $\mathcal{C}$ denoted "$\rightarrowtail$", and the **fibrations** $\mathcal{F}$ denoted "$\twoheadrightarrow$" respectively — so that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ each define weak factorization systems on $\mathcal{M}$.[2]

[2]There is one axiom in standard definition of a model category — the closure of weak equivalences under retracts — that is not obviously included, but this is a consequence of the axioms given here [**50**, 7.8].

Note that Definitions C.2.5 and C.3.1 are self-dual: if $(\mathcal{L}, \mathscr{R})$ defines a weak factorization system on $\mathcal{M}$ then $(\mathscr{R}, \mathcal{L})$ defines a weak factorization system on $\mathcal{M}^{\mathrm{op}}$. Thus all general theorems about the right classes of fibrations $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ of **trivial fibrations** "$\overset{\sim}{\twoheadrightarrow}$," imply the dual results involving the left classes $\mathcal{C}$ of cofibrations and $\mathcal{C} \cap \mathcal{W}$ of **trivial cofibrations** "$\overset{\sim}{\rightarrowtail}$." In particular, by Lemma C.1.4, all of the results proven in §C.1 about a category of fibrant objects hold for the fibrations, trivial fibrations, and weak equivalence between the eponymous fibrant objects in a model category that we now define. Thus, the duals of these results hold for the cofibrations, trivial cofibrations and weak equivalences between cofibrant objects.

C.3.2. Definition. In a model category $\mathcal{M}$ an object X is **fibrant** just when the unique map $X \to 1$ to the terminal object is a fibrant and **cofibrant** just when the unique map $\varnothing \to X$ from the initial object is a cofibration. By factoring the unique maps, any object X has a **cofibrant replacment** QX and a **fibrant replacement** RX constructed as follows:

$$\begin{tikzcd}[column sep=small] & & & & RX \arrow[dr, two heads] \\ \varnothing \arrow[rrr, "!"] \arrow[dr, tail] & & & X \arrow[rr, "!"'] \arrow[ur, tail, "\sim"] & & * \\ & QX \arrow[urr, two heads, "\sim"'] \end{tikzcd}$$

Note also that since either class of a weak factorization system determines the other, the trivial cofibrations can be defined without reference to either the cofibrations or weak equivalences as those maps that have the left lifting property with respect to the fibrations, and dually the trivial fibrations are precisely those maps that have the right lifting property with respect to the cofibrations.

C.3.3. Definition. A functor between model categories is

- **left Quillen** if it preserves cofibrations, trivial cofibrations, and cofibrant objects, and
- **right Quillen** if it preserves fibrations, trivial fibrations, and fibrant objects.

Left Quillen functors admit left derived functors while right Quillen functors admit right derived functors. We leave a full account of this to other authors [**79**, ß2.1-2] so as to avoid defining these terms, but an important component of the "derivability" of Quillen functors is captured by the following result:

C.3.4. Lemma. *Any left Quillen functor between model categories preserves weak equivalences between cofibrant objects, while any right Quillen functor preserves weak equivalences between fibrant objects.*

Proof. For right Quillen functors this follows directly from Lemma C.1.10 and C.1.4. The result for left Quillen functors is dual. ☐

Most left Quillen functors are "cocontinuous," preserving all colimits, while most right Quillen functors are "continuous," preserving all limits; when this is the case there is no need to separately assume that cofibrant or fibrant objects are preserved. This is because Quillen functors commonly occur as an adjoint pair:

C.3.5. Definition. Consider an adjunction between a pair of model categories.

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "U"] \end{tikzcd}$$

By Lemma C.2.6 the following are equivalent, defining a **Quillen adjunction**.

(i) The left adjoint F is left Quillen.
(ii) The right U is right Quillen.
(iii) The left adjoint preserves cofibrations and the right adjoint preserves fibrations.
(iv) The left adjoint preserves trivial fibrations and the right adjoint preserves trivial fibrations.

We now introduce a pair of model structures on diagram categories that are designed to ensure that the diagonal functor $\Delta\colon \mathcal{M} \to \mathcal{M}^{\mathcal{J}}$ is respectively right or left Quillen, so that the colimit and limit functors, respectively, are left or right Quillen. The corresponding left and right derived functors then define the **homotopy colimit** and **homotopy limit** functors.

C.3.6. Definition. Let $\mathcal{M}$ be a model category and let $\mathcal{J}$ be a small category.

(i) The **projective model structure** on $\mathcal{M}^{\mathcal{J}}$ has weak equivalences and fibrations defined pointwise in $\mathcal{M}$.
(ii) The **injective model structure** on $\mathcal{M}^{\mathcal{J}}$ has weak equivalences and cofibrations defined pointwise in $\mathcal{M}$.

When the model category $\mathcal{M}$ is *combinatorial* or more generally *accessible*, the projective and injective model structures always exist. Of course, the projective and injective model structures might happen to exist on $\mathcal{M}^{\mathcal{J}}$, perhaps for particular diagram shapes $\mathcal{J}$, in the absence of these hypotheses.

A Quillen two-variable adjunction is a two-variable adjunction in which the left adjoint is a left Quillen bifunctor while the right adjoints are both right Quillen bifunctors. By Exercise C.2.v, any one of these conditions implying the other two:

C.3.7. Definition. A two-variable adjunction

$$\mathcal{V} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{N}, \quad \mathcal{V}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\{-,-\}} \mathcal{M}, \quad \mathcal{M}^{\mathrm{op}} \times \mathcal{N} \xrightarrow{\hom} \mathcal{V}$$

between model categories $\mathcal{V}$, $\mathcal{M}$, and $\mathcal{N}$ defines a **Quillen two-variable adjunction** if any, and hence all, of the following equivalent conditions are satisfied:

(i) The functor $\hat{\otimes}\colon \mathcal{V}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \to \mathcal{N}^{\mathbf{2}}$ carries any pair comprised of a cofibration in $\mathcal{V}$ and a cofibration in $\mathcal{M}$ to a cofibration in $\mathcal{N}$ and furthermore this cofibration is a weak equivalence if either of the domain cofibrations are.
(ii) The functor $\widehat{\{-,-\}}\colon (\mathcal{V}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$ carries any pair comprised of a cofibration in $\mathcal{V}$ and a fibration in $\mathcal{N}$ to a fibration in $\mathcal{N}$ and furthermore this fibration is a weak equivalence if either of the domain maps are.
(iii) The functor $\widehat{\hom}\colon (\mathcal{M}^{\mathbf{2}})^{\mathrm{op}} \times \mathcal{N}^{\mathbf{2}} \to \mathcal{V}^{\mathbf{2}}$ carries any pair comprised of a cofibration in $\mathcal{M}$ and a fibration in $\mathcal{N}$ to a fibration in $\mathcal{V}$ and furthermore this fibration is a weak equivalence if either of the domain maps are.

C.3.8. Remark. Definition C.3.7, asserts that a two-variable adjunction is Quillen if and only if its left adjoint $\otimes\colon \mathcal{V} \times \mathcal{M} \to \mathcal{N}$ is a **left Quillen bifunctor**: a bifunctor that is left Leibniz with respect to all possible choices of constituent weak factorization systems, except the choice of the trivial cofibrations only for $\mathcal{N}$.

Quillen's axiomatization of the additional properties enjoyed by his model structure on the category of simplicial sets has been generalized by Hovey [**46**, §4.2] to define the notions of monoidal model category and enriched model category. We specialize the former to the cartesian closed categories of §A.1 as those are the only cases we'll need here. If $\mathcal{V}$ has a model structure and also has the structure of a cartesian closed category it is natural to ask that these be compatible in some way.

C.3.9. Definition. A **cartesian closed model category** is a cartesian closed category $(\mathcal{V}, \times, 1)$ with a model structure so that

(i) the cartesian product and internal hom define a Quillen two-variable adjunction and
(ii) furthermore so that the map
$$Q1 \times v \to 1 \times v \cong v$$
are is a weak equivalence whenever v is cofibrant.[3]

Then:

C.3.10. Definition. If $\mathcal{V}$ is a cartesian closed model category a **$\mathcal{V}$-model category** is a model category $\mathcal{M}$ that is

(i) tensored, cotensored, and $\mathcal{V}$-enriched in such a way that $(\otimes, \{,\}, \hom)$ is a Quillen two-variable adjunction and
(ii) the maps
$$Q1 \otimes m \to 1 \otimes m \cong m$$
are weak equivalences if m is cofibrant.[4]

C.3.11. Lemma. *If $\mathcal{M}$ is a $\mathcal{V}$-model category, then for any cofibrant object M and fibrant object N in $\mathcal{M}$, $\hom(M, N)$ is a fibrant object in $\mathcal{V}$. More generally, for any cofibrant object M and fibration $p\colon N \twoheadrightarrow P$, the induced map $p_*\colon \hom(M, N) \to \hom(M, P)$ is a fibration in $\mathcal{V}$.*

Proof. By Proposition A.5.5—which implies, for the terminal object $1 \in \mathcal{M}$ and any $M \in \mathcal{V}$, that $\hom(M, 1) \cong 1$ is terminal in $\mathcal{V}$—the second statement subsumes the first. By Exercise C.2.v, the lifting problem below-left for any trivial cofibration i in $\mathcal{V}$ transposes to the lifting problem below-right
$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "i"', "\wr"] & \hom(M,N) \arrow[d, "p_*"] \\ V \arrow[r] \arrow[ur, dashed] & \hom(M,P) \end{tikzcd} \qquad \begin{tikzcd} U \otimes M \arrow[r] \arrow[d, tail, "i\otimes M"', "\wr"] & N \arrow[d, two heads, "p"] \\ V \otimes M \arrow[r] \arrow[ur, dashed] & P \end{tikzcd}$$
By Exercise C.3.iii, since M is cofibrant, $-\otimes M\colon \mathcal{V} \to \mathcal{M}$ is left Quillen, so $i \otimes M$ is a trivial cofibration in $\mathcal{M}$ and since $p\colon N \twoheadrightarrow P$ is a fibrantion, a solution to the lifting problem exists. ☐

The following result was formulated by Gambino [**39**] in the context of a model category enriched over Quillen's cartesian closed model structure on simplicial sets, but its proof applies in greater generality.

C.3.12. Theorem. *If $\mathcal{M}$ is a $\mathcal{V}$-model category and $\mathcal{J}$ is a small category, then the weighted colimit functor*
$$\operatorname{colim}^{\mathcal{J}}_{-} -\colon \mathcal{V}^{\mathcal{J}} \times \mathcal{M}^{\mathcal{J}^{\mathrm{op}}} \to \mathcal{M}$$
is left Quillen if the domain has the (injective, projective) or (projective, injective) model structure. Similarly, the weighted limit functor
$$\lim^{\mathcal{J}}_{-} -\colon (\mathcal{V}^{\mathcal{J}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{J}} \to \mathcal{M}$$
is right Quillen if the domain has the (projective, projective) or (injective, injective) model structure.

[3] If $1 \in \mathcal{V}$ is cofibrant, it suffices to require only the first condition; see Exercise C.3.iii.

[4] Again, by Exercise C.3.iii this condition is unnecessary if $1 \in \mathcal{V}$ is cofibrant.

Proof. By Definition C.3.7 we can prove both statements in adjoint form. The weighted colimit bifunctor of Definition 7.1.7 has a right adjoint (used to express the defining universal property of the weighted colimit)

$$\hom(-,-)\colon (\mathcal{M}^{\mathcal{J}^{\mathrm{op}}})^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}^{\mathcal{J}}$$

which sends $F \in \mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$ and $m \in \mathcal{M}$ to $\hom(F-, m) \in \mathcal{V}^{\mathcal{J}}$.

To prove the statement when $\mathcal{V}^{\mathcal{J}}$ has the projective and $\mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$ has the injective model structure, we must show that this is a right Quillen bifunctor with respect to the pointwise (trivial) cofibrations in $\mathcal{M}^{\mathcal{J}^{\mathrm{op}}}$, (trivial) fibrations in $\mathcal{M}$, and pointwise (trivial) fibrations in $\mathcal{V}^{\mathcal{J}}$. Because the limits involved in the definition of right Quillen bifunctors are also formed pointwise, this follows immediately from the corresponding property of the simplicial hom bifunctor, which was part of the definition of a simplicial model category. The other cases are similar. $\square$

The upshot of Theorem C.3.12 is that there are two approaches to constructing a homotopy colimit: "fattening up the diagram" — for instance, by requiring that its objects are cofibrant and its morphisms are cofibrations — or "fattening up the weight" — typically by taking a cofibrant replacement of the terminal weight. Lemmas 7.3.9 and 7.3.13 can be understood as examples of the general equivalence between these two approaches.

C.3.13. Corollary. *If $\mathcal{M}$ is a $\mathcal{V}$-model category, then for any diagram $F \in \mathcal{M}^{\mathcal{J}}$ whose objects are all fibrant and any projective cofibrant weight $W \in \mathcal{V}^{\mathcal{J}}$, the weighted limit is a fibrant object.*

Proof. By Theorem C.3.12, the weighted limit functor $\lim^{\mathcal{J}}_{-} -\colon (\mathcal{V}^{\mathcal{J}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{J}} \to \mathcal{M}$ is right Quillen with respect to the projective model structure on the category of weights and the injective model structure on the category of diagrams. Since right Quillen bifunctors preserve fibrant objects, it follows that the limit of a pointwise fibrant diagram weighted by a projective cofibrant weight is fibrant. $\square$

Finally, we will make use of the following theorem which enables the change of base of enrichment for model categories extending the results of §A.6. The premises of Theorem C.3.14 are the obvious extension of the premises of Proposition A.6.8 to the enriched model category context, but the conclusion only allows us to transfer enrichments in the directly of the right adjoint because an enriched model category must also be tensored and cotensored and these properties only transfer in that direction.

The result below is a specialization of a more general theorem proven in [**42**, 3.8] to the cartesian closed bases for enrichment that we have been considering.

C.3.14. Theorem. *Consider a Quillen adjunction between cartesian closed model categories in which the left adjoint preserves finite products:*

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, bend left, "U"] \end{tikzcd}$$

Then any $\mathcal{W}$-model category admits the structure of a $\mathcal{V}$-model category with the same underlying unenriched model category with enriched homs, cotensors, and tensors defined by:

$$\hom_{\mathcal{V}}(M,N) \coloneqq U\hom_{\mathcal{W}}(M,N)\,, \quad V \otimes M \coloneqq FV \otimes M\,, \qquad \textit{and} \qquad M^V \coloneqq M^{FV}.$$

PROOF. By Proposition A.6.10 these definitions make $\mathcal{W}$ into a tensored and cotensored $\mathcal{V}$-enriched category, and Exercise A.6.ii observed that change of base along the right adjoint of a monoidal adjunction preserves underlying 1-categories. It remains only to verify that the functors underlying the $\mathcal{V}$-enriched hom, tensor, and cotensor define a Quillen two-variable adjunction, but this follows easily from the cartesian closure of the model categories $\mathcal{V}$ and $\mathcal{W}$ and the fact that $F \dashv U$ is Quillen. $\square$

Exercises.

C.3.i. EXERCISE. Prove Lemma C.1.4.

C.3.ii. EXERCISE. Verify that a model structure on $\mathcal{M}$, if it exists, is uniquely determined by any of the following data:

(i) The cofibrations and weak equivalences.
(ii) The fibrations and weak equivalences.
(iii) The cofibrations and fibrations.[5]

C.3.iii. EXERCISE.

(i) Prove that if $- \otimes - \colon \mathcal{V} \times \mathcal{M} \to \mathcal{N}$ is a left Quillen bifunctor and $V \in \mathcal{V}$ is cofibrant then $\mathcal{V} \otimes - \colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor.
(ii) Conclude that the second conditions of Definitions C.3.9 and C.3.10 are unnecessary if $\mathbb{1} \in \mathcal{V}$ is cofibrant.

C.3.iv. EXERCISE. In a locally small category $\mathcal{M}$ with products and coproducts the hom bifunctor is part of a two-variable adjunction:

$$- * - \colon \mathcal{S}et \times \mathcal{M} \to \mathcal{M}, \quad \{-,-\} \colon \mathcal{S}et^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}, \quad \mathrm{Hom} \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et.$$

Equipping $\mathcal{S}et$ with the model structure whose weak equivalences are all maps, whose cofibrations are monomorphisms, and whose fibrations are epimorphisms, prove that

(i) $\mathcal{S}et$ is a cartesian monoidal model category.
(ii) Any model category $\mathcal{M}$ is a $\mathcal{S}et$-model category.

C.4. Reedy categories and canonical presentations

In this section we describe a particular structure borne by certain diagram categories $\mathcal{A}$ first exploited by Reedy to prove homotopical results about the category of $\mathcal{A}$-indexed diagrams [**74**]. Our primary examples—$\mathbf{\Delta}$, inverse categories, their opposites, and products of these—are all (strict) *Reedy categories* as defined by Kan, so we confine our attention to this special case. However, we mention for the interested reader, that this theory has been usefully extended by Berger and Moerdijk in such a way as to encompass certain similar categories in which objects are permitted to have non-identity automorphisms [**8**]. Our presentation follows [**82**].

C.4.1. DEFINITION. A **Reedy structure** on a small category $\mathcal{A}$ consists of a **degree function**[6]

$$\deg \colon \mathrm{obj}\,\mathcal{A} \to \omega$$

[5] By a more delicate observation of Joyal [**49**, E.1.10] a model structure is also uniquely determined by

(iv) The cofibrations and fibrant objects.
(v) The fibrations and cofibrant objects.

[6] The degree function can take values in a different ordinal with no substantial effect on the mathematics.

together with a pair of wide subcategories $\vec{\mathcal{A}}$ and $\overleftarrow{\mathcal{A}}$ of **degree-increasing** and **degree-decreasing** arrows respectively so that

(i) For each non-identity morphism in $\vec{\mathcal{A}}$, the degree of its domain is strictly less than the degree of its codomain, and for each non-identity morphism in $\overleftarrow{\mathcal{A}}$, the degree of its domain is strictly greater than the degree of its domain.

(ii) Every $f \in \operatorname{arr}\mathcal{A}$ may be uniquely factored as

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\overleftarrow{\mathcal{A}} \ni \overleftarrow{f}"'] & & \bullet \\ & \bullet \arrow[ur, "\vec{f} \in \vec{\mathcal{A}}"'] & \end{tikzcd} \tag{C.4.2}$$

Axiom (i) implies that $\vec{\mathcal{A}} \cap \overleftarrow{\mathcal{A}} = \operatorname{obj}(\mathcal{A})$, while both conditions together imply that $\mathcal{A}$ contains no non-identity automorphisms; see Exercise C.4.i.

C.4.3. Example. Any inverse category $\mathcal{I}$ as in Definition C.1.15 is a Reedy category, with $\overleftarrow{\mathcal{I}} = \mathcal{I}$ and $\vec{\mathcal{I}} = \operatorname{obj}\mathcal{I}$. Conversely, any Reedy category $\mathcal{A}$ with $\mathcal{A} = \overleftarrow{\mathcal{A}}$ is an inverse category.

C.4.4. Example. The category $\mathbf{\Delta}_+$ is a Reedy categories with $\vec{\mathbf{\Delta}}_+$ the monomorphisms and $\overleftarrow{\mathbf{\Delta}}_+$ the epimorphisms. Here it's convenient to take advantage of the order isomorphism $\mathbb{1} + \omega = \omega$ to define $\deg[n] \coloneqq n$. The subcategories $\mathbf{\Delta}$, $\mathbf{\Delta}_\top$ and $\mathbf{\Delta}_\perp$ all inherit analogous Reedy category structures.

C.4.5. Remark. If $\mathcal{A}$ is a Reedy category, then so is $\mathcal{A}^{\mathrm{op}}$: its Reedy structure has the same degree function but has the degree-increasing and degree-decreasing arrows interchanged. In particular, the Reedy structures of Example C.4.3 dualize to define **direct categories**.

C.4.6. Remark. If $\mathcal{A}$ and $\mathcal{B}$ are Reedy categories, so is $\mathcal{A} \times \mathcal{B}$, with $\deg(a, b) \coloneqq \deg(a) + \deg(b)$. See Exercise C.4.ii.

We refer to the unique factorization (C.4.2) as the **Reedy factorization** of the map f and the degree of the object $\operatorname{cod}\overleftarrow{f} = \operatorname{dom}\vec{f}$ as the **degree** of f. Our next aim is to show that:

(i) It is the minimal degree of an object through which f factors.

(ii) The only factorization of f with this degree is the Reedy factorization.

To prove these assertions, consider the category $\mathcal{F}act\, f$ whose objects are factorizations $a \xrightarrow{g} c \xrightarrow{h} b$ of f and whose morphisms $h \cdot g \to h' \cdot g'$ are maps $k \colon c \to c'$ so that the triangles

$$\begin{tikzcd} & c \arrow[dr, "h"] \arrow[dd, "k"] & \\ a \arrow[ur, "g"] \arrow[dr, "g'"'] & & b \\ & c' \arrow[ur, "h'"'] & \end{tikzcd}$$

commute. Write $\mathcal{F}act_n f \subset \mathcal{F}act\, f$ for the subcategory of factorizations through an object of degree at most n.

C.4.7. Lemma. *The category $\mathcal{F}act\, f$ is connected, and each subcategory $\mathcal{F}act_n f$ is either empty or connected. The minimal n with $\mathcal{F}act_n f$ non-empty is the degree of f, and $\mathcal{F}act_{\deg(f)} f \cong \mathbb{1}$.*

Proof. Consider $h \cdot g \in \mathcal{F}act f$ and their Reedy factorizations:

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[rr, "g"] \arrow[dr, "\overleftarrow{g}"'] & & \bullet \arrow[rr, "h"] \arrow[dr, "\overleftarrow{h}"'] & & \bullet \\
& \bullet \arrow[ur, "\overrightarrow{g}"'] \arrow[rr, "k=\overleftarrow{h}\cdot\overrightarrow{g}"'] \arrow[dr, "\overleftarrow{k}"'] & & \bullet \arrow[ur, "\overrightarrow{h}"'] & \\
& & \bullet \arrow[ur, "\overrightarrow{k}"'] & &
\end{tikzcd} \tag{C.4.8}$$

In this way, we define a zig-zag of morphisms in $\mathcal{F}act f$ connecting $h \cdot g$ to $\overrightarrow{h}\overrightarrow{k} \cdot \overleftarrow{k}\overleftarrow{g}$, which by axiom (ii) must be the Reedy factorization. This shows that $\mathcal{F}act f$ is connected.

Moreover, axiom (i) implies that the degree of $\mathrm{cod}(g) = \mathrm{dom}(h)$ is at least the degree of f. In particular, if $h \cdot g \in \mathcal{F}act_n f$, each of the factorizations in (C.4.8) is as well, proving that $\mathcal{F}act_n f$ is connected if it is non-empty. This diagram also shows that each non-empty category $\mathcal{F}act_n f$ contains the Reedy factorization. Hence, the minimal such n is the degree of f.

Finally, if the degree of $\mathrm{cod}(g) = \mathrm{dom}(h)$ equals the degree of f, then $\overrightarrow{g}$ and $\overleftarrow{h}$ must be identities, from which we deduce that $g \in \overleftarrow{\mathcal{A}}$ and $h \in \overrightarrow{\mathcal{A}}$: i.e., that $h \cdot g$ is the Reedy factorization. Hence $\mathcal{F}act_{\deg(f)} f \cong \mathbb{1}$ is the terminal category as claimed. □

Lemma C.4.7 will be used to establish a "cellular decomposition" for the hom bifunctor $\mathcal{A} \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. That is, we shall use the Reedy structure to present the bifunctor $\mathcal{A}$ as a cell complex in the sense of Definition C.2.4: a sequential composite of pushouts of coproducts of basic "cells" that have a particular form. Lemma C.4.7 implies that the subset of arrows of degree at most n assembles into a sub*functor* of the hom-bifunctor.

C.4.9. Definition (n-skeleton of the hom bifunctor). For any Reedy category $\mathcal{A}$, the n-**skeleton** is the subfunctor

$$\mathrm{sk}_n \mathcal{A} \hookrightarrow \mathcal{A} \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$$

of arrows of degree at most n.

There are obvious inclusions $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$. The colimit of the sequence

$$\varnothing \hookrightarrow \mathrm{sk}_0 \mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{colim} \cong \mathcal{A}$$

is the hom bifunctor $\mathcal{A}$. The morphisms of degree n first appear in $\mathrm{sk}_n \mathcal{A}$. It remains to express each inclusion $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$ as a pushout of a coend of basic "cells" which our next task is to describe.

The external (pointwise) product defines a bifunctor $\mathcal{S}et^{\mathcal{A}} \times \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}} \xrightarrow{-\square-} \mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. For any $a \in \mathcal{A}$, there is a natural "composition" map $\circ \colon \mathcal{A}_a \square \mathcal{A}^a \to \mathcal{A}$ whose domain is the external product of the contravariant $\mathcal{A}^a$ and covariant $\mathcal{A}_a$ representables. By Lemma C.4.7, the composite of any pair of maps that factor through an object a of degree n lies in $\mathrm{sk}_n \mathcal{A}$. Our next task is to describe the subfunctor of the domain of the map

$$\coprod_{\deg(a)=n} \mathcal{A}_a \square \mathcal{A}^a \xrightarrow{\circ} \mathrm{sk}_n \mathcal{A}$$

that factors through $\mathrm{sk}_{n-1} \mathcal{A} \hookrightarrow \mathrm{sk}_n \mathcal{A}$, for which we require some new notation.

C.4.10. Definition (boundaries of representable functors). If $a \in \mathcal{A}$ has degree n, write

$$\begin{aligned} \partial\mathcal{A}_a &:= \mathrm{sk}_{n-1}\mathcal{A}_a && \in \mathcal{S}et^{\mathcal{A}} \quad \text{and} \\ \partial\mathcal{A}^a &:= \mathrm{sk}_{n-1}\mathcal{A}^a && \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}. \end{aligned}$$

By Lemma C.4.7, $\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a$ is the subfunctor of arrows in $\mathcal{A}$ with domain a that do *not* lie in $\overrightarrow{\mathcal{A}}$, while $\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$ is the subfunctor of arrows with codomain a that do *not* lie in $\overleftarrow{\mathcal{A}}$.

In particular, the exterior Leibniz product

$$\mathcal{A}_a \square \partial\mathcal{A}^a \cup \partial\mathcal{A}_a \square \mathcal{A}^a \xhookrightarrow{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\widehat{\square}(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)} \mathcal{A}_a \square \mathcal{A}^a \tag{C.4.11}$$

defines the subfunctor of pairs of morphisms $h \cdot g$ with $\mathrm{dom}\, h = \mathrm{cod}\, g = a$ in which at least one of the morphisms g and h has degree less than the degree of a.

C.4.12. Proposition. *The square*

$$\begin{tikzcd} \coprod_{\deg(a)=n} \partial\mathcal{A}_a \square \mathcal{A}^a \cup \mathcal{A}_a \square \partial\mathcal{A}^a \arrow[r, hook] \arrow[d, "\circ"'] & \coprod_{\deg(a)=n} \mathcal{A}_a \square \mathcal{A}^a \arrow[d, "\circ"] \\ \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n \mathcal{A} \end{tikzcd} \tag{C.4.13}$$

is both a pullback and a pushout in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$.

The fact that (C.4.13) is a pullback is used to facilitate the proof that it is also a pushout.

Proof. An element of the pullback consists of $f \in \mathrm{sk}_{n-1}\mathcal{A}$ together with a factorization $f = h \cdot g$ through an object a of degree n. If both h and g have degree n, then Lemma C.4.7 tells us that $h \cdot g$ is a Reedy factorization, contradicting the fact that f has degree at most $n-1$. So we must have either $h \in \partial\mathcal{A}_a$ or $g \in \partial\mathcal{A}^a$, which tells us that the map from the upper left corner of (C.4.13) surjects onto the pullback. Because the top-horizontal map is monic, the comparison is therefore an isomorphism; i.e., (C.4.13) is a pullback square.

To see that it is a pushout, it suffices now to show that the right-hand vertical is one-to-one on the complement of $\mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A}$. This follows from Lemma C.4.7, which argued that any morphism of degree n has a unique factorization through an object of that degree: namely its Reedy factorization. □

As a corollary of Proposition C.4.12, the two-sided representable $\mathcal{A}$ has a canonical presentation as a cell complex.

C.4.14. Theorem. *The inclusion* $\varnothing \hookrightarrow \mathcal{A}$ *has a canonical presentation as a cell complex:*

$$\begin{tikzcd} & & \coprod_{\deg(a)=n} \partial\mathcal{A}_a \square \mathcal{A}^a \cup \mathcal{A}_a \square \partial\mathcal{A}^a \arrow[r, hook] \arrow[d, "\circ"'] \arrow[dr, phantom, "\ulcorner" very near end] & \coprod_{\deg(a)=n} \mathcal{A}_a \square \mathcal{A}^a \arrow[d, "\circ"] \\ \varnothing \arrow[r, hook] & \mathrm{sk}_0\mathcal{A} \arrow[r, dotted] & \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n\mathcal{A} \arrow[r, dotted] & \mathrm{colim}_n\, \mathrm{sk}_n\mathcal{A} \cong \mathcal{A} \end{tikzcd}$$

i.e., a composite of pushouts of coproducts of cells defined as exterior Leibniz products

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\, \widehat{\square}\, (\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a),$$

where the cell for each $a \in \mathcal{A}$ of degree n is attached at stage n. □

C.4.15. Remark. One meaning of "canonical" should be "functorial." Indeed, a morphism of Reedy categories—a functor preserving degree and the subcategories of degree-increasing and degree-decreasing maps—induces a morphism of generalized cell complexes: given a morphism $\mathcal{A} \to \mathcal{A}'$ of Reedy categories, there is a natural transformation in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}$ between the generalized cell complex presentation for $\mathcal{A}$ and the restriction the generalized cell complex presentation for $\mathcal{A}'$.

As a corollary of Theorem C.4.14, any morphism $f \in \mathcal{M}^{\mathcal{A}}$ is itself a generalized cell complex: the cellular decomposition of $\mathcal{A}$ is translated into a cellular decomposit ion for f by taking weighted colimits. Taking weighted limits instead transforms the cellular decomposition of $\mathcal{A}$ into a "generalized Postnikov presentation" for f as the limit of a countable tower of pullbacks of ends of a particular form. This sort of result is exemplary of the slogan of **[82]** that "it's all in the weights." Before proving this corollary, let us introduce notation for the maps appearing as the generalized cells.

C.4.16. Definition (latching and matching objects). Let $a \in \mathcal{A}$. The **latching** and **matching objects** of diagram $X \in \mathcal{M}^{\mathcal{A}}$ are defined to be the colimits and limits, respectively, weighted by the boundary representables of appropriate variance:

$$L^a X := \operatorname{colim}_{\partial\mathcal{A}^a} X \qquad M^a X := \lim_{\partial\mathcal{A}_a} X.$$

The boundary inclusions $\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$ and $\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a$ induce the **latching** and **matching maps** $L^a X \to X^a$ and $X^a \to M^a X$, on account of the isomorphisms $\operatorname{colim}_{\mathcal{A}^a} X \cong X^a \cong \lim_{\mathcal{A}_a} X$ of Definition 7.1.3(i).

C.4.17. Definition (relative latching and matching maps). The **relative latching** and **relative matching maps** of a natural transformation $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ are defined to be the Leibniz weighted colimits and limits

$$\widehat{\ell}^a f := \widehat{\operatorname{colim}}_{\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a} f \qquad \widehat{m}^a f := \widehat{\lim}_{\partial\mathcal{A}_a\hookrightarrow\mathcal{A}_a} f,$$

i.e., by the pullbacks and pushouts:

$$\begin{tikzcd}
L^a X \arrow[r] \arrow[d, "L^a f"'] & X^a \arrow[d] \arrow[ddr, bend left, "f^a"] & \\
L^a Y \arrow[r] \arrow[drr, bend right] & \ell^a f \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "\widehat{\ell}^a f"] & \\
& & Y^a
\end{tikzcd}
\qquad
\begin{tikzcd}
X^a \arrow[dr, dashed, "\widehat{m}^a f"] \arrow[drr, bend left] \arrow[ddr, bend right, "f^a"'] & & \\
& m^a f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & M^a X \arrow[d, "M^a f"] \\
& Y^a \arrow[r] & M^a Y
\end{tikzcd}$$

of the maps $L^a f := \operatorname{colim}_{\partial\mathcal{A}^a} f$ and $M^a f := \lim_{\partial\mathcal{A}_a} f$.

C.4.18. Notation. For any diagram $X \in \mathcal{M}^{\mathcal{A}}$ let

$$\operatorname{sk}_n X := \operatorname{colim}_{\operatorname{sk}_n \mathcal{A}} X \qquad \text{and} \qquad \operatorname{cosk}_n X := \lim_{\operatorname{sk}_n \mathcal{A}} X$$

denote the results of applying the weighted colimit and weighted limit bifunctors

$$\operatorname{colim}_{-} -\colon \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}} \qquad \text{and} \qquad \lim_{-} -\colon (\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}}$$

to the diagram X with weight $\operatorname{sk}_n \mathcal{A}$.

C.4.19. Corollary. *Let $\mathcal{A}$ be a Reedy category and let $\mathcal{M}$ be bicomplete. Any morphism $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is a cell complex*

$$X \to X \underset{\mathrm{sk}_0 X}{\cup} \mathrm{sk}_0 Y \to \cdots \to X \underset{\mathrm{sk}_{n-1} X}{\cup} \mathrm{sk}_{n-1} Y \to X \underset{\mathrm{sk}_n X}{\cup} \mathrm{sk}_n Y \to \cdots \to \operatorname{colim} \cong Y$$

with the cells

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \mathbin{\widehat{*}} \widehat{\ell}^a f \tag{C.4.20}$$

indexed by objects a of degree n attached at stage n, and also a "Postnikov tower"

$$X \cong \lim \to \cdots \to \operatorname{cosk}_n X \underset{\operatorname{cosk}_n Y}{\times} Y \to \operatorname{cosk}_{n-1} X \underset{\operatorname{cosk}_{n-1} Y}{\times} Y \to \cdots \to \operatorname{cosk}_0 X \underset{\operatorname{cosk}_0 Y}{\times} Y \to Y$$

whose n-th layer is the product of the maps

$$\{\widehat{\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a}, \widehat{m}^a f\} \tag{C.4.21}$$

indexed by the objects a of degree n.

Proof. These dual results follow immediately by applying the weighted colimit and weighted limit bifunctors

$$\operatorname{colim}_{-} -\colon \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}} \qquad \text{and} \qquad \lim_{-} -\colon (\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}}$$

to the cell complex presentations of Theorem C.4.14; recall from Definition 7.1.3(ii) that both bifunctors are cocontinuous in the weight.

To see that the generalized cell complex presentation for f has the asserted form, note that for any diagram $X \in \mathcal{M}^{\mathcal{A}}$ and weight defined by an exterior product of $U \in \mathcal{S}et^{\mathcal{A}}$ and $V \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$, there is a natural isomorphism

$$\operatorname{colim}_{U \square V} X \cong U * \operatorname{colim}_V X,$$

which extends to a natural isomorphism between Leibniz products (Proposition C.2.9(i)).

By the coYoneda lemma, $f \cong \operatorname{colim}_{\mathcal{A}} f \cong \widehat{\operatorname{colim}}_{\varnothing\hookrightarrow\mathcal{A}} f$. By cocontinuity, the Leibniz weighted colimit functor $\widehat{\operatorname{colim}}_{-} f$ preserves generalized cell structures (Proposition C.2.9(vi)). It follows that f admits a canonical presentation as a cell complex with cells

$$\begin{aligned} \widehat{\operatorname{colim}}_{(\partial\mathcal{A}_a\hookrightarrow\mathcal{A}_a)\widehat{\square}(\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a)} f &\cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \mathbin{\widehat{*}} \widehat{\operatorname{colim}}_{\partial\mathcal{A}^a\hookrightarrow\mathcal{A}^a} f \\ &\cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \mathbin{\widehat{*}} \widehat{\ell}^a f. \end{aligned}$$

□

This presentation is most familiar for the Reedy category $\mathbf{\Delta}^{\mathrm{op}}$, in which case it's conventional to use lower subscripts to designate the contravariant indexing. Here we write $\mathbf{\Delta}^n$ for the standard n-simplex $\mathbf{\Delta}[n]$ and $\partial\mathbf{\Delta}^n$ for its boundary to be consistent with the notation of Definition C.4.10

C.4.22. Example. A simplicial object Y taking values in any cocomplete category admits a skeletal filtration

$$\varnothing \to \mathrm{sk}_0 Y \to \cdots \to \mathrm{sk}_{n-1} Y \to \mathrm{sk}_n Y \to \cdots \to Y$$

in which the step from stage $n-1$ to stage n is given by a pushout

$$\begin{tikzcd} \mathbf{\Delta}^n * L_n Y \cup \partial\mathbf{\Delta}^n * Y_n \arrow[r] \arrow[d] & \mathbf{\Delta}^n * Y_n \arrow[d] \\ \mathrm{sk}_{n-1} Y \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{sk}_n Y \end{tikzcd}$$

where $L_n Y \to Y_n$ is the object of "degenerate n-simplices."

Considering the Yoneda embedding as a simplicial object $\mathbf{\Delta} \in (\mathcal{S}et^{\mathbf{\Delta}})^{\mathbf{\Delta}^{\mathrm{op}}}$, this specializes to the "canonical cell complex presentation" of the hom bifunctor of Theorem C.4.14

$$\begin{tikzcd} & & \mathbf{\Delta}^n \times \partial\mathbf{\Delta}_n \cup \partial\mathbf{\Delta}^n \times \mathbf{\Delta}_n \arrow[r, hook] \arrow[d] & \mathbf{\Delta}^n \times \mathbf{\Delta}_n \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] & & \\ \emptyset \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{sk}_{n-1}\mathbf{\Delta} \arrow[r, hook] & \mathrm{sk}_n \mathbf{\Delta} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbf{\Delta} \end{tikzcd}$$

In summary, Corollary C.4.19 tells us that we may express a generic natural transformation between diagrams of shape $\mathcal{A}$ as

(i) a cell complex whose cells are Leibniz tensors built from boundary inclusions of covariant representables and relative latching maps,
(ii) and dually as a Postnikov tower whose layers are Leibniz cotensors built from boundary inclusions of contravariant representables and relative matching maps.

This explains the importance of these maps to Reedy category theory, as we shall discover in the next section.

Exercises.

C.4.i. Exercise. Show that any isomorphism in a (strict) Reedy category is an identity.

C.4.ii. Exercise. Show that the product of two Reedy categories is a Reedy category, with the degree of an object defined to be the sum of the degrees.

C.5. The Reedy model structure

Our aim in this section is to explain how any weak factorization system on $\mathcal{M}$ gives rise to a **Reedy weak factorization system** on $\mathcal{M}^{\mathcal{A}}$. We then prove an inductive result that allows us to prove that the Reedy weak factorization systems associated to a model structure on $\mathcal{M}$ define the **Reedy model structure** on $\mathcal{M}^{\mathcal{A}}$. Finally, we prove that the weighted limit and weighted colimit bifunctors define Quillen bifunctors, as a consequence of a more general algebraic result, and discuss the consequences of this result for the theory of homotopy limits and homotopy colimits indexed by strict Reedy categories.

This work requires one preliminary: a discussion of how the skeleta and coskeleta introduced in the previous section feature in the inductive definition of Reedy-shaped diagrams. For a Reedy category $\mathcal{A}$, write

$$\mathcal{A}_{\leq 0} \subset \mathcal{A}_{\leq 1} \subset \cdots \subset \mathcal{A}_{\leq n-1} \subset \mathcal{A}_{\leq n} \subset \cdots \subset \mathcal{A}$$

for the full subcategories of objects with degree at most the ordinal appearing in the subscript. These categories give us a new way to understand the skeleton and coskeleton functors introduced in Notation C.4.18.

C.5.1. Lemma. *For any bicomplete category $\mathcal{M}$, restriction and left and right Kan extension define an adjoint triple of functors*

$$\begin{tikzcd} \mathcal{M}^{\mathcal{A}} \arrow[r, "\mathrm{res}_n" description] & \mathcal{M}^{\mathcal{A}_{\leq n}} \arrow[l, bend right, "\mathrm{lan}_n"', "\perp"] \arrow[l, bend left, "\mathrm{ran}_n", "\perp"'] \end{tikzcd}$$

with induced comonad $\mathrm{sk}_n := \mathrm{lan}_n \circ \mathrm{res}_n$ *and monad* $\mathrm{cosk}_n := \mathrm{ran}_n \circ \mathrm{res}_n$ *naturally isomorphic to the functors defined by weighted colimit and weighted limit*

$$\mathrm{lan}_n \,\mathrm{res}_n(-) \cong \mathrm{colim}_{\mathrm{sk}_n \mathcal{A}} - \qquad \textit{and} \qquad \mathrm{ran}_n \,\mathrm{res}_n(-) \cong \lim_{\mathrm{sk}_n \mathcal{A}} -.$$

PROOF. Exercise C.5.i. □

Since $\mathrm{sk}_n \mathcal{A} \hookrightarrow \mathcal{A}$ is fully faithful, the functors $\mathrm{lan}_n \dashv \mathrm{res}_n \dashv \mathrm{ran}_n$ of Lemma C.5.1 define a **fully faithful adjoint triple**. For example:

C.5.2. DEFINITION. Specializing the notation of §C.5, write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory of the simplex category of 1.1.1 spanned by the ordinals $[0], \ldots, [n]$. Restriction and left and right Kan extension define adjunctions

$$\begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "\mathrm{res}_n" description] & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \arrow[l, bend right, "\mathrm{lan}_n"'] \arrow[l, bend left, "\mathrm{ran}_n"] \end{tikzcd}$$

inducing an idempotent comonad $\mathrm{sk}_n := \mathrm{lan}_n \circ \mathrm{res}_n$ and an idempotent monad $\mathrm{cosk}_n := \mathrm{ran}_n \circ \mathrm{res}_n$ on $s\mathcal{S}et$ that are adjoint $\mathrm{sk}_n \dashv \mathrm{cosk}_n$. The counit and unit of this comonad and monad define canonical maps

$$\mathrm{sk}_n X \overset{\epsilon}{\rightarrowtail} X \overset{\eta}{\twoheadrightarrow} \mathrm{cosk}_n X$$

relating a simplicial set X with its n-**skeleton** and n-**coskeleton**. We say X is n-**skeletal** or n-**coskeletal** if the former or latter of these maps, respectively, is an isomorphism.

The canonical map from the n-skeleton of a simplicial set to its n-coskeleton can be defined more generally:

C.5.3. LEMMA. *For any fully faithful inclusion* $\mathcal{B} \hookrightarrow \mathcal{A}$ *and bicomplete category* $\mathcal{M}$, *consider the associated adjoint triple:*

$$\begin{tikzcd} \mathcal{M}^{\mathcal{A}} \arrow[r, "\mathrm{res}" description] & \mathcal{M}^{\mathcal{B}} \arrow[l, bend right, "\mathrm{lan}"'] \arrow[l, bend left, "\mathrm{ran}"] \end{tikzcd}$$

(i) The functors $\mathrm{lan}, \mathrm{ran} \colon \mathcal{M}^{\mathcal{B}} \rightrightarrows \mathcal{M}^{\mathcal{A}}$ *are fully faithful; that is, the unit of* $\mathrm{lan} \dashv \mathrm{ran}$ *and the counit of* $\mathrm{res} \dashv \mathrm{ran}$ *are isomorphisms.*

(ii) The common composite in the commutative square below defines a canonical natural transformation

$$\begin{tikzcd} \mathrm{lan} \arrow[r, Rightarrow, "\eta \circ \mathrm{lan}"] \arrow[d, Rightarrow, "\mathrm{lan} \circ \epsilon^{-1}"'] \arrow[dr, Rightarrow, dashed, "\tau"] & \mathrm{ran}\,\mathrm{res}\,\mathrm{lan} \arrow[d, Rightarrow, "\mathrm{ran} \circ \eta^{-1}"] \\ \mathrm{lan}\,\mathrm{res}\,\mathrm{ran} \arrow[r, Rightarrow, "\epsilon \circ \mathrm{ran}"'] & \mathrm{ran} \end{tikzcd}$$

PROOF. It is well-known that a right adjoint functor is fully faithful if and only if the counit is an isomorphism and that the counit of a pointwise right Kan extension along a fully faithful functor is an isomorphism; for proof, specialize the results of Lemma 13.4.3 and Proposition 13.4.4 to the ∞-cosmos $\mathcal{C}at$. These statements and their duals prove (i).

In (ii), τ is defined to be the adjoint transpose of $\eta^{-1} \colon \mathrm{res}\,\mathrm{lan} \Rightarrow \mathrm{id}$ under $\mathrm{res} \dashv \mathrm{ran}$ and also to be the adjoint transpose of $\epsilon^{-1} \colon \mathrm{id} \Rightarrow \mathrm{res}\,\mathrm{ran}$ under $\mathrm{lan} \dashv \mathrm{res}$. To see that this definitions agree, observe that the former asserts that the composite of the right two morphisms below is the

unique right inverse of the left morphism, while the latter asserts that the composite of the left two morphisms below is the unique left inverse of the right morphism:

$$\mathrm{id} \xRightarrow[\cong]{\eta} \mathrm{res}\,\mathrm{lan} \xRightarrow{\mathrm{res}\,\tau} \mathrm{res}\,\mathrm{ran} \xRightarrow[\cong]{\epsilon} \mathrm{id}$$

In other words, both definitions assert exactly that the displayed triple composite is the identity. $\square$

These structures allow us to inductively define Reedy diagrams:

C.5.4. Proposition (inductive definition of diagrams).

(i) A diagram $X \in \mathcal{M}^{\mathcal{A}_{\leq n-1}}$ together with a family of factorizations

$$\begin{tikzcd}
\mathrm{sk}_{n-1} X^a \arrow[rr, "\tau_{n-1}X^a"] \arrow[dr, dashed, "i^a"'] & & \mathrm{cosk}_{n-1} X^a \\
& X^a \arrow[ur, dashed, "p^a"'] &
\end{tikzcd}$$

for each object $a \in \mathcal{A}$ of degree n uniquely determines a diagram $X \in \mathcal{M}^{\mathcal{A}_{\leq n}}$ whose restriction to degree $n-1$ coincides with the original diagram.

(ii) A natural transformation $\phi\colon X \to Y \in \mathcal{M}^{\mathcal{A}_{\leq n-1}}$ together with a family of factorizations

$$\begin{tikzcd}
\mathrm{sk}_{n-1} X^a \arrow[r, dashed, "i^a"'] \arrow[rr, bend left, "\tau_{n-1}X^a"] \arrow[d, "\mathrm{sk}_{n-1}\phi^a"'] & X^a \arrow[r, dashed, "p^a"'] \arrow[d, dashed, "\phi^a"] & \mathrm{cosk}_{n-1} X^a \arrow[d, "\mathrm{cosk}_{n-1}\phi^a"] \\
\mathrm{sk}_{n-1} Y^a \arrow[r, dashed, "i^a"] \arrow[rr, bend right, "\tau_{n-1}Y^a"'] & Y^a \arrow[r, dashed, "p^a"] & \mathrm{cosk}_{n-1} Y^a
\end{tikzcd}$$

for each object $a \in \mathcal{A}$ of degree n uniquely determines a natural transformation $\phi\colon X \to Y \in \mathcal{M}^{\mathcal{A}_{\leq n}}$ whose restriction to degree n coincides with the original natural transformation.

Proof. For (i), it remains to define the action of X on non-identity morphisms whose domain or codomain has degree n. The Reedy factorization of any such morphism $f\colon a \to a'$ is through an object b of degree less than n. By composing the maps in the upper-right or lower-left square, there exist unique dotted-arrow maps making the following diagram commute

$$\begin{tikzcd}
\mathrm{sk}_{n-1} X^a \arrow[r, "i^a"] \arrow[d, "\mathrm{sk}_{n-1} X^{\overleftarrow{f}}"'] & X^a \arrow[r, "p^a"] \arrow[d, dashed, "X^{\overleftarrow{f}}"] & \mathrm{cosk}_{n-1} X^a \arrow[d, "\mathrm{cosk}_{n-1} X^{\overleftarrow{f}}"] \\
\mathrm{sk}_{n-1} X^b \arrow[r, equal, "i^b"] \arrow[d, "\mathrm{sk}_{n-1} X^{\overrightarrow{f}}"'] & X^b \arrow[r, equal, "p^b"] \arrow[d, dashed, "X^{\overrightarrow{f}}"] & \mathrm{cosk}_{n-1} X^b \arrow[d, "\mathrm{cosk}_{n-1} X^{\overrightarrow{f}}"] \\
\mathrm{sk}_{n-1} X^{a'} \arrow[r, "i^{a'}"'] & X^{a'} \arrow[r, "p^{a'}"'] & \mathrm{cosk}_{n-1} X^{a'}
\end{tikzcd}$$

The functoriality of this definition, in a pair of composable maps (f,g) follows from connectedness of the category $\mathcal{F}act_{n-1}(gf)$.

For (ii), apply (i) to the $\mathcal{A}_{\leq n-1}$-shaped diagram $a \mapsto \phi^a$ valued in $\mathcal{M}^{\mathbf{2}}$. $\square$

Now we turn our attention to the main subject of this section. Let $\mathcal{M}$ be a category with a weak factorization system $(\mathcal{L}, \mathscr{R})$ and let $\mathcal{A}$ be a strict Reedy category.

C.5.5. Definition. The **Reedy weak factorization system** $(\mathcal{L}[\mathcal{A}], \mathscr{R}[\mathcal{A}])$ on $\mathcal{M}^{\mathcal{A}}$ defined relative to the weak factorization system $(\mathcal{L}, \mathscr{R})$ on $\mathcal{M}$ has:

- as left class $\mathcal{L}[\mathcal{A}]$ those maps $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ whose relative latching maps $\widehat{\ell}^a f\colon \ell^a f \to Y^a \in \mathcal{M}$ are in $\mathcal{L}$, and
- as right class $\mathscr{R}[\mathcal{A}]$ those maps $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ whose relative matching maps $\widehat{m}^a f\colon X^a \to m^a f \in \mathcal{M}$ are in $\mathscr{R}$.

We say a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is **Reedy in $\mathcal{L}$** or **Reedy in $\mathscr{R}$** if its relative latching or relative matching maps are in $\mathcal{L}$ or $\mathscr{R}$, respectively. The following pair of lemmas, imply that these two classes indeed define a weak factorization system on the category of Reedy diagrams in $\mathcal{M}$.

C.5.6. Lemma. *The maps $i \in \mathcal{L}[\mathcal{A}]$ have the left lifting property with respect to the maps $p \in \mathscr{R}[\mathcal{A}]$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & L \end{tikzcd}$$

Proof. By Corollary C.4.19, to show that $i \boxslash p$ for any pair of morphisms $f, g \in \mathcal{M}^{\mathcal{A}}$, it suffices to solve the lifting problems below-left

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\hat{*}\widehat{\ell}^a i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "\widehat{\ell}^a i"'] & \bullet \arrow[d, "\widehat{\lim}_{\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a} p \cong \widehat{m}^a p"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

in $\mathcal{M}^{\mathcal{A}}$ for each $a \in \mathcal{A}$. By adjunction, it suffices to solve the transposed lifting problem in $\mathcal{M}$ above-right. If $i \in \mathcal{L}[\mathcal{A}]$ and $p \in \mathscr{R}[\mathcal{A}]$, then by definition $\widehat{\ell}^a i \in \mathcal{L}$ and $\widehat{m}^a p \in \mathscr{R}$, so a solution exists. □

C.5.7. Lemma. *Every map $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ can be factored as a map in $\mathcal{L}[\mathcal{A}]$ followed by a map in $\mathscr{R}[\mathcal{A}]$.*

Proof. We define the components of the factorization of $f^a\colon X^a \to Y^a$ inductively in the degree of a. To start, we use the factorization of $(\mathcal{L}, \mathscr{R})$ to factor all components indexed by objects at degree zero. Since the full subcategory $\mathcal{A}_{\leq 0}$ spanned by these objects is discrete, this defines a factorization of the subdiagram $f \in \mathcal{M}^{\mathcal{A}_{\leq 0}}$.

Continuing inductively, suppose we have factored the restriction $f \in \mathcal{M}^{\mathcal{A}_{<n}}$ as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\ell"'] & & Y \\ & Z \arrow[ur, "r"'] & \end{tikzcd} \qquad \in \mathcal{M}^{\mathcal{A}_{<n}}$$

with the relative latching maps $\widehat{\ell}^a \ell \in \mathcal{L}$ and $\widehat{m}^a r \in \mathscr{R}$ for all object a of degree less than n. By Proposition C.5.4, to define the attendant factorization of f^a, it suffices to define an object Z^a of $\mathcal{M}$

together with the dotted arrow maps

$$\begin{tikzcd}
L^aX \arrow[rr] \arrow[dd] & & L^aZ \arrow[rr] \arrow[dl] & & L^nY \arrow[dd] \\
& X_n \underset{L^aX}{\times} L^aZ \arrow[dr, dashed] & & & \\
X^a \arrow[ur] \arrow[rr, dashed, "\ell^a"'] \arrow[dd] & & Z^a \arrow[rr, dashed, "r^a"] \arrow[dr, dashed] & & Y^a \arrow[dd] \\
& & & M^aZ \underset{M^aY}{\times} Y^a \arrow[ur] \arrow[dl] & \\
M^aX \arrow[rr] & & M^aZ \arrow[rr] & & M^aY
\end{tikzcd}$$

We factor the diagonal map from the pushout to the pullback using $(\mathcal{L}, \mathcal{R})$. The diagonal factors become the a-th relative latching map and matching map of the composite morphisms ℓ^a and r^a so-defined, and in particular lie in the classes $\mathcal{L}$ and $\mathcal{R}$, respectively. It follows from the universal properties of the pushout and the pullback that $f^a = r^a \cdot \ell^a$. By Proposition C.5.4 these definitions extend the natural transformations ℓ and r to degree n. $\square$

It follows from Corollary C.4.19 that if the left class of a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ is cofibrantly or cellularly generated, as in Definition C.2.4, then the left class of the Reedy weak factorization system is too:

C.5.8. Proposition. *If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system on $\mathcal{M}$ that is cellularly or cofibrantly generated by the class of maps $\mathcal{J}$, then the Reedy weak factorization system $(\mathcal{L}[\mathcal{A}], \mathcal{R}[\mathcal{A}])$ on $\mathcal{M}^{\mathcal{A}}$ is cellularly or cofibrantly generated, respectively, by the class*

$$\{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{\circledast} j\}_{a\in\mathcal{A}, j\in\mathcal{J}}.$$

Proof. By Corollary C.4.19, any morphism $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ may be presented as a cell complex built from cells

$$\{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{\circledast} \widehat{\ell}^a f\}_{a\in\mathcal{A}}.$$

If $f \in \mathcal{L}[\mathcal{A}]$, then $\widehat{\ell}^a f \in \mathcal{L}$ for each a, and by hypothesis these relative latching maps may be presented as cell complexes or retracts of cell complexes built from the maps in the generating class $\mathcal{J}$. By Proposition C.2.9(vi), the Leibniz tensors $(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{\circledast} \widehat{\ell}^a f$ may then be presented as (retracts of) cell complexes built from the Leibniz tensors of the boundary inclusions and the maps in $\mathcal{J}$, exactly as claimed in the statement. $\square$

For example, the monomorphisms of simplicial sets are cellularly generated by the simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.

C.5.9. Lemma. *The Reedy weak factorization system $(\mathcal{M}[\Delta^{\mathrm{op}}], \mathcal{E}[\Delta^{\mathrm{op}}])$ on $s\mathcal{S}et = \mathcal{S}et^{\Delta^{\mathrm{op}}}$ defined relative to the (monomorphism, epimorphism) weak factorization system on $\mathcal{S}et$ coincides with the (monomorphism, trivial fibration) weak factorization system. Consequently, any monomorphism of simplicial sets decomposes canonically as a sequential composite of pushouts of coproducts of the maps $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq 0$.*

Proof. Monomorphisms of sets are cellularly generated by a single map, the inclusion $!\colon \emptyset \hookrightarrow *$. Consequently, by Proposition C.5.8, the Reedy weak factorization system is cellularly generated as well. In this case, the pushout product functor $-\widehat{*}!\colon \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ is the identity, so the set of generating maps are the familiar simplex boundary inclusions $\{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{[n]\in\mathbf{\Delta}}$, the right lifting property against which is used in Definition 1.1.24 to characterize the trivial fibrations. $\square$

C.5.10. Proposition.

(i) If $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is Reedy in $\mathcal{L}$, that is, if the relative latching maps $\widehat{\ell}^a f$ are in $\mathcal{L}$, then each of the components $f^a\colon X^a \to Y^a$ and each of the latching maps $L^a f\colon L^a X \to L^a Y$ are also in $\mathcal{L}$.

(ii) If $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$ is Reedy in $\mathscr{R}$, that is, if the relative matching maps $\widehat{m}^a f$ are in $\mathcal{L}$, then each of the components $f^a\colon X^a \to Y^a$ and each of the matching maps $M^a f\colon M^a X \to M^a Y$ are also in $\mathscr{R}$.

Proof. We prove the first of these dual statements. The maps f^a and $L^a f$ are the Leibniz weighted colimits of f with the maps $\emptyset \hookrightarrow \mathcal{A}^a$ and $\emptyset \hookrightarrow \partial\mathcal{A}^a$ respectively. Evaluating the covariant variable of the cell complex presentation of Theorem C.4.14 at $a \in \mathcal{A}$, we see that $\emptyset \hookrightarrow \mathcal{A}^a$ is a cell complex whose cells have the form

$$((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x) \mathbin{\widehat{\square}} (\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x), \tag{C.5.11}$$

indexed by the objects $x \in \mathcal{A}$. In fact, it suffices to consider those objects with $\deg(x) \leq \deg(a)$; when $\deg(x) > \deg(a)$ the inclusion $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x$, and hence the cell (C.5.11), is an isomorphism. Similarly, since $\partial\mathcal{A}^a = \mathrm{sk}_{\deg(a)-1}\mathcal{A}^a$, Theorem C.4.14 implies that $\emptyset \hookrightarrow \partial\mathcal{A}^a$ is a cell complex whose cells have the form (C.5.11) with $\deg(x) < \deg(a)$.

By Proposition C.2.9(vi), the maps f^a and $L^a f$ are then cell complexes whose cells, indexed by the objects $x \in \mathcal{A}$ with the degree bounds just discussed, have the form

$$\widehat{\mathrm{colim}}_{((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)\widehat{\square}(\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x)} f \cong ((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x) \mathbin{\widehat{*}} \widehat{\ell}^x f, \tag{C.5.12}$$

the isomorphism arising from Proposition C.2.9(i). By Lemma C.2.12, the Leibniz tensor of a monomorphism with a map in the left class of a weak factorization system is again in the left class. Thus, since $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)$ is a monomorphism and $\widehat{\ell}^x f$ is in $\mathcal{L}$, these cells, and thus the maps f^a and $L^a f$ are in $\mathcal{L}$ as well. $\square$

Recall from Definition C.3.1 that a **model structure** on a category $\mathcal{M}$ with a class of weak equivalences $\mathcal{W}$ satisfying the 2-of-3 property is given by two classes of maps $\mathcal{C}$ and $\mathcal{F}$ so that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ define weak factorization systems. To show that the Reedy weak factorization systems on $\mathcal{M}^{\mathcal{A}}$ relative to a model structure on $\mathcal{M}$ define a model structure on $\mathcal{M}^{\mathcal{A}}$ with the weak equivalences defined pointwise, one lemma is needed.

C.5.13. Lemma. *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathcal{M}$. Then a map $f\colon X \to Y \in \mathcal{M}^{\mathcal{A}}$*

(i) is Reedy in $\mathcal{C} \cap \mathcal{W}$ if and only if f is Reedy in $\mathcal{C}$ and a pointwise weak equivalence, and

(ii) is Reedy in $\mathcal{F} \cap \mathcal{W}$ if and only if f is Reedy in $\mathcal{F}$ and a pointwise weak equivalence.

Proof. We prove the first of these dual statements. If f is Reedy in $\mathcal{C} \cap \mathcal{W}$, then it is obviously Reedy in $\mathcal{C}$, and Proposition C.5.10 implies that its components f^a are also in $\mathcal{C} \cap \mathcal{W}$. Thus f is a pointwise weak equivalence.

For the converse, we make use of the diagram

$$\begin{tikzcd}
L^a X \arrow[r, "L^a f"] \arrow[d] & L^a Y \arrow[d] \arrow[ddr, bend left] & \\
X^a \arrow[r] \arrow[drr, bend right, "f^a"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\widehat{\ell}^a f"] & \\
& & Y^a
\end{tikzcd}$$

which relates the maps $L^a f$, $\widehat{\ell}^a f$, and f^a for any $a \in \mathcal{A}$; this is an instance of Proposition C.2.9(v) applied to $(\varnothing \hookrightarrow \partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \hat{*}_{\mathcal{A}} f$. Suppose that f is Reedy in $\mathcal{C}$ and a pointwise weak equivalence. By Proposition C.5.10, it follows that $L^a f$ is in $\mathcal{C}$. We will show that $L^a f$ is in fact in $\mathcal{C} \cap \mathcal{W}$ and then apply pushout stability of the left class of a weak factorization system and the 2-of-3 property, to conclude that $\widehat{\ell}^a f \in \mathcal{W}$ and hence that f is Reedy in $\mathcal{C} \cap \mathcal{W}$. We argue by induction. If a has degree zero, then $L^a f$ is the identity at the initial object, which is certainly a weak equivalence, and $\widehat{\ell}^a f = f^a$ is in $\mathcal{C} \cap \mathcal{W}$. If a has degree n, we may now assume that $\widehat{\ell}^x f \in \mathcal{C} \cap \mathcal{W}$ for any x with degree less than the degree of a. By the proof of Proposition C.5.10, $L^a f$ may be presented as a cell complex whose cells (C.5.12) are Leibniz tensors of monomorphisms with maps in $\mathcal{C} \cap \mathcal{W}$, and thus lie in $\mathcal{C} \cap \mathcal{W}$. Thus, we conclude that $L^a f \in \mathcal{C} \cap \mathcal{W}$, completing the proof. ☐

Lemmas C.5.6, C.5.7, and C.5.13 assemble to prove:

C.5.14. Theorem (the Reedy model structure). *If $\mathcal{A}$ is a strict Reedy category and $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathcal{M}$, then the Reedy weak factorization systems $(\mathcal{C} \cap \mathcal{W}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F} \cap \mathcal{W}[\mathcal{A}])$ define a model structure on $\mathcal{M}^{\mathcal{A}}$ with pointwise weak equivalences.* ☐

One reason for our interest in the Reedy model structure is it comes equipped with convenient Quillen bifunctors, which arise from the following result, which produces left Leibniz bifunctors in the sense of Definition C.2.10 in the Reedy diagram context.

C.5.15. Theorem. *Let $\mathcal{A}$ be a Reedy category and let $\otimes \colon \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ be a left Leibniz bifunctor with respect to weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathscr{R})$. Then the functor tensor product*

$$\otimes_{\mathcal{A}} \colon \mathcal{K}^{\mathcal{A}^{\mathrm{op}}} \times \mathcal{L}^{\mathcal{A}} \to \mathcal{M}$$

is left Leibniz with respect to the Reedy weak factorization systems $(\mathcal{M}[\mathcal{A}^{\mathrm{op}}], \mathcal{E}[\mathcal{A}^{\mathrm{op}}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{L}, \mathscr{R})$.

Proof. The reasons for the cocontinuity of the functor tensor product are well-understood. We argue that $\otimes_{\mathcal{A}}$ has the Leibniz property. Corollary C.4.19 asserts that the maps $f \in \mathcal{K}^{\mathcal{A}^{\mathrm{op}}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \hat{*} \widehat{\ell}_a f,$$

and the maps $g \in \mathcal{L}^{\mathcal{A}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \hat{*} \widehat{\ell}^b g.$$

By Proposition C.2.9(vi), $f \mathbin{\widehat{\otimes}_{\mathcal{A}}} g$ is then a cell complex whose cells have the form

$$\left((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{*}} \widehat{\ell}_a f\right) \mathbin{\widehat{\otimes}_{\mathcal{A}}} \left((\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \mathbin{\widehat{*}} \widehat{\ell}^b g\right) \\ \cong \left((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\widehat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b)\right) \mathbin{\widehat{*}} (\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g)$$

To say that f is Reedy in $\mathcal{M}$ and g is Reedy in $\mathcal{C}$ means that $\widehat{\ell}_a f \in \mathcal{M}$ and $\widehat{\ell}^b g \in \mathcal{C}$. Since $\otimes$ is left Leibniz, it follows that $\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g \in \mathcal{L}$. The Leibniz functor tensor product

$$(\partial\mathcal{A}^b \hookrightarrow \mathcal{A}^b) \mathbin{\widehat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)$$

of the maps in $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$ and in $\mathcal{S}et^{\mathcal{A}}$ amounts to the inclusion into the hom-set $\mathcal{A}^a_b = \mathcal{A}(b,a)$ of the subset of morphisms from b to a that factor through an object of degree strictly less than a or strictly less than b; in particular, this map is a monomorphism. Now Lemma C.2.12 applies to the weak factorization system $(\mathcal{L},\mathcal{R})$ on $\mathcal{M}$ to prove that the Leibniz tensor of this monomorphism with $\widehat{\ell}_a f \mathbin{\widehat{\otimes}} \widehat{\ell}^b g$ remains in $\mathcal{L}$, completing the proof. $\square$

Applying Theorem C.5.15 to Lemma C.2.12, with (monomorphism, epimorphism) taken as the default weak factorization system on $\mathcal{S}et$, we conclude:

C.5.16. COROLLARY. *For any bicomplete category $\mathcal{M}$ with a weak factorization system $(\mathcal{L},\mathcal{R})$ and any strict Reedy category, the weighted colimit and weighted limit*

$$\operatorname{colim}_{-} - \colon \mathcal{S}et^{\mathcal{A}} \times \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M} \qquad \text{and} \qquad \lim_{-} - \colon (\mathcal{S}et^{\mathcal{A}})^{\mathrm{op}} \times \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$$

define left and right Leibniz bifunctors relative to the Reedy weak factorization systems.

In the setting of a model category, a monoidal model category, or a $\mathcal{V}$-model category (which subsumes the previous two cases by taking $\mathcal{V}$ to be $\mathcal{S}et$ or the model category itself), Corollary C.5.16 specializes to the following result, which helps us understand homotopy limits and colimits of diagrams of Reedy shape.

C.5.17. COROLLARY. *Let $\mathcal{M}$ be a $\mathcal{V}$-model category and let $\mathcal{A}$ be a strict Reedy category. Then for any weight W in $\mathcal{V}^{\mathcal{A}}$ that is Reedy cofibrant*[7]*, the weighted colimit and weighted limit functors*

$$\operatorname{colim}_W - \colon \mathcal{M}^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{M} \qquad \text{and} \qquad \lim_W - \colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$$

are respectively left and right Quillen with respect to the Reedy model structure on $\mathcal{M}^{\mathcal{A}}$.

C.5.18. EXAMPLE (homotopy limits and colimits). Taking the terminal weight 1 in $\mathcal{S}et^{\mathcal{A}}$, the weighted limit reduces to the ordinary limit functor. The functor $1 \in \mathcal{S}et^{\mathcal{A}}$ is Reedy monomorphic just when, for each $a \in \mathcal{A}$, the category of elements for the weight $\partial\mathcal{A}^a$ is either empty or connected. This is the case if and only if $\mathcal{A}$ has **cofibrant constants**, meaning that the constant $\mathcal{A}$-indexed diagram at any cofibrant object in any model category is Reedy cofibrant. Thus, we conclude that if $\mathcal{A}$ has cofibrant constants, then the limit functor $\lim \colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is right Quillen.

Dually, the colimit functor $\operatorname{colim} \colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is a special case of the weighted colimit functor with the terminal weight $1 \in \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$. This is Reedy monomorphic just when each category of elements for the weights $\partial\mathcal{A}_a$ is either empty or connected, which is the case if and only if $\mathcal{A}$ has **fibrant constants**, meaning that the constant $\mathcal{A}$-indexed diagram at any fibrant object in any model category is Reedy fibrant. Thus, we conclude that if $\mathcal{A}$ has fibrant constants, then the colimit functor $\operatorname{colim} \colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is left Quillen. See [**82**, §9] for more discussion.

[7] In the case of $\mathcal{V} = \mathcal{S}et$, "Reedy cofibrant" should be read as "Reedy monomorphic."

The following lemmas can be used to identify Reedy cofibrant weights for use in applications of Corollary C.5.17.

C.5.19. LEMMA. *Any bisimplicial set is Reedy cofibrant as an object of* $s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$.

PROOF. Exercise C.5.ii. $\square$

Reedy cofibrant cosimplicial objects can also be identified, on account of the following lemma.

C.5.20. LEMMA.

(i) Let $X\colon \mathbf{\Delta} \to \mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ *be a cosimplicial object in a presheaf category. If* X *is unaugmentable, in the sense that the equalizer of the pair of coface maps* $\delta^0, \delta^1 \colon X^0 \rightrightarrows X^1$ *is empty, then the latching maps of* X *are all monomorphisms.*

(ii) If X *is an unaugmentable cosimplicial object in a slice category of a presheaf category, then the latching maps of* X *are all monomorphisms.*

PROOF. Since latching objects are defined in terms of certain colimits in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ computed pointwise in $\mathcal{S}et$, we may reduce this result to the corresponding one for cosimplicial sets $X\colon \mathbf{\Delta} \to \mathcal{S}et$. A simplex in a cosimplicial set is "non-degenerate" if it is not in the image of a monomorphism from $\mathbf{\Delta}$. The nth latching map $L^nX \to X^n$ is a monomorphism just when each expression of an n-simplex x as the image of a non-degenerate simplex z under a monomorphism $\sigma\colon [k] \rightarrowtail [n]$ is unique.

So suppose we have two such representations $x = \sigma \cdot z = \sigma' \cdot z'$. Any monomorphism $\sigma \in \mathbf{\Delta}$ has a left inverse τ, so we see that $z = \tau\sigma' \cdot z'$. The map $\tau\sigma'$ can be factored as an epimorphism followed by a monomorphism. Because z is non-degenerate, this monomorphism must be the identity, so $\tau\sigma'$ is an epimorphism. Repeating this argument with an left inverse τ' for σ' we see that $\tau'\sigma$ is an epimorphism, so z and z' have the same degree and both epimorphisms are identities. This proves that $z = z'$.

If the set of left inverses for a monomorphism uniquely characterized that monomorphism, then we could conclude that σ and σ' must be equal, and hence that such decompositions would be fully unique. This is true for nearly all monomorphisms in $\mathbf{\Delta}$, the only exceptions being the face maps $\delta^0, \delta^1 \colon [0] \rightarrowtail [1]$. However, X is assumed to be unaugmentable, so there is no $z \in Z^0$ with $\delta^0 z = \delta^1 \cdot z$, and thus $L^nX \to X^n$ is a monomorphism for all n. This proves the first statement.

For the second statement, we only need to consider slice categories ${}^{A/}\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ under an object, since the slice category $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}{}_{/A}$ is equivalent to the category of presheaves on the category of elements of A. The forgetful functor ${}^{A/}\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ creates monomorphisms and connected colimits, which tells us that the latching maps $L^nX \to X^n$ of a complicial object $X^\bullet$ in ${}^{A/}\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ are calculated in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ for all $n > 1$. The direct calculation given above proves that these are monomorphisms. If X is unaugmentable, the latching map $(\delta^0, \delta^1)\colon L^1X \cong X^0 + X^0 \to X^1$ is a monomorphism: arguing in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ and then ultimately in $\mathcal{S}et$, it is easy to see that the pulllback of this map along itself is A. If X is unaugmentable, this says also that that the equalizer of $(\delta^0, \delta^1)\colon X^0 \rightrightarrows X^1$ in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is $A \to X^0$, so this map is a monomorphism. This tells us that the 0th latching map is a monomorphism, completing the proof that $X^\bullet$ is a Reedy monomorphism. $\square$

C.5.21. EXAMPLE (geometric realization and totalization). By Lemma C.5.20, the Yoneda embedding defines a Reedy cofibrant weight $\mathbf{\Delta}^\bullet \in s\mathcal{S}et^{\mathbf{\Delta}}$. The weighted colimit and weighted limit functors

$$\operatorname{colim}_{\mathbf{\Delta}^\bullet} - \colon \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{M} \qquad \text{and} \qquad \lim_{\mathbf{\Delta}^\bullet} - \colon \mathcal{M}^{\mathbf{\Delta}} \to \mathcal{M}$$

typically go by the names of **geometric realization** and **totalization**. Corollary C.5.17 proves that if $\mathcal{M}$ is a simplicial model category, then these functors are left and right Quillen.

We frequently invoke Lemma C.5.20 in conjunction with the following result to show that various quasi-categories or Kan complexes are equivalent. The geometric idea is as follows: when $X^\bullet \in \mathcal{M}^{\mathbb{\Delta}}$ is a cosimplicial object in $\mathcal{M}$, then for any object $A \in \mathcal{M}$ we may define a simplicial set $\mathcal{M}(X^\bullet, A)$ in which the n-simplices are maps $X^n \to A$ in $\mathcal{M}$. We think of the cosimplicial object $X^\bullet$ as determining the "shape" of the simplices in $\mathcal{M}(X^\bullet, A)$. The following lemma gives us a tool for proving that the simplicial sets constructed via different shapes are equivalent.

C.5.22. Lemma. *Suppose $\mathcal{M}$ is a model category, $\alpha^\bullet \colon X^\bullet \to Y^\bullet \in \mathcal{M}^{\mathbb{\Delta}}$ is a map of cosimplicial objects, and A is a fibrant object in $\mathcal{M}$. On applying the representable functor $\mathcal{M}(-, A) \colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}et$ we obtain a map of quasi-categories*

$$\mathcal{M}(\alpha^\bullet, A) \colon \mathcal{M}(Y^\bullet, A) \to \mathcal{M}(X^\bullet, A)$$

which is:

- *(i) a trivial fibration whenever $\alpha^\bullet$ is a Reedy trivial cofibration in $\mathcal{M}^{\mathbb{\Delta}}$, and*
- *(ii) an equivalence of quasi-categories whenever $\alpha^\bullet$ is a pointwise weak equivalence and $X^\bullet$ and $Y^\bullet$ are Reedy cofibrant.*

Proof. The first statement is a special case of Theorem C.5.15 as we now explain. In the terminology of Definition C.2.10, Lemma C.2.12 shows that for any category with products

$$\mathcal{M}(-,-) \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et$$

is a *right Leibniz bifunctor* relative to the (monomorphism, epimorphism) weak factorization system on $\mathcal{S}et$ and any weak factorization system $(\mathcal{L}, \mathscr{R})$ on $\mathcal{M}$. Essentially what this means is that for any map $\ell \colon K \to L$ in $\mathcal{L}$ and any map $r \colon R \to S$ in $\mathscr{R}$ the function

$$\mathcal{M}(L, R) \xrightarrow{(-\circ \ell, r\circ -)} \mathcal{M}(K, R) \underset{\mathcal{M}(K,S)}{\times} \mathcal{M}(L, S)$$

is an epimorphism; this is another encoding of the lifting property $\ell \boxslash r$ of maps in the left class against maps in the right class.

Now since $\mathbb{\Delta}$ is a strict Reedy category, Theorem C.5.15 tells us that that the corresponding bifunctor

$$\mathcal{M}(-,-) \colon (\mathcal{M}^{\mathbb{\Delta}})^{\mathrm{op}} \times \mathcal{M} \to \mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}}$$

is a right Leibniz bifunctor relative to the (trivial cofibration, fibration) weak factorization system on $\mathcal{M}$, the (Reedy trivial cofibration, Reedy trivial fibration) weak factorization system on $\mathcal{M}^{\mathbb{\Delta}}$, and the (Reedy monomorphism, Reedy epimorphism) weak factorization system on $\mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}}$; by Lemma C.5.9, this latter weak factorization system coincides with the familiar (monomorphism, trivial fibration) weak factorization system. In particular, this bifunctor carries a Reedy trivial cofibration $\alpha^\bullet \colon X^\bullet \to Y^\bullet$ and a fibration $! \colon A \to 1$ to a trivial fibration of simplicial sets, exactly as claimed.

The second statement follows from the first by Ken Brown's Lemma C.1.10. □

C.5.23. Remark. Lemma C.5.22 can be understood as a more primative result in abstract homotopy theory, which does not require a full model structure on $\mathcal{M}$. This will be convenient for its applications in the proofs of Propositions 6.5.3 and 6.5.9. Those categories do admit the desired model structures, but this way we don't need to fully construct them.

In Lemma C.5.22 and the results on which its proof replies, the model category $\mathcal{M}$ may be replaced by a bicomplete category which has:

(i) a class of weak equivalences $\mathcal{W}$ satisfying the 2-of-3 property,
(ii) a weak factorization system $(\mathcal{C}, \mathcal{F}_t)$, with the class $\mathcal{F}_t$ contained among the weak equivalences,
(iii) so that the class $\mathcal{C} \cap \mathcal{W}$ is closed under coproduct, pushout, and sequential composition.

The notion of Reedy cofibrant and Reedy trivial cofibration can be defined relative to the classes $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$. Then for any object A which admits extensions along maps in $\mathcal{C} \cap \mathcal{W}$, the functor $\mathcal{M}(-, A) \colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}et$ carries maps in $\mathcal{C} \cap \mathcal{W}$ to epimorphisms. Since the Reedy trivial cofibrations in $\mathcal{M}^{\Delta}$ are built as cell complexes from maps in $\mathcal{C} \cap \mathcal{W}$, axiom (iii) and the construction in the proof of Theorem C.5.15 shows that $\mathcal{M}(-, A) \colon (\mathcal{M}^{\Delta})^{\mathrm{op}} \to s\mathcal{S}et$ carries Reedy trivial cofibrations to trivial fibrations of simplicial sets. This proves the first statement of Lemma C.5.22.

For the second statement, we revisit the proof of Lemma C.1.10. By factoring the map $(f, \mathrm{id}) \colon X^\bullet + Y^\bullet \to Y^\bullet$ using the Reedy weak factorization system built from (ii) we obtain a factorization of the pointwise weak equivalence $\alpha \colon X^\bullet \to Y^\bullet$ as displayed

$$\begin{tikzcd} & Z^\bullet \arrow[dr, two heads, "q"', "\sim"] & \\ X^\bullet \arrow[ur, tail, "u", "\sim"'] \arrow[rr, "\sim", "f"'] & & Y^\bullet \arrow[ul, tail, bend right, "v"', "\sim"] \end{tikzcd}$$

where the maps u and v are Reedy cofibrations and pointwise weak equivalences and v is a section of q. Axioms (i) and (iii) give us the result proven in Lemma C.5.13, that the maps u and v are Reedy trivial cofibrations. Now the 2-of-3 property assumed in (i) is enough to achieve the conclusion in the second statement of Lemma C.5.22, that $- \circ \alpha \colon \mathcal{M}(Y^\bullet, A) \xrightarrow{\sim} \mathcal{M}(X^\bullet, A)$ is an equivalence of quasi-categories.

Exercises.

C.5.i. Exercise. Prove Lemma C.5.1.

C.5.ii. Exercise ([**79**, 14.3.7]). Prove Lemma C.5.19.

C.5.iii. Exercise ([**79**, 14.3.8]). Prove the relative analog of Lemma C.5.20: if X and Y are both unaugmentable cosimplicial objects in a presheaf category, then any pointwise monomorphism $X \to Y$ is also a Reedy monomorphism, that is, its relative latching maps are monic.

Appendix of Concrete Calculations

APPENDIX D

The combinatorics of (marked) simplicial sets

In this appendix we explore the combinatorics of simplicial sets, proving results alluded to in Chapters 1 and 4. Certain of these results, namely those involving isomorphisms in quasi-categories, are more easily proved in the closely related category of "marked" simplicial sets, where the quasi-categories are identified with those marked simplicial sets that are *1-complicial*. Because the corresponding *n-complicial sets* provide one of the families of examples of ∞-cosmoi appearing in Appendix E, we provide full details of the necessary combinatorial results in that more general context.

D.1. Complicial sets

When a quasi-category is regarded as an $(\infty, 1)$-category, its vertices play the role of the objects and its edges represent the morphisms, with the degenerate edge at a vertex representing its identity. The n-simplices then witness n-ary composition relations. When a complicial set is regarded as an (∞, ∞)-category, its n-simplices must play a dual role: both serving as witnesses for lower-dimensional composition relations and representing a priori non-invertible n-dimensional cells in their own right. To disambiguate between these two interpretations, certain positively-dimensional simplices in a complicial set are *marked* as "thin," indicating that they should be interpreted as "equivalences" witnessing a weak composition relation between their boundary faces. Thus the ambient category in which complicial sets are defined is not the category of ordinary simplicial sets but a closely related category of *marked simplicial sets*[1] that we now introduce.

D.1.1. Definition (marked simplicial sets). A **marked simplicial set** is a simplicial set with a designated subset of **marked** or **thin** positive-dimensional simplices that includes all degenerate simplicies. A map of marked simplicial sets is a simplicial map that preserves marked simplices.

D.1.2. Definition (minimal and maximal marking). The category $s\mathcal{S}et^+$ of marked simplicial sets is equipped with an evident forgetful functor to $s\mathcal{S}et$ admitting both left and right adjoints:

$$\begin{tikzcd} s\mathcal{S}et^+ \arrow[r, "U" description] & s\mathcal{S}et \arrow[l, bend right=50, "(-)^\flat"'] \arrow[l, bend left=20, "(-)^\sharp" description] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=10, "\perp"] \end{tikzcd} \qquad s\mathcal{S}et^+ \xrightarrow[\perp]{\text{core}} s\mathcal{S}et$$

The left adjoint $(-)^\flat$ defines the **minimal marking** of a simplicial set, in which only the degeneracies are marked, while the right adjoint $(-)^\sharp$ defines the **maximal marking**, with all simplices marked. This

[1] In the original sources [**101**, **102**], marked simplicial sets are called *stratified simplicial sets*. To avoid confusing with the increasingly prominent unrelated notion of stratified spaces, we have elected to change the name. Lurie [**61**] uses the term *marked simplicial sets* for a special case of the more general notion we presently introduce.

functor has a further right adjoint, which takes a marked simplicial set to its **core**, the simplicial set with the same vertices comprised of those marked simplices all of whose faces are also marked.

On various occasions, it is convenient to identify $s\mathcal{S}et$ with either of the fully faithful embeddings into $s\mathcal{S}et^+$ just introduced. Unless otherwise specified, the default convention is to identify simplicial sets with their minimal markings. In particular, with this convention, we are free to regard the standard simplices and their subspaces as minimally marked simplicial sets.

To succinctly introduce other marked simplicial sets, the following terminology will be convenient:

D.1.3. Definition. An inclusion $U \hookrightarrow V$ of marked simplicial sets is:

- **regular**, denoted $U \hookrightarrow_r V$, if thin simplices in U are created in V and
- **entire**, denoted $U \hookrightarrow_e V$, if the map is an isomorphism (or more commonly the identity) on underlying simplicial sets, in which case the only difference between U and V is that V has more marked simplices.

Lemma C.5.9 extends to marked simplicial sets as follows:

D.1.4. Lemma. *The momomorphisms in $s\mathcal{S}et^+$ are cellularly generated by*

$$\{\partial\Delta[n] \hookrightarrow_r \Delta[n]\}_{n\geq 0} \cup \{\Delta[n] \hookrightarrow_e \Delta[n]_t\}_{n\geq 1}$$

where the top-dimensional simplex of $\Delta[n]_t$ is marked.

Proof. Exercise D.1.i. □

Let $t\mathbf{\Delta} \hookrightarrow s\mathcal{S}et^+$ denote the full subcategory spanned by the minimally marked simplices $\mathbf{\Delta}[n]$, for $n \geq 0$, together with the marked simplices $\mathbf{\Delta}[n]_t$, for $n \geq 1$. It can be built from the simplex category $\mathbf{\Delta}$ of Notation 1.1.1 by:

- adjoining objects $[n]_t$ for $n \geq 1$,
- adjoining maps $\phi \colon [n] \to [n]_t$ for $n \geq 1$ and $\zeta^i \colon [n+1]_t \to [n]$ for $n \geq 0$ and $0 \leq i \leq n$, and
- imposing relations $\zeta^i\phi = \sigma^i$ and $\sigma^i\zeta^{j+1} = \sigma^j\zeta^i$ for $i \leq j$.[2]

For a marked simplicial set X, the maps $\mathbf{\Delta}[n] \to X$ and $\mathbf{\Delta}[n]_t \to X$ respectively parametrize n-simplices in X and marked n-simplices in X. This defines a canonical embedding $s\mathcal{S}et^+ \hookrightarrow \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$, which is easily seen to be fully faithful. Moreover:

D.1.5. Proposition. *There is a reflective fully faithful embedding*

$$\begin{tikzcd} s\mathcal{S}et^+ \arrow[r, hook, "\perp"'] & \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}} \arrow[l, bend right, dashed] \end{tikzcd}$$

whose

(i) essential image consists of those presheaves F for which the component maps $-\circ\phi \colon F_{[n]_t} \to F_{[n]}$ are monomorphisms, and

(ii) left adjoint is constructed by replacing the set $F_{[n]_t}$ with the image of the map $-\circ\phi \colon F_{[n]_t} \to F_{[n]}$.

Consequently, $s\mathcal{S}et^+$ is a locally finitely presentable category, and in particular is complete and cocomplete, with limits constructed pointwise as presheaves in $\mathcal{S}et$ and with colimits constructed by applying the reflector to the pointwise colimit of presheaves.

[2]Viktoriya Ozornova and Martina Rovelli pointed out to us that this last family of relations was omitted from the original source [**96**] but should have been included. A corrected definition appears in [**68**, 1.1].

Put in more elementary terms, limits and colimits of marked simplicial sets are created by the underlying simplicial set functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$. A simplex in the limit is marked if and only if *each* of its components, defined by composing with the legs of the limit cone, are marked simplices. A simplex in a colimit is marked if *any* of its lifts along any leg of the colimit cone are marked simplices. The reflection in (ii) is a sort of "propositional truncation," remembering which simplices should be marked while forgetting the data that indicates why.

PROOF. The right action of the operators in $t\mathbf{\Delta}$ on a presheaf $F \in \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$ gives the sets of elements of F the structure of a marked simplicial set with the exception of one condition: namely that the marked n-simplices form a subset of the n-simplices. This explains the condition appearing in (i) and the construction appearing in (ii). It follows that marked simplicial sets are the category of models for a finite limit sketch, and hence form a locally finitely presentable category. Any reflective full subcategory of a complete and cocomplete category inherits limits in the manner constructed in the statement; see eg [**80**, 4.5.15]. □

A marked simplicial set is a simplicial set with enough structure to talk about composition of simplices in all dimensions. A complicial set is a marked simplicial set in which composites exist and in which thin witnesses to composition compose to define thin simplices, an associativity condition that will ultimately imply that thin simplices are equivalences in a sense that will be made explicit in §D.7. The following form of the definition of a (nee. *weak*) *complicial set*, due to Verity [**102**], modifies an earlier equivalent presentation due to Street [**91**]. Verity's modification focuses on a particular set of *k-admissible n-simplices*, which are thin n-simplices that exhibit their kth face as a composite of their $(k+1)$th and $(k-1)$th faces, in the case where $0 < k < n$. In the case $k = 0$ or $k = n$, a k-admissible n-simplex witnesses an equivalence between the first or last pair of faces, respectively.

D.1.6. DEFINITION (k-admissible n-simplex). For $n \geq 1$ and $0 \leq k \leq n$, the **k-admissible n-simplex** $\Delta^k[n]$ is the entire superset of the standard n-simplex with certain additional faces marked thin: a non-degenerate m-simplex in $\Delta^k[n]$ is thin if and only if it contains all of the vertices $\{k-1, k, k+1\} \cap [n]$. Thin faces include in particular:

- the top dimensional n-simplex
- all codimension-one faces except for the $(k-1)$th, kth, and $(k+1)$th
- the 2-simplex spanned by $[k-1, k, k+1]$ when $0 \leq k \leq n$ or the edge spanned by $[k-1, k, k+1] \cap [n]$ when $k = 0$ or $k = n$.

When drawing pictures of marked simplicial sets, we use the symbol "$\simeq$" to decorate marked simplices and "$\sim$" to decorate marked edges. Our diagrams will also adopt a convention for the direction of the cells inhabiting an unmarked n-simplex. Following the combinatorics introduced by Street in his "Algebra of oriented simplexes" [**91**], we regard an n-simplex as an n-cell from the pasted composite of its odd-numbered faces to the pasted composite of its even-numbered faces.[3] Note this is compatible with the convention already in use for depicting a 1-simplex in a simplicial set as an arrow from its 1st face (the 0th vertex) to its 0th face (the 1st vertex).

D.1.7. EXAMPLE (admissible simplices in low dimensions).

[3]More exactly, Street defines a strict n-category $\mathcal{O}_n$, which he refers to as the n-th *oriental* to be the free strict n-category generated by an n-simplex and its faces, with the orientation conventions described here.

(i) For $n = 1$, both admissible simplices $\Delta^0[1]$ and $\Delta^1[1]$ equal the thin 1-simplex $\Delta[1]_t = \mathbf{\Delta}[1]^\sharp$. A map $\mathbf{\Delta}[1]^\sharp \to A$ is interpreted as defining an equivalence between the two vertices in its image.

(ii) For $n = 2$, the admissible simplex $\Delta^1[2] = \Delta[2]_t$, the thin 2-simplex. A map $\Delta^1[2] \to A$ is interpreted as specifying that the image of the $\{02\}$-edge is a composite of the images of the $\{01\}$- and $\{12\}$-edges.

By contrast, $\Delta^0[2]$ and $\Delta^2[2]$ each have a marked edge, as well as a marked 2-simplex as indicated by the diagrams:

$$\Delta^0[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur, "\sim"] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \Delta^1[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \Delta^2[2] \coloneqq \begin{tikzcd} & 1 \arrow[dr, "\sim"] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

A map $\Delta^0[2] \to A$ witnesses a homotopy between the image of the $\{12\}$ edge and the image of the $\{02\}$ edge.

(iii) For $n = 3$, the admissible simplices $\Delta^1[3]$ and $\Delta^2[3]$ have their 3rd and 0th faces marked, respectively, as well as the top dimensional 3-simplex, with no other non-degenerate faces marked. We choose to draw admissible 3-simplices in such a way that allows us to see all of their codimension-one faces:

$$\Delta^2[3] \coloneqq \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & 2 \arrow[dr, "k"] \arrow[d, phantom, "\beta\Uparrow"] & \\ 0 \arrow[ur, "f"] \arrow[urr, "h"'] \arrow[rrr, "\ell"'] & {} & {} & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[drr, "kg"'] \arrow[d, phantom, "\Uparrow\alpha*\beta"] & 2 \arrow[dr, "k"] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur, "f"] \arrow[rrr, "\ell"'] & {} & {} & 3 \end{tikzcd}$$

Here we've labelled faces in such a way that allows us to better describe the interpretation of a map $\Delta^2[3] \to A$. Its 0th face, which is itself an admissible simplex $\Delta^1[2]$, witnesses that the edge $\{13\}$ is a composite of the edges $\{12\}$ and $\{23\}$. Note that because the 0th face is thin, its 1st edge is interpreted as a composite kg of g and k, which is needed so that the boundary of the 2-cell appearing in the 2nd face agrees with the boundary of the pasted composite of β and α. On account of this boundary condition and the thin 3-simplex, we interpret the 2nd face as the pasted composite of the 1st and 3rd faces depicted on the right.

The admissible simplex $\Delta^0[3]$ has both its 2nd and 3rd faces marked, as well as the top dimensional 3-simplex, and the edge $\{01\}$. Dually, $\Delta^3[3]$ has its 0th and 1st faces marked, as well as the top dimensional 3-simplex, and the edge $\{23\}$.

$$\Delta^0[3] \coloneqq \begin{tikzcd} & 1 \arrow[r, "fe^{-1}"] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\alpha\Uparrow"] & \\ 0 \arrow[ur, "e", "\sim"'] \arrow[urr, "f"'] \arrow[rrr, "h"'] & {} & {} & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "fe^{-1}"] \arrow[drr, "he^{-1}"'] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\alpha e^{-1} \Uparrow"] & \\ 0 \arrow[ur, "e", "\sim"'] \arrow[rrr, "h"'] & {} & {} & 3 \end{tikzcd}$$

A map $\Delta^0[3] \to A$ is interpreted as witnessing a homotopy between the pair of non-thin 2-simplices occupying the 0th and 1st faces, respectively.

D.1.8. Remark (the odd dual). Recall that the **opposite** of a simplicial set X is the simplicial set obtained by reindexing along the involution $(-)^{\mathrm{op}} \colon \mathbf{\Delta} \to \mathbf{\Delta}$ that reverses the ordering in each ordinal.

This operation may be extended to marked simplicial sets in a natural way: marking an n-simplex in X^{op} just when the corresponding n-simplex in X is marked. Note, however, that under Street's interpretation of an n-simplex as encoding an n-dimensional morphism from the composite of its odd $(n-1)$-dimensional faces to the composite of its even $(n-1)$-dimensional faces, this operation doesn't simply "reverse the direction of all the cells" in a marked simplicial set. Rather, it reverses the direction of all the simplices in the *odd* dimensional cells, while preserving the direction in all of the even dimensional cells. This is because the odd-dimensional simplices have an even number of codimension-one faces, so reversing their labeling exchanges the groups of "odd" and "even" faces, while even-dimensional simplices have an odd number of codimension-one faces, so the "odd" and "even" groupings are preserved. Thus, we refer to the vertex reordering construction as defining the **odd dual** of a marked simplicial set.

D.1.9. DEFINITION. A **complicial set** is a marked simplicial set that admits extensions along the **elementary anodyne extensions**, which are cellularly generated by the following two sets of maps:

(i) The **complicial horn extensions**

$$\Lambda^k[n] \hookrightarrow_r \Delta^k[n] \quad \text{for} \quad n \geq 1,\ 0 \leq k \leq n$$

are regular inclusions of k-**admissible** n-**horns**. An inner admissible n-horn parametrizes "admissible composition" of a pair of $(n-1)$-simplices. The extension defines a composite $(n-1)$-simplex together with a thin n-simplex witness.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta^k[n] \arrow[ur, dashed] \end{tikzcd} \tag{D.1.10}$$

(ii) The **complicial thinness extensions**

$$\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'' \quad \text{for} \quad n \geq 2,\ 0 \leq k \leq n,$$

are entire inclusions of two entire supersets of $\Delta^k[n]$. The stratified simplicial set $\Delta^k[n]'$ is obtained from $\Delta^k[n]$ by also marking the $(k-1)$th and $(k+1)$th faces, while $\Delta^k[n]''$ has all codimension-one faces marked. This extension problem

$$\begin{tikzcd} \Delta^k[n]' \arrow[r] \arrow[d, hook] & A \\ \Delta^k[n]'' \arrow[ur, dashed] \end{tikzcd} \tag{D.1.11}$$

demands that whenever the composable pair of simplices in an admissible horn are thin, then so is any composite.

D.1.12. EXAMPLE (complicial horn extensions). For $\Lambda^2[4] \hookrightarrow_r \Delta^2[4]$ the non-thin codimension-one faces in the horn define the two 3-simplices with a common face displayed on the left, while their composite is a 3-simplex as displayed on the right.

$$\begin{tikzcd}
& 1 \arrow[d] \arrow[ddr] & & & & 1 \arrow[dr] \arrow[dd] & \\
& 2 \arrow[dr] \arrow[dd] & & & 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 4 \\
0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & 4 & & & 3 \arrow[ur] & \\
& 3 \arrow[ur] & & & & &
\end{tikzcd}$$

It makes sense to interpret the right hand simplex, the 2nd face of the 2-admissible 4-simplex, as a composite of the 3rd and 1st faces because the 2-simplex

$$\begin{tikzcd}
& 2 \arrow[dr] \arrow[d, phantom, "\wr"] & \\
1 \arrow[ur] \arrow[rr] & {} & 3
\end{tikzcd}$$

is thin.

D.1.13. Definition. A map of stratified simplicial sets is an **elementary anodyne extension** if it is a sequential composite of pushouts of coproducts of the complicial horn extensions

$$\Lambda^k[n] \hookrightarrow_r \Delta^k[n] \quad \text{for} \quad n \geq 1,\ 0 \leq k \leq n$$

and the complicial thinness extensions

$$\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'' \quad \text{for} \quad n \geq 2,\ 0 \leq k \leq n.$$

For instance, by a mild extension of the argument that solves Exercise 1.1.iv:

D.1.14. Lemma. *Either injection $\mathbb{2}^\sharp \hookrightarrow \mathbb{I}^\sharp$ of the marked 1-simplex into the maximally marked isomorphism is an elementary anodyne extension, as is the injection $\mathbb{1} \hookrightarrow \mathbb{I}^\sharp$.*

Proof. Exercise D.1.iv. ☐

D.1.15. Definition. A map of marked simplicial sets is a **complicial isofibration** if it has the right lifting property with respect to the elementary anodyne extensions and if its domain and codomain are complicial sets.

By Exercise D.1.v, a marked map between complicial sets is a complicial isofibration if and only if it lifts against the complicial horn extensions.

The original meaning of "complicial sets" referred to a particular variety that we now call *strict*.

D.1.16. Definition. A **strict complicial set** is a marked simplicial set that admits unique extensions along the elementary anodyne extensions (D.1.10) and (D.1.11).

In the manuscript [**101**], Verity proves that the strict complicial sets are precisely those marked simplicial sets that are Street nerves of strict ω-categories, resolving a conjecture of Street and Roberts. In this manuscript, we will primarily utilize marked simplicial sets to streamline the proofs of results concerning isomorphisms in quasi-categories of equivalences between them. We will discuss this topic more explicitly in §D.4 and §D.7 after developing some combinatorial constructions we will require in the interim. We conclude this section with one final definition.

D.1.17. DEFINITION. A **marked homotopy** between a pair of maps $f, g\colon X \rightrightarrows Y$ is given by a map $\alpha\colon X\times\Delta[1]^\sharp \to Y$ that restricts along the endpoint inclusions $X+X \hookrightarrow_r X\times\Delta[1]^\sharp$ to the maps f and g, respectively. In the case where X and Y are minimally marked simplicial sets, a map $X\times\Delta[1] \to Y$ extends to a map $X \times \Delta[1]^\sharp \to Y$ just when for each 0-simplex $x \in X$, the 1-simplex $(x\cdot\sigma^0, \mathrm{id}_{[1]}) \in X \times \Delta[1]^\sharp$ maps to a degenerate and hence marked 1-simplex of Y.[4]

A **marked homotopy equivalence** consists of:

- a pair of marked maps $f\colon X \to Y$ and $g\colon Y \to X$ and
- a pair of marked homotopies $\alpha\colon X \times \Delta[1]^\sharp \to X$ and $\beta\colon Y \to \Delta[1]^\sharp \to Y$ between id_X and gf and fg and id_Y, respectively.

This definition works best when X and Y are complicial sets, in which case marked homotopies can be reserved and composed. Even when this is not the case, we permit ourselves the reverse the direction of the marked homotopies that comprise a marked homotopy equivalence without comment.

D.1.18. DIGRESSION (the Verity model structure for complicial sets). The category of marked simplicial sets bears a cartesian closed, cofibrantly generated model structure whose fibrant objects are exactly the complicial sets and whose cofibrations are the monomorphisms [**102**, §6.2-4]. The fibrations and weak equivalences between fibrant objects are precisely the classes of complicial isofibrations and marked homotopy equivalences defined above. In the following sections, we verify many of these properties for the category of fibrant objects directly, leaving only the verification of the actual model structure, which follows from Jeff Smith's theorem, to the literature.

Exercises.

D.1.i. EXERCISE. Prove Lemma D.1.4.

D.1.ii. EXERCISE. Prove that a maximally marked simplicial set defines a complicial set if and only if the underlying simplicial set is a Kan complex.

D.1.iii. EXERCISE. Prove that the underlying simplicial set of any complicial set in which all simplices of dimension greater than 1 are marked is a quasi-category.[5]

D.1.iv. EXERCISE. Prove Lemma D.1.14.

D.1.v. EXERCISE. Let $f\colon A \to B$ be any map of stratified simplicial sets whose domain A is a complicial set. Prove that f has the (unique) right lifting property against the complicial thinness extensions.

D.2. The join and slice constructions

In this section, we revisit Joyal's join and slice constructions in considerably more detail than given in Definition 4.2.4 and discuss their extension to marked simplicial sets. We prove that Leibniz joins of monomorphisms and various classes of anodyne maps again define monomorphisms of the same type. The combinatorics are slightly easier if we work with *augmented* simplicial sets in place of ordinary simplicial sets, an approach that follows the original definition of the simplicial join by Ehlers and Porter [**36**].

D.2.1. DEFINITION (ordinal sum). The algebraists' skeletal category $\mathbf{\Delta}_+$ of finite ordinals and order preserving maps — with objects $[n] = \{0 \le 1 \le \cdots \le n\}$ and $[-1] = \varnothing$ — supports a strict (non-symmetric) monoidal structure $(\mathbf{\Delta}_+, \oplus, [-1])$ in which $\oplus$ denotes the **ordinal sum** given

[4] The cartesian product of marked simplicial sets is described in more detail in Proposition D.3.4.

[5] A converse of sorts to this result will appear in Theorem D.4.13.

- for objects $[n], [m] \in \mathbf{\Delta}_+$ by $[n] \oplus [m] := [n+1+m]$,
- for arrows $\alpha\colon [n] \to [n'], \beta\colon [m] \to [m']$ by $\alpha \oplus \beta\colon [n+1+m] \to [n'+1+m']$ defined by

$$\alpha \oplus \beta(i) = \begin{cases} \alpha(i) & \text{if } i \leq n, \\ \beta(i-n-1)+n'+1 & \text{otherwise.} \end{cases}$$

By Day convolution [29], the join bifunctor $\oplus\colon \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \to \mathbf{\Delta}_+$ extends to a (non-symmetric) monoidal closed structure

$$(s\mathcal{S}et_+, \star, \Delta[-1], \mathsf{dec}_l, \mathsf{dec}_r)$$

on the category of augmented simplicial sets $s\mathcal{S}et_+ := \mathcal{S}et^{\mathbf{\Delta}_+^{\mathrm{op}}}$.

D.2.2. Definition (join of augmented simplicial sets). The **join** $X \star Y$ of augmented simplicial sets X and Y may be described explicitly as follows:

- it has simplices pairs $(x, y) \in (X \star Y)_{r+1+s}$ with $x \in X_r$, $y \in Y_s$,
- if (x, y) is a simplex of $X \star Y$ with $x \in X_r$ and $y \in Y_s$ and $\alpha\colon [n] \to [r+1+s]$ is a simplicial operator in $\mathbf{\Delta}_+$, then α may be uniquely decomposed as $\alpha = \alpha_1 \star \alpha_2$ with $\alpha_1\colon [n_1] \to [r]$ and $\alpha_2\colon [n_2] \to [s]$, and we define $(x, y) \cdot \alpha := (x \cdot \alpha_1, y \cdot \alpha_2)$.

Note by construction that $\Delta[n] \star \Delta[m] \cong \Delta[n+1+m]$, since $[n] \oplus [m] = [n+1+m]$.

D.2.3. Definition (décalage of augmented simplicial sets). The closures dec_l and dec_r, known as the **left and right décalage constructions**, respectively, are defined as the parametrized right adjoints to the join: to fix handedness, we denote the adjoints, for each augmented simplicial set X, by

$$\begin{tikzcd} s\mathcal{S}et_+ \arrow[r, bend left, "X\star-"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_+ \arrow[l, bend left, "\mathsf{dec}_l(X,-)"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} s\mathcal{S}et_+ \arrow[r, bend left, "-\star X"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et_+ \arrow[l, bend left, "\mathsf{dec}_r(X,-)"] \end{tikzcd}$$

Collectively, the bifunctors

$$s\mathcal{S}et_+ \times s\mathcal{S}et_+ \xrightarrow{-\star-} s\mathcal{S}et_+\,, \quad s\mathcal{S}et_+^{\mathrm{op}} \times s\mathcal{S}et_+ \xrightarrow{\mathsf{dec}_l(-,-)}: s\mathcal{S}et_+\,, \quad s\mathcal{S}et_+^{\mathrm{op}} \times s\mathcal{S}et_+ \xrightarrow{\mathsf{dec}_r(-,-)} s\mathcal{S}et_+$$

define a two-variable adjunction and as such preserve limits and colimits in each variable separately.

D.2.4. Observation (simplicial sets vs augmented simplicial sets). The evident functor that forgets the augmentation $U\colon s\mathcal{S}et_+ \to s\mathcal{S}et$ admits both left and right adjoints

$$\begin{tikzcd} s\mathcal{S}et_+ \arrow[r, "U" description] & s\mathcal{S}et \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "*"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

where the left adjoint augments a simplicial set X with its set of path components $\pi_0 X$, defined by the coequalizer

$$X_1 \underset{\delta^0}{\overset{\delta^1}{\rightrightarrows}} X_0 \twoheadrightarrow \pi_0 X$$

and the right adjoint augments a simplicial set "trivially" by adding a single -1-simplex. The unit of $\pi_0 \dashv U$ and counit of $U \dashv *$ are both isomorphisms; hence either adjoint defines a fully faithful embedding $s\mathcal{S}et \hookrightarrow s\mathcal{S}et_+$.

Any augmented simplicial set is canonically a coproduct of its terminally-augmented "components":

D.2.5. LEMMA. *Let X be an augmented simplicial set and let $x \in X_{-1}$.*

(i) The subset comprised of those simplices in any dimension whose -1-simplex face is x forms a terminally augmented simplicial subset of X.
(ii) The disjoint union of these components is isomorphic to X.

Thus, any augmented simplicial set X admits a canonical decomposition into a coproduct of terminally augmented simplicial sets indexed by the set X_{-1}.

PROOF. Exercise D.2.i. □

D.2.6. DEFINITION (join of simplicial sets). By convention, the **join** of a pair of simplicial sets is defined to be the underlying simplicial set of the trivially augmented simplicial sets. Thus, the join bifunctor is the composite

$$\begin{tikzcd} s\mathcal{S}et \times s\mathcal{S}et \arrow[r, dashed, "- \star -"] \arrow[d, hook, "* \times *"'] & s\mathcal{S}et \\ s\mathcal{S}et_+ \times s\mathcal{S}et_+ \arrow[r, "- \star -"] & s\mathcal{S}et_+ \arrow[u, "U"'] \end{tikzcd}$$

Explicitly, n-simplices of $X \star Y$ are pairs comprised of a j-simplex of X and a k-simplex of Y where $j + k = n - 1$, where in the case $j = -1$ such a "pair" consists of a single n-simplex of Y and in the case $k = -1$ such a "pair" consists of a single n-simplex of X. This recovers the definition given in 4.2.4.

As observed in Definition 4.2.4, the join of simplicial sets X and Y admits canonical embeddings

$$X \hookrightarrow X \star Y \hookleftarrow Y$$

which can be understood as the maps obtained by applying $X \star -$ or $- \star Y$ respectively to the maps $\Delta[-1] \to Y$ and $\Delta[-1] \to X$ in $s\mathcal{S}et_+$ that pick out the unique -1-simplices in the trivial augmentations.

D.2.7. LEMMA.

(i) The join bifunctor $- \star -\colon s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$ preserves connected colimits in each variable separately.
(ii) For any simplicial set X, the join functors

$$s\mathcal{S}et \xrightarrow{X \star -} {}^{X/}s\mathcal{S}et \qquad \text{and} \qquad s\mathcal{S}et \xrightarrow{- \star X} {}^{X/}s\mathcal{S}et$$

preserve all colimits.

PROOF. In Definition D.2.6, the join of simplicial sets is defined as the composite of three functors, two of which possess right adjoints and hence preserve all colimits. The third functor $*\colon s\mathcal{S}et \to s\mathcal{S}et_+$ does not possess a right adjoint but nevertheless preserves *connected* components as is clear from the following definition: an indexing 1-category J is connected just when the colimit of the constant J-indexed diagram valued at the singleton set is a singleton. This proves (i).

Now the forgetful functor ${}^{X/}s\mathcal{S}et \to s\mathcal{S}et$ strictly creates connected colimits [**80**, 3.3.8], so the join functors of (ii) preserve connected components. Arbitrary colimits may be built from connected colimits and coproducts, so to prove (ii) it remains only to argue that these functors preserve coproducts. While $X \star (\coprod_i Y_i) \ncong \coprod_i (X \star Y_i)$ if the latter coproduct is interpreted in $s\mathcal{S}et$, it can be directly

verified that $X \star (\coprod_i Y_i)$ is the quotient of $\coprod_i (X \star Y_i)$ modulo the identification of the image of each inclusion $X \hookrightarrow X \star Y_i$ with a single copy of X, which is exactly the construction of the coproduct in the category ${}^{X/}s\mathcal{S}et$. □

D.2.8. Definition (slice of simplicial sets). The categories $s\mathcal{S}et$ and ${}^{X/}s\mathcal{S}et$ are locally presentable, so the cocontinuous functors of Lemma D.2.7(ii) have right adjoints

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "X\star-"] \arrow[r, phantom, "\perp"] & {}^{X/}s\mathcal{S}et \arrow[l, bend left, "-/\_"] \end{tikzcd} \qquad \begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star X"] \arrow[r, phantom, "\perp"] & {}^{X/}s\mathcal{S}et \arrow[l, bend left, "\_/-"] \end{tikzcd}$$

the values of which at $f\colon X \to A$ define Joyal's sliced simplicial sets ${}^{f/}A$ and $A_{/f}$ characterized by the universal properties

$$\left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow {}^{f/}A \right\} \quad \text{and} \quad \left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] \\ Y \star X \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow A_{/f} \right\}.$$

See Proposition 4.2.5.

We think of the slice ${}^{f/}A$ as being the simplicial set of **cones under the diagram** f and we think of the dual slice $A_{/f}$ as being the simplicial set of **cones over the diagram** f. This terminology will be reconciled with the terminology of Definition 4.2.1 in Proposition D.6.5. We can also recover these sliced simplicial sets from the décalage construction of Definition D.2.3 via Lemma D.2.5:

D.2.9. Lemma. *For any map $f\colon X \to A$ of simplicial sets, the simplicial sets ${}^{f/}A$ and $A_{/f}$ are the terminally augmented components of $\mathsf{dec}_l(X, A)$ and $\mathsf{dec}_r(X, A)$, respectively, indexed by the -1-simplex $f\colon X \to A$.*

Proof. We identify simplicial sets X and A with their terminally augmented simplicial sets. Recall that $\Delta[-1]$ is the monoidal unit for the join bifunctor on $s\mathcal{S}et_+$. Consequently, by adjunction, maps $\Delta[-1] \to \mathsf{dec}_l(X, A)$ or $\Delta[-1] \to \mathsf{dec}_r(X, A)$ correspond to maps $X \to A$. For another terminally augmented simplicial set Y, transposing across the adjunction of Definition D.2.3 provides a correspondence:

$$\left\{ \begin{tikzcd} & X \star \Delta[-1] \cong X \arrow[dl, hook', "X\star !"'] \arrow[dr, "f"] \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} & \Delta[-1] \arrow[dl, hook', "!"'] \arrow[dr, "f"] \\ Y \arrow[rr] & & \mathsf{dec}_l(X, A) \end{tikzcd} \right\}$$

which shows that the simplicial subset of $\mathsf{dec}_l(X, A)$ comprised of those simplices whose -1-simplex face is f has the universal property that defines ${}^{f/}A$. The dual argument proves that the simplicial subset of $\mathsf{dec}_r(X, A)$ comprised of those simplices whose -1-simplex face is f has the universal property that defines $A_{/f}$. In other words, these décalages admit the following canonical decompositions as disjoint unions of (terminally augmented) slices:

$$\mathsf{dec}_r(X, A) = \coprod_{f\colon X \to A} A_{/f} \qquad \mathsf{dec}_l(X, A) = \coprod_{f\colon X \to A} {}^{f/}A$$ □

D.2.10. Definition ((left-/right-/inner-)anodyne extensions).

- The set of **horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$ cellularly generates the **anodyne extensions**.

- The set of **left horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k < n$ cellularly generates the **left anodyne extensions**.
- The set of **right horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 < k \leq n$ cellularly generates the **right anodyne extensions**.
- The set of **inner horn inclusions** $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 2$ and $0 < k < n$ cellularly generates the **inner anodyne extensions**.

By an easy direct calculation:

D.2.11. Lemma. *The Leibniz join of a horn inclusion and a boundary inclusion is isomorphic to a single horn inclusion:*

$$(\Lambda^k[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \Lambda^k[n+1+m] \hookrightarrow \Delta[n+1+m]$$
$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\Lambda^k[m] \hookrightarrow \Delta[m]) \cong \Lambda^{n+k+1}[n+1+m] \hookrightarrow \Delta[n+1+m]$$

Proof. Since the join bifunctor is the Day convolution of the ordinal sum $[n]\oplus[m] = [n+1+m]$, $\Delta[n] \star \Delta[m] \cong \Delta[n+1+m]$. The domain of the first Leibniz tensor is the simplicial set

$$\Lambda^k[n] \star \Delta[m] \underset{\Lambda^k[n]\star\partial\Delta[m]}{\cup} \Delta[n] \star \partial\Delta[m].$$

From the explicit description of the simplices contained in the join given in Definition D.2.2, is plainly a simplicial subset of $\Delta[n+1+m]$. Since $\partial\Delta[m]$ contains all the codimension-one faces of $\Delta[m]$, the $\Delta[n] \star \partial\Delta[m]$ component contains the jth face of $\Delta[n+1+m]$ for each index $j > n$. Similarly, since $\Lambda^k[n]$ contains all codimension-one faces of $\Delta[n]$-except one, the $\Lambda^k[n] \star \Delta[m]$ component contains the ith face of $\Delta[n+1+m]$ for each index $i \leq n$ except $i = k$. Thus, we see that only the kth face and the $n+1+m$-simplex are missing, which allows us to identify the domain of this Leibniz join with the horn $\Lambda^k[n+1+m]$. The dual argument proves the second claimed isomorphism. □

Lemma D.2.11 reveals that the Leibniz join of an inner horn with a boundary inclusion is an inner horn. Consequently:

D.2.12. Corollary. *If $f \colon X \to A$ is any simplicial map and A is a quasi-category, then ${}^{f/}A$ and $A_{/f}$ are quasi-categories.*

Proof. By Proposition C.2.9(vi) and Lemma C.5.9, the Leibniz join of an inner horn inclusion with a monomorphism gives a map in the class that is cellularly generated by the inner horn inclusions. By Proposition C.2.9(ii), for any quasi-category A and simplicial set X, the augmented simplicial sets $\mathrm{dec}_l(X, A)$ and $\mathrm{dec}_r(X, A)$ then admit fillers for all inner horns, considered as trivially augmented simplicial sets. By Lemma D.2.9, it follows that for any simplicial map $f \colon X \to A$, the slices ${}^{f/}A$ and $A_{/f}$ are quasi-categories. □

Our next aim is to prove that the slice quasi-categories are equivalent to the quasi-categories of cones introduced in §4.2. As sketched there, this result hinges on a suitable equivalence between the join construction and the so-called "fat join" construction of Definition 4.2.2, which we now extend to augmented simplicial sets. Recall from Lemma D.2.5 that an augmented simplicial set X canonically decomposes into a coproduct $X \cong \sqcup_{i\in X_{-1}} X^i$ of terminally augmented simplicial sets, indexed by the set of -1-simplices.

D.2.13. Definition (fat join and décalage of augmented simplicial sets). For augmented simplicial sets $X \cong \sqcup_{i \in X_{-1}} X^i$ and $Y \cong \sqcup_{j \in Y_{-1}} Y^j$ their **fat join** is constructed by the pushout:

$$\begin{tikzcd} (X \times Y) \sqcup (X \times Y) \arrow[r, "\pi_X \sqcup \pi_Y"] \arrow[d, hook] & (X \times Y_{-1}) \sqcup (X_{-1} \times Y) \arrow[d, hook] \\ X \times \mathbb{2} \times Y \arrow[r] & X \diamond Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \text{i.e.,} \qquad X \diamond Y \coloneqq \bigsqcup_{(i,j) \in X_{-1} \times Y_{-1}} X^i \diamond Y^j$$

where $X^i \diamond Y^j$ is the terminally augmented simplicial set corresponding to the fat join of Definition 4.2.2. This construction is arranged so that the bifunctor $- \diamond - \colon s\mathcal{S}et_+ \times s\mathcal{S}et_+ \to s\mathcal{S}et_+$ preserves all colimits in each variable, not simply the connected ones preserved by the bifunctor $- \diamond - \colon s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$.

Explicitly, the set of n-simplices $(X \diamond Y)_n$ is the quotient of the set $X_n \times \mathbf{\Delta}([n],[1]) \times Y_n$ of n-simplices of $X \times \mathbb{2} \times Y$ modulo the relation that identifies triples

- $(x, 0, y) \sim (x, 0, y')$ where $0\colon [n] \to [1]$ is the constant operator and y and y' are in the same component of $Y \cong \sqcup_{j \in Y_{-1}} Y^j$ and
- $(x, 1, y) \sim (x', 1, y)$ where $1\colon [n] \to [1]$ is the constant operator and x and x' are in the same component of $X \cong \sqcup_{i \in X_{-1}} X^i$.

By cocontinuity and the adjoint functor theorems, the fat join bifunctor on $s\mathcal{S}et_+$ has both left and right closures $\mathsf{fatdec}_l(X, A)$ and $\mathsf{fatdec}_r(X, A)$, called **left and right fat décalage** respectively, which notation we fix by declaring that if X is an augmented simplicial set then $X \diamond - \dashv \mathsf{fatdec}_l(X, -)$ and $- \diamond X \dashv \mathsf{fatdec}_r(X, -)$.

There is a canonical comparison map from the fat join to the join as previewed in the discussion surrounding Proposition 4.2.7.

D.2.14. Lemma. *There exists a canonical map of augmented simplicial sets*

$$s^{X,Y} \colon X \diamond Y \to X \star Y$$

natural in X and Y that in particular defines a natural transformation

$$s^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \quad \in s\mathcal{S}et_+^{\mathbf{\Delta}_+ \times \mathbf{\Delta}_+}$$

that is an isomorphism if n or m equals -1 and otherwise arises as a quotient of the map $\bar{s}^{n,m} \colon \Delta[n] \times \Delta[1] \times \Delta[m] \to \Delta[n+1+m]$ defined by its order-preserving action on vertices:

$$\bar{s}^{n,m}(i,j,k) = \begin{cases} i & \text{if } j = 0, \quad \text{and} \\ k + n + 1 & \text{if } j = 1. \end{cases} \tag{D.2.15}$$

Proof. Identifying the set X_{-1} with the augmented simplicial set $\sqcup_{X_{-1}} \Delta[-1]$, the Yoneda lemma supplies a canonical map $X_{-1} \to X$ of augmented simplicial sets, which gives rise to a canonical map

$$\begin{tikzcd} (X \times Y) \sqcup (X \times Y) \arrow[r, two heads, "\pi \sqcup \pi"] \arrow[d, hook] & (X \times Y_{-1}) \sqcup (X_{-1} \times Y) \arrow[d, hook] \arrow[dr, hook] & \\ X \times \mathbb{2} \times Y \arrow[r, two heads] \arrow[dr, two heads, "\pi"'] & X \diamond Y \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "s^{X,Y}"] \arrow[d, two heads] & X \star Y \arrow[dl, two heads] \\ & \mathbb{2} & \end{tikzcd} \tag{D.2.16}$$

Note that the fibers of both $X \diamond Y$ and $X \star Y$ over the endpoints $0, 1$ of $\mathbb{2}$ are $X \times Y_{-1}$ and $X_{-1} \times Y$ respectively, and the map $s^{X,Y}$ commutes with the inclusions of these fibers.

The map $s^{X,Y}$ is defined on those n-simplices over $X \diamond Y$ that map surjectively onto $\mathbb{2}$ by sending a triple $(\sigma \in X_n, \alpha\colon [n] \twoheadrightarrow [1], \tau \in Y_n)$ representing an n-simplex of $X \diamond Y$ to the pair $(\sigma|_{\{0,\ldots,k\}} \in X_k, \tau|_{\{k+1,\ldots,n\}} \in Y_{n-k-1})$ representing an n-simplex of $X \star Y$, where $k \in [n]$ is the maximal vertex in $\alpha^{-1}(0)$.

In the case of a pair of standard simplices, we can be even more explicit. Note that $X \diamond \Delta[-1] \cong X \star \Delta[-1] \cong X \cong \Delta[-1] \diamond X \cong \Delta[-1] \star X$, so it suffices to consider $n, m \geq 0$. In this case, we may describe $s^{n,m}$ as the quotient of a map $\bar{s}^{n,m}\colon \Delta[n] \times \Delta[1] \times \Delta[m] \to \Delta[n+1+m]$ defined by its order-preserving action on vertices, as described in the statement. Note this definition takes simplices related under the congruence described in Definition D.2.13 to the same simplex and thus induces a unique map $s^{n,m}\colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m]$ on the quotient simplicial set, which can easily be checked to coincide with the definition given in (D.2.16). ☐

We now prove that the natural comparison between the fat join of simplices and the join of simplices defines a component of a marked homotopy equivalence, in the sense introduced in Definition D.1.17.

D.2.17. Proposition. *For each $n, m \geq -1$, the map of augmented simplicial sets*

$$s^{n,m}\colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \quad \in s\mathcal{S}et_+^{\mathbf{\Delta}_+ \times \mathbf{\Delta}_+}$$

is a marked homotopy retract equivalence which is an isomorphism in the case $n = -1$ or $m = -1$.

Proof. To define a section and left homotopy inverse to $s^{n,m}$, we consider a map $\bar{t}^{n,m}\colon \Delta[n+1+m] \to \Delta[n] \times \Delta[1] \times \Delta[m]$ determined by its order-preserving action on vertices:

$$\bar{t}^{n,m}(i) = \begin{cases} (i, 0, 0) & \text{if } i \leq n \quad \text{and} \\ (n, 1, i-n-1) & \text{if } i > n \end{cases}$$

and note immediately that that $\bar{s}^{n,m} \circ \bar{t}^{n,m} = \mathrm{id}$. The obverse composite $\bar{t}^{n,m} \circ \bar{s}^{n,m}$ is given by the explicit formula:

$$(\bar{t}^{n,m} \circ \bar{s}^{n,m})(i, j, k) = \begin{cases} (i, 0, 0) & \text{if } j = 0 \quad \text{and} \\ (n, 1, k) & \text{if } j = 1. \end{cases}$$

Now we may define a related order preserving endo-map $\bar{u}^{n,m}$ on $[n] \times [1] \times [m]$ by

$$\bar{u}^{n,m}(i, j, k) = \begin{cases} (i, 0, 0) & \text{if } j = 0 \text{ and} \\ (i, 1, k) & \text{if } j = 1 \end{cases}$$

which is of interest because in the pointwise ordering on such maps we have $\bar{u}^{n,m} \leq \bar{t}^{n,m} \circ \bar{s}^{n,m}$ and $\bar{u}^{n,m} \leq \mathrm{id}_{[n]\times[1]\times[m]}$, representing simplicial homotopies:

$$\begin{tikzcd}
\Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dr, "\delta^0"] \arrow[ddrr, bend right, "\bar{t}^{n,m}\circ\bar{s}^{n,m}"'] & & \Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dl, "\delta^1"'] \arrow[dd, "\bar{u}^{n,m}"] \arrow[dr, "\delta^1"] & & \Delta[n]\times\Delta[1]\times\Delta[m] \arrow[dl, "\delta^0"'] \arrow[ddll, bend left, "\mathrm{id}_{\Delta[n]\times\Delta[1]\times\Delta[m]}"] \\
& (\Delta[n]\times\Delta[1]\times\Delta[m])\times\Delta[1] \arrow[dr, "\bar{h}^{n,m}"'] & & (\Delta[n]\times\Delta[1]\times\Delta[m])\times\Delta[1] \arrow[dl, "\bar{k}^{n,m}"] & \\
& & \Delta[n]\times\Delta[1]\times\Delta[m] & &
\end{tikzcd}$$

Passing to quotients under the congruence defined in Definition D.2.13, these maps induce simplicial maps $t^{n,m}\colon \Delta[n] \star \Delta[m] \to \Delta[n] \diamond \Delta[m]$ and $u^{n,m}\colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \diamond \Delta[m]$ so that $s^{n,m} \circ t^{n,m} = \mathrm{id}_{\Delta[n]\star\Delta[m]}$, and simplicial homotopies $h^{n,m}$ and $k^{n,m}$ that assemble into a diagram:

$$\begin{tikzcd}
\Delta[n]\diamond\Delta[m] \arrow[dr, "\delta^0"] \arrow[ddrr, bend right, "t^{n,m}\circ s^{n,m}"'] & & \Delta[n]\diamond\Delta[m] \arrow[dl, "\delta^1"'] \arrow[dd, "u^{n,m}"] \arrow[dr, "\delta^1"] & & \Delta[n]\diamond\Delta[m] \arrow[dl, "\delta^0"'] \arrow[ddll, bend left, "\mathrm{id}_{\Delta[n]\diamond\Delta[m]}"] \\
& (\Delta[n]\diamond\Delta[m])\times\Delta[1] \arrow[dr, "h^{n,m}"'] & & (\Delta[n]\diamond\Delta[m])\times\Delta[1] \arrow[dl, "k^{n,m}"] & \\
& & \Delta[n]\diamond\Delta[m] & &
\end{tikzcd}$$

To see that the maps $h^{n,m}$ and $k^{n,m}$ define marked homotopies, Definition D.1.17 tells us that we must verify that for each 0-simplex $[i,j,k]_\sim$ of $\Delta[n] \diamond \Delta[m]$ the 1-simplex $([i,j,k]_\sim \cdot \sigma^0, \mathrm{id}_{[1]})$ of $(\Delta[n] \diamond \Delta[n]) \times \Delta[1]$ is mapped by $h^{n,m}$ and $k^{n,m}$ to degenerate, and thus marked, simplices in $\Delta[n] \diamond \Delta[m]$. We argue by cases in the index j. If $j = 0$, then $\bar{u}^{n,m}(i,0,k) = (i,0,0) \sim (i,0,k) = \bar{t}^{n,m} \circ \bar{s}^{n,m}(i,0,k)$, so the components of both $h^{n,m}$ and $k^{n,m}$ are degenerate. If $j = 1$, then $\bar{u}^{n,m}(i,1,k) = (i,1,k) \sim (n,i,k) = \bar{t}^{n,m} \circ \bar{s}^{n,m}(i,1,k)$, so again the components of both $h^{n,m}$ and $k^{n,m}$ are degenerate. Thus, $s^{n,m}$ extends to a marked homotopy retract equivalence with equivalence inverse $t^{n,m}$. ☐

The marked simplicial homotopy equivalence constructed in Proposition D.2.17 witnesses a pointwise weak equivalence in a suitable sense between two diagrams in $s\mathcal{S}et_+^{\Delta_+\times\Delta_+}$ considered in Lemma D.2.14. This is the key ingredient in the proof that the canonical map of augmented simplicial sets $s^{X,Y}\colon X \diamond Y \to X \star Y$ is also a weak equivalence in a suitable sense, but this conclusion will require an exploration of the connection between the homotopy theory of marked simplicial sets and the homotopy theory of quasi-categories. We make this connection in §D.4 and then resume this line of reasoning in §D.6.

We close this section with one final result, a marked analogue of Lemma D.2.11. The join construction of Definition D.2.6 is extended to marked simplicial sets in [**101**].

D.2.18. Definition (join of marked simplicial sets). The simplicial join lifts to a join bifunctor

$$s\mathcal{S}et^+ \times s\mathcal{S}et^+ \xrightarrow{\star} s\mathcal{S}et^+$$

in which a simplex $\Delta[n] \to A \star B$, with components $\Delta[k] \to A$ and $\Delta[n-k-1] \to B$, is marked in $A \star B$ if and only if at least one of the simplices in A or B is marked.

D.2.19. LEMMA.

(i) The Leibniz join of a complicial horn inclusion and a boundary inclusion is isomorphic to a single complicial horn inclusion:

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \cong \Lambda^k[n+1+m] \hookrightarrow_r \Delta^k[n+1+m]$$

unless $k = n$, in which case the Leibniz join $(\Lambda^n[n] \hookrightarrow_r \Delta^n[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m])$ is a pushout of the complicial horn inclusion $\Lambda^n[n+1+m] \hookrightarrow_r \Delta^n[n+1+m]$.

(ii) The Leibniz joins

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \cong \Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$$

$$(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \cong \Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$$

unless $k = n$, in which case the Leibniz joins are instead pushout of $\Delta^k[n+1+m]' \hookrightarrow_e \Delta^k[n+1+m]''$, while $(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$ is the identity map.

The last case of this result is generalized in Exercise D.2.iii.

PROOF. The underlying map of simplicial sets in (i) is identified in Lemma D.2.11, so it remains only to consider the markings. Similarly, in each of the three Leibniz joins considered in (ii), the underlying map of simplicial sets is a Leibniz join of a monomorphism with an identity, and is thus an identity, so it remains only to consider the markings in the resulting entire inclusion. Since a simplex in a join of marked simplicial sets is marked if and only if either of its components are, this description lends itself readily to a case analysis. We leave the details to Exercise D.2.ii or to [**102**, 38]. □

The slice construction of Definition D.2.8 also extends to marked simplicial sets. We adopt new notation for this construction because it is not always the case that the underlying simplicial set of the marked join is the slice of corresponding map of underlying simplicial sets.

D.2.20. LEMMA. *For any map of marked simplicial sets $f \colon X \to A$, there exist marked simplicial sets ${}^{f/\!/}A$ and $A_{/\!/f}$ characterized by the universal properties*

$$\left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] & \\ X \star Y \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow {}^{f/\!/}A \right\} \quad \text{and} \quad \left\{ \begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] & \\ Y \star X \arrow[rr] & & A \end{tikzcd} \right\} \cong \left\{ Y \longrightarrow A_{/\!/f} \right\}.$$

PROOF. Exercise D.2.iv. □

Exercises.

D.2.i. EXERCISE. Prove Lemma D.2.5.

D.2.ii. EXERCISE ([**102**, 38]). Finish the proof of Lemma D.2.19.

D.2.iii. EXERCISE. Generalize the last case of Lemma D.2.19(ii) by showing that the Leibniz join of two entire inclusions is an identity.

D.2.iv. EXERCISE. Prove Lemma D.2.20.

D.3. Leibniz stability of cartesian products

We now turn our attention to analogous Leibniz constructions defined with respect to the cartesian product, which in the context of marked simplicial sets is called the *Gray tensor product* in **[102]** for reasons we shall explain. To warm up, let us demonstrate the following basic result about the combinatorics of simplicial sets.

D.3.1. Lemma. *For any $n, m \geq 0$, the Leibniz product of the simplex boundary inclusions*

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \partial\Delta[n] \times \Delta[m] \underset{\partial\Delta[n]\times\partial\Delta[m]}{\cup} \Delta[n] \times \partial\Delta[m] \hookrightarrow \Delta[n] \times \Delta[m]$$

is a monomorphism.

Proof. Products, pushouts, and monomorphisms in $s\mathcal{S}et$ are determined pointwise in the category of sets, so this result follows from the fact that for monomorphisms $S \hookrightarrow T$ and $U \hookrightarrow V$ of sets, the Leibniz product

$$(S \hookrightarrow T) \mathbin{\widehat{\times}} (U \hookrightarrow V) \cong (S \times V \underset{S\times U}{\cup} T \times U \hookrightarrow T \times V)$$

is a monomorphism, which is clear by inspection. □

D.3.2. Remark (why "Leibniz"). The domain of the inclusion of Lemma D.3.1 defines the boundary of the prism $\partial(\Delta[n] \times \Delta[m])$ and the identification

$$\partial(\Delta[n] \times \Delta[m]) \cong \partial\Delta[n] \times \Delta[m] \underset{\partial\Delta[n]\times\partial\Delta[m]}{\cup} \Delta[n] \times \partial\Delta[m]$$

is formally similar to various identities that are commonly called "the Leibniz rule."

In this section, we prove a number of combinatorial results along the lines of Lemma D.3.1, each of which have corollaries along the following lines:

D.3.3. Corollary.

- *(i) The Leibniz product of any pair of monomorphisms of simplicial sets is again a monomorphism.*
- *(ii) The Leibniz exponential of any trivial fibration of simplicial sets with any monomorphism is again a trivial fibration.*

Proof. By Lemma C.5.9 any monomorphism of simplicial sets can be built as a cell complex from the simplex boundary inclusions. By Proposition C.2.9(vi) it follows that pushout products of monomorphisms can be built as a cell complex out of pushout products of simplex boundary inclusions. Lemma D.3.1 verifies that these maps are monomorphisms, and since the monomorphisms are closed under coproduct, pushout, and sequential composition, the result of (i) follows.

For (ii), recall from Definition 1.1.24 that the trivial fibrations are characterized by the right lifting property against the monomorphisms. To see that the Leibniz exponential $\widehat{\{i,p\}}$ of a trivial fibration p with a monomorphism i is a trivial fibration, it suffices to show that for any other monomorphism j, j has the left lifting property with respect to $\widehat{\{i,p\}}$. By Proposition C.2.9(ii) this lifting problem transposes to one between $i \mathbin{\widehat{\times}} j$ and p, which can be solved by (i). □

D.3.4. Proposition. *The category of marked simplicial sets is cartesian closed with*

- *cartesian product, called the **Gray tensor product**, giving by marking every simplex in the cartesian product of the underlying simplicial sets just when both components are marked simplices*

- *internal hom Y^X defined to be the simplicial set whose n-simplices $\sigma \in Y^X$ are marked simplicial maps $\sigma\colon X \times \Delta[n] \to Y$ that are marked just when σ extends to a marked simplicial map*

$$\begin{tikzcd} X \times \Delta[n] \arrow[r, "\sigma"] \arrow[d, hook, "e"'] & Y \\ X \times \Delta[n]_t \arrow[ur, dashed] \end{tikzcd}$$

Proof. It's clear from the universal property of the product and its closure that the cartesian product and internal hom must be defined in this way if these objects exist. To verify the adjunction, recall from Proposition D.1.5 that marked simplicial sets embed as a reflexive full subcategory of a category of presheaves $\mathcal{S}et^{t\Delta^{op}}$, and this embedding preserves the products and internal homs as just defined. Now we conclude that these define the functors of a two-variable adjunction on $s\mathcal{S}et^+$ by restricting the corresponding natural isomorphisms from $\mathcal{S}et^{t\Delta^{op}}$. □

The result of Lemma D.3.1 can easily be extended to the marked context. Our proof will use a simple observation that will also be deployed elsewhere.

D.3.5. Lemma. *The Leibniz product of any map of marked simplicial sets with an entire inclusion is an entire inclusion.*

Proof. By definition the Leibniz product of maps $X \to Y$ and $U \hookrightarrow_e V$ is the induced map of marked simplicial sets

$$\begin{tikzcd} X \times U \arrow[r, hook, "e"'] \arrow[d] & X \times V \arrow[d] \arrow[ddr, bend left] \\ Y \times U \arrow[r, hook, "e"'] \arrow[drr, bend right, "e"'] & \bullet \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner" very near start] \\ & & Y \times V \end{tikzcd}$$

Note that the forgetful functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$ preserves products and pushouts, and recall that a map of marked simplicial sets is entire just when the underlying map is an isomorphism. Since the product of a simplicial set with an isomorphism is an isomorphism, the maps $X \times U \hookrightarrow_e X \times V$ and $Y \times U \hookrightarrow_e Y \times V$ are entire. Since pushouts of isomorphisms are isomorphisms, it follows that the remaining horizontal map is also entire. Finally, since isomorphisms obey the 2-of-3 property, the Leibniz product map must also be entire. □

D.3.6. Lemma. *For any $n, m \geq 0$, the Leibniz products*

$$(\partial\Delta[n] \hookrightarrow_r \Delta[n]) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \qquad \text{and} \qquad (\Delta[n] \hookrightarrow_e \Delta[n]_t) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$

are monomorphisms of marked simplicial sets.

Proof. By Lemma D.3.5, both Leibniz products are entire maps of marked simplicial sets. Note that every entire map of marked simplicial sets is a monomorphism. □

D.3.7. Lemma. *The Leibniz product of two regular inclusions is again a regular inclusion.*

Proof. By Corollary D.3.3, the underlying simplicial set of the Leibniz product of two regular inclusions $A \hookrightarrow_r B$ and $C \hookrightarrow_r D$ is the monomorphism

$$A \times D \underset{A\times C}{\cup} B \times C \hookrightarrow B \times D.$$

Note, in particular, that the inclusions of the components $A \times D$ and $B \times C$ jointly surject onto the domain of this map. Our task is to show that any n-simplex in $A \times D \underset{A\times C}{\cup} B \times C$ that is marked in $B \times D$ is marked in $A \times D \underset{A\times C}{\cup} B \times C$. We argue by cases and assume without loss of generality that the n-simplex is in the image of the inclusion from $B \times C$. In this case, the regularity of the map $B \times C \hookrightarrow_r B \times D$ implies that it is marked in $A \times D \underset{A\times C}{\cup} B \times C$ as claimed. □

It follows easily from Lemma D.3.5 that the Leibniz product of a regular inclusion with a non-invertible entire inclusion is entire but not regular; see Exercise D.3.i.

Considerably harder is to show the "Leibniz stability" of the class of marked anodyne extensions with the class of marked monomorphisms. We prove a slightly more specific result that also describes the cases of "inner," "left," or "right" marked anodyne extensions, which restrict the inequalities $0 \leq k \leq n$ to $0 < k < n$, $k < n$, or $0 < k$, respectively.

D.3.8. Proposition. *For $n \geq 1$, $m \geq 0$, and $0 < k \leq n$ each of the Leibniz products*

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \qquad (\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$
$$(\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow_r \Delta[m]) \qquad (\Delta^k[n]' \hookrightarrow_e \Delta^k[n]'') \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow_e \Delta[m]_t)$$

is a right marked anodyne extension and is an inner marked anodyne extension if $k < n$.

Note that, on account of the symmetry of the cartesian product—in contrast to the antisymmetry $(A \star B)^{\mathrm{op}} \cong B^{\mathrm{op}} \star A^{\mathrm{op}}$ of the join—whether the horn inclusion or the simplex boundary inclusion appears on the left or right is immaterial. The proof of this result will require some special notation to describe the cartesian product of simplices.

D.3.9. Digression (on shuffles). By the Yoneda lemma, an r-simplex in $\Delta[n]$ may be represented by a map $i\colon \Delta[r] \to \Delta[n]$. Since $\Delta[n]$ is the nerve of the poset $[n]$, an r-simplex may equally be represented by the ordered sequence of vertices $i_0 \leq \cdots \leq i_r \in [n]$ appearing in its image.

By the universal property of the product, an r-simplex in $\Delta[n]\times\Delta[m]$ is given by a pair $i\colon \Delta[r] \to \Delta[n]$ and $j\colon \Delta[r] \to \Delta[m]$ comprised of an r-simplex in $\Delta[n]$ together with an r-simplex in $\Delta[m]$. Since $\Delta[n]\times\Delta[m]$ is the nerve of the poset $[n]\times[m]$, such simplices correspond bijectively to ordered sequences of pairs

$$(i_0, j_0) \leq (i_1, j_1) \leq \cdots \leq (i_r, j_r) \tag{D.3.10}$$

with each $i_s \in [n]$ and each $j_t \in [m]$.

The non-degenerate $n+m$-simplices of the simplicial set $\Delta[n] \times \Delta[m]$ are called **shuffles**. An $n+m$-simplex (i, j) defines a shuffle just when $i_t + j_t = t$ for all $t \in [n+m]$. If the objects of $[n]\times[m]$ are arranged in a rectangle grid, the shuffles are those maximal-length non-degenerate paths that start from $(0, 0)$ and end with (n, m), by taking steps which change exactly one coordinate at a time.

The first case of the following proof is an adaptation of an argument of Dugger and Spivak [**31**, A.1] to the marked context.

Proof. By Lemma D.3.7, the Leibniz product $(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \widehat{\times} (\partial\Delta[m] \hookrightarrow_r \Delta[m])$ is the regular inclusion

$$\Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m] \hookrightarrow_r \Delta^k[n] \times \Delta[m].$$

A non-degenerate r-simplex (D.3.10) of $\Delta^k[n]\times\Delta[m]$ is missing from the domain of the Leibniz product inclusion just when

- its component $\{i_0, \ldots, i_r\} \supset [n]\backslash\{k\}$ and
- its component $\{j_0, \ldots, j_r\} \supset [m]$.

We will filter this inclusion as a sequence of regular inclusions

$$\Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m] =: Y^{-1} \hookrightarrow_r Y^0 \hookrightarrow_r \cdots \hookrightarrow_r Y^m = \Delta^k[n] \times \Delta[m]$$

and argue that each $Y^t \hookrightarrow Y^{t+1}$ is left or inner marked anodyne, as appropriate.

Starting from $Y^{-1} := \Lambda^k[n] \times \Delta[m] \underset{\Lambda^k[n]\times\partial\Delta[m]}{\cup} \Delta^k[n] \times \partial\Delta[m]$, we define Y^t to be the smallest regular simplicial subset of $\Delta^k[n] \times \Delta[m]$ containing Y^{t-1} together with every simplex (D.3.10) that contains the vertex (k, t). Since every missing simplex is a face of a simplex that contains one of the vertices $(k, 0), \ldots, (k, m)$, it is clear from this description that $Y^m = \Delta^k[n] \times \Delta[m]$.

It remains only to analyze the regular inclusions $Y^{t-1} \hookrightarrow_r Y^t$, which we do by producing another filtration

$$Y^{t-1} = Y^{t,n-1} \hookrightarrow_r Y^{t,n} \hookrightarrow_r \cdots \hookrightarrow_r Y^{t,n+m} = Y^t.$$

Note that every simplex of $\Delta^k[n] \times \Delta[m]$ that contains the vertex (k, t) and has dimension $n - 1$ or less is contained in Y^{-1}, so the simplices containing the vertex (k, t) that are attached to Y^{t-1} to form Y^t have dimensions between n and $n + m$. With this in mind, we define $Y^{t,r}$ to be the smallest regular simplicial subset of $\Delta^k[n] \times \Delta[m]$ containing $Y^{t,r-1}$ and all simplices of dimension r that contain the vertex (k, t). In particular, $Y^{t,n-1} = Y^{t-1}$ and $Y^{t,n+m} = Y^t$.

We now argue that each regular inclusion in this filtration is a pushout of a coproduct of complicial horn inclusions

$$\begin{tikzcd}
\coprod\limits_{\tau\in S^{t,r}} \Lambda^{\ell_\tau}[r] \arrow[r, hook, "r"'] \arrow[d] & \coprod\limits_{\tau\in S^{t,r}} \Delta^{\ell_\tau}[r] \arrow[d] \\
Y^{t,r-1} \arrow[r, hook, "r"'] & Y^{t,r} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

indexed by the set $S^{t,r}$ of r-simplices containing the vertex (k, t) and not already present in $Y^{t,r-1}$. Moreover, for each $\tau \in S^{t,r}$, we will see that that $0 < \ell_\tau$ and if $k < n$ then $\ell_\tau < r$. This will show that the Leibniz product is a right marked anodyne extension, which is an inner marked anodyne extension if $k < n$.

To see this, let $\tau \in S^{t,r}$ be the r-simplex

$$(i_0, j_0) \leq \cdots \leq (i_{\ell-1}, j_{\ell-1}) \leq (i_\ell, j_\ell) = (k, t) \leq \cdots \leq (i_r, j_r)$$

containing (k, t) as its ℓ_τth vertex. Since the set $\{i_0, \ldots, i_r\} \supset [n]$ and $0 < k$ we must also have $0 < \ell_\tau$, an if $k < n$, we must also have $\ell_\tau < r$. We will argue that:

- Each face of τ except the ℓ_τth is contained in $Y^{t,r-1}$.
- The ℓ_τth face of τ is not in $Y^{t,r-1}$.
- The r-simplex τ is a ℓ_τ-admissible simplex of $\Delta^k[n] \times \Delta[m]$.

Thus, the union of τ with $Y^{t,r-1}$ may be formed as a pushout of a complicial horn extension $\Lambda^{\ell_\tau}[r] \hookrightarrow_r \Delta^{\ell_\tau}[r]$ as claimed.

For the first item, note that each codimension-one face except for $\tau \cdot \delta^{\ell_\tau}$ has dimension $r-1$ and contains the vertex (k,t) and thus lies in $Y^{t,r-1}$ as claimed. To see that $Y^{t,r-1}$ does not also contain the face $\tau \cdot \delta^{\ell_\tau}$, we consider the vertex $(i_{\ell-1}, j_{\ell-1})$. If $i_{\ell-1} = k$ then by non-degeneracy, $j_{\ell-1} < t$, in which case we would have $\tau \in Y^{t-1}$, a contradiction. Thus $i_{\ell-1} < k$. Now if $i_{\ell-1} \leq k-2$, then we would have $\tau \in Y^{-1}$, again a contradiction. So it must be that $i_{\ell-1} = k-1$. Now if $j_{\ell-1} < t$, then τ would be a face of the $r+1$-dimensional simplex

$$(i_0, j_0) \leq \cdots \leq (i_{\ell-1}, j_{\ell-1}) \leq (k, t-1) \leq (i_\ell, j_\ell) = (k,t) \leq \cdots \leq (i_r, j_r)$$

which is contained in Y^{t-1}, a contradiction. So we conclude that $(i_{\ell-1}, j_{\ell-1}) = (k-1, t)$.

From this computation we see that the vertices of $\tau \cdot \delta^{\ell_\tau}$ satisfy $\{i_0, \ldots, i_{\ell-1} = k-1, i_{\ell+1}, \ldots, i_r\} \supset [n]\backslash\{k\}$ and $\{j_0, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_r\} \supset [m]$. Thus, $\tau \cdot \delta^{\ell_\tau}$ is not in Y^{-1}. Furthermore, $\tau \cdot \delta^{\ell_\tau}$ was not added in along the way to $Y^{t,r-1}$, since it is not a face of a simplex containing the vertex (k,s) for any $s < t$. This completes our second task.

We have shown that it is possible to attach τ to $Y^{t,r-1}$ along with its ℓ_τth face by filling a suitable horn. It remains only to argue that the horn $\Lambda^{\ell_\tau}[r] \to Y^{t,r-1}$ along which we are attaching τ is admissible. Since the inclusion $Y^{t,r-1} \hookrightarrow_r \Delta^k[n] \times \Delta[m]$ is regular it suffices to show that each simplex containing the vertices $(i_{\ell_\tau-1}, j_{\ell_\tau-1})$, $(i_{\ell_\tau}, j_{\ell_\tau})$, and $(i_{\ell_\tau+1}, j_{\ell_\tau+1})$ — or just the first two of these in the case $\ell_\tau = r$ — is marked in $\Delta^k[n] \times \Delta[m]$. We've seen above that $(i_{\ell_\tau-1}, j_{\ell_\tau-1}) = (k-1, t)$ and $(i_{\ell_\tau}, j_{\ell_\tau}) = (k,t)$. In the case $\ell_\tau < r$, it's easy to see that $i_{\ell_\tau+1}$ equals k or $k+1$. But now the component in $\Delta[m]$ of this simplex is degenerate, containing the sequence $t \leq t$, while the component in $\Delta^k[n]$ is either degenerate, contains the sequence $k-1 \leq k \leq k+1$, or contains the sequence $k-1 \leq k$ in the case $\ell_\tau = r$ in which case $k = n$. Thus both components are marked simplices, which means that their product is marked in $\Delta^k[n] \times \Delta[m]$ as required.

By Lemma D.3.5, the remaining three Leibniz products are entire inclusions, so all that is required is to verify that the additional markings present in the codomains are the results of complicial thinness extensions. We treat simultaneously the two cases involving Leibniz products

$$(\Delta^k[n+1]' \hookrightarrow_e \Delta^k[n+1]'') \mathbin{\widehat{\times}} (A \hookrightarrow B)$$

of a complicial horn inclusion with a marked monomorphism. The only marked simplex of $\Delta^k[n+1]''$ that is not marked in $\Delta^k[n+1]''$ is the face $\delta^k \colon \Delta[n]_t \to \Delta^k[n+1]''$, which implies that we have a pullback

$$\begin{tikzcd} \coprod_{\tau \in S} \Delta[n] \arrow[r] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^k[n+1]' \times B \underset{\Delta^k[n+1]' \times A}{\cup} \Delta^k[n+1]'' \times A \arrow[d, hook, "e"'] \\ \coprod_{\tau \in S} \Delta[n]_t \arrow[r, "(\delta^k, \tau)"] & \Delta^k[n+1]'' \times B \end{tikzcd}$$

where S is the set of marked n-simplices in B that are not present or not marked in A. We argue that for any marked n-simplex $\tau \in B$, the degenerate $n+1$-simplex $\tau\sigma^k$ admits the indicated markings:

$$\begin{tikzcd}
\Delta[n+1] \arrow[r, hook, "e"'] \arrow[d, "\sigma^{k-1}"'] & \Delta^{k-1}[n+1]'' \arrow[d, dashed] \\
\Delta[n]_t \arrow[r, "\tau"] & B
\end{tikzcd}$$

because the $k-1$th and kth faces equal τ, and so are marked, and any face that contains the vertices $k-1$ and k is degenerate and so is also marked; these conditions cover all of the required marked faces. Now it is clear that the pullback square above factors through the right-hand pushout diagram

$$\begin{tikzcd}
\coprod\limits_{\tau \in S} \Delta[n] \arrow[r] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod\limits_{\tau \in S} \Delta^k[n+1]' \arrow[r] \arrow[d, hook, "e"'] & \Delta^k[n+1]' \times B \underset{\Delta^k[n+1]' \times A}{\cup} \Delta^k[n+1]'' \times A \arrow[d, hook, "e"'] \\
\coprod\limits_{\tau \in S} \Delta[n]_t \arrow[r, "\delta^k"] & \coprod\limits_{\tau \in S} \Delta^k[n+1]'' \arrow[r, "(\mathrm{id}, \tau\sigma^{k-1})"] \arrow[ur, phantom, "\ulcorner", very near end] & \Delta^k[n+1]'' \times B
\end{tikzcd}$$

demonstrating that the Leibniz product inclusion is a pushout of coproducts of suitable elementary thinness extensions.

The final case of

$$(\Lambda^k[n] \hookrightarrow_r \Delta^k[n]) \widehat{\times} (\Delta[m] \hookrightarrow_e \Delta[m]_t) \cong \Lambda^k[n] \times \Delta[m]_t \underset{\Lambda^k[n] \times \Delta[m]}{\cup} \Delta^k[n] \times \Delta[m] \hookrightarrow_e \Delta^k[n] \times \Delta[m]_t$$

is again an entire inclusion. Since the only simplex that is marked in $\Delta[m]_t$ but not in $\Delta[m]$ is the top-dimensional m-simplex, the only simplices that are marked in the codomain but not in the domain are m-simplices $(\tau, \mathrm{id}) \colon \Delta[m]_t \to \Delta^k[n] \times \Delta[m]_t$ in which the image of τ either

- contins $[n]$ or
- contains $[n] \backslash \{k\}$ but not $\{k\}$ and is degenerate.

In particular, this Leibniz product inclusion is an identify if $m < n$. In the case $m = n$, there are $m+1$-simplices that are marked in $\Delta^k[m] \times \Delta[m]_t$ but not in the domain, corresponding to the m m-simplices that are degenerate on the kth face of $\Delta^k[m]$ and the top-dimensional m-simplex.

We will factor the inclusion as a finite composite of pushouts of coproducts of maps $\Delta^i[m+1]' \hookrightarrow_e \Delta^s[m+1]''$ for varying $0 < s \leq m+1$, where each $s < m+1$ if $k < n$. This will prove that this Leibniz product is a complicial thinness extension of the appropriate kind.

We can classify the missing marked simplices in terms of their component $\tau \colon \Delta[m]_t \to \Delta^k[n]$, which we may represent as a sequence $i_0, \ldots, i_m$ of vertices of $[n]$ that either contains $[n]$ or contains $[n] \backslash \{k\}$ and has repetitions. We partially order these simplices in decreasing order of the sum $\sum_{t=0}^m i_t$.[6]

For a simplex τ with maximal vertex sum $\sum_{t=0}^m i_t$ let s be minimal so that $i_s \geq k$; when $k = n$ it is possible that all $i_t < k = n$, which gives a second case that we will consider in a moment. Then we consider the $m+1$-simplex:

$$\Delta^s[m+1]'' \xrightarrow{(\tau \cdot \sigma^s, \sigma^{s-1})} \Delta^k[n] \times \Delta[m]_t.$$

[6]Here the vertex sum of an m-simplex τ is greater than the vertex sum of an m-simplex τ' if and only if τ has greater "depth" in the sense defined in [**102**, 68]. The induction of [**102**, §5.2] involves Leibniz products of inner or left horn inclusions and starts by considering simplices of lowest depth; ours involves an inner or right horn inclusion and starts by considering simplices of highest depth.

By construction, the $s+1$th face is marked in $\Delta^k[n]\times\Delta[m]$, while the $s-1$th face has strictly greater vertex sum, and so is marked by the inductive hypothesis. The faces containing the $s-1$, s, and $s+1$ vertices are all degenerate and thus marked. This proves that the face τ can be marked by forming an extension $\Delta^s[m+1]' \hookrightarrow_e \Delta^s[m+1]''$.

In the case where τ is a simplex where all $i_t < k = n$, then we consider the $m+1$-simplex

$$\Delta^{m+1}[m+1]'' \xrightarrow{(\chi,\sigma^m)} \Delta^n[n]\times\Delta[m]_t.$$

where $\chi\colon \Delta[m+1] \to \Delta^n[n]$ is the simplex spanned by the vertices $i_0, \ldots, i_m = n-1, n$. Here the mth face has strictly greater sum, and so is makred by the inductive hypothesis. The faces containing the vertices mth and $m+1$th vertices have a degenerate component in $\Delta[m]$ and have a component in $\Delta^n[n]$ that contains the last two vertices $n-1$ and n. Thus, all such simplices are marked. This proves that the face τ can be marked by forming an extension $\Delta^{m+1}[m+1]' \hookrightarrow_e \Delta^{m+1}[m+1]''$. ☐

Proposition D.3.8 of course implies its unmarked analogue, refining the result proven in Corollary D.3.3(i):

D.3.11. Corollary. *Let $i\colon A \hookrightarrow B$ and $j\colon K \hookrightarrow L$ be monomorphisms of simplicial sets. If either i or j is also anodyne (or, respectively, left-, right-, or inner-anodyne), then so is the pushout product*

$$A\times L \cup_{A\times K} B\times K \xhookrightarrow{i\hat{\times}j} B\times L$$

Proof. The forgetful functor $U\colon s\mathcal{S}et^+ \to s\mathcal{S}et$ preserves products and colimits and carries the (left/right/inner) marked anodyne extensions to the classes of anodyne extensions introduced in Definition D.2.10. Thus, this result follows immediately from the first case of Proposition D.3.8. ☐

Recall from Definition D.1.15 that a marked map between complicial sets is a **complicial isofibration** if it has the right lifting property with respect to the complicial horn inclusions and complicial thinness extensions of Definition D.1.9.

D.3.12. Corollary.

(i) For any quasi-category A and simplicial set X, A^X is again a quasi-category.
(ii) For any complicial set A and marked simplicial set X, A^X is again a complicial set.
(iii) For any complicial isofibration $p\colon E \twoheadrightarrow B$ and any monomorphism of marked simplicial sets $i\colon X \hookrightarrow Y$, the Leibniz exponential $\widehat{\{i,p\}}\colon E^Y \to E^X \underset{B^X}{\times} B^Y$ is a complicial isofibration.

Proof. The second statement is a special case of the third statement, which follows by transposing the result of Proposition D.3.8 across the Leibniz version of the two variable adjunction of Proposition D.3.4. See Proposition C.2.9. The first statement follows similarly by applying Corollary D.3.11 in place of Proposition D.3.8. ☐

We would like to prove the analogous statement to Corollary D.3.12(iii) for isofibrations between quasi-categories, which requires analogous statements to Proposition D.3.8 and Corollary D.3.11 analyzing Leibniz products of monomorphisms of simplicial sets with the inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$. We shall deduce this by considering the relationship between isomorphisms in quasi-categories and marked edges in complicial sets, which is the subject of the next sect ion.

Exercises.

D.3.i. Exercise. Prove that the Leibniz product of a regular inclusion with a non-invertible entire inclusion will be entire but not regular.

D.4. Isomorphisms and naturally marked quasi-categories

Our aim in this section is to explain the relevance of Proposition D.3.8 to the theory of quasi-categories. In particular, this will finally enable us to complete the combinatorial work required to supply proofs of the results stated in §1.1, which we dispense with in §D.5.

The connection between the theory of complicial sets and the theory of quasi-categories is established by the two main results proven in this section, Theorems D.4.13 and D.4.18, the first of which explains that any quasi-category can be equipped with a canonically-defined "natural" marking in such a way that it defines a 1-trivial complicial set. The markings on the 1-simplices cannot be arbitrarily assigned: every marked edge must be an equivalence in a sense that we now introduce:

D.4.1. Definition. A 1-simplex f in a marked simplicial set is an **equivalence** if there exist a pair of thin 2-simplices as displayed

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\wr"] & \\ w \arrow[ur] \arrow[rr, equal] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] \arrow[d, phantom, "\wr"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & z \end{tikzcd}$$

Note the notion of equivalence is defined relative to the choice of markings on the 2-simplices. A very similar notion is defined for the edges of a quasi-category in Definition 1.1.13 under the name "isomorphism."

D.4.2. Lemma. *Every marked edge in a complicial set is an equivalence.*

Proof. If f is a marked edge in any complicial set A, then the $\Lambda^2[2]$-horn with 0th face f and 1st face degenerate is admissible, so f has a right equivalence inverse. A dual construction involving a $\Lambda^0[2]$-horn shows that f has a left equivalence inverse:

$$\begin{tikzcd} & x \arrow[dr, "f", "\sim"'] \arrow[d, phantom, "\wr"] & \\ w \arrow[ur, dashed, "\sim"] \arrow[rr, equal] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, dashed, "\sim"] \arrow[d, phantom, "\wr"] & \\ x \arrow[ur, "f", "\sim"'] \arrow[rr, equal] & {} & z \end{tikzcd}$$

Note also that the elementary thinness extensions imply further than these one-sided inverses are also marked, so they admit further inverses of their own. ☐

This result suggests two ways to mark the edges in the nerve of a 1-category.

D.4.3. Lemma. *The nerve of a 1-category defines a complicial set by marking all simplices in dimension greater than one and then either defining:*

(i) the marked edges to be the identity arrows only or
(ii) the marked edges to be all isomorphisms.

Proof. Exercise D.4.i. ☐

This motivates a definition of the canonical marking of a quasi-category, which is called the "natural marking" in [**61**].

D.4.4. Definition. For any quasi-category A, its **natural marking** is defined by:

- marking all simplices in dimension greater than one

- marking exactly those edges that are isomorphisms, in the sense of Definition 1.1.13, or equivalently marking all those edges that are equivalences, in the sense of Definition D.4.1.

The natural marking for quasi-categories is convenient for stating and proving an important combinatorial result due to Joyal:

D.4.5. PROPOSITION. *Any naturally marked quasi-category A admits fillers for outer complicial horn inclusions for $n \geq 1$:*

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d, hook, "r"'] & A \\ \Delta^0[n] \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d, hook, "r"'] & A \\ \Delta^n[n] \arrow[ur, dashed] \end{tikzcd}$$

In the original [**48**], the result is stated without reference to markings as follows: a quasi-category admits fillers for **special outer horns**, left horns $\Lambda^0[n] \to A$ whose initial $\{01\}$-edge is mapped to an isomorphism in A and right horns $\Lambda^n[n] \to A$ whose final $\{n-1n\}$-edge is mapped to an isomorphism in A.

Many proofs of Proposition D.4.5 are possible; see [**31**, §B] or the original [**48**]. We choose to use a combinatorial result of Verity [**102**, §4.2] which uses an alternate (a posteriori equivalent) notion of homotopy coherent isomorphism, which a homotopy type theorist would recognize by the name "half adjoint equivalence."

D.4.6. DIGRESSION (subcomplexes of the coherent isomorphism). Recall the **coherent isomorphism** is the simplicial set $\mathbb{I}$ defined as the nerve of the free-living isomorphism. It has exactly two non-degenerate simplices in each dimension. If we label its vertices as "$-$" and "$+$," then its remaining non-degenerate simplices are uniquely determined by their vertices, which are given by alternating sequences of "$-$" and "$+$" starting from either vertex. Following the notation introduced in [**102**, 42], we write $E_n^+, E_n^- \subset \mathbb{I}$ for the simplicial subsets generated by the n-simplices $-+-\cdots\pm$ and $+-+\cdots\mp$ respectively. Both simplicial subsets include uniquely into both E_{n+1}^+ and E_{n+1}^- and these inclusions can be realized as pushouts

$$\begin{tikzcd} \Lambda^0[n+1] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta^0[n+1] \arrow[d] \\ E_n^\pm \arrow[r, hook] & E_{n+1}^\pm \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{n+1}[n+1] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta^{n+1}[n+1] \arrow[d] \\ E_n^\pm \arrow[r, hook] & E_{n+1}^\mp \end{tikzcd}$$

as suggested by the hints given to Exercise 1.1.iv.

As marked simplicial sets, we give $\mathbb{I}$ and its subcomplexes the maximal marking.

The following result gives a criterion under which an "inner complicial fibration" — a marked map that is only assumed to have the right lifting property against the inner complicial horn inclusions and inner complicial thinness extensions — whose codomain is a complicial set in fact defines a complicial isofibration in the sense of Definition D.1.15.

D.4.7. PROPOSITION. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then p is a complicial isofibration if and only if p admits lifts against the inclusions $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$.*

PROOF. Clearly any complicial isofibration admits lifts against the complicial anodyne extensions $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$. Thus, the heart of this result, and the only part that remains to be proven,

is the assertion that any inner complicial fibration whose codomain is a complicial set that admits lifts against this pair of inclusions also admits fillers for outer complicial horn inclusions and outer complicial thinness extensions.

We begin by arguing that an inner complicial fibration $p\colon A \twoheadrightarrow B$ satisfying the conditions of the statement also admits lifts against the dual inclusion $E_0^- \hookrightarrow E_1^+$. Given a lifting problem such as presented by the maps a and b in the square below right:

$$\begin{tikzcd}
& E_0^- \arrow[r, "a"] \arrow[dl] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\
E_1^- \arrow[dr] & E_1^+ \arrow[r, "b"] \arrow[d, hook] & B \\
& E_3^- \arrow[ur, dashed] \arrow[uur, dotted] &
\end{tikzcd}$$

there exists the dashed extension of b along $E_1^+ \hookrightarrow E_3^-$, since this inclusion factors as a composite of pushouts of outer complicial horn extensions and B is a complicial set. Now the inclusion $E_0^- \hookrightarrow E_3^-$ factors as indicated $E_0^- \hookrightarrow E_1^- \hookrightarrow E_3^-$ and since p is assumed to lift against both maps, the dotted lift exists as well, which restricts to define a solution to the original lifting problem

With this result in hand, it follows that the odd dual of $p\colon A \twoheadrightarrow B$ satisfies the same lifting properties as $p\colon A \twoheadrightarrow B$ does, since the odd dual of $E_0^- \hookrightarrow E_1^-$ is $E_0^- \hookrightarrow E_1^+$, while the odd dual of $E_1^- \hookrightarrow E_3^-$ is itself. So it suffices to show that p admits lifts against left complicial horn inclusions $\Lambda^0[n] \hookrightarrow_r \Delta^0[n]$ and left complicial thinness extensions $\Delta^0[n]' \hookrightarrow_e \Delta^0[n]''$ since its odd dual will then share these properties, which implies that p also admits lifts again right complicial horn inclusions and right complicial thinness extensions. The case $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ is the map $E_0^- \hookrightarrow E_1^-$ so it suffices also to assume $n > 1$, in which case we have an isomorphism

$$\Lambda^0[n] \hookrightarrow \Delta^0[n] \cong (\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\partial\Delta[n-2] \hookrightarrow \Delta[n-2])$$

by Lemma D.2.19. Writing $m = n-2$ for concision, consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[rrr, "a"] \arrow[dr, hook, "i"] \arrow[dd, hook] & & & A \arrow[dd, two heads, "p"] \\
& E_2^- \star \Delta[m] \cup E_3^- \star \partial\Delta[m] \arrow[r, hook] \arrow[urr, dotted] & E_3^- \star \Delta[m] \arrow[dr, dashed] & \\
E_1^- \star \Delta[m] \arrow[rrr, "b"'] \arrow[ur, hook] \arrow[urr, hook, bend right, "j"] & & & B
\end{tikzcd}$$

By Lemma D.2.19 and Proposition C.2.9(vi), the map j is a complicial anodyne extension, so since B is a complicial set, the dashed extension exists. To show that the dotted lift exists as well, we argue that the map i is cellularly generated by the inner complicial horn extensions and the map $E_1^- \hookrightarrow E_3^-$. To see this, factor the map i as the composite of the three vertical maps in the middle column of the

diagram of pushout squares

$$\begin{tikzcd}
E_1^- \arrow[r, hook] \arrow[d, hook] & E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[d, hook] \\
E_3^- \arrow[r, hook] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[d, hook] & E_1^- \star \partial\Delta[m] \cup E_3^- \star \varnothing \arrow[l, hook'] \arrow[d, hook] \\
E_0^- \star \Delta[m] \cup E_2^- \star \partial\Delta[m] \arrow[r, hook] \arrow[d, hook] & \bullet \arrow[d] \arrow[ur, phantom, very near start, "\urcorner"] & E_3^- \star \partial\Delta[m] \arrow[l, hook'] \\
E_2^- \star \Delta[m] \arrow[r, hook] \arrow[ur, phantom, very near end, "\ulcorner"] & E_2^- \star \Delta[m] \cup E_3^- \star \partial\Delta[m]
\end{tikzcd} \tag{D.4.8}$$

The first attached cell is the map $E_1^- \hookrightarrow E_3^-$ itself. By Digression D.4.6, $E_1^- \hookrightarrow E_3^-$ is a right anodyne extension, so by Lemma D.2.19 the second attached cell $(E_1^- \hookrightarrow E_3^-) \mathbin{\widehat{\star}} (\varnothing \hookrightarrow \partial\Delta[m])$ is an inner anodyne extension. Similarly $E_0^- \hookrightarrow E_2^-$ is a right anodyne extension, so the final attached cell $(E_0^- \hookrightarrow E_2^-) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$ is also an inner anodyne extension. Thus, p admits lifts against left complicial horn inclusions as claimed.

To see that $p\colon A \twoheadrightarrow B$ also admits outer complicial thinness extensions we make use of the isomorphism

$$(\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow \Delta[m]_t) \cong \Delta^0[m+2]' \hookrightarrow \Delta^0[m+2]''$$

of Lemma D.2.19, recalling that $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ is the inclusion $E_0^- \hookrightarrow E_1^-$. So we may consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m]_t \cup E_1^- \star \Delta[m] \arrow[rrr, "a"] \arrow[dr, hook, "i"] \arrow[dd, hook] & & & A \arrow[dd, two heads, "p"] \\
& E_2^- \star \Delta[m] \cup E_3^- \star \Delta[m] \arrow[r, hook] \arrow[urr, dotted] & E_3^- \star \Delta[m]_t \arrow[dr, dashed] & \\
E_1^- \star \Delta[m]_t \arrow[ur, hook] \arrow[urr, hook, bend right, "j"] \arrow[rrr, "b"'] & & & B
\end{tikzcd}$$

By Lemma D.2.19, the map j is a complicial anodyne extension, so since B is a complicial set, the dashed extension exists. To show that the dotted lift exists as well, we argue that the map i is cellularly generated by the inner complicial horn inclusions, the inner complicial thinness extensions, and the map $E_1^- \hookrightarrow E_3^-$. To see this, factor the map i as the composite of the three vertical maps in the middle

column of the diagram of pushout squares

$$\begin{tikzcd}
E_1^- \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & E_0^- \star \Delta[m]_t \cup E_1^- \star \Delta[m] \arrow[d, hook] & \\
E_3^- \arrow[r, hook] & \bullet \arrow[d, hook] & E_1^- \star \Delta[m] \cup E_3^- \star \varnothing \arrow[l, hook'] \arrow[d, hook] \arrow[dl, phantom, "\urcorner" very near end] \\
E_0^- \star \Delta[m]_t \cup E_2^- \star \Delta[m] \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near end] & \bullet \arrow[d] & E_3^- \star \Delta[m] \arrow[l, hook'] \\
E_2^- \star \Delta[m]_t \arrow[r, hook] & E_2^- \star \Delta[m]_t \cup E_3^- \star \Delta[m] &
\end{tikzcd}$$

The first attached cell is the map $E_1^- \hookrightarrow E_3^-$ itself. By Digression D.4.6, $E_1^- \hookrightarrow E_3^-$ is a right anodyne extension, so by Lemma D.2.19 the second attached cell $(E_1^- \hookrightarrow E_3^-) \mathbin{\widehat{\star}} (\varnothing \hookrightarrow \Delta[m])$ is an inner complicial anodyne extension. Similarly $E_0^- \hookrightarrow E_2^-$ is a right anodyne extension, so the final attached cell $(E_0^- \hookrightarrow E_2^-) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow \Delta[m]_t)$ is also an inner complicial thinness extension. Thus, p admits lifts against left complicial thinness extensions as claimed. $\square$

Since the inclusions $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$ are left complicial anodyne extensions, Proposition D.4.7 has the following immediate corollary.

D.4.9. Corollary. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then if p is a left complicial fibration or a right complicial fibration, then p is a complicial isofibration.* $\square$

D.4.10. Remark. Note that in the proof of Proposition D.4.7, lifts against the inclusion $E_0^- \hookrightarrow E_1^-$ are only needed to construct lifts for the outer complicial horn extensions $\Lambda^0[1] \hookrightarrow \Delta^0[1]$ and $\Lambda^1[1] \hookrightarrow \Delta^1[1]$. Thus, even if this lifting condition is dropped, the outer complicial horn extensions in higher dimensions can still be constructed.

The argument just given supplies a proof of a special case of "special outer horn filling" that will be useful in proving the general version.

D.4.11. Lemma. *Let $p\colon A \twoheadrightarrow B$ be an inner complicial fibration whose codomain B is a complicial set. Then p admits fillers for left horns*

$$\begin{tikzcd}
\Lambda^0[n] \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\
\Delta^0[n] \arrow[r, "b"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

with $n > 1$ provided a carries the $\{01\}$ edge of the horn $\Lambda^0[n]$ to a degenerate simplex in A.

Proof. Writing $m = n - 2$ we have an isomorphism

$$\Lambda^0[m+2] \hookrightarrow \Delta^0[m+2] \cong (\Lambda^0[1] \hookrightarrow \Delta^0[1]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$$

by Lemma D.2.19, so once more we are asked to consider a lifting problem as presented by the maps a and b:

$$\begin{tikzcd}
E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \arrow[rrr, "a"] \arrow[dr, hook, "k"] \arrow[ddr, hook, "i"'] \arrow[ddd, hook] & & & A \arrow[ddd, two heads, "p"] \\
& E_0^- \star \Delta[m] \cup E_1^- \star \partial\Delta[m] \underset{E_1^-}{\cup} E_3^- \arrow[d, hook, "\ell"'] \arrow[urr, dashed] & & \\
& E_2^- \star \Delta[m] \cup E_3^- \star \partial\Delta[m] \arrow[r, hook] \arrow[uurr, dotted] & E_3^- \star \Delta[m] \arrow[dr, dashed] & \\
E_1^- \star \Delta[m] \arrow[ur, hook] \arrow[urr, hook, bend right=10, "j"] \arrow[rrr, hook, "b"'] & & & B
\end{tikzcd}$$

The map j is a complicial anodyne extension, so since B is a complicial set, the lower dashed extension exists, and we are left to solve a lifting problem between the map i and the map p. To do so, we factor the map i as a composite of the three middle vertical morphisms displayed in (D.4.8) and let k denote the first of these morphisms while ℓ denotes the composite of the second two. We next solve the lifting problem between k and p by defining the image of the attached E_3^- to be a degenerate 3-simplex; note that k is constructed by attaching this E_3^- to the E_1^- that corresponds to the initial edge of the horn $\Lambda^0[m+2]$, which a maps to a degenerate edge.

Now to construct the dotted lift, it remains only to solve the lifting problem between ℓ and p, and the diagram (D.4.8) reveals that this can be done, as it expresses the map ℓ as the composite of pushouts of inner complicial horn inclusions. ☐

D.4.12. Corollary. *A marked simplicial set A that admits fillers for inner complicial horn extensions and inner complicial thinness extensions is a complicial set if and only if it admits extensions along the map $\Delta[1]^\sharp \hookrightarrow \mathrm{sk}_2\, \mathbb{I}^\sharp$.*

Proof. Applying Proposition D.4.7 to the inner complicial fibration $A \to 1$, to conclude that A is a complicial set we need only construct extensions along the maps $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$. The former is automatic, since $E_0^- \hookrightarrow E_1^-$ is isomorphic to $\Delta[0] \hookrightarrow \Delta[1]^\sharp$, which admits a retraction, so to complete our proof we will show that if A admits extensions along the map $\Delta[1]^\sharp \hookrightarrow \mathrm{sk}_2\, \mathbb{I}^\sharp$ as well as fillers for inner complicial horns, then A admits extensions along the map $E_1^- \hookrightarrow E_3^-$.

The domain map $\Delta[1]^\sharp \cong E_1^- \to A$ defines a marked 1-simplex $f\colon x \to y$ in A, why by hypothesis we may extend to a map $\mathrm{sk}_2\, \mathbb{I}^\sharp \to A$, which specifies marked 2-simplices

$$\begin{tikzcd}[column sep=small]
& y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, equal] & {} & x
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& x \arrow[dr, "f"] \arrow[d, phantom, "\alpha'"] & \\
y \arrow[ur, "f^{-1}"] \arrow[rr, equal] & {} & y
\end{tikzcd}$$

It won't necessarily be the case that the pair of 2-simplices α and α' form the two non-degenerate faces of a map $E_3^- \to A$ but we will construct a replacement β of α' so that α and β form the non-degenerate simplices of $E_3^- \to A$. The 3-simplex E_3^- will be constructed as the 2nd face of a horn $\Lambda^2[4] \to A$ that we now construct. The 4th face is $\alpha\sigma^1$. The 3rd face is constructed by filling the horn $\Lambda^1[3] \to A$

depicted below

$$
\begin{tikzcd}
& y \arrow[dd, "f^{-1}" near end] \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f"' near start] \arrow[dr, equal] & \arrow[l, phantom, "\alpha" near end, shift left=3] \arrow[r, phantom, "\alpha'" near end, shift left=3] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}
$$

in which the back face is degenerate. Writing γ for the face defined by filling this horn, the 1st face is constructed by filling a $\Lambda^0[3] \hookrightarrow \Delta^0[3]$ whose 2nd face is γ and 1st and 3rd faces are degenerate, as permitted by Lemma D.4.11. The 0th face is constructed by filling the horn $\Lambda^1[3] \to A$ depicted below

$$
\begin{tikzcd}
& x \arrow[dd, equal] \arrow[dr, "f"] & \\
y \arrow[ur, "f^{-1}"] \arrow[rr, equal] \arrow[dr, "f^{-1}"'] & \arrow[r, phantom, "\gamma" near end, shift left=3] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}
$$

in which the unlabeled front face is degenerate and the unlabeled back face is α'. The face defined by filling this horn is this replacement 2-simplex β. These four 3-simplices define a map $\Lambda^2[4] \to A$ whose filler defines a 2-simplex face as depicted below:

$$
\begin{tikzcd}
& y \arrow[dd, "f^{-1}" near end] \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f"' near start] \arrow[dr, equal] & \arrow[l, phantom, "\alpha" near end, shift left=3] \arrow[r, phantom, "\beta" near end, shift left=3] & y \\
& x \arrow[ur, "f"'] &
\end{tikzcd}
$$

in which both the back and bottom faces are degenerate. This defines the required extension $E_3^- \to A$. $\square$

With Corollary D.4.12 in hand, we can now prove Joyal's special outer horn filling result.

Proof of Proposition D.4.5. Let A be a naturally marked quasi-category. We must show that the unique map $!\colon A \to \mathbb{1}$ satisfies the hypotheses of Corollary D.4.12. For the inner complicial horn extensions, all of the non-degenerate marked simplices are in dimension two and higher, so A admits fillers for these as well, simply because quasi-categories are simplicial sets that admit fillers for all inner horns. The complicial thinness extensions of (D.1.11) are entire inclusions that differ only in markings of simplices in dimension at least two; since all such simplices are thin in the natural marking, A admits these extensions as well.

To conclude, we need only argue that A admits extensions of the form

$$
\begin{tikzcd}
\Delta[1]^\sharp \arrow[r, "f"] \arrow[d, hook] & A \\
\mathrm{sk}_2\, \mathbb{I}^\sharp \arrow[ur, dashed] &
\end{tikzcd}
$$

Since A is naturally marked, the attaching map $f\colon x \to y$ defines an isomorphism in A. By Definition 1.1.13, this means there exist 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\alpha'"] & \\ y \arrow[ur, "f^{-1}"] \arrow[rr, equal] & & y \end{tikzcd}$$

which provide the data of the required extension $\mathrm{sk}_2\, \mathbb{I}^\sharp \to A$. $\square$

This proves the hard direction of the following characterization of quasi-categories as complicial sets:

D.4.13. Theorem. *The natural marking of a quasi-category is a complicial set and indeed is the maximal marking that turns a quasi-category into a complicial set. Conversely, the underlying simplicial set of any complicial set with all simplices above dimension one marked is a quasi-category.*

Proof. In the proof of Proposition D.4.5, we have shown that the natural marking of a quasi-category A defines a complicial set: the inner extensions are straightforward, while Proposition D.4.5 proves that A admits the outer complicial horn and thinness extensions of (D.1.10) and (D.1.11). By Lemma D.4.2, it is not possible to mark any additional edges in A and retain the property of being a complicial set.

The converse is elementary, and left to the reader in Exercise D.1.iii. $\square$

We now give a few sample applications of Theorem D.4.13, revisiting some results that were proven in §1.1 using Proposition D.4.5. For instance:

D.4.14. Corollary. *A quasi-category A is a Kan complex if and only if its homotopy category is a groupoid.*

Proof. It's clear that the homotopy category of a Kan complex is a groupoid, so we focus our attention on the converse. By Theorem D.4.13, a quasi-category may be regarded as a complicial set, with every simplex above dimension 1 marked, and where the marked edges are exactly the isomorphisms defined to be those 1-simplices that represent isomorphisms in the homotopy category. If the homotopy category of A is a groupoid, then this tells us that A is maximally marked, and Exercise D.1.ii observes that a maximally marked complicial set defines a Kan complex. $\square$

For our next result, we revisit Corollary 1.1.16 and eliminate the reference to the maximal Kan complex spanned by the isomorphisms in a quasi-category — the existence of which follows from special outer horn lifting — from the proof given there.

D.4.15. Corollary. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a* ***homotopy coherent isomorphism***

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. When the quasi-category A is regarded as a naturally marked complicial set, the isomorphism f defines a marked map $f\colon \mathbb{2}^\sharp \to A$. By Lemma D.1.14, the injection $\mathbb{2}^\sharp \hookrightarrow \mathbb{I}^\sharp$ is an elementary

anodyne extension, and thus a lift

$$\begin{tikzcd} \mathbb{2}^\sharp \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I}^\sharp \arrow[ur, dashed] \end{tikzcd}$$

exists. Forgetting markings, this proves that every isomorphism in A extensions to a "homotopy coherent isomorphism." The converse is obvious from Definition 1.1.13. ☐

D.4.16. Lemma. *If A is a naturally marked quasi-category and X is a minimally marked simplicial set, then A^X is a naturally marked quasi-category.*

Proof. By Proposition D.3.4, n-simplices in A^X correspond to maps $X \times \Delta[n] \to A$. Since X is minimally marked, it follows that the underlying simplicial set of A^X coincides with the exponential of the underlying simplicial sets and hence, by Corollary D.3.12(i) defines a quasi-category.

To see that every simplex of dimension greater than one is marked in A^X consider an extension problem

$$\begin{tikzcd} X \times \Delta[n] \arrow[r] \arrow[d, hook, "e"'] & A \\ X \times \Delta[n]_t \arrow[ur, dashed] \end{tikzcd}$$

for $n > 1$. By Proposition D.3.4, the only simplices that are marked in $X \times \Delta[n]_t$ but not in $X \times \Delta[n]$ are n-simplices, and since all n-simplices in A are marked, it is clear that the desired extension exists.

It remains only to show that every isomorphism $f \colon \Delta[1] \to A^X$ in the quasi-category A^X is marked, admitting an extension as indicated below-right:

$$\begin{tikzcd} \coprod_{x \in X_0} \Delta[1] \arrow[r] \arrow[d, hook, "e"'] \arrow[rr, bend left, "f_x"] \arrow[dr, phantom, "\lrcorner", very near start] & X \times \Delta[1] \arrow[r, "f"] \arrow[d, hook, "e"'] & A \\ \coprod_{x \in X_0} \Delta[1]_t \arrow[r, "{(x\sigma^0, \mathrm{id})}"] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[urr, dotted, bend right] & X \times \Delta[1]_t \arrow[ur, dashed] \end{tikzcd}$$

By Proposition D.3.4, the only simplices that are marked in $X \times \Delta[1]_t$ but not in $X \times \Delta[1]$ are 1-simplices whose component in X is degenerate and whose component in $\Delta[1]_t$ is the non-degenerate 1-simplex, as indicated by the square above-left that is both a pullback and a pushout. The images of such simplices in A define the "components" of f, as indicated by the top composite above.

If f is an isomorphism in A^X then each of its components f_x, the image of f under the evaluation function $\mathrm{ev}_x \colon A^X \to A$, are clearly also isomorphisms, and in particular marked in A. Thus the dotted lift exists, and by the universal property of the pushout, so does the dashed one, as required. ☐

As a consequence of Lemma D.4.16, we can show:

D.4.17. Corollary. *For any quasi-category A and simplicial set X, an edge in A^X is an isomorphism if and only if each of its components in A, indexed by the vertices of X, are isomorphisms.*

PROOF. Regarding A as a naturally marked quasi-category and X as a minimally marked simplicial set, by Lemma D.4.16 a 1-simplex in A^X is marked if and only if it defines an isomorphism in the quasi-category A^X. By the definition of the markings in the exponential, a 1-simplex $\Delta[1] \to A^X$ is marked if and only if its components, defined by evaluating at a vertex of X are marked. Since A is a naturally marked quasi-category, this is the case if and only if each of these components are isomorphisms in A as claimed. $\square$

A similar argument to the one used to prove Proposition D.4.5 that also employs the conclusion of that result, allow us to extend the special outer horn lifting property to isofibrations between naturally marked quasi-categories:

D.4.18. THEOREM. *An isofibration between naturally marked quasi-categories admits fillers for outer complicial horn inclusions for $n \geq 1$:*

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d, hook, "r"'] & A \arrow[d, two heads] \\ \Delta^0[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d, hook, "r"'] & A \arrow[d, two heads] \\ \Delta^n[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

and an inner fibration between naturally marked quasi-categories admits fillers for outer complicial horn inclusions for $n \geq 2$. Consequently, an isofibration between naturally marked quasi-categories is a complicial isofibration.

PROOF. By Theorem D.4.13, the codomain B is a complicial set, so to apply Proposition D.4.7 we must argue that an isofibration between naturally marked quasi-categories has the right lifting property against the inner complicial horn extensions, inner complicial thinness extensions, and the two maps $E_0^- \hookrightarrow E_1^-$ and $E_1^- \hookrightarrow E_3^-$. By Remark D.4.10, the first of these maps is only needed to construct the outer complicial horn inclusions for $n = 1$, so the proof given below will establish fillers for the complicial horn inclusions in dimension $n \geq 2$ under the weaker hypothesis where $A \twoheadrightarrow B$ is only assumed to be an inner fibration. We'll concentrate on the case where $A \twoheadrightarrow B$ is an isofibration henceforth.

The inner complicial horn and thinness extensions are straightforward, as in the proof of Proposition D.4.5. To construct a lift

$$\begin{tikzcd} E_0^- \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ E_1^- \arrow[r, "b"] \arrow[d, hook] & B \\ \mathbb{I} \arrow[uur, dotted] \arrow[ur, dashed] & \end{tikzcd}$$

we use Corollary D.4.15 to extend the codomain to a homotopy coherent isomorphism and then solve the composite lifting problem.

The construction of the lift against $E_1^- \hookrightarrow E_3^-$ is considerably more laborious. To begin, we argue that since A and B are complicial sets and p admits lifts against $E_0^- \hookrightarrow E_1^-$ and is an inner complicial fibration, then p admits lifts against the complicial horn inclusion $\Lambda^0[2] \to \Delta^0[2]$. To see this, we identify the codomain $\Delta^0[2]$ with the second face of the 3-simplex $E_2^- \star \Delta[0]$ and consider the lifting

problem presented by the exterior diagram

$$\begin{tikzcd}
\Lambda^0[2] \arrow[r, hook, "i_{013}"'] \arrow[d, hook] \arrow[rrr, bend left, "a"] & E_0^- \star \Delta[0] \cup E_2^- \star \varnothing \arrow[rr, dashed] \arrow[d, hook] \arrow[dr, "k"] & & A \arrow[d, two heads, "p"] \\
\Delta^0[2] \arrow[r, hook] \arrow[rr, bend right, "\delta^2"'] \arrow[rrr, bend right=40, "b"'] \arrow[ur, phantom, "\ulcorner" very near end] & \bullet \arrow[r, hook, "j"] \arrow[rr, dotted, bend right] & E_2^- \star \Delta[0] \arrow[r, dashed] \arrow[ur, dotted] & B
\end{tikzcd}$$

The inclusion i_{013} is complicially anodyne, so the top dashed extension exists since A is a complicial set by Theorem D.4.13. This induces the dotted map by the universal property of the pushout. Since the map j is also complicially anodyne, the bottom dashed extension exists since B is a complicial set by Theorem D.4.13. These dashed maps define a new lifting problem between the composite map k and p and since k is an inner complicial anodyne extension, the dotted lift exists, solving the original lifting problem.

Now we can use the fact that p admits lifts along $\Lambda^0[2] \hookrightarrow \Delta^0[2]$ to construct lifts along the horizontal composite map

$$\begin{tikzcd}
& E_1^+ \arrow[r, hook] \arrow[d, hook] & E_2^+ \arrow[d, hook] \\
E_1^- \arrow[r, hook] & E_2^- \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & E_2^- \underset{E_1^+}{\cup} E_2^+
\end{tikzcd}$$

since Digression D.4.6 reveals that both maps $E_1^\pm \hookrightarrow E_2^\pm$ are pushouts of $\Lambda^0[2] \hookrightarrow \Delta^0[2]$. Lifts agains the composite $E_1^- \hookrightarrow E_2^- \underset{E_1^+}{\cup} E_2^+$ have the effect of giving the data of a right inverse and also a right inverse to the right inverse to an isomorphism in A, lifting the corresponding data in B. To solve a lifting problem

$$\begin{tikzcd}
& E_1^- \arrow[r, "f"] \arrow[dl, hook'] \arrow[d, hook, "{(\alpha,\alpha')}"] & A \arrow[d, two heads, "p"] \\
E_2^- \underset{E_1^+}{\cup} E_2^+ \arrow[r] \arrow[urr, dashed] & E_3^- \arrow[r, "\tau"'] & B
\end{tikzcd}$$

we first construct the outer lift. The lift defines a pair of 2-simplices

$$\begin{tikzcd}
& y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\alpha"] & \\
x \arrow[ur, "f"] \arrow[rr, equal] & {} & x
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dr, "(f^{-1})^{-1}"] \arrow[d, phantom, "\alpha'"] & \\
y \arrow[ur, "f^{-1}"] \arrow[rr, equal] & {} & y
\end{tikzcd}$$

in A.

We'll now extend the data of the lifted $E_2^- \underset{E_1^+}{\cup} E_2^+ \to A$ to construct a map $E_3^- \to A$ lifting $E_3^- \to B$ using a mild modification of the construction in the proof of Proposition D.4.5 can be used to solve the lifting problem.

As above, we'll define a complicial inner horn horn

$$\begin{tikzcd} \Lambda^2[4] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \Delta^2[4] \arrow[r, "\tau\sigma^2"'] \arrow[ur, dashed] & B \end{tikzcd}$$

so that the lift of the 2nd face defines the simplex $E_3^- \to A$ that we seek. It remains only to define an appropriate horn $\Lambda^2[4] \to A$ over $\tau\sigma^2 \colon \Delta^2[4] \to B$. The 4th face is $\alpha\sigma^1$. The 3rd face is constructed by lifting along a horn $\Lambda^1[3] \hookrightarrow \Delta^1[3]$ whose 3rd face is α, whose 0th face is α', and whose 2nd face is degenerate. Writing γ for the face defined by filling this horn, the 1st face is constructed by filling a $\Lambda^0[3] \hookrightarrow \Delta^0[3]$ whose 2nd face is γ and 1st and 3rd faces are degenerate; the argument required to justify this, as permitted by Lemma D.4.11. The 0th face is constructed by filling the horn $\Lambda^1[3] \to \Delta^1[3]$ whose faces have all already been described. The face defined by filling this horn is this replacement 2-simplex β which witnesses that f is a right inverse to its right inverse f^{-1}. These four 3-simplices define a map $\Lambda^2[4] \to A$ over $\tau\sigma^2$ whose filler defines a 2-simplex face as depicted below:

$$\begin{tikzcd} & y \arrow[dd, "f^{-1}"] \arrow[dr, equal] \arrow[dr, phantom, "\beta" near end, shift right=4] & \\ x \arrow[ur, "f"] \arrow[rr, "f"' near start] \arrow[dr, equal] \arrow[ur, phantom, "\alpha", shift left=-4] & & y \\ & x \arrow[ur, "f"'] & \end{tikzcd}$$

in which both the back and bottom faces are degenerate. This defines the required lift along $E_1^- \hookrightarrow E_3^-$. Now Proposition D.4.7 completes the proof that isofibrations lift against outer complicial horn inclusions.

It follows that an isofibration between naturally marked quasi-categories is a complicial isofibration: we've demonstrated that it has the required lifting property against the complicial horn inclusions. By Exercise D.1.v, the fact that A is a complicial set suffices to show that $p \colon A \twoheadrightarrow B$ also lifts against the complicial thinness extensions. □

Theorems D.4.13 and D.4.18 permit the use of complicial techniques to solve lifting problems involving isofibrations between quasi-categories. The results of this section suggest that these techniques are particularly fruitful when isomorphisms are involved. We conclude this section by developing a few specific applications of this principle.

To that end we consider a pair of cosimplicial marked simplicial sets

$$\mathbf{\Delta}[\bullet]^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp \quad \in (s\mathcal{S}et^+)^{\mathbf{\Delta}}$$

the former of which is given by the maximally marked simplices $\Delta[n]^\sharp$ and the latter of which is given by the maximally marked contractible groupoids $\mathbb{I}[n]^\sharp$ on objects $0, 1, \ldots, n$.

D.4.19. Lemma. *The natural inclusion $\mathbf{\Delta}[\bullet]^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp$ is a Reedy monomorphism in $(s\mathcal{S}et^+)^{\mathbf{\Delta}}$ between Reedy monomorphic cosimplicial objects that is moreover a pointwise weak equivalence.*

Proof. The first half of the statement follows from a general principle: any pointwise monomorphism between "unaugmentable" cosimplicial objects is a Reedy monomorphism between Reedy monomorphic cosimplicial objects, where a cosimplicial object $X^\bullet$ is **unaugmentable** if the equalizer of the face

maps $\delta^0, \delta^1 \colon X^0 \rightrightarrows X^1$ is empty [**79**, 14.3.8]. The idea is that in the presence of this condition, any simplex in $X^\bullet$ is uniquely expressable as $\alpha \cdot x$, where $\alpha \in \mathbf{\Delta}$ is a monomorphism and $x \in X^\bullet$ is not in the image of any monomorphism. The proof of this relies on the observation that nearly every monomorphism in $\mathbf{\Delta}$ is uniquely determined by its set of left-inverses, the only exceptions being $\delta^0, \delta^1 \colon [0] \rightrightarrows [1]$; hence the "unaugmentable" condition. From this definition it is clear that both $\mathbf{\Delta[\bullet]}^\sharp$ and $\mathbb{I}[\bullet]^\sharp$ are unaugmentable, so we conclude that these simplicial objects are Reedy monomorphic and the natural inclusion is a Reedy monomorphism.

Finally to prove that $\Delta[n]^\sharp \to \mathbb{I}[n]^\sharp$ is a pointwise weak equivalence, we appeal to the 2-of-3 property and argue that both $\Delta[n]^\sharp$ and $\mathbb{I}[n]^\sharp$ are contractible in the sense of being marked homotopy equivalent to $\mathbb{1} \in s\mathcal{S}et^+$. The inverse homotopy equivalences are given by $0 \colon \mathbb{1} \to \Delta[n]^\sharp$ and $0 \colon \mathbb{1} \to \mathbb{I}[n]^\sharp$ and the marked homotopies $\Delta[n]^\sharp \times \Delta[1]^\sharp \to \Delta[n]^\sharp$ and $\mathbb{I}[n]^\sharp \times \Delta[1]^\sharp \to \mathbb{I}[n]^\sharp$ are both defined by the map on objects $(i, 0) \mapsto 0$ and $(i, 1) \mapsto i$. ☐

Our intent is to use the simplicial objects $\mathbf{\Delta[\bullet]}^\sharp$ and $\mathbb{I}[\bullet]^\sharp$ to "freely invert" the simplices of a simplicial set K. To see how this works, consider also the cosimplicial object $\mathbf{\Delta[\bullet]} \in s\mathcal{S}et^{\mathbf{\Delta}}$ defined by the Yoneda embedding. By the Yoneda lemma, the weighted colimit $\mathrm{colim}_K \mathbf{\Delta[\bullet]} \cong K$ recovers the original simplicial set K. Similarly, since the maximal marking functor $(-)^\sharp \colon s\mathcal{S}et \to s\mathcal{S}et^+$ is a left adjoint, the weighted colimit $\mathrm{colim}_K \mathbf{\Delta[\bullet]}^\sharp \cong K^\sharp$ equips the simplicial set K with the maximal marking. Finally, we define $\tilde{K}^\sharp := \mathrm{colim}_K \mathbb{I}[\bullet]^\sharp$ using the weighted colimit bifunctor. The idea of this functor is that it replaces each n-simplex of K by $\mathbb{I}[n]^\sharp$, a "homotopy coherent composite of n-isomorphisms." As the notation suggests, $\tilde{K}^\sharp$ is also maximally marked.

D.4.20. Proposition. *For any simplicial set K, the natural map $K^\sharp \to \tilde{K}^\sharp$ is a trivial cofibration of marked simplicial sets.*

Proof. Any simplicial set K is Reedy monomorphic when considered as an object of $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$. Hence, by Corollary C.5.17, the weighted colimit functor

$$\mathrm{colim}_K - \colon (s\mathcal{S}et^+)^{\mathbf{\Delta}} \to s\mathcal{S}et^+$$

is left Quillen with respect to the Reedy model structure on $(s\mathcal{S}et^+)^{\mathbf{\Delta}}$. Lemmas C.5.13 and D.4.19 prove that $\mathbf{\Delta[\bullet]}^\sharp \hookrightarrow \mathbb{I}[\bullet]^\sharp$ is a Reedy trivial cofibration, so it follows that $K^\sharp \to \tilde{K}^\sharp$ is a trivial cofibration as claimed. ☐

We refer to the functor $(\tilde{-}) \colon s\mathcal{S}et \to s\mathcal{S}et$ defined by applying $\mathrm{colim}_{-} \mathbb{I}[\bullet]^\sharp$ and forgetting the markings as the "free-inversion functor." Proposition D.4.20 has the following consequence:

D.4.21. Corollary. *Any diagram $K \to A$ in a quasi-category A whose edges are sent to isomorphisms extends to a diagram indexed by the free inversion of K. More generally, if $K \to A$ sends the edges of K to isomorphisms of A, and $p \colon A \twoheadrightarrow B$ is an isofibration between quasi-categories, then any lifting problem*

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \tilde{K} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution

Proof. The hypothesis is that our given diagram extends to a map $K^\sharp \to A^\natural$ of marked simplicial sets. By Theorem D.4.13, the natural marking of A defines a complicial set and by Theorem D.4.18 $p\colon A^\natural \twoheadrightarrow B^\natural$ is a complicial isofibration, so Proposition D.4.20 implies that the lift

$$\begin{tikzcd} K^\sharp \arrow[r] \arrow[d, hook] & A^\natural \arrow[d, two heads, "p"] \\ \tilde{K}^\sharp \arrow[r] \arrow[ur, dashed] & B^\natural \end{tikzcd}$$

exists. □

D.4.22. Example. Proposition D.4.20 can be applied to the anodyne extension $\Lambda^1[2] \hookrightarrow \Delta[2]$ to prove that composites of homotopy coherent isomorphisms can be lifted along isofibrations of quasi-categories. By definition $\widetilde{\Delta[2]} \cong \mathbb{I}[2]$, the contractible groupoid on three vertices 0, 1, and 2, so we adopt similar notation $\widetilde{\Lambda^1[2]} =: \Lambda^1[\mathbb{I}[2]]$ for the freely-inverted horn. Since $\Lambda^1[2]$ is built by gluing two 1-simplices along a common vertex and the free inversion functor preserves colimits, we see that $\Lambda^1[\mathbb{I}[2]]$ is the union of two homotopy coherent isomorphisms between 0 and 1 and between 1 and 2. Giving these simplicial sets their maximal markings, it follows from the 2-of-3 property applied to the square

$$\begin{tikzcd} \Lambda^1[2]^\sharp \arrow[r, hook] \arrow[d, hook] & \Lambda^1[\mathbb{I}[2]]^\sharp \arrow[d, hook] \\ \Delta[2]^\sharp \arrow[r, hook] & \mathbb{I}[2]^\sharp \end{tikzcd}$$

that the inclusion $\Lambda^1[\mathbb{I}[2]]^\sharp \hookrightarrow \mathbb{I}[2]^\sharp$ is a trivial cofibration of marked simplicial sets. Applying Theorem D.4.18, we conclude that composites of homotopy coherent isomorphisms can be lifted along isofibrations between quasi-categories

$$\begin{tikzcd} \Lambda^1[\mathbb{I}[2]] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads] \\ \mathbb{I}[2] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

A similar example is left as Exercise D.4.ii. In fact, Joyal proves that $(\tilde{-})\colon s\mathcal{S}et \to s\mathcal{S}et$ defines a left Quillen functor from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories, whose right adjoint is the functor that sends a simplicial set X to the simplicial set whose n-simplices are maps $\mathbb{I}[n] \to X$. His proof, which is replicated in [**79**, 17.6.1], makes use of a lemma that is also of interest.

D.4.23. Lemma. *For any simplicial set K, the natural inclusion $K \hookrightarrow \tilde{K}$ is an anodyne extension.*

Proof. This follows from Proposition D.4.20 and the fact that the functor $s\mathcal{S}et^+ \to s\mathcal{S}et$ that forgets the markings carries elementary anodyne extensions to anodyne extensions but a direct proof is also possible. To see that $K \hookrightarrow \tilde{K}$ is an anodyne extension, consider a Kan fibration between Kan

complexes $p\colon A \twoheadrightarrow B$. By Corollary D.4.21 a lift

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "p"] \\ \tilde{K} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

exists from which we conclude that $K \hookrightarrow \tilde{K}$ is an anodyne extension. $\square$

D.4.24. PROPOSITION. *The free inversion functor and its left adjoint defines a Quillen adjunction from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories.*

PROOF. Lemma D.4.19 proves that the functor $\mathbb{I}[\bullet]^\sharp$ is Reedy monomorphic in $(s\mathcal{S}et^+)^{\mathbf{\Delta}}$. Hence, by Theorem C.5.15, the weighted colimit functor

$$\operatorname{colim}_{-} \mathbb{I}[\bullet]^\sharp\colon s\mathcal{S}et \cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to s\mathcal{S}et^+$$

carries Reedy monomorphisms to monomorphisms. Since the Reedy monomorphisms in $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ are just the monomorphisms in $s\mathcal{S}et$, in this way we see that the left adjoint $\tilde{(-)}\colon s\mathcal{S}et \to s\mathcal{S}et$ preserves monomorphisms.

It remains to argue that $\tilde{(-)}$ carries weak equivalences in the Quillen model structure to weak equivalences in the Joyal model structure. If $K \xrightarrow{\sim} L$ is a weak equivalence in the Quillen model structure then by the 2-of-3 property applied to the square

$$\begin{tikzcd} K \arrow[r, "\sim"] \arrow[d, "\wr"] & L \arrow[d, "\wr"] \\ \tilde{K} \arrow[r] & \tilde{L} \end{tikzcd}$$

the map $\tilde{K} \xrightarrow{\sim} \tilde{L}$ is as well. By construction of $\tilde{K}$ and $\tilde{L}$, their homotopy categories of Definition 1.1.10 are groupoids. If we fibrantly replace each object in the Joyal model structure, we obtain quasi-categories $\tilde{K}^\dagger$ and $\tilde{L}^\dagger$ which are weakly equivalent in the Joyal model structure to $\tilde{K}$ and $\tilde{L}$ and whose homotopy categories are thus also groupoids. In particular, by Corollary D.4.14, $\tilde{K}^\dagger$ and $\tilde{L}^\dagger$ are Kan complexes. This gives us a commutative diagram

$$\begin{tikzcd} \tilde{K} \arrow[r] \arrow[d] & \tilde{L} \arrow[d] \\ \tilde{K}^\dagger \arrow[r] & \tilde{L}^\dagger \end{tikzcd}$$

in which the vertical maps are weak equivalences in the Joyal model structure, and hence also the Quillen model structure. Since the top horizontal map is also a weak equivalence is the Quillen model structure, it follows from the 2-of-3 property that $\tilde{K}^\dagger \xrightarrow{\sim} \tilde{L}^\dagger$ is as well, but any weak homotopy equivalence between Kan complexes is also an equivalence of quasi-categories. By the 2-of-3 property again, we conclude that $\tilde{K} \xrightarrow{\sim} \tilde{L}$ is a weak equivalence in the Joyal model structure, as desired. $\square$

Exercises.

D.4.i. EXERCISE. Prove Lemma D.4.3.

D.4.ii. Exercise. Extend the result of Example D.4.22 to show that the maximally marked "$\Lambda^1[3]$-horn of homotopy coherent isomorphisms" $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$, whose codomain is the contractible groupoid on four vertices $0, 1, 2, 3$ and whose domain is the union of the three copies of $\mathbb{I}[2]$ spanned by the subsets of three of these four vertices that include the vertex 1, is a complicial anodyne extension.

D.5. Isofibrations between quasi-categories

Our first aim in this section is to integrate the class of isofibrations between quasi-categories into the results proven in §D.3. We start by proving Propositions 1.1.19 and 1.1.28, restated here for convenience.

D.5.1. Proposition.

(i) There is a solution to any lifting problem between the pushout product of a monomorphism $i\colon X \hookrightarrow Y$ and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ and any isofibration $f\colon A \twoheadrightarrow B$.

(ii) If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is again an isofibration.

(iii) If $i\colon X \hookrightarrow Y$ is a monomorphism and $f\colon A \xrightarrow{\sim} B$ is a trivial fibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is again an trivial fibration.

(iv) If $i\colon X \hookrightarrow Y$ is in the class generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ and $f\colon A \twoheadrightarrow B$ is an isofibration, then the induced Leibniz exponential map

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

is a trivial fibration.

Proof. It suffices to construct the lift of (i) in marked simplicial sets and then forget the markings. By Lemma D.1.14, $\mathbb{1} \hookrightarrow \mathbb{I}$ is a complicial anodyne extension, when $\mathbb{I}$ is assigned its natural maximal marking. Thus by Proposition D.3.8, the Leibniz product of the minimally marked monomorphism i with this map is again a complicial anodyne extension. By Theorem D.4.18, an isofibration defines a complicial isofibration between naturally marked quasi-categories, so the postulated lift exists.

Parts (ii) and (iv) follow from the conclusion of (i) and a similar result, Corollary D.3.11 that shows that the pushout product of a monomorphism and an inner horn inclusion can be factored as a sequence of pushouts of inner horn inclusions. By transposing across the two-variable adjunction between the pushout product and the Leibniz exponential, these results imply that the map $\widehat{i \pitchfork f}$ lifts against the inner horn inclusions and against the map $\mathbb{1} \hookrightarrow \mathbb{I}$. Part (iii) follows by a similar argument from an easier observation: that the pushout product of two monomorphisms is again a monomorphism. □

D.5.2. Proposition. *For a map $f\colon A \to B$ of quasi-categories the following are equivalent:*

(i) f is at trivial fibration

(ii) f is both an isofibration and an equivalence

(iii) *f is a* ***split fiber homotopy equivalence***: *an isofibration admitting a section s that is also an equivalence inverse via a homotopy from id_A to sf that composes with f to the constant homotopy from f to f.*

Proof. For (i)⇒(ii), observe that the simplex boundary inclusions generate the monomorphisms of simplicial sets under coproduct, pushout, and sequential composition (see Lemma C.5.9), so the lifting property of (1.1.25) implies that the trivial fibrations lift against all monomorphisms of simplicial sets, and in particular against the monomorphisms that detect the class of isofibrations. Thus, trivial fibrations are isofibrations. By the same lifting property, every trivial fibration admits a section

$$\begin{tikzcd} & A \arrow[d, two heads, "f", "\wr"'] \\ B \arrow[ur, dashed, "s"] \arrow[r, equal] & B \end{tikzcd}$$

To show that s defines an inverse homotopy equivalence to f, observe that the other rectangle built from the constant homotopy $\pi\colon A \times \mathbb{I} \to A$

$$\begin{tikzcd} A + A \arrow[rr, "{(\mathrm{id}_A, sf)}"] \arrow[d, hook] & & A \arrow[d, two heads, "f", "\wr"'] \\ A \times \mathbb{I} \arrow[urr, dashed, "\alpha"] \arrow[r, "\pi"'] & A \arrow[r, two heads, "f", "\sim"'] & B \end{tikzcd}$$

commutes since $fsf = f$. The lift defines a homotopy between id_A and sf completing the proof that trivial fibrations are equivalences. And note in fact that the equivalence just constructed is a split fiber homotopy equivalence, proving that (i)⇒(iii).

To prove (ii)⇒(iii), suppose that f is an isofibration with equivalence inverse g. By Lemma D.5.3 below, the homotopies α from id_A to gf and β from fg to id_B may be chosen so as to define a "half-adjoint equivalence," meaning that there exists a map $\Phi\colon A \times \mathbb{I}[2] \to B$, where $\mathbb{I}[2]$ is the contractible groupoid on three objects, whose boundary is formed by $f\alpha$, βf, and the constant homotopy $\mathrm{id}_f :=$ $f\pi$.[7]

Applying Proposition D.5.1(i) to the monomorphism $\varnothing \hookrightarrow B$, we find that we can lift the homotopy β between fg and id_B along f

$$\begin{tikzcd} & B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & A \arrow[d, two heads, "f"] \\ B \arrow[r, hook, "i_1"] \arrow[urr, dotted, "s"] \arrow[rr, equal, bend right] & B \times \mathbb{I} \arrow[r, "\beta"] \arrow[ur, dashed, "\gamma"] & B \end{tikzcd}$$

[7]See Theorem 1.4.7 and Proposition 2.1.11 for an explanation of this terminology.

The composite map s defines a strict section of f, while the lift defines a homotopy γ from g to s. Applying Proposition D.3.8, Theorem D.4.18, and Example D.4.22, we can solve the lifting problem

$$\begin{tikzcd}
 & A \times \Lambda^1[\mathbb{I}[2]] \arrow[r, "(\alpha,\gamma f)"] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\
A \times \mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\eta"] \arrow[dr, two heads, "\pi"'] & A \times \mathbb{I}[2] \arrow[r, "\Phi"'] \arrow[ur, dashed, "\Psi"'] & B \\
 & A \arrow[ur, two heads, "f"'] &
\end{tikzcd}$$

The lift defines a composite homotopy η from id_A to sf so that $f\eta = f\pi$ is the constant homotopy. This data exhibits f as a split fiber homotopy equivalence.

Finally, for (iii)⇒(i) to prove that $f\colon A \twoheadrightarrow B$ is a trivial fibration it suffices to show that the Leibniz exponential constructed in the diagram below is surjective on vertices (a vertex in its codomain defining a lifting problem that any lift would solve):

$$\begin{tikzcd}
A^{\Delta[n]} \arrow[dr, dashed, two heads] \arrow[drr, bend left] \arrow[ddr, bend right, two heads, "f^{\Delta[n]}"'] & & \\
 & \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\partial\Delta[n]} \arrow[d, two heads, "f^{\partial\Delta[n]}"] \\
 & B^{\Delta[n]} \arrow[r] & B^{\partial\Delta[n]}
\end{tikzcd}$$

By Proposition 1.1.19, the maps displayed with two heads are all isofibrations. Moreover, equivalences are preserved by exponentiation, so the outer vertical maps are also equivalences.

The point of promoting the equivalence f to a split fiber homotopy equivalence is that surjective equivalences are also stable under pullback, which defines a simplicial functor between slice categories and surjective equivalences live fully in these slice categories. Since the equivalences satisfy the 2-of-3 property (Exercise 1.1.vii) it now follows that the Leibniz exponential is both an isofibration and equivalence. By (ii)⇒(iii), this map admits a section and in particular is surjective on vertices. Thus, we conclude that split fiber homotopy equivalences are trivial fibrations. □

D.5.3. Lemma. *Any equivalence of quasi-categories*

$$\begin{tikzcd}
A \arrow[d, hook, "i_0"'] \arrow[dr, equal] & \\
A \times \mathbb{I} \arrow[r, "\alpha"] & A \\
A \arrow[u, hook', "i_1"] \arrow[r, "f"'] & B \arrow[u, "g"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & A \arrow[d, "f"] \\
B \times \mathbb{I} \arrow[r, "\beta"] & B \\
B \arrow[u, hook', "i_1"] \arrow[ur, equal] &
\end{tikzcd}$$

can be extended to a ***half-adjoint equivalence of quasi-categories****, with an additional coherence homotopy* $\Phi\colon A \times \mathbb{I}[2] \to B$ *whose boundary is comprised of the three homotopy coherent isomorphisms:*

$$\begin{tikzcd} & fgf \arrow[dr, dash, "\beta f"] \arrow[d, phantom, "\Phi"] & \\ f \arrow[ur, dash, "f\alpha"] \arrow[rr, dash, "\mathrm{id}_f"'] & {} & f \end{tikzcd}$$

at the cost of replacing one of the homotopies α *or* β*.*

The proof is by a simplicial reinterpretation of the 2-categorical argument that proves Proposition 2.1.11.

Proof. Consider an equivalence of quasi-categories as in the statement. By Example D.4.22, the homotopies $f\alpha$ and βf admit some composite defined by solving the lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[2]] \arrow[r, "(f\alpha,\beta f)"] \arrow[d, hook] & B \\ A \times \mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\psi"] & A \times \mathbb{I}[2] \arrow[ur, dashed, "\Psi"'] & \end{tikzcd}$$

Restriction along the non-identity involution $\mathbb{I} \to \mathbb{I}$ defines the inverse of any homotopy, denoted by $(-)^{-1}$. We will replace α by the composite $\alpha' \coloneqq g\psi^{-1} \cdot \alpha$ defined by solving the lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[2]] \arrow[r, "(\alpha,g\psi^{-1})"] \arrow[d, hook] & A \\ A \times \mathbb{I} \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\alpha'"] & A \times \mathbb{I}[2] \arrow[ur, dashed, "\Xi"'] & \end{tikzcd}$$

and show that this homotopy defines a half-adjoint equivalence with β.

The witness to the half-adjoint equivalence will be obtained by solving a final lifting problem

$$\begin{tikzcd} & A \times \Lambda^1[\mathbb{I}[3]] \arrow[r, "\Gamma"] \arrow[d, hook] & B \\ A \times \mathbb{I}[2] \arrow[r, hook, "\delta^1"'] \arrow[urr, dotted, "\Phi"] & A \times \mathbb{I}[3] \arrow[ur, dashed] & \end{tikzcd}$$

involving an extension along $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$, whose codomain is the contractible groupoid with four objects $0, 1, 2, 3$ and whose domain is the union of the three faces $\mathbb{I}[2]$ which contain the vertex 1. This is permitted by Exercise D.4.ii.

It remains to define the faces of the "horn of homotopy coherent isomorphisms" Γ which is built from the homotopy coherent isomorphisms

$$\begin{tikzcd} & fgf \arrow[dr, dash, "f\alpha^{-1}"] \arrow[d, dash] & \\ f \arrow[ur, dash, "f\alpha"] \arrow[r, equal] \arrow[dr, dash, "f\alpha'"'] & fg\phi^{-1} \arrow[r, equal] \arrow[d, dash] & f \\ & fgf \arrow[ur, dash, "\beta f"'] & \end{tikzcd}$$

The 3rd face is $f\Xi\colon A \times \mathbb{I}[2] \to B$ while the second face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A\times q} A \times \mathbb{I} \xrightarrow{f\alpha} B$$

where $q\colon \mathbb{I}[2] \to \mathbb{I}$ is the unique map defined by $0, 2 \mapsto 0$ and $1 \mapsto 1$. It remains to define the 0th face. For this, we first extend along another horn $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$ of homotopy coherent isomorphisms in B^A as depicted below

$$\begin{tikzcd} & f \arrow[dr, dash, "\psi^{-1}"] \arrow[dd, dash, "\beta^{-1}f" near end] & \\ fgf \arrow[ur, dash, "\beta f"] \arrow[rr, dash, "\delta" near start] \arrow[dr, equal] & & f \\ & fgf \arrow[ur, dash, "f\alpha^{-1}"'] & \end{tikzcd}$$

Here the 0th face is Ψ^{-1}, the 3rd face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A\times q} A \times \mathbb{I} \xrightarrow{\beta f} B$$

and the 2nd face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A\times d} A \times \mathbb{I}[2] \times \mathbb{I} \xrightarrow{\Psi\times\mathbb{I}} B \times \mathbb{I} \xrightarrow{\beta} B$$

where $d\colon \mathbb{I}[2] \to \mathbb{I}[2] \times \mathbb{I}$ is the unique map defined by $0 \mapsto (2, 0)$, $1 \mapsto (2, 1)$, and $2 \mapsto (0, 1)$. The δ^1-face of this horn can be used to define a final horn $\Lambda^1[\mathbb{I}[3]] \hookrightarrow \mathbb{I}[3]$ of homotopy coherent isomorphisms in B^A as depicted below

$$\begin{tikzcd} & fgf \arrow[dr, dash, "\delta"] \arrow[dd, dash, "fg\psi^{-1}" near end] & \\ fgf \arrow[ur, equal] \arrow[rr, dash, "f\alpha^{-1}" near start] \arrow[dr, dash, "fg\psi^{-1}"'] & & f \\ & fgf \arrow[ur, dash, "\beta f"'] & \end{tikzcd}$$

whose 3rd face is degenerate, whose 3nd face is an inversion that swaps the first two vertices of the δ^1-face $A \times \mathbb{I}[2] \to B$ just defined, and whose 0th face is the composite

$$A \times \mathbb{I}[2] \xrightarrow{A\times c} A \times \mathbb{I}[2] \times \mathbb{I} \xrightarrow{\Psi\times\mathbb{I}} B \times \mathbb{I} \xrightarrow{\beta} B$$

where $c\colon \mathbb{I}[2] \to \mathbb{I}[2] \times \mathbb{I}$ is the unique map defined by $0 \mapsto (2, 0)$, $1 \mapsto (0, 0)$, and $2 \mapsto (0, 1)$. The filler defines the desired 0th face

$$\begin{tikzcd} & fgf \arrow[dr, dash, "\beta f"] \arrow[d, phantom, "\Sigma"] & \\ fgf \arrow[ur, dash, "fg\psi^{-1}"] \arrow[rr, dash, "f\alpha^{-1}"'] & {} & f \end{tikzcd}$$

that completes the horn $\Gamma\colon A \times \Lambda^1[\mathbb{I}[3]] \to B$ whose filler defines the witness for the half-adjoint equivalence between α' and β. $\square$

D.5.4. Proposition. *A Kan fibration is a trivial fibration if and only if its fibers are contractible.*

Proof. By pullback stability of the class of trivial fibrations, it is clear that the fibers of a trivial fibration are contractible Kan complexes. For the converse, our task is to show that any lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

has a solution. The inclusion of the n-sphere into n-simplex factors as

$$\partial\Delta[n] \xrightarrow{\partial\Delta[n]\times i_1} \partial\Delta[n]\times\Delta[1] \hookrightarrow \Delta[n]\times\Delta[1] \xrightarrow{q} \Delta[n]$$

where q is the quotient map defined on vertices by $(j,i) \mapsto j \cdot i$. The inclusion $\partial\Delta[n]\times i_1$ is an anodyne extension; hence, there is a lift

$$\begin{tikzcd} \partial\Delta[n] \arrow[rrrr, "e"] \arrow[d, hook, "\partial\Delta[n]\times i_1"'] & & & & E \arrow[d, two heads, "p"] \\ \partial\Delta[n]\times\Delta[1] \arrow[r, hook] \arrow[urrrr, dashed, "\ell"] & \Delta[n]\times\Delta[1] \arrow[r, "q"'] & \Delta[n] \arrow[rr, "b"'] & & B \end{tikzcd}$$

This lift in turn produces a new lifting problem, which we restrict to the sphere sitting over the other endpoint of the cylinder as in the following diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\partial\Delta[n]\times i_0"] \arrow[d, hook] & \partial\Delta[n]\times\Delta[1] \arrow[rr, "\ell"] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times i_0"'] \arrow[urrr, dashed, "k"] & \Delta[n]\times\Delta[1] \arrow[r, "q"'] & \Delta[n] \arrow[r, "b"'] & B \end{tikzcd} \tag{D.5.5}$$

By construction, the sphere in this restricted lifting problem lives in the fiber over the vertex $\{0\}$ of the n-simplex b in B; hence, there exists a solution to this lifting as displayed since that fibre is contractible by assumption. Taking the pushout of left-hand square of (D.5.5) we obtain the lifting problem that is the right-hand square in the following diagram:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook, "\partial\Delta[n]\times i_1"] \arrow[d, hook] & \Delta[n]\times\{0\} \underset{\partial\Delta[n]\times\{0\}}{\cup} \partial\Delta[n]\times\Delta[1] \arrow[rr, "m"] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times i_1"'] & \Delta[n]\times\Delta[1] \arrow[r, "q"'] \arrow[urr, dashed] & \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

Here the horizontal map m is induced by maps ℓ and k in (D.5.5). The middle vertical is the Leibniz product of a monomorphism and anodyne extension; hence this map is an anodyne extension, and the displayed lift exists. Finally, restriction to the endpoint over the vertex $\{1\}$ of the cylinder, that is composing with the square on the left in the diagram above, defines a solution to the original lifting problem. ☐

D.6. Equivalence between slices and cones

The work of §D.4 also enables us to supply full proofs of the results sketched in §4.2, which follow easily from the combinatorial work done in §D.2. Recall in particular Proposition D.2.17, which shows

that the map of augmented simplicial sets

$$s^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m] \qquad \in s\mathcal{S}et_+^{\Delta_+ \times \Delta_+}$$

is a marked homotopy retract equivalence. We argue now that such maps induce equivalences upon mapping into quasi-categories.

D.6.1. Lemma. *Let A be a quasi-category and let $I \to J$ be a map of simplicial sets that extends to a marked homotopy equivalence. Then the induced map $A^J \xrightarrow{\sim} A^I$ is an equivalence of quasi-categories.*

Proof. Equip A with its natural marking so that by Theorem D.4.13 it defines a complicial set. The marked homotopy equivalence defines maps

$$A^J \to A^I, \quad A^I \to A^J, \quad A^I \to (A^I)^{\Delta[1]^\sharp}, \quad A^J \to (A^J)^{\Delta[1]^\sharp}.$$

By Lemma D.4.16, the complicial sets A^I and A^J are also naturally marked quasi-categories. By Exercise 1.1.iv, the inclusion $\Delta[1]^\sharp \hookrightarrow \mathbb{I}^\sharp$ is a complicial anodyne extension so by Proposition D.3.8, the restriction maps $(A^I)^{\mathbb{I}^\sharp} \xrightarrow{\sim} (A^I)^{\Delta[1]^\sharp}$ are trivial fibrations of marked simplicial sets. In particular, there exists lifts

$$\begin{tikzcd} & (A^I)^{\mathbb{I}^\sharp} \arrow[d, two heads, "\wr"] \\ A^I \arrow[r] \arrow[ur, dashed] & (A^I)^{\Delta[1]^\sharp} \end{tikzcd} \qquad \begin{tikzcd} & (A^J)^{\mathbb{I}^\sharp} \arrow[d, two heads, "\wr"] \\ A^J \arrow[r] \arrow[ur, dashed] & (A^J)^{\Delta[1]^\sharp} \end{tikzcd}$$

Forgetting markings, this data defines an inverse equivalence of categories to $A^J \to A^I$. □

In order to use this result to verify that the canonical map of augmented simplicial sets $s^{X,Y} \colon X \diamond Y \to X \star Y$ is also a weak equivalence in a suitable sense, we first prove that these diagrams are Reedy monomorphic.

D.6.2. Lemma. *The latching maps for the diagrams $F_\diamond, F_\star \in s\mathcal{S}et_+^{\Delta_+ \times \Delta_+}$ defined by*

$$F_\diamond^{n,m} := \Delta[n] \diamond \Delta[m] \qquad \text{and} \qquad F_\star^{n,m} := \Delta[n] \star \Delta[m]$$

are the maps

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\diamond}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \qquad \text{and} \qquad (\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]), \tag{D.6.3}$$

which are both monomorphisms. Hence, both $F_\diamond$ and $F_\star$ are Reedy monomorphic.

Proof. By direct calculation,

$$(\partial\Delta[n] \hookrightarrow \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m]) \cong \partial\Delta[n+1+m] \hookrightarrow \Delta[n+1+m],$$

which is clearly a monomorphism. It follows by Proposition C.2.9(vi) and Lemma C.5.9 that Leibniz joins of monomorphisms are monomorphisms.

For the analogous result for the fat join, observe first that $(\partial\Delta[-1] \hookrightarrow \Delta[-1]) \mathbin{\widehat{\diamond}} (X \hookrightarrow Y) \cong (X \hookrightarrow Y)$, since $\partial\Delta[-1]$ is the initial object and $\Delta[-1]$ is the unit for the fat join. Thus, it suffices to consider the case of terminally augmented simplicial sets X and Y, where we make use of the fact

observed in Definition 4.2.2 that for $n \geq 0$, $(X \diamond Y)_n \cong X_n \sqcup (\coprod_{[n]\twoheadrightarrow[1]} X_n \times Y_n) \sqcup Y_n$ Thus, we see that for any monomorphisms of simplicial sets $X \hookrightarrow Y$ and $U \hookrightarrow V$, the square

$$\begin{tikzcd} (U \diamond X)_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & (V \diamond X)_n \arrow[d, hook] \\ (U \diamond Y)_n \arrow[r, hook] & (V \diamond Y)_n \end{tikzcd}$$

is a pullback in the category of sets. For any pullback square comprised of monomorphisms in the category of sets, the pushout inside the square is constructed by the joint image of the lower and right-hand legs: in particular the map $(U \diamond Y)_n \cup_{(U \diamond X)_n} (V \diamond X)_n \hookrightarrow (V \diamond Y)_n$ is a monomorphism. Thus, the Leibniz fat join of two monomorphisms

$$(U \hookrightarrow V) \mathbin{\hat{\diamond}} (X \hookrightarrow Y)$$

is a monomorphism as claimed. By Proposition C.2.9(vi) this argument extends also to non-terminally augmented simplicial sets, since these can be built as cell complexes from the cell boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \geq -1$.

Now by Definition C.4.16, the latching map at the object $([n],[m]) \in \mathbf{\Delta}_+ \times \mathbf{\Delta}_+$ is the map on weighted colimits induced by the maps (D.6.3) of weights. Applying Corollary C.5.16 to the weak factorization system on $s\mathcal{S}et_+$ whose left class is the monomorphisms, it follows that the latching maps for $F_\diamond$ and $F_\star$ are monomorphisms as claimed, proving that these diagrams are Reedy monomorphic. □

Recall the natural comparison map of Lemma D.2.14 from the fat join of a pair of simplicial sets to the join of the pair of simplicial sets.

D.6.4. Proposition. *For all simplicial sets X and Y, the natural map $s\colon X \diamond Y \to X \star Y$ induces an equivalence of quasi-categories $A^{X\star Y} \xrightarrow{\sim} A^{X\diamond Y}$ for all quasi-categories A.*

Proof. For any pair of terminally augmented simplicial sets X and Y we define their **external product** $X\Box Y \in \mathcal{S}et^{\mathbf{\Delta}_+^{op}\times\mathbf{\Delta}_+^{op}}$ to be the functor that takes an object $([n],[m])$ to the set $X_n \times Y_m$. We can view this functor as a weight for the diagrams $F_\diamond$ and $F_\star$ introduced in Lemma D.6.2 and use Definition 7.1.4 and cocontinuity of the join and fat join bifunctors (in the full subcategory $s\mathcal{S}et \hookrightarrow s\mathcal{S}et_+$ of terminally augmented simplicial sets) to compute the weighted colimits:

$$\begin{aligned}\mathrm{colim}^{\mathbf{\Delta}_+^{op}\times\mathbf{\Delta}_+^{op}}_{X\Box Y} F_\diamond &\cong \int^{([n],[m])\in\mathbf{\Delta}_+\times\mathbf{\Delta}_+} \coprod_{X_n\times Y_m} \Delta[n]\diamond\Delta[m] \cong \left(\int^{[n]\in\mathbf{\Delta}_+}\coprod_{X_n}\Delta[n]\right)\diamond\left(\int^{[m]\in\mathbf{\Delta}_+}\coprod_{Y_m}\Delta[m]\right)\\ &\cong X\diamond Y\end{aligned}$$

Similarly, $\mathrm{colim}^{\mathbf{\Delta}_+^{op}\times\mathbf{\Delta}_+^{op}}_{X\Box Y} F_\star \cong X \star Y$.

A direct verification shows that the latching maps of the augmented bisimplicial set $X\Box Y \in \mathcal{S}et^{\mathbf{\Delta}_+^{op}\times\mathbf{\Delta}_+^{op}}$ are monomorphisms.[8] Hence by Corollary C.5.16, the weighted colimit functor

$$\mathrm{colim}^{\mathbf{\Delta}_+^{op}\times\mathbf{\Delta}_+^{op}}_{X\Box Y} -\colon s\mathcal{S}et^{\mathbf{\Delta}_+\times\mathbf{\Delta}_+} \to s\mathcal{S}et$$

is a left Leibniz bifunctor with respect to any weak factorization systems of our choosing on $s\mathcal{S}et$.

[8] This follows because $\mathbf{\Delta}_+ \times \mathbf{\Delta}_+$ is an **elegant Reedy category**: every element of a presheaf indexed by this category is a degeneracy of some non-degenerate element in a unique way [**13**].

By Lemma D.6.1, the components $s^{n,m} \colon \Delta[n] \diamond \Delta[m] \to \Delta[n] \star \Delta[m]$ are weak equivalences in the Joyal model structure, as these are characterized as those maps that induce equivalences upon mapping into an arbitrary quasi-category. So Proposition D.2.17 and Lemma D.6.2 establish that the natural transformation $s \colon F_\diamond \to F_\star$ is a pointwise weak equivalence between Reedy monomorphic objects in $s\mathcal{S}et^{\Delta_+ \times \Delta_+}$. By the dual of Ken Brown's Lemma C.1.10, the induced map on weighted colimits $s^{X,Y} \colon X \diamond Y \to X \star Y$ is then a weak equivalence in the Joyal model structure, which means exactly that it induces an equivalence of quasi-categories $A^{X \star Y} \xrightarrow{\sim} A^{X \diamond Y}$ for all quasi-categories A as claimed. $\square$

In particular, for any quasi-category A, there are natural equivalences $A^{\mathbb{1} \star J} \xrightarrow{\sim} A^{\mathbb{1} \diamond J}$ and $A^{J \star \mathbb{1}} \xrightarrow{\sim} A^{J \diamond \mathbb{1}}$ over $A \times A^J$. By Lemma 4.2.3 the codomains pullback to define the quasi-categories of cones under or over a diagram $d \colon J \to A$, respectively. However, as discussed in Warning 4.2.10, the domains do not pull back to the slice quasi-categories $A_{/d}$ and ${}^{d/}A$ of Definition D.2.8. Nonetheless, we can use the equivalence between the join and fat join constructions to prove that $A_{/d} \simeq \mathsf{Hom}_{A^J}(\Delta, d)$ and ${}^{d/}A \simeq \mathsf{Hom}_{A^J}(d, \Delta)$ over A and do so now.

D.6.5. Proposition. *For any diagram $d \colon J \to A$ indexed by a simplicial set J and valued in a quasi-category A, there are natural equivalences*

$$\begin{tikzcd} A_{/d} \arrow[rr, "\sim"] \arrow[dr, two heads, "\mathrm{res}"'] & & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[dl, two heads, "\mathrm{res}"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} {}^{d/}A \arrow[rr, "\sim"] \arrow[dr, two heads, "\mathrm{res}"'] & & \mathsf{Hom}_{A^J}(d, \Delta) \arrow[dl, two heads, "\mathrm{res}"] \\ & A & \end{tikzcd}$$

between the slice quasi-categories and the quasi-categories of cones.

Proof. We appeal to Lemma C.5.22 reprising the strategy used to prove Proposition 6.5.3.

To begin, recall the adjunction of Definition D.2.8

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star J"] \arrow[r, phantom, "\perp"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "{}^{-/}-"] \end{tikzcd}$$

which gives a correspondence, for a simplicial set I and a map $d \colon J \to A$, between maps $I \star J \to A$ under J and maps of simplicial sets $I \to {}^{A/}d$. A right adjoint to the fat join functor $- \diamond J \colon s\mathcal{S}et \to {}^{J/}s\mathcal{S}et$ can be calculated similarly. By the defining pushout of Definition 4.2.2, the data of a map $I \diamond J \to A$ under J displayed below-left transposes to the data displayed below-right

$$\begin{tikzcd} (I \times J) \sqcup (I \times J) \arrow[r, "\pi_I \sqcup \pi_J"] \arrow[d, hook] & I \sqcup J \arrow[d, hook] \arrow[dr, "- \sqcup d"] & \\ I \times \mathbb{2} \times J \arrow[r] \arrow[rr, bend right] & I \diamond J \arrow[ul, phantom, "\ulcorner" very near start] \arrow[r, dashed] & A \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2} \times J} \arrow[d, two heads] \\ & 1 \times A \arrow[r, "d \times \Delta"'] & A^J \times A^J \end{tikzcd}$$

whence we see that that the value of the right adjoint to $- \diamond J$

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\diamond J"] \arrow[r, phantom, "\perp"] & {}^{J/}s\mathcal{S}et \arrow[l, bend left, "\mathsf{Hom}_{-^J}(\Delta, -)"] \end{tikzcd}$$

at $d\colon J \to A$ defines the ∞-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ over d.

The natural map $s^{X,Y}\colon X \diamond Y \to X \star Y$ of Lemma D.2.14 defines a natural transformation

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend left, "{\Delta[\bullet]\diamond J}"] \arrow[r, bend right, "{\Delta[\bullet]\star J}"'] \arrow[r, phantom, "\Downarrow s"] & {}^{J/}s\mathcal{S}et \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{\Delta} \arrow[r, "よ"] & s\mathcal{S}et \arrow[r, bend left, "-\diamond J"] \arrow[r, bend right, "-\star J"'] \arrow[r, phantom, "\Downarrow s"] & {}^{J/}s\mathcal{S}et \end{tikzcd}$$

that Proposition D.6.4 is a pointwise weak equivalence in the (sliced) Joyal model structure. Note that by adjunction, $\mathsf{Hom}_{A^J}(\Delta, d)$ is isomorphic to the simplicial set defined by mapping from the cosimplicial object $\Delta[\bullet] \diamond J$ to $d\colon J \to A$ in ${}^{J/}s\mathcal{S}et$, and $A_{/d}$ is isomorphic to the simplicial set defined by mapping from the cosimplicial object $\Delta[\bullet] \star J$ to $d\colon J \to A$ in ${}^{J/}s\mathcal{S}et$. In particular, the mate of the natural transformation s defines a natural comparison map $\hat{s}\colon A_{/d} \to \mathsf{Hom}_{A^J}(\Delta, d)$; since $s^{\Delta[n],\varnothing}$ is the identity, this natural comparison map lies over A.

To apply Lemma C.5.22 to prove that this natural comparison is an equivalence, we need only verify that both cosimplicial objects are Reedy cofibrant, which can be verified by a direct calculation along the lines of that given in the proof of Lemma D.6.2. Lemma C.5.22 now proves that the map $\hat{s}\colon A_{/d} \to \mathsf{Hom}_{A^J}(\Delta, d)$ defines an equivalence of quasi-categories over A, as desired. ☐

In the case $J = \mathbb{1}$, a diagram $a\colon 1 \to A$ defines an **element** of the quasi-category A, and we have the same result with different notion.

D.6.6. Corollary. *Let $a\colon 1 \to A$ define an element of a quasi-category A. Then there are canonical equivalences $A_{/a} \simeq \mathsf{Hom}_A(A, a)$ and ${}^{a/}A \simeq \mathsf{Hom}_A(a, A)$ over A.*

Proof. When $J = \mathbb{1}$, the constant diagram functor $\Delta\colon A \to A^J$ appearing in Proposition D.6.5 reduces to the identity functor on A. ☐

D.7. Equivalences and saturation

Having proven the required results about quasi-categories and isofibrations we indulge in a final section that develops a bit more of the general theory of complicial sets.

In Lemma D.4.2, we observed that the marked edges in a complicial set should be interpreted as equivalences, in a suitable sense. Below we demonstrate that a similar interpretation is appropriate for the higher-dimensional marked simplices as well. Consequently, if a complicial set has a property such as included in Definition D.4.4 of the natural marking that all simplices in dimension $r > n$ are marked—i.e., is *n-trivial* in terminology we will presently introduce—we may interpret that condition as demanding that all simplices in dimension $r > n$ are weakly invertible. Having understood that every marked simplex in a complicial set is an equivalence, we are lead to consider complicial sets that satisfy the converse of this condition, in which every equivalence is marked. Such *saturated* complicial sets are especially important, and we introduce the terminology *n-complicial set* to describe an n-trivial saturated complicial set. The Kan complexes are precisely the *0-complicial sets* while the quasi-categories are precisely the *1-complicial sets*. As this pattern suggests, the *n-complicial sets* define a well-behaved model for (∞, n)-categories in the sense that the full subcategory of such defines a cartesian closed ∞-cosmos, as we prove Proposition E.3.9.

To start our exploration of the higher-dimensional invertible simplices in a complicial set, we introduce an equivalence relation on simplices in a complicial set. To define it, we extend the notation of Definition D.1.6: for $n \geq 1$ and $0 \leq j < k \leq n$ write $\Delta^{j,k}[n]$ for the pushout of $\Delta^j[n]_e \hookleftarrow \Delta[n] \hookrightarrow_e \Delta^k[n]$.

D.7.1. DEFINITION. For every marked simplicial set A, we define an equivalence relation $\sim_{A_n}$ on the set of n-simplices that is generated by the following relation: two n-simplices α, β in a marked simplicial set A are **complicial companions** if there exists an $(n+1)$-simplex τ in A and $0 \leq k \leq n$ so that

- τ is both k-admissible and $(k+1)$-admissible.
- $\alpha = \tau \cdot \delta^k$ and $\beta = \tau \cdot \delta^{k+1}$.

$$\begin{tikzcd}
\Delta[n] \arrow[r, hook, "\delta^k"] \arrow[dr, "\alpha"'] & \Delta^{k,k+1}[n+1] \arrow[d, "\tau"] & \Delta[n] \arrow[l, hook', "\delta^{k+1}"'] \arrow[dl, "\beta"] \\
& A &
\end{tikzcd}$$

For example, vertices of a marked simplicial set are complicial companions just when they are connected by a finite zig-zag of marked edges.

D.7.2. LEMMA. *If A is a complicial set and if $\alpha \simeq_{A_n} \beta$ are n-simplices of A that are complicial companions, then α is marked if and only if β is marked.*

PROOF. By induction it suffices to consider the generating relation. Suppose α and β appear, respectively, as the kth and $(k+1)$th faces of a k-admissible and $(k+1)$-admissible $(n+1)$-simplex τ; in particular, these conditions imply apriori that all of the codimension-one faces of τ except perhaps α and β are marked. If β is marked, then the markings of the k-admissible $(n+1)$-simplex τ extend as indicated in the solid arrow diagram

$$\begin{tikzcd}
\Delta^k[n+1] \arrow[r, hook, "e"'] \arrow[rr, bend left, "\tau"] & \Delta^k[n+1]' \arrow[r] \arrow[d, hook, "e"'] & A \\
& \Delta^k[n+1]'' \arrow[ur, dashed] &
\end{tikzcd}$$

and hence the dashed extension exists by Definition D.1.9. Since all codimension faces of $\Delta^k[n+1]''$ are marked, this tells us that α is marked also. If α is assumed to be marked instead, we apply the analogous argument, regarding τ as an $(k+1)$-admissible $(n+1)$-simplex. □

D.7.3. LEMMA. *In a complicial set A, every n-simplex α is the complicial companion of a pair of n-simplices $\hat{\alpha}$ and $\check{\alpha}$ with the property that $\hat{\alpha} \cdot \delta^0$ is maximally marked and $\check{\alpha} \cdot \delta^n$ is maximally marked.*

In fact, as the proof will show, the face $\hat{\alpha} \cdot \delta^0$ can be taken to be degenerate at the final vertex of α, while the face $\check{\alpha} \cdot \delta^n$ can be arranged to be degenerate at the initial vertex of α.

PROOF. We explain the construction of the complicial companion n-simplex $\check{\alpha}$ with $\check{\alpha} \cdot \delta^n$ degenerate at the initial vertex of α; the construction of $\hat{\alpha}$ is the odd dual of Remark D.1.8. To prove that

$\alpha \simeq_{A_n} \hat{\alpha}$ we build a diagram starting from the left:

$$\begin{tikzcd}
\Delta[n] \arrow[dr, "\delta^1"] \arrow[ddrrrr, dotted, bend right, "\alpha=\alpha^1"'] & & \Delta[n] \arrow[dl, "\delta^2"'] \arrow[dr, "\delta^2"] \arrow[ddrr, dotted, "\alpha^2"'] & & \Delta[n] \arrow[dl, "\delta^3"'] \arrow[dd, dotted, "\alpha^3"] & \cdots & \Delta[n] \arrow[dr, "\delta^{n-1}"] \arrow[ddll, dotted, "\alpha^{n-1}"] & & \Delta[n] \arrow[dl, "\delta^n"'] \arrow[ddllll, dotted, bend left, "\alpha^n=\check{\alpha}"] \\
& \Delta^{1,2}[n+1] \arrow[drrr, bend right, "\tau^2"] & & \Delta^{2,3}[n+1] \arrow[dr, "\tau^3"] & & \cdots & & \Delta^{n-1,n}[n+1] \arrow[dlll, bend left, "\tau^n"'] & \\
& & & & A & & & &
\end{tikzcd}$$

To explain the idea of the construction of the sequence of complicial companion simplices

$$\alpha = \alpha^1 \simeq_{A_n} \alpha^2 \simeq_{A_n} \alpha^3 \cdots \alpha^{n-1} \simeq_{A_n} \alpha^n = \check{\alpha}$$

consider the sequence of n composable edges $f_{n-1,n}, \dots, f_{1,2}$ along the spine of $\alpha = \alpha^1$. In α^{n-1}, the last $n-2$ composable edges along its spine are same, the 2nd edge is a composite $f_{1,2} \cdot f_{0,1}$, and the initial edge is degenerate. In α^2, the last $n-3$ composable edges are the same, the 3rd edge is a composite $f_{2,3} \cdot f_{1,2} \cdot f_{0,1}$, and the first two edges are degenerate. In $\alpha^n = \check{\alpha}$, the last edge is a composite of the original sequence of n edges, while the initial $n-1$-edges are degenerate.

Inductively starting from the left, we will build the simplex τ^i by filling an admissible horn $\Lambda^i[n+1]$ whose $i-1$th face is the simplex α^{i-1} defined in the previous stage and whose ith face is the simplex α^i used in the next stage. The admissible horn is itself built by extending a "generalized admissible horn" that is defined by gluing α^{i-1} to a degenerate simplex of dimension i. To extend from the generalized admissible horn to the admissible horn $\Lambda^i[n+1]$, a series of admissible simplices are attached to admissible inner horns $\Lambda^k[m]$ starting in dimension $m = 2$. For each of these admissible inner horns, the vertex k is mapped to the vertex i of the simplex τ^i being built, which is to say that all of the composites are being formed at a single vertex.

Writing α^1 for the original n-simplex α, the simplex τ^2 is built by extending a generalized admissible horn defined by the pushout

$$\begin{tikzcd}
& \Delta[1] \arrow[dl, "\{0,1\}"'] \arrow[dr, "\delta^1"] \arrow[ddr, equal] \arrow[dd, phantom, "\ulcorner" very near end] & \\
\Delta[n] \arrow[dr] \arrow[ddr, bend right, "\alpha^1"'] & & \Delta[2] \arrow[dl] \arrow[d, two heads, "\sigma^0"] \\
& H^2 \arrow[d, dashed] & \Delta[1] \arrow[dl, "\alpha^1 \cdot \{0,1\}"] \\
& A &
\end{tikzcd}$$

To extend this generalized admissible horn to a horn $\Lambda^2[n+1]$ whose filler defines τ^2 and α^2, we first attach fillers for each injection $\Lambda^1[2] \to H^2$ that is not already filled in H^2 that sends the inner vertex to the vertex 2 of H^2, which is the final vertex of the $\Delta[1]$ along which the $\Delta[n]$ and $\Delta[2]$ are being glued. We then attach fillers for each injection of a not-already-filled inner k-admissible 3-horn that sends the vertex k to the vertex 2 of H^2. Continuing inductively, we attach fillers for each injection of

a not already filled k-admissible m simplex that sends k to the vertex 2. After attaching simplices in dimension $m = n - 1$, we will have built a 2-admissible $n + 1$ horn extending the original generalized admissible horn H^2. Its filler defines τ^2 and α^2. Note by construction that the initial edge of α^2 is degenerate.

Assume now we've defined the simplex α^{i-1} as the $i - 1$th face of $\tau^{i-1} \colon \Delta^{i-2,i-1}[n + 1] \to A$. The simplex τ^i is built by extending a generalized complicial horn defined by the pushout

$$
\begin{tikzcd}
& \Delta[i-1] \arrow[dl, "\text{first}"'] \arrow[dr, "\delta^1"] \arrow[ddr, equals] & \\
\Delta[n] \arrow[dr] \arrow[ddr, bend right, "\alpha^{i-1}"'] & & \Delta[i] \arrow[dl] \arrow[d, "\sigma^0"] \\
& H^i \arrow[d, dashed] \arrow[u, phantom, "\ulcorner" near start] & \Delta[i-1] \arrow[dl, "\alpha^i \cdot \delta^{\text{first}}"] \\
& A &
\end{tikzcd}
$$

We extend H^i to an admissible horn $\Lambda^i[n + 1]$ by attaching fillers for each injection of a k-admissible m-horn that is not filled already which sends the vertex k to the vertex i. The filler for this admissible horn defines τ^i and α^i. Note that by construction the image of the face of H^i spanned by the vertices $0, \dots, i-1$ is degenerate on the face spanned by the vertices $0, \dots, i-2$ of α^{i-1}, which is degenerate at the first vertex of α. Hence, the face of α^i spanned by the vertices $0, \dots, i-1$ is degenerate at the first vertex of α. This constructs the desired sequence of complicial companion simplices connecting α to an n-simplex $\check{\alpha}$ whose initial codimension-1 face is degenerate, and in particular maximally marked. ☐

D.7.4. Definition. A marked simplicial set X is **n-trivial** if all r-simplices are marked for $r > n$.

The full subcategory of n-trivial marked simplicial sets is reflective and coreflective

$$
\begin{tikzcd}
s\mathcal{S}et^+_{n\text{-tr}} \arrow[r, hook] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & s\mathcal{S}et^+ \arrow[l, bend right=40, "\mathrm{trv}_n"'] \arrow[l, bend left=40, "\mathrm{core}_n"]
\end{tikzcd}
$$

in the category of marked simplicial sets. That is **n-trivialization** defines an idempotent monad on $s\mathcal{S}et^+$ with unit the entire inclusion

$$X \hookrightarrow_e \mathrm{trv}_n X$$

of a marked simplicial set X into the marked simplicial set $\mathrm{trv}_n X$ with the same marked simplices in dimensions $1, \dots, n$, and with all higher simplices "made thin." A complicial set is **n-trivial** if this map is an isomorphism.

The **n-core** $\mathrm{core}_n X$, defined by restricting to those simplices whose faces above dimension n are all thin in X, defines an idempotent comonad with counit the regular inclusion

$$\mathrm{core}_n X \hookrightarrow_r X.$$

Again, a complicial set is n-trivial just when this map is an equivalence. As is always the case for a monad-comonad pair arising in this way, these functors are adjoints: $\mathrm{trv}_n \dashv \mathrm{core}_n$.

The subcategories of n-trivial marked simplicial sets assemble to define a string of inclusions with adjoints

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, hook, "(-)^\sharp", "\cong"'] & s\mathcal{S}et^+_{0\text{-trv}} \arrow[r, hook, phantom, "\perp"] \arrow[r, hook] & s\mathcal{S}et^+_{1\text{-trv}} \arrow[l, bend right, "\mathrm{trv}_1"'] \arrow[l, bend left, "\mathsf{core}_1"] \arrow[r, hook] & \cdots \arrow[r, hook] & s\mathcal{S}et^+_{(n-1)\text{-trv}} \arrow[r, hook] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^+_{n\text{-trv}} \arrow[l, bend right, "\mathrm{trv}_{n-1}"'] \arrow[l, bend left, "\mathsf{core}_{n-1}"] \arrow[r, hook] & \cdots \arrow[r, hook] & s\mathcal{S}et^+ \end{tikzcd}$$

that filter the inclusion of simplicial sets, considered as maximally marked marked simplicial sets, into the category of all marked simplicial sets.

D.7.5. Lemma. *The n-core of a complicial set is a complicial set.*

Proof. Exercise D.7.i. □

By contrast, the n-trivialization functor does not necessarily preserve complicial sets; see Exercise D.7.ii.

D.7.6. Remark. By contrast, the left adjoint, which just marks simplices in the appropriate dimension without changing the underlying simplicial set, does not preserve complicial structure; this construction is too naive to define the "freely invert n-arrows" functor from (∞, n)-categories to $(\infty, n-1)$-categories, whose construction for $n = 1$ is given by Proposition D.4.24. For instance, the minimally marked 1-simplex $\Delta[1]$ defines a 1-trivial complicial set, but $0\text{-trv}(\Delta[1]) = \Delta[1]^\sharp$ is not a 0-trivial complicial set because its underlying simplicial set is not a Kan complex; see Exercise D.1.ii.

Recall Lemma D.4.3, which described two markings for the nerve of a 1-category which make it into a 1-trivial complicial set: marking only the identity arrows or marking all isomorphisms. Clearly the first of these options defines the minimal 1-trivial marking that makes the nerve of a 1-category into a complicial set. By Lemma D.4.2, the latter option defines the maximal marking that makes the nerve of a 1-category into a complicial set. We now introduce terminology to describe "maximally marked" 1-trivial complicial sets.

D.7.7. Definition. A complicial set is 1-**saturated** if every equivalence it contains among its edges is marked.[9]

If a quasi-category is given a 1-saturated 1-trivial marking, then it is necessarily the natural marking of Definition D.4.4. Conversely, Theorem D.4.13 proves that the underlying simplicial set of any 1-trivial saturated complicial set is a quasi-category and the markings recover the natural markings.

Our aim is now to extend these notions to higher dimensions. To define equivalences and saturation in any dimension, it is helpful to reformulate the notion of 1-equivalence as a lifting property. To state it, we make use of a 3-simplex $\Delta[3]_{\text{eq}}$ in which the edges $\{02\}$ and $\{13\}$ are marked as well as all simplices in dimension greater than 1.

D.7.8. Proposition.

(i) A 1-simplex in a complicial set A is an equivalence if and only if defines the $\{12\}$-edge of a 3-simplex $\Delta[3]_{\text{eq}} \to A$.

[9] Since the characterization of equivalences among edges requires prior argreement about which 2-simplices are marked, we typically apply Definition D.7.7 to 1-trivial complicial sets, in which case we say a 1-saturated 1-trivial complicial set is simply "saturated." This terminology will agree with Definition D.7.9.

(ii) A complicial set is 1-saturated if and only if it admits extensions along the entire inclusion

$$\begin{tikzcd}
\Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook, "e"'] & A \\
\Delta[3]^\sharp \arrow[ur, dashed]
\end{tikzcd}$$

Proof. If f is an equivalence, then the witnessing 2-simplices of Definition D.4.1 define the 3rd and 0th faces of an admissible horn $\Lambda^{1,2}[3] \to A$ that fills to define a thin 3-simplex define a thin 3-simplex

$$\begin{tikzcd}
& 1 \arrow[r, "f"] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] \\
0 \arrow[ur, "e"] \arrow[urr, equal] \arrow[rrr, "h"'] & & {} & 3
\end{tikzcd}
\simeq
\begin{tikzcd}
& 1 \arrow[r, "f"] \arrow[drr, equal] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] \\
0 \arrow[ur, "e"] \arrow[rrr, "h"'] & & {} & 3
\end{tikzcd}$$

of the form stipulated in (i). Conversely, the middle edge of any such 3-simplex then admits left and right inverses in the weaker sense of composing to marked edges, as witnessed by the 3rd and 0th faces:

$$\begin{tikzcd}
& 1 \arrow[dr, "f"] \arrow[d, phantom, "\alpha\ \wr"] \\
0 \arrow[ur, "e"] \arrow[rr, "\sim", "j"'] & {} & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[dr, "g"] \arrow[d, phantom, "\beta\ \wr"] \\
1 \arrow[ur, "f"] \arrow[rr, "\sim", "k"'] & {} & 3
\end{tikzcd}$$

By Lemma D.4.2, the marked edge j admits a right equivalence inverse i, and from this data we may build a 1-admissible 3-horn in which all 2-simplex faces are marked.

$$\begin{tikzcd}
& 1 \arrow[r, "e"] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "f"] \arrow[d, phantom, "\wr"] \\
0 \arrow[ur, "i", "\sim"'] \arrow[urr, dashed, "e\cdot i"'] \arrow[rrr, equal] & & {} & 3
\end{tikzcd}
\simeq
\begin{tikzcd}
& 1 \arrow[r, "e"] \arrow[drr, "\sim", "j"'] \arrow[d, phantom, "\wr"] & 2 \arrow[dr, "f"] \arrow[d, phantom, "\alpha\ \wr"] \\
0 \arrow[ur, "i", "\sim"'] \arrow[rrr, equal] & & {} & 3
\end{tikzcd}$$

By a complicial anodyne extension followed by a complicial thinness extension, we produce a thin 2-simplex that witnesses that f admits a right equivalence inverse in the stricter sense of of Definition D.4.1. The dual argument constructs the desired left equivalence inverse.

Now for (ii) recall that a complicial set is 1-saturated if every 1-equivalence is marked. By (i), this implies that for any $\Delta[3]_{\mathrm{eq}} \to A$, the image of the middle edge should be marked. But in this case the complicial thinness extensions $\Delta^2[2]' \hookrightarrow_e \Delta^2[2]''$ and $\Delta^0[2]' \hookrightarrow_e \Delta^0[2]''$ imply that the first and last edges must be marked as well. This proves that a complicial set is 1-saturated if and only if it enjoys the postulated lifting property. □

There are similar extension problems that can be used to define equivalences and detect saturation in any dimension, which are defined by forming the join of the inclusion $\Delta[3]_{\mathrm{eq}} \hookrightarrow_e \Delta[3]^\sharp$ with simplices on one side or the other. It is somewhat subtle to characterize the k-equivalences in a complicial set unless that complicial set is n-trivial for some n because the notion of k-equivalence depends upon composites of k-simplices, which are witnessed by $k+1$-dimensional equivalences. In a general complicial set, the class of equivalences is defined *coinductively* rather than *inductively*. Despite this, there

is a surprisingly simple characterization of the saturated complicial sets, in which all equivalences (yet to be defined) are marked:

D.7.9. Definition (saturated complicial set). A complicial set is **saturated** if it admits extensions along the set of entire inclusions

$$\{\Delta[m] \star \Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow_e \Delta[m] \star \Delta[3]^\sharp \star \Delta[n] \mid n, m \geq -1\}.$$

In fact, it suffices to require only extensions

$$\begin{tikzcd} \Delta[3]_{\mathrm{eq}} \star \Delta[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[3]^\sharp \star \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Delta[n] \star \Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \star \Delta[3]^\sharp \arrow[ur, dashed] & \end{tikzcd}$$

along inclusions of one-sided joins of the inclusion $\Delta[3]_{\mathrm{eq}} \hookrightarrow_e \Delta[3]^\sharp$ with an n-simplex for each $n \geq -1$, and as it turns out only the left-handed joins or right-handed joins are needed.

By Proposition D.7.8, the $n = -1$ case of Definition D.7.9 asserts that every 1-equivalence in A, defined relative to the marked 2-simplices and marked 3-simplices, is marked. By Proposition D.7.8 again, the general extension property

$$\begin{tikzcd} \Delta[n] \arrow[r, hook] \arrow[rr, bend left, "\sigma"] & \Delta[3]_{\mathrm{eq}} \star \Delta[n] \arrow[r] \arrow[d, hook] & A \\ & \Delta[3]^\sharp \star \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \Delta[3]_{\mathrm{eq}} \arrow[r] \arrow[d, hook] & A_{/\sigma} \\ \Delta[3]^\sharp \arrow[ur, dashed] & \end{tikzcd}$$

asserts that every 1-equivalence in the slice complicial set $A_{/\sigma}$ is marked.

At first blush, Definition D.7.9 does not seem to be general enough. In the case of a vertex $\sigma\colon \Delta[0] \to A$, 1-equivalences in $A_{/\sigma}$ define 2-simplices in A whose $\{01\}$-edge is a 1-equivalence. In particular, a generic 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow \alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

with no 1-equivalence edges along its boundary, does not define a 1-equivalence in any slice complicial set. However, there are admissible 3-horns that can be filled to define the pasted composites of α with 1_f and 1_g, respectively:

$$\begin{tikzcd} & 1 \arrow[r, "f"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow \alpha"] & \\ 0 \arrow[ur, equal] \arrow[urr, "f"'] \arrow[rrr, "h"'] \arrow[r, phantom, "="] & {} & {} & 3 \end{tikzcd} \quad \simeq \quad \begin{tikzcd} & 1 \arrow[r, "f"] \arrow[drr, "gf"'] \arrow[d, phantom, "\Uparrow \check{\alpha}"] & 2 \arrow[dr, "g"] \arrow[d, phantom, "\wr"] & \\ 0 \arrow[ur, equal] \arrow[rrr, "h"'] & {} & {} & 3 \end{tikzcd}$$

$$\begin{tikzcd} & 1 \arrow[r, "g"] \arrow[dr, phantom, "\wr"] & 2 \arrow[dr, equals] \\ 0 \arrow[ur, "f"] \arrow[urr, "gf"'] \arrow[rrr, "h"'] & & \hat{\alpha}\Uparrow & 3 \end{tikzcd} \simeq \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[drr, "g"'] & 2 \arrow[dr, equals] \arrow[d, phantom, "\shortparallel"] \\ 0 \arrow[ur, "f"] \arrow[rrr, "h"'] & \Uparrow\alpha & & 3 \end{tikzcd}$$

By the complicial thinness extension property, if any of α, $\hat{\alpha}$, or $\check{\alpha}$ are marked, then all of them are. That is, α, $\hat{\alpha}$, and $\check{\alpha}$ are complicial companions in the sense of Definition D.7.1. Lemma D.7.3 generalizes this construction to simplices of arbitrary dimensions.

D.7.10. Definition. In an n-trivial complicial set, an n-simplex $\sigma\colon \Delta[n] \to A$ is an n-**equivalence** if it admits an extension

$$\begin{tikzcd} \Delta[n] \arrow[r, "\sigma"] \arrow[d, hook] & A \\ \Delta[3]_{\mathrm{eq}} \star \Delta[n-2] \arrow[ur, dashed] \end{tikzcd}$$

along the map $\Delta[n] \hookrightarrow \Delta[3]_{\mathrm{eq}} \star \Delta[n-2]$ whose image includes the edge $\{12\}$ of $\Delta[3]_{\mathrm{eq}}$ and all of the vertices of $\Delta[n-2]$, or if it is the complicial companion of an n-simplex which admits such an extension.

The set of n-equivalences identified by Definition D.7.10 depends on the marked $(n+1)$-simplices, which is the reason we have only stated this definition for an n-trivial complicial set. The n-equivalences in a generic complicial set are characterized by an inductive definition, the formulation of which we leave to the reader.

Finally, we introduce special terminology for those complicial sets that most closely represent (∞, n)-categories.

D.7.11. Definition (n-complicial set). A marked simplicial set is an n-**complicial set** if and only if it is a complicial set that is n-trivial and saturated.

D.7.12. Example. For instance, by Exercise D.1.ii, the 0-complicial sets are precisely the Kan complexes, with their maximal marking. By Theorem D.4.13, the 1-complicial sets are precisely the quasi-categories, with their natural marking, which is the largest marking which makes a quasi-category into a complicial set.

D.7.13. Digression (the Verity model structure for n-complicial sets). The category of marked simplicial sets bears a cartesian closed, cofibrantly generated model structure whose fibrant objects are exactly the n-complicial sets and whose cofibrations are the monomorphisms, which is obtained as a left Bousfield localization of the model structure described in Digression D.1.18. In [**102**, §6.2-4] Verity describes a general paradigm for obtaining model structures that localize the model structure for complicial sets. The verification in this particular case is completed by Ozornova and Rovelli [**68**, 1.25].

The closure properties of complicial sets extend to saturated complicial sets and n-complicial sets. The proof, which appears in Corollary D.7.16, depends on two technical lemmas, the first and harder of which was first proven by Ozornova and Rovelli [**68**, B.5-6].

D.7.14. Lemma. *The Leibniz products*

(i) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \mathbin{\widehat{\times}} (\partial\Delta[m] \hookrightarrow \Delta[m])$ *and*

(ii) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \mathbin{\widehat{\times}} (\Delta[m] \hookrightarrow \Delta[m]_t)$

are in the class of maps that is cellularly generated by the complicial anodyne extensions together with the saturation extensions of Definition D.7.9.

Proof. When $m = 0$, the map considered in (i) above reduces to the inclusion $\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]$ which is one of the generating saturation extensions, so we may assume that $m > 0$. We begin by observing that the only k-simplices that are marked in the codomain of

$$(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m] \underset{(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \partial\Delta[m]}{\cup} (\Delta[3]^\sharp \star \Delta[n]) \times \partial\Delta[m] \hookrightarrow_e (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]$$

but not in the domain are k-simplices for $m < k \leq n + 2$ of the form

$$\Delta[1] \star \Delta[k-2] \xrightarrow{(\gamma \star \alpha, \beta)} (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]$$

where

- γ is one of the four edges $\{01\}$, $\{12\}$, $\{23\}$, $\{03\}$ of $\Delta[3]_{\mathrm{eq}}$,
- $\alpha \colon \Delta[k-2] \rightarrowtail \Delta[n]$ is injective, and
- $\beta \colon \Delta[k] \twoheadrightarrow \Delta[m]$ is surjective.

These conditions explain the bounds on k given above.

Since $\beta \colon [k] \twoheadrightarrow [m]$ is surjective and order-preserving, $\beta(0) = 0$ and $\beta(1)$ equals either 0 or 1. We factor the inclusion as a sequence of two pushouts

$$\begin{tikzcd}
\coprod \Delta[3]_{\mathrm{eq}} \star \Delta[k-2] \arrow[rr, "{(\mathrm{id} \star \alpha, \beta\sigma^0\sigma^0)}"] \arrow[d, hook, "e"'] & & (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m] \cup (\Delta[3]^\sharp \star \Delta[n]) \times \partial\Delta[m] \arrow[d, hook, "e"'] & \\
\coprod \Delta[3]^\sharp \star \Delta[k] \arrow[rr] \arrow[urr, phantom, "\ulcorner" very near end] & & \bullet \arrow[d, hook, "e"'] & \coprod \Delta^1[k+1]' \arrow[l, "{((\gamma \star \alpha)\sigma^1, \beta\sigma^0)}"'] \arrow[d, hook, "e"] \arrow[dl, phantom, "\urcorner" very near end] \\
& & (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] & \coprod \Delta^1[k+1]'' \arrow[l]
\end{tikzcd}$$

where the left-hand coproduct is over all pairs (α, β) as above with $\beta(1) = 0$ and the right-hand coproduct is over all triples (γ, α, β) with $\beta(1) = 1$. A detailed argument that the pair of attaching maps preserves the required marked simplices is given in [**68**, B.5]. The top pushout marks all triples (γ, α, β) with $\beta(1) = 0$, while the bottom pushout marks all triples (γ, α, β) with $\beta(1) = 1$.

For the map considered in (i), observe that the only simplices that are marked in the codomain of

$$(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]_t \underset{(\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]}{\cup} (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] \hookrightarrow_e (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]_t$$

but not in the domain are m-simplices

$$\Delta[1] \star \Delta[m-2] \xrightarrow{(\gamma \star \alpha, \mathrm{id})} (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]$$

where

- γ is one of the four edges $\{01\}$, $\{12\}$, $\{23\}$, $\{03\}$ of $\Delta[3]_{\mathrm{eq}}$ and
- $\alpha \colon \Delta[m-2] \rightarrowtail \Delta[n]$ is injective,

which implies that we must have $m-2 \leq n$. To mark the simplices $(\gamma \star \alpha, \mathrm{id})$ we take a pushout along a suitable complicial thinness extension:

$$\begin{tikzcd} \coprod \Delta^1[m+1]' \arrow[r, "{(\gamma\star\alpha)\sigma^1,\sigma^0)}"] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\ulcorner" very near end] & (\Delta[3]_{\mathrm{eq}} \star \Delta[n]) \times \Delta[m]_t \cup (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m] \arrow[d, hook, "e"'] \\ \Delta^1[m+1]'' \arrow[r] & (\Delta[3]^\sharp \star \Delta[n]) \times \Delta[m]_t \end{tikzcd}$$

where the coproduct is over all unmarked m-simplices $(\gamma \star \alpha, \mathrm{id})$. A detailed argument that the attaching map preserves the required marked simplices is given in [**68**, B.6]. This completes the verification that the two Leibniz products are in the saturated class generated by the complicial anodyne extensions and the saturation maps. $\square$

D.7.15. Lemma. *The Leibniz joins*

(i) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \mathbin{\widehat{\star}} (\partial\Delta[m] \hookrightarrow \Delta[m])$,
(ii) $(\partial\Delta[m] \hookrightarrow \Delta[m]) \mathbin{\widehat{\star}} (\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n])$,
(iii) $(\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n]) \mathbin{\widehat{\star}} (\Delta[m] \hookrightarrow \Delta[m]_t)$, *and*
(iv) $(\Delta[m] \hookrightarrow \Delta[m]_t) \mathbin{\widehat{\star}} (\Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow \Delta[3]^\sharp \star \Delta[n])$

are in the class of maps that is cellularly generated by the complicial anodyne extensions together with the saturation extensions of Definition D.7.9.

Proof. The Leibniz join considered in (i) is evidentally a pushout

$$\begin{tikzcd} \Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m] \arrow[r, "\mathrm{id}"] \arrow[d, hook, "e"'] \arrow[dr, phantom, "\ulcorner" very near end] & \Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m] \cup \Delta[3]^\sharp \star \Delta[n] \star \partial\Delta[m] \arrow[d, hook, "e"'] \\ \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \arrow[r, "\mathrm{id}"] & \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \end{tikzcd}$$

of a generating saturation extension. Similarly, the Leibniz join considered in (ii) is evidentally a pushout of the saturation extension

$$\Delta[m] \star \Delta[3]_{\mathrm{eq}} \star \Delta[n] \hookrightarrow_e \Delta[m] \star \Delta[3]^\sharp \star \Delta[n].$$

The Leibniz join

$$\Delta[3]_{\mathrm{eq}} \star \Delta[n] \star \Delta[m]_t \cup \Delta[3]^\sharp \star \Delta[n] \star \Delta[m] \xhookrightarrow[e]{} \Delta[3]^\sharp \star \Delta[n] \star \Delta[m]_t$$

considered in (iii) is the identity (and not just entire), as is the Leibniz join considered in (iv). $\square$

Lemmas D.7.14 and D.7.15 enable us to expeditiously prove the following result, extending Corollary D.3.12.

D.7.16. Corollary.

(i) For any saturated complicial set A and marked simplicial set, A^X is again a saturated complicial set.
(ii) For any n-complicial set A and marked simplicial set, A^X is again an n-complicial set.
(iii) For any complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!/f}$ and ${}^{f/\!/}A$ are complicial sets. Moreover, for any complicial isofibration $p\colon A \twoheadrightarrow B$ of complicial sets, any

monomorphism of marked simplicial sets $i\colon X \hookrightarrow Y$, and any map $f\colon Y \to A$, the Leibniz marked slices

$$A_{/\!\!/ f} \to A_{/\!\!/ fi} \underset{B_{/\!\!/ pfi}}{\times} B_{/\!\!/ pf} \qquad \text{and} \qquad {}^{f/\!\!/}A \to {}^{fi/\!\!/}A \underset{{}^{pfi/\!\!/}B}{\times} {}^{pf/\!\!/}B$$

are complicial isofibrations.

(iv) For any saturated complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!\!/ f}$ and ${}^{f/\!\!/}A$ are saturated complicial sets.

(v) For any n-complicial set A and map of marked simplicial sets $f\colon X \to A$, the marked slices $A_{/\!\!/ f}$ and ${}^{f/\!\!/}A$ are n-complicial sets.

Proof. The analogue of (i) and (ii) for a complicial set A is proven in Corollary D.3.12, so we pick up where that argument left off. In light of Definition D.7.9, (i) follows immediately from Lemma D.7.14. To extend this result to prove (ii), we need only verify that we may solve lifting problems

$$\begin{tikzcd} \Delta[m] \times X \arrow[r] \arrow[d, hook, "e"'] & A \\ \Delta[m]_t \times X \arrow[ur, dashed] \end{tikzcd}$$

for $m > n$ under the hypothesis that A is an n-trivial saturated complicial set. On account of the pushout diagram

$$\begin{tikzcd} \coprod\limits_{\sigma \in tX_m} \Delta[m] \arrow[r, "(\mathrm{id},\sigma)"] \arrow[d, hook, "e"'] & \Delta[m] \times X \arrow[r] \arrow[d, hook, "e"'] & A \\ \coprod\limits_{\sigma \in tX_m} \Delta[m]_t \arrow[r] \arrow[urr, dotted] & \Delta[m]_t \times X \arrow[ul, phantom, "\ulcorner" very near start] \arrow[ur, dashed] \end{tikzcd}$$

where the coproduct is over the set of thin m-simplices in X, this is clear.

The final three statements refer to the marked slices defined by Lemma D.2.20. Lemma D.2.19 proves (iii), while this result combines with Lemma D.7.15 to prove (iv). For the final statement (iv), we need to verify that if A is n-trivial, then the lifting problems

$$\begin{tikzcd} \Delta[m] \star X \arrow[r] \arrow[d, hook, "e"'] & A \\ \Delta[m]_t \star X \arrow[ur, dashed] \end{tikzcd} \qquad \begin{tikzcd} X \star \Delta[m] \arrow[r] \arrow[d, hook, "e"'] & A \\ X \star \Delta[m]_t \arrow[ur, dashed] \end{tikzcd}$$

can be solved for all $m > n$. Since the only simplices that are marked in the codomain of the entire extensions but not in the domain have dimension greater than n, this is clear. ☐

Exercises.

D.7.i. Exercise ([**102**, 25]). Prove Lemma D.7.5.

D.7.ii. Exercise. Find another example of a complicial set whose n-trivialization is no longer a complicial set.

D.8. Marked homotopy coherent realization and the marked homotopy coherent nerve

In this section, we extend the construction of the homotopy coherent realization and homotopy coherent nerve functors

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

described in §6.3 to construct an analogous adjunction

$$\begin{tikzcd} s\mathcal{S}et^+ \arrow[r, bend left, "\mathfrak{C}^+"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^+\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}^+"] \end{tikzcd}$$

between the category of marked simplicial sets and *a* category of categories enriched in marked simplicial sets, as we shall explain. These constructions were first introduced in [**100**] in a slightly different context.[10] We refer to the left adjoint as the *marked homotopy coherent realization* of a marked simplicial set and the right adjoint as the *marked homotopy coherent nerve*.

To explain our strategy to define the functors $\mathfrak{C}^+$ and $\mathfrak{N}^+$, we perform a thought experiment. Note that our sought-for adjunction composes with the adjunction of Proposition D.1.5 to define an adjunction

$$\begin{tikzcd} & t\mathbf{\Delta} \arrow[dl, "\text{よ}"'] \arrow[dr, "{\mathfrak{C}^+[\bullet]}"] & \\ \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^+ \arrow[l, bend left] \arrow[r, bend left, "\mathfrak{C}^+"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^+\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}^+"] \end{tikzcd}$$

in which the domain of the left adjoint is a category of presheaves. Since the Yoneda embedding $\text{よ}\colon t\mathbf{\Delta} \hookrightarrow \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$ is the free cocompletion of $t\mathbf{\Delta}$, to specify this composite adjunction, it is necessary and sufficient to define a functor $\mathfrak{C}^+\Delta[\bullet]\colon t\mathbf{\Delta} \to s\mathcal{S}et^+\text{-}\mathcal{C}at$. The composite left adjoint is then the left Kan extension of this functor along the Yoneda embedding, while the composite right adjoint is the associated "nerve functor," which we abusively also call $\mathfrak{N}^+$. However, in general, the image of a right adjoint $\mathfrak{N}^+\colon s\mathcal{S}et^+\text{-}\mathcal{C}at \to \mathcal{S}et^{t\mathbf{\Delta}^{\mathrm{op}}}$ obtained in this way might not land in the reflective full subcategory of marked simplicial sets.

To see this, note that by this construction, for any marked simplicial set enriched category $\mathcal{E}$, the n-simplices in the presheaf $\mathfrak{N}^+\mathcal{E}$ will necessarily be enriched functors $\mathfrak{C}^+\Delta[n] \to \mathcal{E}$, while marked n-simplices in the presheaf $\mathfrak{N}^+\mathcal{E}$ will be enriched functors $\mathfrak{C}^+\Delta[n]_t \to \mathcal{E}$. By precomposing with the natural map $[n] \to [n]_t$ in $t\mathbf{\Delta}$, any marked n-simplex is naturally regarded as an n-simplex, but from the description of the right adjoint of the adjunction of Proposition D.1.5 in order for the functor $\mathfrak{N}^+$ to take values in the subcategory $s\mathcal{S}et^+$ the marked n-simplices must be characterized among the n-simplices by an additional condition, not by any additional *data*. As we shall see below, for $n > 1$ there are natural definitions of the categories $\mathfrak{C}^+\Delta[n]$ and $\mathfrak{C}^+\Delta[n]_t$ so that an extension along $\mathfrak{C}^+\Delta[n] \to \mathfrak{C}^+\Delta[n]_t$ does not require any additional data in the cases where it exists. These natural

[10]Here we define our enrichment relative to the Gray tensor product, which is the cartesian product in the cartesian closed category of marked simplicial sets; see Proposition D.3.4. In the original [**100**], Verity works with categories enriched over the *lax Gray tensor product*. As the identity defines a lax monoidal functor from the Gray tensor product to the lax Gray tensor product, the categories enriched over the former can be regarded as categories enriched over the latter.

definitions extend to $n = 1$ if we allow ourselves to mark 0-arrows in the homs of a category enriched over marked simplicial sets, which is why the adjunction appearing in Proposition D.8.6 below takes this modified form and why we qualified our discussion of "a" category of categories enriched in marked simplicial sets above. This technical point has little practical effect, since we will be interested in marked homotopy coherent nerves of categories enriched over complicial sets, in which case it will be easy to say which 0-arrows should be marked in the homs as we shall discover.

With this aim in mind, we first define the cosimplicial object $\mathfrak{C}^+\Delta[\bullet]\colon \mathbf{\Delta} \to s\mathcal{S}et^+\text{-}\mathcal{C}at$ on the full subcategory $\mathbf{\Delta} \subset t\mathbf{\Delta}$. In parallel with Definition 6.2.10, the objects in this diagram are full enriched subcategories of a marked simplicial set enriched category we now introduce.

D.8.1. Definition (the marked homotopy coherent ω-simplex). The **marked homotopy coherent ω-simplex** $\mathfrak{C}^+\Delta[\omega]$ is a category enriched in marked simplicial sets whose underlying simplicial category is the homotopy coherent ω-simplex $\mathfrak{C}\Delta[\omega]$ introduced in Definition 6.2.6. In particular, the objects of $\mathfrak{C}^+\Delta[\omega]$ are natural numbers $k \geq 0$ and the hom-spaces are the simplicial cubes

$$\mathfrak{C}^+\Delta[\omega](j,k) \coloneqq \begin{cases} \varnothing & j > k \\ \Delta[0] & j = k \\ \Delta[1]^{\times(k-j-1)} & j < k \end{cases}$$

marked in a manner we now describe.

An r-simplex in $\Delta[1]$ can be identified with an order preserving map $[r] \to [1]$, which can be identified element of the set $\{-, 1, 2, \ldots, r, +\}$; here $+$ represents the constant function at 0, $-$ represents the constant function at 1, and a numerical index i represents the epimorphism where $i \in [r]$ is the smallest element that maps to $1 \in [1]$.[11] Consequently, an r-simplex in $\Delta[1]^{k-j-1}$ can be identified with a function $\rho\colon \{1, \ldots, k-j-1\} \to \{-, 1, 2, \ldots, r, +\}$, which describes each of its component r-simplices. The r-simplex ρ is *not* marked if there exists an ordered subsequence $\{i_1 < \ldots < i_r\} \subset \{1, \ldots, k-j-1\}$ so that

(i) $\rho(i_s) = s$ for all $1 \leq s \leq r$
(ii) for each $i_1 < t < i_r$, $\rho(t) \neq -$.

and the r-simplex ρ is marked otherwise.

D.8.2. Remark. In the notation of Definition 6.2.6, an r-simplex in $\mathfrak{C}^+\Delta[\omega](j,k) = \Delta[1]^{\times(k-j-1)}$, for $j < k \in \omega$, is represented by a sequence of subsets

$$\{j,k\} \subset T^0 \subset \cdots \subset T^r \subset [j,k]$$

of the interval $[j,k] = \{j \leq t \leq k\}$ containing both endpoints. In this notation, conditions (i) and (ii) assert that the r-simplex $T^\bullet$ is not marked if there exists an ordered subsequence $\{t_1 < \ldots < t_r\} \subset [j,k]$ so that

(i) $t_i \in T^i \backslash T^{i-1}$ for all $1 \leq i \leq r$
(ii) $[t_1, t_r] \cap T^0 = \varnothing$

and is marked otherwise.

D.8.3. Digression. The description of the markings in the marked homotopy coherent ω-simplex is due to [**100**, §5], who arrives at this definition in the following manner. He first equips the simplicial sets $\mathfrak{C}^+\Delta[\omega](j,k) = \Delta[1]^{\times(k-j-1)}$ with the markings of the lax Gray tensor product $\Delta[1]^{\otimes(k-j-1)}$ of the

[11] This characterizes $\rho\colon [r] \to [1]$ via the "interval representation" described in Digression 9.1.8.

unmarked 1-simplex. An r-simplex is marked in $\Delta[1]^{\otimes(k-j-1)}$ exactly when it does not satisfy condition (i) of Definition D.8.1. In the present context, we consider $\mathfrak{C}^+\Delta[\omega]$ as a category enriched over marked simplicial sets with the cartesian product, as opposed to over the category of marked simplicial sets with the lax Gray tensor product. The lax Gray tensor product defines an entire subset $X \otimes Y \hookrightarrow_e X \times Y$ of the cartesian product, so when $\mathfrak{C}^+\Delta[\omega]$ is presented as a category enriched over the cartesian closed category of marked simplicial sets this necessitates additional markings of the homs: namely marking all simplices that arise as horizontal composites of marked simplices. This explains condition (ii) which asserts that a simplex is not marked if and only if it is not marked in the lax Gray tensor product $\Delta[1]^{\otimes(k-j-1)}$ and also is not a composite of marked simplices.

The combinatorics of the markings of the lax Gray tensor product are described in a different manner in [**100**, §5]. For more details about the comparison between these presentations, see Exercise D.8.i.

Recall that the finite ordinals define full subcategories of $\boldsymbol{\omega}$.

D.8.4. Definition (the marked homotopy coherent n-simplex). For any $n \geq 0$, the **marked homotopy coherent n-simplex** $\mathfrak{C}^+\Delta[n]$ is the full subcategory of the marked homotopy coherent ω-simplex $\mathfrak{C}^+\Delta[\omega]$ spanned by the objects $0, \ldots, n$.

The **marked homotopy coherent marked n-simplex** $\mathfrak{C}^+\Delta[n]_t$ is defined analogously, with a single additional marked $n-1$-simplex: the unique top-dimensional unmarked simplex in $\mathfrak{C}^+\Delta[n](0,n)$ is marked in $\mathfrak{C}^+\Delta[n]_t(0,n)$.

The categories $\mathfrak{C}^+\Delta[n]$ for $n \geq 0$ and $\mathfrak{C}^+\Delta[n]_t$ for $n \geq 2$ are enriched over the usual category of marked simplicial sets. The remaining category $\mathfrak{C}^+\Delta[1]_t$, which has two objects and a single non-identity zero arrow $0 \to 1$, which is marked, the enrichment is over an extended category $s\mathcal{S}et^{+_0}$ of marked simplicial sets that also allows for marked vertices.

D.8.5. Lemma. *The marked homotopy coherent simplices and marked homotopy coherent marked simplices assemble into a simplicial functor*

$$\mathfrak{C}^+[\bullet] \colon t\boldsymbol{\Delta} \to s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at.$$

Proof. Exercise D.8.ii. ☐

These definitions are arranged so that the claimed adjunction follows formally.

D.8.6. Proposition. *There is an adjoint pair of functors*

$$\begin{tikzcd} s\mathcal{S}et^+ \arrow[r, bend left, "\mathfrak{C}^+"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}^+"] \end{tikzcd}$$

Proof. There is an adjoint pair of functors as displayed below-left.

$$\begin{tikzcd} \mathcal{S}et^{t\boldsymbol{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\mathfrak{C}^+"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}^+"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{S}et^{t\boldsymbol{\Delta}^{\mathrm{op}}} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^+ \arrow[l, bend left] \arrow[r, bend left, "\mathfrak{C}^+"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at \arrow[l, bend left, "\mathfrak{N}^+"] \end{tikzcd}$$

whose left adjoint is defined by left Kan extending the functor $\mathfrak{C}^+[\bullet] \colon t\boldsymbol{\Delta} \to s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at$ of Lemma D.8.5 along the Yoneda embedding and whose right adjoint is the associated nerve functor; see [**80**,

6.5.9] for more details. It remains to argue that this adjunction factors as displayed above right through the adjunction that characterizes $s\mathcal{S}et^+$ as a full reflective subcategory of $\mathcal{S}et^{t\Delta^{\mathrm{op}}}$.

By Proposition D.1.5 a presheaf on $t\mathbf{\Delta}$ is a marked simplicial set, just when the map from its marked n-simplices to its n-simplices is a monomorphism, for all $n \geq 1$. By construction, the marked n-simplices in $\mathfrak{N}^+\mathcal{E}$ for $\mathcal{E} \in s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at$ are enriched functors $\mathfrak{C}^+\Delta[n]_t \to \mathcal{E}$, while the unmarked n-simplices are enriched functors $\mathfrak{C}^+\Delta[n] \to \mathcal{E}$. Since the canonical map $\mathfrak{C}^+\Delta[n] \to \mathfrak{C}^+\Delta[n]_t$ is an epimorphism in $s\mathcal{S}et^{+_0}\text{-}\mathcal{C}at$, this property holds, so the adjunction restricts as claimed. □

We now introduce the nerve functor of Proposition D.8.6 more explicitly.

D.8.7. Definition. Let $\mathcal{E}$ be a category enriched over marked simplicial sets. Then the **marked homotopy coherent nerve** of $\mathcal{E}$ is the marked simplicial set $\mathfrak{N}^+\mathcal{E}$ whose

- n-simplices are marked simplicially enriched functors $\mathfrak{C}^+\Delta[n] \to \mathcal{E}$
- marked n-simplices are marked simplicially enriched functors $\mathfrak{C}^+\Delta[n] \to \mathcal{E}$ that send the unique (top dimensional) unmarked $n-1$ arrow in $\mathfrak{C}^+\Delta[n](0,n)$ to a marked $n-1$-arrow in $\mathcal{E}$

D.8.8. Remark. If $\mathcal{E}$ is a category enriched over complicial sets, there is a canonical choice of marking for its 0-arrows which enables us to apply Definition D.8.7. We mark the 0-arrows $f \in \mathcal{E}(x,y)$ that are *equivalences* in the sense that there exists a 0-arrow $g \in \mathcal{E}(y,x)$ together with a pair of marked 1-arrows $\mathrm{id}_x \to gf \in \mathcal{E}(x,x)$ and $\mathrm{id}_y \to fg \in \mathcal{E}(y,y)$.[12]

D.8.9. Observation (marked homotopy coherent nerve data). Explicitly:

- 0-simplices x in $\mathfrak{N}^+\mathcal{E}$ are objects $x \in \mathcal{E}$.
- 1-simplices $f_{01}\colon x_0 \to x_1$ in $\mathfrak{N}^+\mathcal{E}$ are 0-arrows $f_{01} \in \mathcal{E}(x_0,x_1)$, which are marked just when f_{01} is marked.
- A marked 1-simplex is a 1-simplex, as above, in which the 0-arrow f_{01} is marked.
- 2-simplices $\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] \arrow[rr, phantom, bend left=20, "\Uparrow\alpha_{012}"] & & x_2 \end{tikzcd}$ in $\mathfrak{N}^+\mathcal{E}$ are given by
  - objects $x_0, x_1, x_2 \in \mathcal{E}$;
  - 0-arrows $f_{01} \in \mathcal{E}(x_0,x_1)$, $f_{12} \in \mathcal{E}(x_1,x_2)$, $f_{02} \in \mathcal{E}(x_0,x_2)$; and
  - a 1-arrow $\alpha_{012}\colon f_{02} \to f_{12}f_{01} \in \mathcal{E}(x_0,x_2)$.
- A marked 2-simplex is a 2-simplex, as above, in which the 1-arrow α_{012} is marked in $\mathcal{E}(x_0,x_2)$.
- 3-simplices

$$\begin{tikzcd} & x_1 \arrow[r, "f_{12}"] \arrow[d, phantom, "\Uparrow\alpha_{012}"] & x_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, "\alpha_{023}\Uparrow"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[urr, "f_{02}"'] \arrow[rrr, "f_{03}"'] & {} & {} & x_3 \end{tikzcd} \quad \overset{\Phi_{0123}}{\Rrightarrow} \quad \begin{tikzcd} & x_1 \arrow[r, "f_{12}"] \arrow[drr, "f_{13}"'] \arrow[d, phantom, "\Uparrow\alpha_{013}"] & x_2 \arrow[dr, "f_{23}"] \arrow[d, phantom, "\Uparrow\alpha_{123}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rrr, "f_{03}"'] & {} & {} & x_3 \end{tikzcd}$$

  in $\mathfrak{N}^+\mathcal{E}$ are given by
  - objects $x_0, x_1, x_2, x_3 \in \mathcal{E}$;
  - 0-arrows $f_{01}, f_{12}, f_{23}, f_{02}, f_{13}, f_{03}$ in $\mathcal{E}$;

[12] Since the homs of $\mathcal{E}$ are complicial sets, the specification of a direction for the marked 1-arrows is immaterial.

– 1-arrows $\alpha_{012} \colon f_{02} \to f_{12}f_{01} \in \mathcal{E}(x_0, x_2)$, $\alpha_{023} \colon f_{03} \to f_{23}f_{02} \in \mathcal{E}(x_0, x_3)$, $\alpha_{013} \colon f_{03} \to f_{13}f_{01} \in \mathcal{E}(x_0, x_3)$, $\alpha_{123} \colon f_{13} \to f_{23}f_{12} \in \mathcal{E}(x_1, x_3)$, and $\beta_{0123} \colon f_{03} \to f_{23}f_{12}f_{01} \in \mathcal{E}(x_0, x_3)$; and

– a pair of 2-arrows in $\mathcal{E}(x_0, x_3)$ with boundary as displayed below, one of which is marked:

$$\begin{tikzcd}
f_{03} \arrow[r, "\alpha_{023}"] \arrow[d, "\alpha_{013}"'] \arrow[dr, "\beta_{0123}" description] \arrow[dr, phantom, "\simeq", shift left=3] & f_{23}f_{02} \arrow[d, "f_{23}\alpha_{012}"] \\
f_{13}f_{01} \arrow[r, "\alpha_{123}f_{01}"'] \arrow[ur, phantom, "\swarrow\Phi_{0123}", near start] & f_{23}f_{12}f_{01}
\end{tikzcd}$$

- A marked 3-simplex is a 3-simplex, as above, in which the 2-arrow Φ_{0123} is marked in $\mathcal{E}(x_0, x_3)$.
- A 4-simplex Ξ_{01234} in $\mathcal{E}$ is given by
  – objects $x_0, x_1, x_2, x_3, x_4 \in \mathcal{E}$;
  – 0-arrows $f_{ij} \in \mathcal{E}(x_i, x_j)$ for all $0 \le i < j \le 4$;
  – 1-arrows $\alpha_{ijk} \colon f_{ik} \to f_{jk}f_{ij} \in \mathcal{E}(x_i, x_k)$ for all $0 \le i < j < k \le 4$;
  – 1-arrows $\beta_{ijk\ell} \colon f_{i\ell} \to f_{k\ell}f_{jk}f_{ij} \in \mathcal{E}(x_i, x_\ell)$ for all $0 \le i < j < k < \ell \le r$;
  – a 1-arrow $\gamma_{01234} \colon f_{04} \to f_{34}f_{23}f_{12}f_{01} \in \mathcal{E}(x_0, x_4)$;
  – for all $0 \le i < j < k < \ell \le 4$, a pair of 2-arrows in $\mathcal{E}(x_i, x_\ell)$ with boundary as displayed below, one of which is marked:

$$\begin{tikzcd}
f_{i\ell} \arrow[r, "\alpha_{ik\ell}"] \arrow[d, "\alpha_{ij\ell}"'] \arrow[dr, "\beta_{ijk\ell}" description] \arrow[dr, phantom, "\simeq", shift left=3] & f_{k\ell}f_{ik} \arrow[d, "f_{k\ell}\alpha_{ijk}"] \\
f_{j\ell}f_{ij} \arrow[r, "\alpha_{jk\ell}f_{ij}"'] \arrow[ur, phantom, "\swarrow\Phi_{ijk\ell}", near start] & f_{k\ell}f_{jk}f_{ij}
\end{tikzcd}$$

  – a 3-arrow in $\mathcal{E}(x_0, x_4)$

$$\begin{tikzcd}
& f_{14}f_{01} \arrow[r, "\alpha_{124}f_{01}"] & f_{24}f_{12}f_{01} \arrow[dr, "\alpha_{234}f_{12}f_{01}"] & & & & f_{14}f_{01} \arrow[r, "\alpha_{124}f_{01}"] \arrow[drr, "\beta_{1234}f_{01}" description] & f_{24}f_{12}f_{01} \arrow[dr, "\alpha_{234}f_{12}f_{01}"] \arrow[d, phantom, "\Uparrow\Phi_{1234}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{014}"] \arrow[urr, "\beta_{0124}" description] \arrow[rrr, "\gamma_{01234}"'] \arrow[ur, phantom, "\Leftarrow\Phi_{0124}", shift left=4] \arrow[rr, phantom, "\Uparrow\widetilde{\alpha_{012} * \Phi_{0234}}", shift left=3] & & & f_{34}f_{23}f_{12}f_{01} & \overset{\Xi_{01234}}{\Rrightarrow} & f_{04} \arrow[ur, "\alpha_{014}"] \arrow[rrr, "\gamma_{01234}"'] \arrow[r, phantom, "\Uparrow\widetilde{\Phi_{0134} \cdot \Phi_{0123}}", shift left=3] & & & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}$$

in which the two 2-arrow faces that lie on the exterior of the cube $\Delta[1]^{\times 3}$ have been specified previously;

– five marked 3-arrows in $\mathcal{E}(x_0, x_4)$

$$
\begin{tikzcd}
& f_{24}f_{02} \arrow[r, "f_{24}\alpha_{012}"] & f_{24}f_{12}f_{01} \arrow[dr, "\alpha_{234}f_{12}f_{01}"] \arrow[d, phantom, "\Uparrow\widetilde{\alpha_{012}*\Phi_{0234}}"] & \\
f_{04} \arrow[ur, "\alpha_{024}"] \arrow[urr, "\wr", "\beta_{0124}"'] \arrow[rrr, "\gamma_{01234}"'] & & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
\simeq
\begin{tikzcd}
& f_{24}f_{02} \arrow[r, "f_{24}\alpha_{012}"] \arrow[drr, "\wr", "\alpha_{234}*\alpha_{012}"'] \arrow[d, phantom, "\Uparrow\widetilde{\alpha_{012}*\Phi_{0234}}"] & f_{24}f_{12}f_{01} \arrow[dr, "\alpha_{234}f_{12}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{024}"] \arrow[rrr, "\gamma_{01234}"'] & {} & & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& f_{24}f_{02} \arrow[r, "\alpha_{234}f_{02}"] \arrow[d, phantom, "\Phi_{0234}\,\Uparrow"] & f_{34}f_{23}f_{02} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] \arrow[d, phantom, "\wr"] & \\
f_{04} \arrow[ur, "\alpha_{024}"] \arrow[urr, "\beta_{0234}"'] \arrow[rrr, "\gamma_{01234}"'] & {} & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
\simeq
\begin{tikzcd}
& f_{24}f_{02} \arrow[r, "\alpha_{234}f_{02}"] \arrow[drr, "\wr", "\alpha_{234}*\alpha_{012}"'] \arrow[d, phantom, "\Uparrow\widetilde{\alpha_{012}*\Phi_{0234}}"] & f_{34}f_{23}f_{02} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{024}"] \arrow[rrr, "\gamma_{01234}"'] & {} & & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& f_{14}f_{01} \arrow[r, "\alpha_{134}f_{01}"] \arrow[d, phantom, "\Phi_{0134}\,\Uparrow"] & f_{34}f_{13}f_{01} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] \arrow[d, phantom, "\Uparrow\widetilde{\Phi_{0123}*\alpha_{034}}"] & \\
f_{04} \arrow[ur, "\alpha_{014}"] \arrow[urr, "\beta_{0134}"'] \arrow[rrr, "\gamma_{01234}"'] & {} & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
\simeq
\begin{tikzcd}
& f_{14}f_{01} \arrow[r, "\alpha_{134}f_{01}"] \arrow[drr, "\wr", "\beta_{1234}f_{01}"'] \arrow[d, phantom, "\Uparrow\widetilde{\Phi_{0134}\cdot\Phi_{0123}}"] & f_{34}f_{13}f_{01} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{014}"] \arrow[rrr, "\gamma_{01234}"'] & {} & & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& f_{34}f_{03} \arrow[r, "f_{34}\alpha_{013}"] & f_{34}f_{13}f_{01} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] \arrow[d, phantom, "\Uparrow\widetilde{\Phi_{0123}*\alpha_{034}}"] & \\
f_{04} \arrow[ur, "\alpha_{034}"] \arrow[urr, "\wr", "\beta_{0134}"'] \arrow[rrr, "\gamma_{01234}"'] & & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
\simeq
\begin{tikzcd}
& f_{34}f_{03} \arrow[r, "f_{34}\alpha_{013}"] \arrow[drr, "f_{34}\beta_{0123}"'] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "f_{34}\Phi_{0123}\,\Uparrow"] & f_{34}f_{13}f_{01} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{034}"] \arrow[rrr, "\gamma_{01234}"'] & {} & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& f_{34}f_{03} \arrow[r, "f_{34}\alpha_{023}"] & f_{34}f_{23}f_{02} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] \arrow[d, phantom, "\wr"] & \\
f_{04} \arrow[ur, "\alpha_{034}"] \arrow[urr, "\wr", "\beta_{0234}"'] \arrow[rrr, "\gamma_{01234}"'] & & {} & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
\simeq
\begin{tikzcd}
& f_{34}f_{03} \arrow[r, "f_{34}\alpha_{023}"] \arrow[drr, "\wr", "f_{34}\beta_{0123}"'] \arrow[d, phantom, "\wr"] & f_{34}f_{23}f_{02} \arrow[dr, "f_{34}\alpha_{123}f_{01}"] & \\
f_{04} \arrow[ur, "\alpha_{034}"] \arrow[rrr, "\gamma_{01234}"'] & {} & & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

in which the ten 2-arrow faces that lie on the exterior of the cube $\Delta[1]^{\times 3}$ have been specified previously.[13]

[13]This is least obvious in the case of the cubical face

$$
\begin{tikzcd}
f_{24}f_{02} \arrow[r, "f_{24}\alpha_{012}"] \arrow[d, "\alpha_{234}f_{02}"'] \arrow[dr, "\alpha_{234}\alpha_{012}", "\simeq"'] \arrow[r, phantom, bend left=0, ""] & f_{24}f_{12}f_{01} \arrow[d, "\alpha_{234}f_{12}f_{01}"] \\
f_{34}f_{23}f_{02} \arrow[r, "f_{34}f_{23}\alpha_{012}"'] & f_{34}f_{23}f_{12}f_{01}
\end{tikzcd}
$$

In particular, we can regard a quasi-categorically enriched category $\mathcal{E}$ as a category enriched over marked simplicial sets by assigning each hom quasi-category $\mathcal{E}(x,y)$ the natural marking of Definition D.4.4. As in Remark D.8.8, we also mark the 0-arrows $f \in \mathcal{E}(x,y)$ that are *equivalences* in the sense that there exists a 0-arrow $g \in \mathcal{E}(y,x)$ together with a pair of marked 1-arrows $\mathrm{id}_x \to gf \in \mathcal{E}(x,x)$ and $\mathrm{id}_y \to fg \in \mathcal{E}(y,y)$. Note that the marked 1-arrows of $\mathcal{E}$ play no role in the definition of the marked homotopy coherent nerve $\mathfrak{N}^+\mathcal{E}$, except in identifying which 1- and 2-simplices should be marked. Thus, in this case, Definition D.8.7 specializes to the following.

D.8.10. Definition (homotopy coherent nerves of quasi-categorically enriched categories). Let $\mathcal{E}$ be a quasi-categorically enriched category. The **natural marking of its homotopy coherent nerve** $\mathfrak{N}\mathcal{E}$, defined in the classical sense of Definition 6.3.1, may be regarded as a marked simplicial set in which:

- all n-simplices with $n > 2$ are marked;
- those 2-simplices $\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & \arrow[u, phantom, "\Uparrow\alpha_{012}"] & x_2 \end{tikzcd}$ in which the 1-arrow $\alpha_{012} \colon f_{02} \to f_{12}f_{01}$ is an isomorphism in the quasi-category $\mathcal{E}(x_0,x_2)$ are marked; and
- those 1-simplices $f_{01} \colon x_0 \to x_1$ whose corresponding 0-arrow $f_{01} \in \mathcal{E}(x_0,x_1)$ is an **equivalence**, in the sense that there exists a 0-arrow $f_{01}^{-1} \in \mathcal{E}(x_1,x_0)$ together with a pair of invertible 1-arrows $\mathrm{id}_{x_0} \to f_{01}^{-1}f_{01}$ and $\mathrm{id}_{x_1} \to f_{01}f_{01}^{-1}$ in the quasi-categories $\mathcal{E}(x_0,x_0)$ and $\mathcal{E}(x_1,x_1)$ respectively, are marked.

Our main theorem, the proof of which will occupy the bulk of this section, extends a previous theorem which demonstrates that the marked homotopy coherent nerve of a complicially-enriched category is a complicial set [**100**, 40].

D.8.11. Theorem. *Let $\mathcal{E}$ be a category enriched in marked simplicial sets.*

(i) If the homs of $\mathcal{E}$ are complicial sets, then the homotopy coherent complicial nerve $\mathfrak{N}^+\mathcal{E}$ is a complicial set.

(ii) If the homs of $\mathcal{E}$ are saturated complicial sets, then the homotopy coherent complicial nerve $\mathfrak{N}^+\mathcal{E}$ is a saturated complicial set.

(iii) If the homs of $\mathcal{E}$ are n-complicial sets, then the homotopy coherent complicial nerve $\mathfrak{N}^+\mathcal{E}$ is a $n+1$-complicial set.

By Example D.7.12, the $n = 0$ case of the third statement recovers Cordier and Porter's result reviewed in Theorem 6.3.13, that the homotopy coherent nerve of a Kan complex-enriched category is a quasi-category. The proof will occupy the remainder of this section. For now, see [**100**, 40] for a proof of (i). A proof of (ii) is forthcoming.

The statements ask us to verify that $\mathfrak{N}^+\mathcal{E}$ satisfies a right lifting problem described in Definition D.1.9 for (i), Definition D.7.9 for (ii), and Definition D.7.11 for (iii). To define the marked homotopy coherent nerve, we adopt the canonical marking of the 0-arrows that are equivalences, as described in Remark D.8.8. On account of the adjunction described in Proposition D.8.6, we may transpose these lifting properties across $\mathfrak{C}^+ \dashv \mathfrak{N}^+$ and instead verify the lifting property at the level of categories enriched in marked simplicial sets, and this is generally what we shall do.

---

but note here that both 2-simplices are horizontal composites of degenerate 2-simplices.

One of the three statements is easily dealt with. Assuming (ii) has been proven, it's straightforward to verify (iii). We're asked to verify the lifting property

$$\begin{tikzcd} \mathfrak{C}^+\Delta[m] \arrow[r] \arrow[d, hook, "e"'] & \mathcal{E} \\ \mathfrak{C}^+\Delta[m]_t \arrow[ur, dashed] & \end{tikzcd}$$

for $m > n+1$. By Definition D.8.7, this means we must show that every marked simplicially enriched functors $\mathfrak{C}^+\Delta[m] \to \mathcal{E}$ sends the unique top dimensional unmarked $m-1$ arrow in $\mathfrak{C}^+\Delta[m](0,m)$ to a marked $m-1$-arrow in $\mathcal{E}$. Since $m-1 > n$ and the homs of $\mathcal{E}$ are n-trivial, every $m-1$ arrow in $\mathcal{E}$ is marked, so this lifting property holds.

To verify (i), we appeal to Corollary D.4.12 to reduce the problem from constructing extensions for all complicial horn and complicial thinness extensions to the problem of constructing extensions only for *inner* complicial horn and *inner* complicial thinness extensions, after constructing a lift

$$\begin{tikzcd} \mathfrak{C}^+\Delta[1]^\sharp \arrow[r] \arrow[d, hook] & \mathcal{E} \\ \mathfrak{C}^+\mathrm{sk}_2\,\mathbb{I}^\sharp \arrow[ur, dashed] & \end{tikzcd}$$

Since $\Delta[1]^\sharp \cong \Delta[1]_t$, the domain map $\mathfrak{C}^+\Delta[1]^\sharp \to \mathcal{E}$ picks out a 0-arrow equivalence $e\colon a \to b$ in the complicial set enriched category $\mathcal{E}$. Thus, by Remark D.8.8, there exists an equivalence inverse $f\colon b \to a$, together with marked 1-arrows $\alpha\colon \mathrm{id}_a \to fe \in \mathcal{E}(a,a)$ and $\beta\colon \mathrm{id}_b \to ef \in \mathcal{E}(b,b)$. This exactly describes the data of a map $\mathfrak{C}^+(\mathrm{sk}_2\,\mathbb{I}^\sharp) \to \mathcal{E}$.

Now by Corollary D.4.12 to prove that $\mathfrak{N}^+\mathcal{E}$ is a complicial set it remains only to construct extensions

$$\begin{tikzcd} \mathfrak{C}^+\Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathcal{E} & \mathfrak{C}^+\Delta^k[n]' \arrow[r] \arrow[d, hook] & \mathcal{E} \\ \mathfrak{C}^+\Delta^k[n] \arrow[ur, dashed] & & \mathfrak{C}^+\Delta^k[n]'' \arrow[ur, dashed] & \end{tikzcd}$$

for $n \geq 2$ and $0 < k < n$. By Lemma 6.3.7, the left-hand lifting problem amounts to filling cubical horns in a hom complicial set

$$\begin{tikzcd} \mathfrak{C}^+\Lambda^k[n](0,n) \cong \sqcap^{n-1,k}_1 \arrow[r] \arrow[d, hook] & \mathcal{E}(e_0,e_n) \\ \mathfrak{C}^+\Delta^k[n](0,n) \cong \square^{n-1} \arrow[ur, dashed] & \end{tikzcd}$$

with certain markings that we now describe.

D.8.12. Definition. For any $n \geq 1$ and $1 \leq k \leq n$ define $\square^k[n] := \mathfrak{C}^+\Delta^k[n+1](0,n+1)$ to be the marked simplicial set whose underlying simplicial set is the n-cube $\Delta[1]^n$, the poset of subsets of $\{1,\ldots,n\}$. An r-simplex $\emptyset \subset T^0 \subset T^1 \subset \cdots \subset T^r \subset \{1,\ldots,n\}$ of $\square^k[n]$ is not marked if if there exists an ordered subsequence $\{t_1 < \ldots < t_r\} \subset \{1,\ldots,n\}$, which is only allowed to include k if at least one element of $\{k-1,k+1\} \cap \{1,\ldots,n\}$ is missing from T^r, so that

(i) $t_i \in T^i \backslash T^{i-1}$ for all $1 \leq i \leq r$

(ii) $[t_1, t_r] \cap T^0 = \varnothing$

and is marked otherwise. Define $\sqcap^k_1[n] \subset_r \square^k[n]$ to be the regular subset whose underlying simplicial set is the cubical horn $\sqcap^{n,k}_1$ missing the 0th face in direction k.

This takes the marked simplices of Remark D.8.2 and adds markings to data in the homotopy coherent nerve corresponding to the faces in the n-simplex that contain the vertices $\{k-1, k, k+1\}$. See [**100**, 34]. For each $i \in \{k-1, k, k+1\}$, the marked simplicial set $\square^k[n]$ contains a unique $(n-1)$-simplex on the exterior of the cube corresponding to the ith face of $\Delta^k[n]$. When $1 \le i \le n$, this simplex ω_i is given by the sequence of subsets

$$\varnothing \subset \{1\} \subset \{1,2\} \subset \cdots \subset \{1, \ldots, i-1\} \subset \{1, \ldots, i-1, i+1\} \subset \cdots \subset \{1, \ldots, \hat{i}, \ldots, n\}.$$

If $i = 0$, this simplex ω_0 is given by the sequence of subsets

$$\{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \cdots \subset \{1, \ldots, n\},$$

while if $i = n+1$ this simplex ω_n is given by the sequence of subsets

$$\{n\} \subset \{1,n\} \subset \{1,2,n\} \subset \{1,2,3,n\} \subset \cdots \subset \{1, \ldots, n\}.$$

D.8.13. Definition. For any $n \ge 1$ and $1 \le k \le n$ let

$$\square^k[n] \hookrightarrow_e \square^k[n]' \hookrightarrow_e \square^k[n]''$$

be the marked supersets of $\square^k[n]$ with certain additional $n-1$-simplices marked:

- Define $\square^k[n]'$ by also marking ω_{k-1} and ω_{k+1}.
- Define $\square^k[n]''$ by also marking ω_k.

These definitions are arranged so that:

D.8.14. Lemma. *Lifting problems*

$$\begin{tikzcd} \mathfrak{C}^+\Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathcal{E} \\ \mathfrak{C}^+\Delta^k[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C}^+\Delta^k[n]' \arrow[r] \arrow[d, hook] & \mathcal{E} \\ \mathfrak{C}^+\Delta^k[n]'' \arrow[ur, dashed] & \end{tikzcd}$$

can be solved in a category $\mathcal{E}$ enriched over marked simplicial sets for $n \ge 2$ and $0 < k < n$ if and only if the corresponding hom-spaces of $\mathcal{E}$ admits extensions

$$\begin{tikzcd} \sqcap^k_1[n-1] \arrow[r] \arrow[d, hook, "r"'] & \mathcal{E}(e_0, e_n) \\ \square^k[n-1] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \square^k[n-1]' \arrow[r] \arrow[d, hook, "e"'] & \mathcal{E}(e_0, e_n) \\ \square^k[n-1]'' \arrow[ur, dashed] & \end{tikzcd} \qquad \square$$

Before constructing such extensions in all dimensions, we anchor the reader's intuition through some low-dimensional calculations:

D.8.15. Lemma. *Any complicial set A admits admissible cubical horn and admissible cubical thinness extensions*

$$\begin{tikzcd} \sqcap^k_1[n] \arrow[r] \arrow[d, hook, "r"'] & A & \square^k[n]' \arrow[r] \arrow[d, hook, "e"'] & A \\ \square^k[n] \arrow[ur, dashed] & & \square^k[n]'' \arrow[ur, dashed] & \end{tikzcd}$$ ◻

for $n = 1, 2, 3$ and $1 \le k \le n$.

Proof. The cubical horn inclusion $\sqcap^1_1[1] \hookrightarrow_r \square^1[1]$ is isomorphic to the inclusion of the codomain vertex in the marked edge. This map has a retract, so any marked simplicial set A admits extensions along $\sqcap^1_1[1] \hookrightarrow_r \square^1[1]$. Note that $\square^1[1]' \hookrightarrow_e \square^1[1]''$ is the identity so there is nothing to argue in this case.

The cubical horn inclusions $\sqcap^k_1[2] \hookrightarrow_r \square^k[2]$ for $k = 1, 2$ are displayed below-left and below-right

$$\begin{tikzcd} \varnothing \arrow[r] & 1 \arrow[d] & \subset_r & \varnothing \arrow[r] \arrow[d] \arrow[dr, "\simeq", "\simeq"'] & 1 \arrow[d] & \varnothing \arrow[d] & 1 \arrow[d, "\wr"] & \subset_r & \varnothing \arrow[r] \arrow[d] \arrow[dr, "\simeq", "\simeq"'] & 1 \arrow[d, "\wr"] \\ 2 \arrow[r, "\sim"'] & 12 & & 2 \arrow[r, "\sim"'] & 12 & 2 \arrow[r] & 12 & & 2 \arrow[r] & 12 \end{tikzcd}$$

where the subsets of $\{1, 2\}$ are depicted without parentheses for readability. In each case, an extension may be constructed in A by first extending along the a horn $\Lambda^1[2] \hookrightarrow \Delta^1[2]$ formed by the unmarked edges and then extending along an outer admissible horn $\Lambda^2[2] \hookrightarrow \Delta^2[2]$. To see that A also admits extensions along $\square^k[2]' \hookrightarrow_e \square^k[2]''$ note in the case $k = 1$ that if the edges $\omega_2 = \varnothing \to 1$ and $\omega_0 = 1 \to 12$ are marked in A then so is the diagonal edge, and hence so is the edge $\omega_1 = \varnothing \to 2$. The argument in the case $k = 2$ is similar.

Finally, to extend along the cubical horns $\sqcap^k_1[3] \hookrightarrow \square^k[3]$ for $k = 1, 2, 3$ we must attach the six 3-simplices that triangulate the cube along admissible horn inclusions.

$$\begin{tikzcd} & \varnothing \arrow[rr] \arrow[dl] \arrow[dd] & & 1 \arrow[dl] \arrow[dd] \\ 2 \arrow[rr, crossing over] \arrow[dd] & & 12 & \\ & 3 \arrow[rr] \arrow[dl] & & 13 \arrow[dl] \\ 23 \arrow[rr] & & 123 \arrow[from=uu, crossing over] & \end{tikzcd}$$

The spines of these 3-simplices define maximal length paths through the poset of subsets of $\{1, 2, 3\}$ containing the endpoints. As such they are in bijection with elements of S_3, with each permutation rst representing the order in which the three missing elements are included. Two permutations agree up to an adjacent transposition $srt \leftrightarrow rst \leftrightarrow rts$ if and only if the corresponding 3-simplices share a codimension-2 face.

The sequence of attachments can be described similarly for each of $k = 1, 2, 3$. Two of the six 3-simplices share faces with the open face of the cubical horn $\sqcap^k_1[3]$ and one of these two missing faces is marked in $\square^k[3]$. This is the 3-simplex that should be attached first, while the other should be attached last. The adjacent transpositions then describe a circuit that indicates in which order each of the six 3-simplices should be attached: the first is via a $\Lambda^{1,2,3}[3]$-horn (only the 3rd face is present; the others must be filled), the second through fifth are attached via $\Lambda^1[3]$ or $\Lambda^2[3]$-horns in

an alternating sequence. The final simplex is attached via a $\Lambda^3[3]$-horn, which is admissibile because the edge $\{1,2,3\}\backslash k \to \{1,2,3\}$ from the final vertex in the missing face to the final vertex in the cube is marked.

Finally, we argue that A admits a cubical thinness extension $\square^3[3]' \hookrightarrow_e \square^3[3]''$. Here the task is to argue that the 2-simplex $\omega_3 = \varnothing \to 1 \to 12$ is marked in A assuming $\omega_2 = \varnothing \to 1 \to 13$ and $\omega_4 = 3 \to 13 \to 123$ are. The simplex ω_3 arises as the 3rd face of a 3-simplex $\Delta^3[3]$ in $\square^3[3]$ whose 0th and 1st faces are always marked. The 2nd face of this simplex, $\varnothing \to 1 \to 123$, arises as the 2nd face of a simplex $\Delta^2[3]$ in $\square^3[3]$ whose 0th face is marked and whose 3rd face is ω_2, which is marked by hypothesis. The 1st face of this simplex, $\varnothing \to 13 \to 123$, is the 1st face of a simplex $\Delta^1[3]$ in $\square^3[3]$ whose 2nd and 3rd faces are marked and whose 0th face is ω_4, which is marked by hypothesis. Consequently, the image of $\varnothing \to 13 \to 123$ is also marked in A, and hence so is the image of $\varnothing \to 1 \to 123$, and hence so is the image of ω_3, as required. The remaining two cases are similar. $\square$

We will now reduce the higher dimensional cubical extension problems of Lemma D.8.14 to the low-dimensional extensions constructed in Lemma D.8.14 by employing the following auxiliary structures.

D.8.16. Definition. For a marked simplicial set X define $X \boxtimes \Delta[1]$ to be the entire subset of $X \times \Delta[1]$ in which a non-degenerate r-simplex is marked if and only if either

(i) it is in the image of one of the endpoint inclusions $X \sqcup X \hookrightarrow X \boxtimes \Delta[1]$,

(ii) its δ^r face is a marked $(r-1)$-simplex in X and its final edge has the form $(x,0) \to (x,1)$ for some $x \in X$, or

(iii) it contains the edge $(x,0) \to (x,1)$ as the edge from its $i-1$th to its ith vertex for some $0 < i < r$.

Similarly define $(\Delta[1] \boxtimes X)^{\mathrm{op}} := X^{\mathrm{op}} \boxtimes \Delta[1]^{\mathrm{op}}$, which is to say that $\Delta[1] \boxtimes X$ is the entire superset of $\Delta[1] \times X$ in which a non-degenerate r-simplex is marked if and only if either

(i) it is in the image of one of the endpoint inclusions $X \sqcup X \hookrightarrow \Delta[1] \boxtimes X$,

(ii) its δ^0 face is a marked $(r-1)$-simplex in X and its initial edge has the form $(0,x) \to (1,x)$ for some $x \in X$,

(iii) it contains the edge $(0,x) \to (1,x)$ as the edge from its $i-1$th to its ith vertex for some $0 < i < r$.

That is, $X \boxtimes \Delta[1]$ contains $X \sqcup X$ as a regular subset, and the only additional non-degenerated marked r simplices are those that contain a "vertical edge" $(x,0) \to (x,1)$ along their spine and either this edge lies in the interior of the simplex or this is the final edge and the initial face is marked in X. A more general pretensor product $X \boxtimes Y$ of marked simplicial sets is defined in [**101**, 135] specializing to Definition D.8.16. Here we require only products with $\Delta[1]$ and its simplicial subsets, which we define to be the regular subset.

D.8.17. Example. The markings for $\Delta[1] \boxtimes \Delta[1]$ and $\Delta[1]_t \boxtimes \Delta[1]$ are depicted below-left and below-right:

$$\Delta[1] \boxtimes \Delta[1] := \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, "\simeq"'] & 10 \arrow[d] \\ 01 \arrow[r] & 11 \end{tikzcd} \qquad \Delta[1]_t \boxtimes \Delta[1] := \begin{tikzcd} 00 \arrow[r, "\sim"] \arrow[d] \arrow[dr, "\simeq", "\simeq"'] & 10 \arrow[d] \\ 01 \arrow[r, "\sim"'] & 11 \end{tikzcd}$$

To be continued.

Exercises.

D.8.i. Exercise. Verify that the marked simplices of Definition D.8.1 can also be characterized as follows: the marked r-simplices in $\mathfrak{C}^{+}\Delta[\omega](j,k) \cong \Delta[1]^{\times(k-j-1)}$ include:

(i) every r-simplex with the property that for every partition of r as a sum $m = d_1 + \cdots + d_{k-j-1}$ with $d_i \in \{0,1\}$ at least one of the simplices

$$\begin{array}{ccccccc} \Delta[d_i] & \hookrightarrow & \Delta[r] & \longrightarrow & \Delta[1]^{\times(k-j-1)} & \xrightarrow{\pi_i} & \Delta[1] \\ 0 & \longmapsto & d_1 + \cdots + d_{i-1} & & & & \\ 1 & \longmapsto & d_1 + \cdots + d_i & & & & \end{array}$$

with $d_i = 1$ is degenerate and

(ii) also mark any r-simplex that is a composite of marked r-simplices relative to (i).

D.8.ii. Exercise. Prove Lemma D.8.5.

D.8.iii. Exercise. Verify that Definition D.8.7 specializes to Definition D.8.10.

APPENDIX E

∞-cosmoi found in nature

In this chapter, we establish concrete examples of ∞-cosmoi found in nature. Typically, the objects of these ∞-cosmoi are infinite-dimensional categories as instantiated by some particular non-algebraic model and the functors between them are also morphisms of such. In most cases, there is an accompanying model structure which lends us appropriate classes of isofibrations, equivalences, and trivial fibrations. It would take far too long to re-prove this model structures here, but in each case we provide an appropriate literature citation. The only work that remains for us is to transfer the enrichment found in the model category literature to an enrichment over Joyal's model structure for quasi-categories on simplicial sets.

The general theory of what we might call "quasi-categorically enriched model categories" is discussed in §E.1. In §E.2, we apply these results to establish the familiar ∞-cosmoi of $(\infty, 1)$-categories $\mathcal{CSS}$, $\mathcal{S}egal$, and $1\text{-}\mathcal{C}omp$ together with their accompanying biequivalences to and from $\mathcal{Q}Cat$. In §E.3, we turn our attention to what might be called *higher ∞-categories*, establishing ∞-cosmoi whose objects are (∞, n)- or even (∞, ∞)-categories in some model. More exotic examples are considered in §E.4. Finally, in §E.5, we discuss a natural generalization of the notion of ∞-cosmos intended to expand the scope of the formal method to develop the theory of ∞-categories even further.

E.1. Quasi-categorically enriched model categories

Many examples of ∞-cosmoi arise as categories of fibrant objects in a model category that is enriched over Joyal's model structure for quasi-categories on simplicial sets described in Digression 1.1.29 and Theorem **??**—at least if all fibrant objects are cofibrant as is surprisingly often the case.[1]

E.1.1. Proposition. *Let $\mathcal{M}$ be any model category that is enriched over the Joyal model structure and in which every fibrant object is cofibrant. Then the full subcategory of fibrant objects $\mathcal{M}_f$ inherits the structure of an ∞-cosmos in which the isofibrations are the fibrations between fibrant objects, the equivalences are the weak equivalences between fibrant objects, and the trivial fibrations are the trivial fibrations between fibrant objects.*

Proof. By Lemma C.3.11, since the fibrant objects in $\mathcal{M}$ are also cofibrant, the simplicially enriched homs between fibrant-cofibrant objects of $\mathcal{M}$ are quasi-categories, which we denote by $\mathsf{Fun}(A, B)$. The same result also implies that any fibration $f : A \twoheadrightarrow B$ between fibrant objects, the induced map $f_* : \mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

By Lemma C.1.4, the fibrant objects and fibrations and weak equivalences between them define a category of fibrant objects in the sense of Definition C.1.1. In particular, the unenriched category $\mathcal{M}_f$ possesses a terminal object, small products, pullbacks of isofibrations, and limits of countable towers of isofibrations, with each of these limits created in $\mathcal{M}$. Since $\mathcal{M}$ admits simplicial cotensors,

[1]This hypothesis, that all fibrant objects are cofibrant, is not essential for the development of ∞-category theory. A more general notion of ∞-cosmos that accommodates possibly non-cofibrant objects is introduced in Section E.5.

Proposition A.5.5 implies that these 1-categorical limits are conical, and thus $\mathcal{M}_f$ possesses the conical limits of axiom 1.2.1(i). By hypothesis $\mathcal{M}$ is also cotensored over simplicial sets, and by Definition C.3.7, since all objects in the Joyal model structure are cofibrant, the fibrant objects are closed under simplicial cotensors. Thus $\mathcal{M}_f$ possesses all the limits of 1.2.1(i).

By Lemma C.1.4, the class of fibrations between fibrant objects contains the isomorphisms and all maps to the terminal object and is closed under all the 1-categorical limits of axiom 1.2.1(ii). Leibniz stability a special case of Definition C.3.7, and thus all the required axioms have been verified. ☐

Furthermore:

E.1.2. Corollary. *Any simplicially enriched right Quillen adjoint between quasi-categorically enriched model categories with all fibrant objects cofibrant defines a cosmological functor that is a cosmological biequivalence whenever the Quillen adjoint defines a Quillen equivalence.*

Proof. Exercise E.1.i. ☐

With Proposition E.1.1 in hand, the next question is where do model categories enriched over the Joyal model structure come from? Unsurprisingly, this question has not attracted much attention in the literature, but the mathematical community has done us a considerable favor, in many cases, by providing model categories of infinite-dimensional categories that are enriched over some other cartesian closed model category. Our strategy will be to apply Theorem C.3.14 to convert a known enrichment to an enrichment over Joyal's model structure for quasi-categories. Combining that result with Proposition E.1.1, we obtain the following immediate corollary.

E.1.3. Corollary. *Let $\mathcal{V}$ be a cartesian closed model category equipped with a Quillen adjunction*

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "U"] \end{tikzcd}$$

whose right adjoint is valued in the Joyal model structure and whose left adjoint preserves finite products.

(i) Then for any $\mathcal{V}$-model category $\mathcal{M}$ in which every fibrant object is cofibrant, the full subcategory of fibrant objects $\mathcal{M}_f$ defines an ∞-cosmos in which the isofibrations are the fibrations between fibrant objects, the equivalences are the weak equivalences between fibrant objects, the trivial fibrations are the trivial fibrations between fibrant objects, the functor spaces are defined by $\mathsf{Fun}(M,N) := U\mathcal{M}(M,N)$, where $\mathcal{M}(M,N)$ is the hom-object in $\mathcal{V}$, and the simplicial cotensor of $M \in \mathcal{M}_f$ with $S \in s\mathcal{S}et$ are defined by the $\mathcal{V}$-cotensor M^{FS}.

(ii) Moreover, any $\mathcal{V}$-enriched right Quillen adjoint between $\mathcal{V}$-model categories of this form defines a cosmological functor that is a cosmological biequivalence whenever the Quillen adjoint is a Quillen equivalence. ☐

E.1.4. Remark. In particular, in the context of the statement of Corollary E.1.3, if the fibrant objects of $\mathcal{V}$ are cofibrant, then $\mathcal{V}$ itself is an ∞-cosmos and U defines a cosmological functor $U\colon \mathcal{V}_f \to \mathcal{QC}at$. Indeed, by Remark A.1.9, the change of base functor U is naturally isomorphic to the underlying quasi-category functor $\mathsf{Fun}(1,-)\colon \mathcal{V}_f \to \mathcal{QC}at$ for the ∞-cosmos structure it induces on $\mathcal{V}_f$.

Corollary E.1.3 inspires the following trivial examples of ∞-cosmoi.

E.1.5. Example (1-categories as ∞-cosmoi). Any complete locally small 1-category C can be made into an ∞-cosmos in which $\mathsf{Fun}(A,B)$ is just the set of morphisms from A to B. By the Yoneda lemma,

the equivalences are then the isomorphisms in $\mathcal{C}$ and so by Lemma 1.2.13 all maps must necessarily be isofibrations. The cotensor of an object $A \in \mathcal{C}$ with a simplicial set A is defined by

$$A^S := A^{\pi_0 S} := \prod_{\pi_0 S} A.$$

Ignoring the fact that model categories are typically assumed to have colimits as well as limits, this construction can be seen as a special case of Corollary E.1.3 applied to the adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathcal{S}et \arrow[l, bend left, "\mathrm{sk}_0"] \end{tikzcd}$$

whose right adjoint embeds $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ as the subcategory of 0-skeletal simplicial sets; see Definition C.5.2. Here the cartesian closed model structure on $\mathcal{S}et$ is not the one considered in Exercise C.3.iv but rather the one in which the weak equivalences are the isomorphisms and all maps are taken to be both cofibrations and fibrations. To see that this adjunction is Quillen, note that π_0 vacuously preserves cofibrations, while sk_0 carries any map to an isofibration of quasi-categories: This latter claim follows by adjunction since the defining lifting-properties below-left transpose to the lifting properties below-right:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, tail, "\wr"'] & \mathrm{sk}_0 A \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{sk}_0 B \end{tikzcd} \quad \begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, tail, "\wr"'] & \mathrm{sk}_0 A \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & \mathrm{sk}_0 B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \pi_0\Lambda^k[n] \arrow[r] \arrow[d, equal] & A \arrow[d] \\ \pi_0\Delta[n] \arrow[r] \arrow[ur, dashed, "\exists!"'] & B \end{tikzcd} \quad \begin{tikzcd} \pi_0\mathbb{1} \arrow[r] \arrow[d, equal] & A \arrow[d] \\ \pi_0\mathbb{I} \arrow[r] \arrow[ur, dashed, "\exists!"'] & B \end{tikzcd}$$

Famously, π_0 preserves finite products, so the conditions of change-of-base theorem apply. The homotopy 2-category of an ∞-cosmos arising in this way will have no non-identity 2-cells.

E.1.6. Example (2-categories as ∞-cosmoi). Categorifying the previous example, any 2-category $\mathcal{C}$ with sufficient limits defines an ∞-cosmos where $\mathsf{Fun}(A, B)$ is the nerve of the hom-category of morphisms from A to B in $\mathcal{C}$. By Theorem 1.4.7, the equivalences are necessarily the equivalences in the 2-category. Inspired by Proposition 1.4.10, we take the isofibrations to be the isofibrations in the 2-category.

Interpreting "sufficient limits" to mean the limits of axiom 1.2.1(i), the remaining axiom 1.2.1(ii) can be verified by hand. Alternatively, again ignoring the fact that model categories are typically assumed to have colimits as well as limits, we may apply Corollary E.1.3 to the homotopy category $\dashv$ nerve adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathsf{h}"] \arrow[r, phantom, "\perp"] & \mathcal{C}at \arrow[l, bend left] \end{tikzcd}$$

of Proposition 1.1.11 and make use of the "trivial" $\mathcal{C}at$-enriched model structure of Lack [**58**], whose weak equivalences and fibrations are exactly the equivalences and isofibrations just described.

It remains to unpack the meaning of the weaselly phrase "sufficient limits." By Corollary 7.3.3, the 2-category $\mathcal{C}$ is required to have all PIE-limits, that is 2-categorical products, inserters, and equifiers discussed in Digression 7.2.6. This implies that $\mathcal{C}$ admits pseudopullbacks of all maps, by the construction of Definition 7.3.5, but this doesn't quite imply that $\mathcal{C}$ admits 2-pullbacks of isofibrations. Instead, the proof of Lemma 7.3.9 constructs a *bipullback* of an isofibration, with the usual hom-category isomorphism replaced by a hom-category equivalence. Similar remarks apply to limits of towers of

isofibrations. But in practice, the 2-categories that admit PIE limits such as those considered in **[14]** do seem to admit 2-pullbacks of isofibrations and 2-limits of towers of isofibrations and thus define examples of ∞-cosmoi.

In particular, Example E.1.6 specializes to recover the ∞-cosmos structure on $\mathcal{C}at$ discussed in Example 1.2.22. Intriguingly, it also defines an ∞-cosmos structure on $\mathcal{C}at^{\mathrm{op}}$ in which the "isofibrations" are those functors that are injective on objects.[2] Combining these observations with the dual ∞-cosmos construction of Definition 1.2.23, we see that the four 2-categorical duals $\mathcal{C}at$, $\mathcal{C}at^{\mathrm{op}}$, $\mathcal{C}at^{\mathrm{co}}$, and $\mathcal{C}at^{\mathrm{coop}}$ are all ∞-cosmoi.

The ∞-cosmoi of Example E.1.6 admit an abstract characterization as those ∞-cosmoi that are isomorphic (as quasi-categorically enriched categories) to their homotopy 2-categories. In this case, the weak 2-limits of Chapter 3 are actually strict and many of our results specialize to known theorems in the 2-categorical literature.

E.1.7. Example (simplicial model categories as ∞-cosmoi). The identity functor $\mathrm{id}\colon s\mathcal{S}et \to s\mathcal{S}et$ defines a right Quillen adjoint from Quillen's model structure for Kan complexes to Joyal's model structure for quasi-categories; evidently its left adjoint preserves products. Hence, any Kan complex enriched model category—or **simplicial model category** in the usual parliance—may be regarded as a quasi-categorically enriched model category in which each of the mapping-spaces between fibrant-cofibrant objects happens to be a Kan complex. Thus, any simplicial model category whose fibrant objects are cofibrant may be regarded as an ∞-cosmos.

The homotopy 2-categories of ∞-cosmoi arising in this manner are all "$(2,1)$-categories, with every natural transformation defining a natural isomorphism.

Exercises.

E.1.i. Exercise. Extend the proof of Proposition E.1.1 to prove Corollary E.1.2.

E.1.ii. Exercise. Prove the assertions made in Remark E.1.4 if you find them unconvincing.

E.2. ∞-cosmoi of $(\infty,1)$-categories

The ∞-cosmos for quasi-categories was introduced in Proposition 1.2.9. In this section, we establish three other ∞-cosmoi whose objects define $(\infty,1)$-categories in some model and construct the biequivalences between them displayed in (14.0.1):

$$\begin{tikzcd}
\mathcal{CSS} \arrow[rr, "\mathrm{disc}"] \arrow[dr, shift left, "\mathrm{row}_0"] & & \mathcal{S}egal \arrow[dl, shift right, "\mathrm{row}_0"'] \\
& \mathcal{QC}at \arrow[ul, shift left, "\mathrm{cylinder}"] \arrow[ur, shift right, "\mathrm{prism}"'] \arrow[d, shift left, "\natural"] & \\
& 1\text{-}\mathcal{C}omp \arrow[u, shift left, "U"] &
\end{tikzcd}$$

A complete Segal space, as defined by Charles Rezk in **[75]**, is firstly a bisimplicial set $X \in \mathcal{S}et^{\mathbb{\Delta}^{\mathrm{op}}\times\mathbb{\Delta}^{\mathrm{op}}}$, where we regard $X_{m,n}$ as the set of n-simplices in the m^{th} space of a simplicial space

[2] In the "folk" model structure on $\mathcal{C}at$, the fibrations are the isofibrations, the weak equivalences are the equivalences, and the cofibrations are the injective-on-objects functors. Injective-on-objects functors satisfy an isomorphism extension property dual to the isomorphism lifting property that defines the 2-category notion of isofibration.

$X_\bullet \in s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$. It is conventional to regard the simplicial sets $X_m = X_{m,\bullet}$ as the "columns" of the bisimplicial set X, while the simplicial sets $X_{\bullet,n}$ define the "rows."

In a complete Segal space, the diagram

$$X_0 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; X_2 \qquad\qquad \cdots$$

defines a simplicial object in the category of Kan complexes, with each space X_m defining the "total space" of a Kan fibration whose base is the space M_mX of "boundary data" associated with the m-simplex. The spaces X_0 and X_1 are the "spaces of objects and arrows" for the complete Segal space. The so-called "Segal condition" implies that the space X_n may be regarded as the "space of n-composable arrows." A Segal space, satisfying the conditions enumerated thus far, is then something like a "category object up to homotopy." The final "completeness" condition relates the spatial structure of X_0 with the categorical structure just defined, expressing the idea that paths in X_0 should correspond to isomorphisms in X.

A bisimplicial set defines a **complete Segal space** just when it is Reedy fibrant and satisfies the Segal and completeness conditions. All three of these conditions are most easily defined in terms of the weighted limits bifunctor

$$(\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathrm{op}} \times s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \xrightarrow{\lim_{-} -} s\mathcal{S}et$$

where $s\mathcal{S}et$ is regarded as a $\mathcal{S}et$-enriched category. Note that the weights for $\mathbf{\Delta}^{\mathrm{op}}$-indexed diagrams in $s\mathcal{S}et$ are $\mathbf{\Delta}^{\mathrm{op}}$-indexed diagrams in $\mathcal{S}et$, i.e., simplicial sets. In more detail:

E.2.1. Definition (complete Segal space).

(i) A simplicial object $X_\bullet \in s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ is **Reedy fibrant** just when the induced map on weighted limits

$$X_m \cong \lim_{\Delta[m]} X \to \lim_{\partial\Delta[m]} X =: M_mX$$

is a Kan fibration of simplicial sets for all $m \geq 0$.

(ii) A Reedy fibrant simplicial object $X_\bullet$ is a **Segal space** just when the induced map on weighted limits

$$X_n \cong \lim_{\Delta[n]} X \to \lim_{\Lambda^k[n]} X$$

is a trivial fibration of simplicial sets for all $n \geq 2$ and $0 < k < n$.[3]

(iii) A Segal space $X_\bullet$ is a **complete Segal space**, just when the induced map on weighted limits

$$\lim_{\mathbb{I}} X \to \lim_{\Delta[0]} X \cong X_0$$

[3]By Reedy fibrancy, the induced map is already a Kan fibration, so to demand that it is a trivial fibration is equivalent to demanding that it is a weak homotopy equivalence. A priori, this definition is stronger than the usual **Segal condition**, which requires that the map induced on weighted limits by the inclusion of the spine of the n-simplex for each $n \geq 2$ is a trivial fibration. The spine inclusions are in the class cellularly generated by the inner horn inclusions, so by Exercise C.2.v applied to the two-variable adjunction involving the weighted limit, our condition clearly implies the classical Segal condition. The proof of the converse is more subtle and can be found as [**50**, 3.4].

is a trivial fibration of simplicial sets, asserting that the "space of isomorphisms in X"[4] is equivalent to the space X_0.[5]

The category of bisimplicial sets, as a presheaf category, is cartesian closed and hence enriched over itself. Among the great supply of product-preserving functors $\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to \mathcal{S}et^{\Delta^{\mathrm{op}}}$ that may be used to convert this to a simplicial enrichment, there are two of particular interest: $\mathrm{column}_0\colon \mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to \mathcal{S}et^{\Delta^{\mathrm{op}}}$, which sends a bisimplicial set X to its space X_0 of 0-simplices and $\mathrm{row}_0\colon \mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to \mathcal{S}et^{\Delta^{\mathrm{op}}}$, which passes to set the set of vertices in each space in the simplicial object. As observed by Joyal and Tierney [**50**], the former construction carries a complete Segal space to a Kan complex, while the latter construction carries a complete Segal space to a quasi-category and will be used to prove:construction that we will use to prove:

E.2.2. Proposition. *The full subcategory $\mathcal{CSS} \hookrightarrow \mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos in which the functor space $\mathrm{Fun}(A,B)$ is defined to be the* ***underlying quasi-category****, formed by the vertices in each space of internal hom B^A. With respect to this ∞-cosmos structure:*

(i) The underlying quasi-category functor $(-)_0 \coloneqq \mathrm{row}_0\colon \mathcal{CSS} \to \mathcal{QC}at$ is a cosmological biequivalence.
(ii) There is a second cosmological biequivalence cylinder$\colon \mathcal{QC}at \to \mathcal{CSS}$, *which carries a quasi-category A to the bisimplicial set whose (m,n)-simplices are simplicial maps $\Delta[m]\times\widetilde{\Delta[n]} \to A$ indexed by the product of the ordinal category with the ordinal groupoid.*

Proof. By a theorem of Rezk, the complete Segal spaces form the fibrant objects in a cartesian closed model structure borne by the category of bisimplicial sets in which all objects are cofibrant [**75**]. Precomposing with the adjoint pair of functors defined by $\pi_1([m]\times[n]) = [m]$ and $\iota_0([m]) = [m]\times[0]$ induces an adjunction as below-right:

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\iota_0"'] \arrow[r, phantom, "\perp"] & \mathbf{\Delta}\times\mathbf{\Delta} \arrow[l, bend right, "\pi_1"'] \end{tikzcd} \quad\rightsquigarrow\quad \begin{tikzcd} \mathcal{S}et^{\Delta^{\mathrm{op}}} \arrow[r, bend left, "\pi_1^*"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \arrow[l, bend left, "\iota_0^*"] \end{tikzcd} \tag{E.2.3}$$

Joyal and Tierney prove that this pair of functors defines a Quillen equivalence between the model structure for quasi-categories and the model structure for complete Segal spaces [**50**, 4.11]. By inspection, the left adjoint preserves finite products, so Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $\mathcal{CSS}$. By Lemma A.6.9, this makes the adjunction $\pi_* \dashv \iota_0^*$ into a simplicially enriched adjunction. Thus, it follows immediately from Corollary E.1.2 that $(-)_0 \coloneqq \mathrm{row}_0 \coloneqq \iota_0^*$ is a cosmological biequivalence.

A second adjunction between simplicial sets and bisimplicial sets pointing in the opposite direction has a left adjoint defined as the left Kan extension of the functor

$$\begin{array}{ccc} \mathbf{\Delta}\times\mathbf{\Delta} & \longrightarrow & \mathcal{S}et^{\Delta^{\mathrm{op}}} \\ [m]\times[n] & \longmapsto & \Delta[m]\times\widetilde{\Delta[n]} \end{array}$$

[4]Other choices of weight may be used to define the "space of isomorphisms" such as the colimit of the diagram

$$\Delta[2] \xleftarrow{d^0} \Delta[1] \xrightarrow{d^2} \Delta[2]$$

See [**75**, §11] for a discussion.

[5]By the 2-of-3 property, this is equivalent to the arguably more natural condition that the map $\Delta\colon X_0 \to \lim_{\mathbb{I}} X$, induced by $!\colon \mathbb{I} \to \Delta[0]$ is a weak homotopy equivalence.

along the Yoneda embedding $\mathbf{\Delta}\times\mathbf{\Delta}\hookrightarrow\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$; here $\widetilde{\Delta[n]}$ is the nerve of the groupoid with $n+1$ objects and one exactly morphism in each hom-set, obtained by freely inverting the morphisms in the ordinal category $\mathbb{n}+\mathbb{1}$. The right adjoint is the corresponding "nerve" functor described in the statement of (ii). Joyal and Tierney also prove that the adjunction

$$\begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\mathrm{lan}"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[l, bend left, "\mathrm{cylinder}"] \end{tikzcd}$$

is a Quillen equivalence with respect to the model structures for complete Segal spaces and quasi-categories [**50**, 4.12]. To conclude from Corollary E.1.2 that cylinder: $\mathcal{QC}at\to\mathcal{CSS}$ is a cosmological biequivalence it remains only to show that this functor is simplicially enriched and preserves simplicial cotensors, or equivalently, by Proposition A.4.6, that the adjunction $\mathrm{lan}\dashv$ cylinder is simplicially enriched.

To verify this, we make use of the external product bifunctor:

$$\begin{aligned} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}\times\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} &\xrightarrow{\quad\square\quad} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \\ (A,B) &\longmapsto (A\square B)_{m,n} := A_m\times B_n \end{aligned}$$

Since any bisimplicial set X may be recovered as a conical colimit of representables, which the left adjoint of course preserves, it suffices to consider maps from a representable bisimplicial set $\Delta[m]\square\Delta[n]$ to a simplicial set A. In the simplicial enrichment of $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ just defined, the simplicial set of maps from $\Delta[m]\square\Delta[n]$ to cylinder(A) has k-simplices defined to be the set of $(k,0)$-simplices in the bisimplicial set cylinder$(A)^{\Delta[m]\square\Delta[n]}$. Now

$$(\mathrm{cylinder}(A)^{\Delta[m]\square\Delta[n]})_{k,0} := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\square\Delta[n])\times(\Delta[k]\square\Delta[0]),\mathrm{cylinder}(A))$$

by the definition of the cartesian closed structure on bisimplicial sets

$$\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\times\Delta[k])\square\Delta[n],\mathrm{cylinder}(A))$$

by the definition of the external product

$$\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{lan}((\Delta[m]\times\Delta[k])\square\Delta[n]),A)$$

by adjunction. Joyal and Tierney prove in [**50**, 2.11] that the left Kan extension acts on the external tensor product by $\mathrm{lan}(B\square\Delta[n])\cong B\times\widetilde{\Delta[n]}$. So we have

$$\begin{aligned} &\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\times\Delta[k])\times\widetilde{\Delta[n]},A) \\ &\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}((\Delta[m]\times\widetilde{\Delta[n]})\times\Delta[k],A) \\ &\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{lan}(\Delta[m]\square\Delta[n])\times\Delta[k],A) \\ &\cong (A^{\mathrm{lan}(\Delta[m]\square\Delta[n])})_k \end{aligned}$$

by the definition of the cartesian closed structure on simplicial sets. This proves that the adjunction is compatible with the simplicial enrichments, so it follows from Corollary E.1.2 and [**50**, 4.12] that cylinder: $\mathcal{QC}at\to\mathcal{CSS}$ is a cosmological biequivalence. ☐

A second model of $(\infty, 1)$-categories is closely related.

E.2.4. Definition (Segal categories). A **Segal precategory** is a bisimplicial set $X_\bullet \in s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ whose space of 0-simplices X_0 is 0-skeletal on the set $X_{0,0}$ of its vertices. A **Segal category** is a Segal category that is Reedy fibrant and satisfying the Segal condition of Definition E.2.1.

Definition E.2.4 is mildly stronger than the usual definition first introduced by Dwyer, Kan, and Smith [**34**] and further developed by Hirschowitz and Simpson [**45**], which states that a Segal precategory $X_\bullet$ is a **Segal category** so that for each $n \geq 2$, the map

$$X_n \cong \lim_{\Delta[n]} X \to X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1 \cong \lim_{\Gamma[n]} \Delta[n]$$

induced on weighted limits by the inclusion $\Gamma[n] \hookrightarrow \Delta[n]$ of the spine of the n-simplex is a weak homotopy equivalence of simplicial sets without requiring Reedy fibrancy. We prefer to include Reedy fibrancy in our notion of Segal category so that the Segal categories are precisely the fibrant objects in an appropriate model structure on the category $\mathcal{PC}at$ of Segal precategories, which then gives rise to an ∞-cosmos.

Before we introduce the ∞-cosmos $\mathcal{S}egal$, we explain how to transform a complete Segal space into a Segal category.

E.2.5. Lemma. *There is a functor* disc: $s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ *defined by the pullback*

$$\begin{tikzcd} \mathrm{disc}(X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d] \\ \mathrm{cosk}_0(X_{0,0}) \arrow[r, hook] & \mathrm{cosk}_0(X_0) \end{tikzcd}$$

that lands in the subcategory of Segal precategories, and indeed is right adjoint to the inclusion $\mathcal{PC}at \hookrightarrow s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$. Moreover, the discretization of a Reedy fibrant Segal space is a Segal category.

Proof. Since the "vertex evaluation" map $X \to \mathrm{cosk}_0(X_0)$ is bijective on the 0th column, the pullback $\mathrm{disc}(X) \to \mathrm{cosk}_0(X_{0,0})$ must we as well. Hence $\mathrm{disc}(X)_0 \cong X_{0,0}$, which proves that $\mathrm{disc}(X)$ is a Segal precategory. To prove the adjointness, note that for any Segal precategory Y and bisimplicial map $f \colon Y \to X$, the component $f_0 \colon Y_0 \to X_0$ factors uniquely through $X_{0,0} \hookrightarrow X_0$ by discreteness of Y. This induces the required unique factorization of f through $\mathrm{disc}(X) \hookrightarrow X$.

Finally, any simplicial space that is 0-coskeletal is automatically Reedy fibrant and a Segal space: the maps of Definition E.2.1(i) and (ii) are both isomorphisms. When X is Reedy fibrant, the map $X_n \to \mathrm{cosk}_0(X_0)_n \cong X_0^n$ is a Kan fibration, so the pullback that defines the simplicial set $\mathrm{disc}(X)_n$ is a homotopy pullback. Applying Lemma C.1.11 to Quillen's model structure for Kan complexes on simplicial sets, the Segal maps (ii) for X pull back to define analogous weak homotopy equivalences for $\mathrm{disc}(X)$. ☐

E.2.6. Proposition. *The full subcategory $\mathcal{S}egal \hookrightarrow \mathcal{PC}at$ defines a cartesian closed ∞-cosmos in which the functor space $\mathrm{Fun}(A, B)$ is defined to be the* ***underlying quasi-category****, formed by the vertices in each space of internal hom B^A. With respect to this ∞-cosmos structure:*

(i) The underlying quasi-category functor $(-)_0 \coloneqq \mathrm{row}_0 \colon \mathcal{S}egal \to \mathcal{QC}at$ is a cosmological biequivalence.

(ii) There is a cosmological biequivalence disc: $\mathcal{CSS} \to \mathcal{S}egal$ *that "discretizes" a complete Segal space into a Segal category.*

(iii) *There is a second cosmological biequivalence* prism: $\mathcal{QC}at \to \mathcal{S}egal$*, which carries a quasi-category* A *to the bisimplicial set whose* (m, n)*-simplices are simplicial maps* $\Delta[m] \times \Delta[n] \to A$ *whose components at each vertex of* $\Delta[m]$ *are constant.*

Proof. By Pellissier and Bergner, the (Reedy fibrant) Segal category form the fibrant objects in a cartesian closed model structure borne by the category of Segal precategories in which all objects are cofibrant [**69**, **9**, **11**]. The cartesian closed structure on $\mathcal{PC}at$ can be defined explicitly, or deduced from the observation that $\mathcal{PC}at$ is a category of presheaves; see Exercise E.2.i.

The adjoint functors of (E.2.3) restrict to an adjunction between simplicial sets and Segal precategories, which Joyal and Tierney again prove define a Quillen equivalence between the model structure for quasi-categories and the model structure for Segal categories [**50**, 5.6]. Arguing as in Proposition **??**, Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $\mathcal{S}egal$, and Lemma A.6.9 and Corollary E.1.2 imply that $(-)_0 \coloneqq \mathrm{row}_0 \coloneqq \iota_0^*$ is a cosmological biequivalence.

By a theorem of Bergner, the inclusion $\dashv$ discretization adjunction of Lemma E.2.5 defines a Quillen equivalence between the model structure for complete Segal spaces and the model structure for Segal categories [**11**, §6]. To conclude from Corollary E.1.2 that disc: $\mathcal{CSS} \to \mathcal{S}egal$ is a cosmological biequivalence it remains only to show that this functor is simplicially enriched and preserves simplicial cotensors, or equivalently, by Proposition A.4.6, that the adjunction is simplicially enriched. This follows from the fact that this adjunction commutes with the underlying quasi-category adjunctions for $\mathcal{CSS}$ and $\mathcal{S}egal$; see Remark E.2.7. In particular, since the inclusion $\mathcal{PC}at \hookrightarrow s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ preserves binary products, for any bisimplicial set C and Segal precategory S, $\mathrm{disc}(C^S) \cong \mathrm{disc}(C)^S$. A similar argument shows that the simplicial cotensors are preserved. Passing to underlying quasi-categories, this induces the desired simplicially enriched adjunction, which makes disc: $\mathcal{CSS} \to \mathcal{S}egal$ simplicial and hence cosmological.

A second adjunction between simplicial sets and Segal precategories pointing in the opposite direction has left adjoint given by restriction along the diagonal functor $\Delta\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ and right adjoint, which we call "prism," given by right Kan extension along the same followed by discretization. Joyal and Tierney also prove that this adjunction defines a Quillen equivalence with respect to the model structures for complete Segal spaces and quasi-categories [**50**, 5.7]. As above, to conclude that prism: $\mathcal{QC}at \to \mathcal{S}egal$ is a cosmological biequivalence it remains only to argue that this adjunction is simplicially enriched. Since the prism functor is the composite of the right adjoint to the diagonal functor diag: $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ followed by discretization and we have already argued that the latter adjunction is simplicially enriched, it suffices to show that diag $\dashv$ ran is simplicially enriched.

To that end, consider a bisimplicial set X and a simplicial set A. By definition

$$\begin{aligned}
(\mathrm{ran}(A)^X)_k &\coloneqq (\mathrm{ran}(A)^X)_{k,0} \\
&\coloneqq \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}(X \times (\Delta[k] \square \Delta[0]), \mathrm{ran}(A)) \\
&\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}(\mathrm{diag}(X \times (\Delta[k] \square \Delta[0])), A) \\
&\cong \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}(\mathrm{diag}(X) \times \Delta[k], A) \\
&\coloneqq (A^{\mathrm{diag}(X)})_k,
\end{aligned}$$

which is what we wanted to show. $\square$

E.2.7. REMARK. This discretization functor commutes with the underlying quasi-category functors:

$$\begin{tikzcd} \mathcal{CSS} \arrow[rr, "\mathrm{disc}"] \arrow[dr, "(-)_0"'] & & \mathcal{S}egal \arrow[dl, "(-)_0"] \\ & \mathcal{QC}at & \end{tikzcd}$$

as can most easily be seen by considering the left adjoints to these functors at the level of model categories. However, discretization does not commute with the cylinder and prism constructions on the nose, only up to equivalence. For a quasi-category A, prism(A) is the Segal category with (m,n)-simplices given by the set of simplicial maps $\Delta[m] \times \Delta[n] \to A$ whose components at each vertex of $\Delta[m]$ are constant. By contrast, disc(cylinder(A)) is the Segal category with (m,n)-simplices given by the set of simplicial maps $\Delta[m] \times \widetilde{\Delta[n]} \to A$ whose components at each vertex of $\Delta[m]$ are constant.

E.2.8. PROPOSITION. *The full subcategory* $1\text{-}\mathcal{C}omp \hookrightarrow 1\text{-}\mathcal{S}trat$ *defines a cartesian closed* ∞*-cosmos in which the functor space* $\mathsf{Fun}(A,B)$ *is defined to be the* ***underlying quasi-category*** *of the internal hom* B^A*. With respect to this* ∞*-cosmos structure, both the underlying quasi-category functor* $(-)_0 \colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$ *and the natural marking functor* $(-)^\natural \colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ *are cosmological.*

PROOF. By independent theorems of Lurie [**61**, §3.1.3-4] and Verity [**102**, §6.5], the naturally marked quasi-categories, which we call **1-complicial sets**, form the fibrant objects in a cartesian closed model structure borne by the category of stratified simplicial sets in which all objects are cofibrant. There is an adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "(-)^\flat"] \arrow[r, phantom, "\perp"] & 1\text{-}\mathcal{S}trat \arrow[l, bend left, "U"] \end{tikzcd}$$

in which the right adjoint forgets the marking and the left adjoint assigns each simplicial set the minimal marking, which Lurie proves defines a Quillen equivalence between the model structure for quasi-categories and the model structure for 1-complicial sets [**61**, 3.1.5.1]. By inspection, the left adjoint preserves finite products, so Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on the full subcategory $1\text{-}\mathcal{C}omp$; the fact that the model category of 1-complicial sets is enriched over the model structure for quasi-categories via this construction is observed already in [**61**, 3.1.4.5]. As in the proofs of Proposition E.2.2 and E.2.6, it follows that the forgetful functor defines a cosmological biequivalence $U \colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$ the coincides with the underlying quasi-category functor; see Remark 1.3.9.

For any quasi-categories A and B, observe that there is a natural isomorphism $\mathsf{Fun}(A,B) \cong \mathsf{Fun}(A^\natural, B^\natural)$ between the functor quasi-category in $\mathcal{QC}at$ and the just-defined functor space in $1\text{-}\mathcal{C}omp$ between their natural markings; the point is that simplicial maps $A \to B$ preserve isomorphisms and hence the natural markings. Verity shows that the natural marking functor $(-)^\sharp \colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ creates the fibrations between fibrant objects [**102**, 114-118], and hence, since limits in $1\text{-}\mathcal{C}omp$ are created in $\mathcal{QC}at$, it follows that the functor $(-)^\natural \colon \mathcal{QC}at \to 1\text{-}\mathcal{C}omp$ is a cosmological biequivalence, and indeed an inverse isomorphism to $(-)_0 \colon 1\text{-}\mathcal{C}omp \to \mathcal{QC}at$. □

Exercises.

E.2.i. Exercise. Joyal and Tierney identify the subcategory $\mathcal{PC}at \hookrightarrow \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ with the category of presheaves indexed by the 1-categorical quotient $\mathbf{\Delta}|_2$ of $\mathbf{\Delta}\times\mathbf{\Delta}$ defined by inverting the maps in the image of the functor $[0]\times\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}\times\mathbf{\Delta}$ [**50**, 5.4]. Redefine the three adjunctions between $\mathcal{PC}at$, $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$ and $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ appearing in the proof of Proposition E.2.6 from this point of view.

E.2.ii. Exercise. Show that each of the cosmological biequivalences cylinder: $\mathcal{QC}at \rightrightarrows \mathcal{CSS}$ and prism: $\mathcal{QC}at \rightrightarrows \mathcal{S}egal$ are, respectively, sections of the underlying quasi-category functors.

E.3. ∞-cosmoi of (∞, n)-categories

In this section, we introduce a variety of ∞-cosmoi whose objects define models of (∞, n)-categories for $1 < n \leq \infty$. In these cases, the ∞-cosmos describes the $(\infty, 2)$-category of ∞-categories, ∞-functors, and ∞-natural transformations, omitting higher-dimensional transformations. In individual cases, internal homs or generalized elements may allow access to higher-dimensional non-invertible morphisms.

Because the combinatorics entailed in precisely specifying a model of (∞, n)-categories can be rather involved, to save space, we do not define every one of the higher categorical notions discussed here, instead providing external references to where such definitions can be found.

A few of our models of (∞, n)-categories are defined as presheaves indexed by a 1-category $\mathbf{\Theta}_n$ first introduced by Joyal in an unpublished note [**47**], which we present in an equivalent form due to Berger [**7**].

E.3.1. Definition. For $0 \leq n \leq \infty$, define a family of 1-categories $\mathbf{\Theta}_n$ inductively as follows.

- $\mathbf{\Theta}_0 \coloneqq \mathbb{1}$ is the terminal category and $\mathbf{\Theta}_1 \coloneqq \mathbf{\Delta}$ is the category of finite non-empty ordinals and order-preserving maps.
- $\mathbf{\Theta}_n \coloneqq \mathbf{\Delta}\wr\mathbf{\Theta}_{n-1}$, where $\mathbf{\Delta}\wr- \colon \mathcal{C}at \to \mathcal{C}at$ is the **categorical wreath product** construction.Explicitly, for a 1-category C, $\mathbf{\Delta}\wr C$ is the category whose:
  - objects are tuples $[n](c_1, \ldots, c_n)$ where $[n] \in \mathbf{\Delta}$ and $c_i \in C$.
  - morphisms $(\alpha; \vec{f}) \colon [n](c_1, \ldots, c_n) \to [m](c'_1, \ldots, c'_m)$ are given by a simplicial map $\alpha \colon [n] \to [m] \in \mathbf{\Delta}$ together with morphisms $f_{i,j} \colon c_i \to c'_j \in C$ for all $0 < i \leq n$ and $\alpha(i-1) < j \leq \alpha(i)$.

The objects of $\mathbf{\Theta}_n$ define pasting diagrams of k-cells for $0 \leq k \leq n$ while the morphisms define projection, composition, and degeneracy maps. The functor $\mathbf{\Theta}_n \hookrightarrow n\text{-}\mathcal{C}at$ that sends a pasting diagram to the free strict n-category that it generates is full and faithful [**7**, 3.7].

For instance, the morphism in $\mathbf{\Theta}_2$

$$[2]([1],[1]) \xrightarrow{(\delta^2;(\delta^1,!,\mathrm{id}))} [3]([2],[0],[1])$$

corresponds to the 2-functor between the free 2-categories generated by the pasting diagrams

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} 0 \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=50, ""{name=D, above}] \arrow[from=U, to=M, phantom, "\Downarrow"] \arrow[from=M, to=D, phantom, "\Downarrow"] & 1 \arrow[r] & 2 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 3 \end{tikzcd}$$

that sends 0 to 0, 1 to 1, and 2 to 3, and sends the left 2-cell of the domain to the vertical composite of the leftmost 2-cells of the codomain and the right 2-cell of the domain to the whiskered composite of the rightmost 2-cell of the codomain.

E.3.2. LEMMA. *For any 1-category with a terminal element t, the adjunction below-left induces an adjunction below-right:*

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "!"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathbf{\Delta} \cong \mathbf{\Delta} \wr \mathbb{1} \arrow[r, bend right, "\mathbf{\Delta}\wr t"'] \arrow[r, phantom, "\perp"] & \mathbf{\Delta} \wr C \arrow[l, bend right, "\mathbf{\Delta}\wr !"'] \end{tikzcd}$$

PROOF. The categorical wreath product construction defines a 2-functor $\mathbf{\Delta} \wr - \colon \mathcal{C}at \to \mathcal{C}at$. □

Ara introduced a model of (∞, n)-categories for each $1 \leq n < \infty$ called n-**quasi-categories** as presheaves on $\mathbf{\Theta}_n$ characterized by a particular right lifting property described in [**1**, §5]. The case of 1-quasi-categories coincides with the usual notion of quasi-categories.

E.3.3. PROPOSITION. *For each $n \geq 1$, the full subcategory $n\text{-}\mathcal{QC}at \hookrightarrow \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos of n-quasi-categories.*

PROOF. Ara constructs a cartesian closed model structure on the category $\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ generalizing the Joyal model structure in the case $n = 1$ in which the fibrant objects are exactly the n-quasi-categories and in which the cofibrations are the monomorphisms [**1**]; in particular, all objects are cofibrant. Hence, to induce a cartesian closed ∞-cosmos structure on the full subcategory $n\text{-}\mathcal{QC}at$ it suffices to find a Quillen adjunction between this model structure and the model structure for quasi-categories whose left adjoint $\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ preserves binary products.

To that end, note that $[0] \in \mathbf{\Theta}_{n-1}$ is terminal for all $n > 1$, so Lemma E.3.2 provides an adjunction as below-left and hence an adjunction as below-right

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\mathbf{\Delta}\wr[0]"'] \arrow[r, phantom, "\perp"] & \mathbf{\Theta}_n \arrow[l, bend right, "\mathbf{\Delta}\wr !"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "(\mathbf{\Delta}\wr !)^*"] \arrow[r, phantom, "\perp"] & \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \arrow[l, bend left, "(\mathbf{\Delta}\wr[0])^*"] \end{tikzcd}$$

The right adjoint $\mathbf{\Delta} \wr [0] \colon \mathbf{\Delta} \hookrightarrow \mathbf{\Theta}_n$ induces $\mathbf{\Delta}$ as the subcategory of "pasting diagrams comprised of only 1-cells"; identifying $\mathbf{\Theta}_n$ with its image in $n\text{-}\mathcal{C}at$, the left adjoint $\mathbf{\Delta}\wr!$ then discards all the cells in dimension greater than 1 in each object of $\mathbf{\Theta}_n$.[6] Hence, the forgetful functor $(\mathbf{\Delta} \wr [0])^* \colon \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \to \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ forgets higher-dimensional cells. Its left adjoint, as a restriction functor between categories of presheaves, has its own left adjoint, and so clearly preserves products.

Indeed, for the same reason, the left adjoint preserves all limits and hence also preserves monomorphisms (which can be characterized as those maps whose kernel pair is given by identities). By a result of Joyal and Tierney [**50**, 7.15], to prove that an adjunction is Quillen, it suffices to show that the left adjoint preserves cofibrations, as we've just done, and the right adjoint preserves fibrations between fibrant objects. By Lemma C.2.6, this means that we need only verify that the left adjoint carries the inner horn inclusions $\{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n}$ and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ to trivial cofibrations in Ara's model structure. In fact, by [**50**, 3.5], it suffices to consider the spine inclusions $\{\Gamma[n] \hookrightarrow \Delta[n]\}_{n\geq 2}$ in place of the inner horn inclusions, which we shall.

[6] More precisely, the left adjoint to the inclusion $\mathcal{C}at \hookrightarrow n\text{-}\mathcal{C}at$, restricts to define the functor $\mathbf{\Delta}\wr! \colon \mathbf{\Theta}_t \to \mathbf{\Delta}$.

To see this, its helpful to note, as observed in [**1**, §6], that that the left adjoint commutes with the nerve embeddings of strict 1-categories and strict n-categories:

$$\begin{tikzcd} \mathcal{C}at \arrow[r, hook] \arrow[d, hook] & n\text{-}\mathcal{C}at \arrow[d] \\ \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, "(\Delta\wr !)^*"] & \mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \end{tikzcd}$$

In particular, the left adjoint carries the 1-categorical nerve of $\mathbb{1} \hookrightarrow \mathbb{I}$ to the strict n-categorical nerve of this map and, since the left adjoint also preserves colimits, it carries the inner horn inclusion $\Gamma[n] \hookrightarrow \Delta[n]$ to the corresponding "spine inclusion" for the object $[n]([0], \ldots, [0]) \in \mathbf{\Theta}_n$. As both types of maps are among Ara's "localizer of n-quasi-categories" of [**1**, 5.17], they are certainly trivial cofibrations. Hence, the adjunction is Quillen, as claimed, and Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on $n\text{-}\mathcal{QC}at$. ☐

Another model of (∞, n)-categories, for $0 \leq n < \infty$ is due to Rezk [**76**]. A **$\mathbf{\Theta}_n$-space** is a simplicial presheaf on $\mathbf{\Theta}_n$-satisfying Reedy fibrancy, Segal, and completeness conditions analogous to those of Definition E.2.1. A $\mathbf{\Theta}_1$-space is exactly a complete Segal space, while a $\mathbf{\Theta}_0$-space is just a Kan complex.

E.3.4. Proposition. *For each $n \geq 1$, the full subcategory $\mathbf{\Theta}_n\text{-}\mathcal{S}p \hookrightarrow s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ defines a cartesian closed ∞-cosmos of $\mathbf{\Theta}_n$-spaces for which the underlying complete Segal space functor $U \colon \mathbf{\Theta}_n\text{-}\mathcal{S}p \to \mathbf{\Theta}_1\text{-}\mathcal{S}p \cong \mathcal{CSS}$ is comological.*

Proof. Rezk constructs a cartesian closed model structure on the category $s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ generalizing his model structure for complete Segal spaces in the case $n = 1$, in which the fibrant objects are exactly the $\mathbf{\Theta}_n$-spaces and in which the cofibrations are the monomorphisms [**76**]; in particular, all objects are cofibrant. Hence, to induce a cartesian closed ∞-cosmos structure on the full subcategory $\mathbf{\Theta}_n\text{-}\mathcal{S}p$ it suffices to find a Quillen adjunction between this model structure and the model structure for complete Segal spaces whose left adjoint $s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \to s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}}$ preserves binary products. We then apply Corollary E.1.3 to the composite of this adjunction with the adjunction (E.2.3).

As in the proof of Proposition E.3.3, we obtain the desired adjunction from Lemma E.3.2 applied to the terminal object $[0] \in \mathbf{\Theta}_{n-1}$.

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, bend right, "\Delta\wr[0]"'] \arrow[r, phantom, "\perp"] & \mathbf{\Theta}_n \arrow[l, bend right, "\Delta\wr !"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} s\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "(\Delta\wr !)^*"] \arrow[r, phantom, "\perp"] & s\mathcal{S}et^{\mathbf{\Theta}_n^{\mathrm{op}}} \arrow[l, bend left, "(\Delta\wr[0])^*"] \end{tikzcd} \tag{E.3.5}$$

The left adjoint has a further left adjoint given by left Kan extension, and so preserves products.

It remains only to argue that this adjunction is Quillen. The model structure for $\mathbf{\Theta}_n$-spaces, and by specialization, also the model structure for complete Segal spaces, are defined as left Bousfield localiations of the injective (or, equivalently, Reedy) model structures on simplicial presheaves. In the injective model structures, the cofibrations and trivial cofibrations are defined objectwise in $s\mathcal{S}et$, so the left adjoint restriction functor is manifestly left Quillen with respect to these model structures. Consequently, the adjunction is Quillen for the localized model structures, if and only if the right adjoint, which Rezk refers to as the "underlying simplicial space" functor, preserves fibrant objects, because in that the case the left adjoint will preserve the new trivial fibrations, which are defined in

terms of these. A functor $X \in s\mathcal{S}et^{\Theta_n^{op}}$ is fibrant if and only if it satisfies Reedy, Segal, and completeness conditions. Since the adjunction (E.3.5) is Quillen for the injective/Reedy model structure, the Reedy fibrancy condition is preserved, and Rezk proves that the Segal condition is preserved as well [**76**, 7.2]. By definition, the completeness condition for Θ_n-spaces is created from the completeness condition for underlying simplicial spaces [**76**, §7], so this is preserved as well. Hence, the right adjoint (E.3.5) restricts to a functor $U\colon \Theta_n\text{-}\mathcal{S}p \to \mathcal{CSS}$, which we call the **underlying complete Segal space functor**.

Corollary E.1.3 applies to create a cartesian closed ∞-cosmos structure on $\Theta_n\text{-}\mathcal{S}p$. By Lemma A.6.9, the adjunction (E.3.5) is enriched over bisimplicial sets, and so the second part of Corollary E.1.3 applies to prove that the underlying complete Segal space functor is cosmological. □

There is another model for (∞, n)-categories that generalizes the complete Segal space model for $(\infty, 1)$-categories, which makes use of the notion of a *Rezk object* valued in a model category.

E.3.6. Definition (Rezk). Let $\mathcal{M}$ be a model category.

(i) A simplicial object $X_\bullet \in \mathcal{M}^{\Delta^{op}}$ is **Reedy fibrant** just when the induced map on weighted limits
$$X_m \cong \lim_{\Delta[m]} X \to \lim_{\partial\Delta[m]} X =: M_m X$$
is a fibration for all $m \geq 0$.

(ii) A Reedy fibrant simplicial object $X_\bullet$ is a **Segal object** just when the induced map on weighted limits
$$X_n \cong \lim_{\Delta[n]} X \to \lim_{\Lambda^k[n]} X$$
is a trivial fibration for all $n \geq 2$ and $0 < k < n$.

(iii) A Segal object $X_\bullet$ is a **Rezk object**, just when the induced map on weighted limits
$$\lim_{\mathbb{I}} X \to \lim_{\Delta[0]} X \cong X_0$$
is a trivial fibration.

A map $p\colon X_\bullet \to Y_\bullet \in \mathcal{M}^{\Delta^{op}}$ is a **Rezk isofibration** if the relative analogues of the maps appearing in (i), (ii), and (iii) formed by the Leibniz weighted limit of p with the appropriate maps of weights, are respectively fibrations, trivial fibrations, and a trivial fibration.

Our formulation of Definition E.2.1 was designed to make it clear that the complete Segal spaces are precisely the Rezk objects valued in Quillen's model structure for Kan complexes.

E.3.7. Proposition. *Suppose $\mathcal{M}$ is a Cisinski model category.*[7] *Then the full subcategory $\mathscr{R}ezk_{\mathcal{M}} \hookrightarrow \mathcal{M}^{\Delta^{op}}$ of Rezk objects defines an ∞-cosmos.*

Proof. We prove this result directly from Proposition E.1.1 by proving that a left Bousfield localization of the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ defines a Cisinski model structure in which the fibrant objects are exactly the Rezk objects that is enriched over the model structure for quasi-categories.

To begin, observe that the category $\mathcal{M}^{\Delta^{op}}$ is enriched, tensored, and cotensored over simplicial sets, with homs suggestively denoted by "Fun," in such a way that the Leibniz tensors of monomorphisms of simplicial sets with (trivial) Reedy cofibrations are (trivial) Reedy cofibrations [**30**, 4.4]. We will apply Jeff Smith's theorem **??** to prove that $\mathcal{M}^{\Delta^{op}}$ admits a model structure in which

[7]A **Cisinski model structure** is a combinatorial model structure on a Grothendieck topos in which the cofibrations are exactly the monomorphisms. It follows that the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ coincides with the injective model structure, and in particular that all objects are cofibrant [**23, 13**].

- the cofibrations are the Reedy/injective cofibrations, these being the monomorphisms,
- the fibrant objects are the Rezk objects,
- the fibrations between fibrant objects are the Rezk isofibrations, and
- weak equivalences are the **Rezk weak equivalences**, though maps $U \to V$ that induce equivalences of quasi-categories $\mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ for all Rezk objects X.

Note that by adjunction, a map $p\colon X \to Y \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ is a Rezk isofibration if and only if for all monomorphisms $m\colon A \to B \in \mathcal{M}$, the induced map

$$\mathcal{M}(B, X) \xrightarrow{\widehat{\mathcal{M}(m,p)}} \mathcal{M}(A, X) \underset{\mathcal{M}(A,Y)}{\times} \mathcal{M}(B, Y)$$

of simplicial sets is an isofibration of quasi-categories; see Exercise C.2.v. By Propositions **??** and **??**, this is the case if and only if this map has the right lifting property with respect to maps in the set

$$\mathcal{I}\hat{\times}\mathcal{J}, \quad \text{where} \quad \mathcal{I} \coloneqq \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n\geq 0} \quad \text{and} \quad \mathcal{J} \coloneqq \{\Lambda^k[n] \hookrightarrow \Delta[n]\}_{n\geq 2, 0<k<n} \cup \{\mathbb{1} \hookrightarrow \mathbb{I}\}.$$

By adjunction again, and Proposition C.2.9(i), p is a Rezk isofibration if and only if it has the right lifting property with respect to the sets of maps $(i\hat{\times}j)\hat{*}m \cong j\hat{\otimes}(i\hat{*}m)$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and m among the generating cofibrations in $\mathcal{M}$, where $*$ denotes the pointwise tensor $*\colon s\mathcal{S}et \times \mathcal{M} \to \mathcal{M}^{\Delta^{\mathrm{op}}}$ and $\otimes\colon s\mathcal{S}et \times \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}^{\Delta^{\mathrm{op}}}$ denotes the simplicial tensor. Since the Reedy cofibrations in $\mathcal{M}^{\Delta^{\mathrm{op}}}$ are generated by the set of maps $i\hat{*}m$ for $i \in \mathcal{I}$ and as m ranges over the generating cofibrations in $\mathcal{M}$ [**82**, 7.7], we conclude, again by adjunction, that p is a Rezk isofibration between Rezk objects if and only if

$$\mathsf{Fun}(V, X) \xrightarrow{\widehat{\mathsf{Fun}(c,p)}} \mathsf{Fun}(U, X) \underset{\mathsf{Fun}(U,Y)}{\times} \mathsf{Fun}(V, Y)$$

is an isofibration of quasi-categories for all monomorphisms $c\colon U \to V$ in $\mathcal{M}^{\Delta^{\mathrm{op}}}$.

Now it's easy to verify the conditions of Jeff Smith's theorem. The Rezk weak equivalences are accessible and satisfy the 2-of-3 property. We argue that the Rezk weak equivalences contain all Reedy weak equivalences and hence the Reedy trivial fibrations, characterized by the right lifting property against the monomorphisms. Transposing the observations already made in [**30**, 4.4] about the Reedy model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$, we see that for any Reedy trivial cofibration $w\colon U \to V$ and Rezk object X, $w^*\colon \mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ is an equivalence of quasi-categories. By Ken Brown's lemma C.1.10, the same is true when w is a mere Reedy weak equivalence. Note that a map $w\colon U \to V$ is both a Rezk weak equivalence and a cofibration just when $w^*\colon \mathsf{Fun}(V, X) \to \mathsf{Fun}(U, X)$ is a trivial fibration between quasi-categories. This characterization proves that the class of Rezk weak equivalences and cofibrations is stable under pushout and transfinite composition. Smith's theorem now implies that the model structure for Rezk objects exists.

To see that the model structure for Rezk objects is enriched over the model structure for quasi-categories, we must verify the three conditions for a Quillen two variable adjunction

$$(\otimes, \{,\}, \mathsf{Fun})\colon s\mathcal{S}et \times \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}^{\Delta^{\mathrm{op}}}.$$

The fact that Leibniz tensors of cofibrations are cofibrations was verified already for the Reedy model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ and the localized model structure for Rezk objects has the same cofibrations. To verify the remaining 2/3rds of this axiom, we appeal to a result of Dugger [**30**, 3.2], which tells us that in the presence of the first 1/3rd, to verify that Leibniz tensors of monomorphisms of simplicial sets with trivial cofibrations are trivial cofibrations, it suffices to show that the simplicial cotensor $(-)^K\colon \mathcal{M}^{\Delta^{\mathrm{op}}} \to \mathcal{M}^{\Delta^{\mathrm{op}}}$ preserves fibrations between fibrant objects. For left Bousfield localizations,

Rezk fibrations between Rezk objects coincide with Reedy fibrations between Rezk objects [**44**, 3.3.16]. It's easy to verify directly that $(-)^K$ preserves Rezk objects, and the preservation of Reedy fibrations is one of the facts we knew already.

For the final 1/3 of the Quillen two-variable adjunction, we use the second part of Dugger's [**30**, 3.2], which tells us that in the presence of the first 2/3rds, we need only verify that for all Rezk objects Z and trivial cofibrations of simplicial sets $j\colon J \to K$, the map $Z^j\colon Z^K \to Z^J$ is a Rezk weak equivalence (assuming $\mathcal{M}^{\Delta^{\mathrm{op}}}$ is left proper, which is the case here since all objects are cofibrant). In fact, we can show that this map is a trivial fibration, by checking the right lifting property against the monomorphisms $c\colon U \to V \in \mathcal{M}^{\Delta^{\mathrm{op}}}$. Transposing, we see that $c \boxslash Z^j$ if and only if $j \boxslash c^*\colon \mathsf{Fun}(V, Z) \to \mathsf{Fun}(U, V)$. But we verified three paragraphs above that c^* is an isofibration between quasi-categories, so the desired lifting property holds. □

Barwick's n-**fold complete Segal space** model of (∞, n)-categories is formed by iterating the Rezk objects construction n times [**3**]. For this to make sense, note that the model structure for Rezk objects on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ remains a Cisinski model structure, so this construction can be iterated. Specialization Proposition E.3.7, we conclude that for all $n \geq 1$, there exist ∞-cosmoi $\mathcal{CSS}_n$ of n-fold complete Segal spaces.

E.3.8. Remark. If $\mathcal{M}$ is a left proper combinatorial model category, the proof just given constructs a model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ whose fibrant objects are the Rezk objects that is enriched as a model category over the model structure for quasi-categories. The only hitch is that without the Cisinski condition, it's possible that not all fibrant objects are cofibrant. Nonetheless, this generalization can be understood as defining an ∞-cosmos of a sort to be discussed in §E.4.

Verity constructs a general family of cartesian model structures on the category of stratified simplicial sets whose fibrant objects are complicial sets of various flavors and whose fibrations are the corresponding notions of complicial fibration [**102**, §9.3]. A restriction of one of these model structures to the case of *1-trivial stratified simplicial sets*, with all simplices above dimension 1 marked, underlies the ∞-cosmos of Proposition E.2.8. Here, we consider model structures whose fibrant objects model (∞, n)-categories for $0 \leq n \leq \infty$, in which case we write **complicial set** to mean ∞-complicial set. The definitions are arranged so that a 0-complicial set is a (maximally marked) Kan complex, a 1-complicial set is a (naturally marked) quasi-category, and a k-complicial set is an n-complicial set whenever $m < n$.

E.3.9. Proposition. *For each $0 \leq n \leq \infty$, the full subcategory in the category of stratified simplicial sets spanned by the complicial sets defines a cartesian closed ∞-cosmos n-$\mathcal{C}omp$. Moreover, whenever $m < n$, the functor* $\mathsf{core}\colon n\text{-}\mathcal{C}omp \to m\text{-}\mathcal{C}omp$ *that discards all simplices in dimension $k > m$ that are not marked is cosmological.*

Proof. For a suitable class of monomorphisms $\mathcal{K}$, Verity defines a cartesian closed model structure on the category of stratified simplicial sets whose fibrant objects and fibrations between them are the $\mathcal{K}$-complicial sets and $\mathcal{K}$-complicial fibrations, characterized by a right lifting property against $\mathcal{K}$ [**102**, §9.3]. The cofibrations are the monomorphisms so in particular all objects are cofibrant. In more detail, for the m-complicial sets, the class of monomorphisms is defined to be

$$\begin{aligned}\mathcal{K}_m := &\left\{\Lambda^k[n] \hookrightarrow_r \Delta^k[n]\right\}_{n\geq 1, k\in[n]} \cup \left\{\Delta^k[n]' \hookrightarrow_e \Delta^k[n]''\right\}_{n\geq 2, k\in[n]}\\ &\cup \{\Delta[r] \hookrightarrow_e \Delta[r]_t\}_{r>m} \cup \left\{\Delta[j] \star \Delta[3]_{\mathrm{eq}} \star \Delta[k] \hookrightarrow \Delta[j] \star \Delta[3]^\sharp \star \Delta[k]\right\}_{j,k\geq -1}\end{aligned}$$

See [**81**] for an explanation of this notation. The first set of maps are referred to as the **complicial horn extensions** while the second set define the **complicial thinness extensions**. The third set imposes the condition that all simplices in dimension greater than m are marked, while the final condition is **saturation**, which in the presence of the other conditions, implies that all equivalences are marked. To apply Verity's theorem, the sets $\mathcal{K}_m$ must satisfy some technical conditions spelled out in [**102**, 91-92]. In this case, these conditions have been verified in forthcoming work of Viktoriya Ozornova and Martina Rovelli. By construction, the n-complicial sets live in the subcategory $n\text{-}\mathcal{S}trat$ of n-trivial stratified simplicial sets (with all simplices in dimension greater than n marked), and we may restrict the cartesian closed model structures to these subcategories.

The ∞-cosmoi $0\text{-}\mathcal{C}omp$ and $1\text{-}\mathcal{C}omp$ are isomorphic to the ∞-cosmoi $\mathcal{K}an$ and $\mathcal{QC}at$ respectively, so for now we consider $2 \leq n \leq \infty$. To define the ∞-cosmos $n\text{-}\mathcal{C}omp$, we apply Corollary E.1.3 to convert these self enrichments into an enrichment over quasi-categories via a string of Quillen adjunctions whose left adjoints preserve binary products:

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "(-)^\flat"] \arrow[r, phantom, "\perp"] & 1\text{-}\mathcal{S}trat \arrow[l, bend left, "U"] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & 2\text{-}\mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_1"] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, bend left] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & (n-1)\text{-}\mathcal{S}trat \arrow[l, bend left] \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & n\text{-}\mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_{n-1}"] & \cdots \end{tikzcd}$$

In the limiting case, we also consider adjunctions

$$\begin{tikzcd} n\text{-}\mathcal{S}trat \arrow[r, bend left, hook] \arrow[r, phantom, "\perp"] & \mathcal{S}trat \arrow[l, bend left, "\mathsf{core}_n"] \end{tikzcd}$$

where $\mathsf{core}_n X \hookrightarrow X$ is the simplicial subset containing only those simplices in dimension greater than n that are marked. By adjunction it is easy to verify that these functors carry $(n+1)$-complicial sets to n-complicial sets. Since the left adjoints preserve monomorphisms and products, this is enough to verify that the adjunctions are Quillen. Corollary E.1.3 now induces the desired ∞-cosmoi and cosmological core functors. $\square$

Building on past work of Hirschowitz-Simpson [**45**] and Pellissier [**69**], Simpson iterates the construction of the model structure for Segal categories [**89**, §19.2-4]. When the base model category is taken to be Quillen's model structure for Kan complexes, the n-th iteration defines the notion of **Segal n-categories**, the Simpson considers more general model categorical bases. Under suitable hypotheses, satisfied in the case of Segal n-categories, the model structure so produced is cartesian closed and has all objects cofibrant, which strongly suggests that there exists an ∞-cosmos spanned by its fibrant objects: the Reedy fibrant Segal n-categories. We leave the confirmation of this as an exercise for the interested reader.

Exercises.

E.3.i. Exercise. Given an explicit formulation of the "relative analogue" of the conditions (i), (ii), and (iii) used in Definition E.3.6 to define the notion of **Rezk isofibration**.

E.3.ii. Exercise. Investigate potential ∞-cosmos structures on the Segal n-categories of Hirschowitz and Simpson [**45**].

E.3.iii. Exercise. Search for cosmological biequivalences between the ∞-cosmoi constructed in this section (and please share your discoveries with the authors).

E.4. Other examples

In this section, we present a few additional examples of ∞-cosmoi to complement those found in Chapter 8. Left as an exercise for now.

E.5. ∞-cosmoi with non-cofibrant objects

See [**84**, §2] for now.

APPENDIX F

Compatibility with the analytic theory of quasi-categories

The aim in this section is to prove that the synthetic theory of quasi-categories is compatible with the analytic theory pioneered by André Joyal, Jacob Lurie, and many others.

F.1. Initial and terminal elements

In this section, we complete the argument sketched in Digression 4.3.11 and prove that the synthetic definition of a terminal element in a quasi-category coincides with the analytic definition first introduced by Joyal [**48**, 4.1]. In the following result, we prove the equivalence between three synthetic definitions of a terminal element—(i), which appeared first in Definition 2.2.1, (ii), which is essentially contained in Lemma 2.2.2, and (iii), which appeared as Proposition 4.3.10—and two analytic definitions of a terminal element (iv) and (v), which Joyal proves are equivalent [**48**, 4.2].

F.1.1. Proposition. *For a quasi-category A and element $t\colon 1 \to A$ the following are equivalent:*

(i) The element t defines a right adjoint to the unique functor:

$$\begin{tikzcd} 1 \arrow[r, bend right, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$

(ii) There exists a natural transformation

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "!"'] & \Downarrow\eta & A \\ & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

so that the component ηt is an isomorphism.

(iii) The domain-projection functor

$$\mathrm{Hom}_A(A,t) \xrightarrow[\sim]{p_0} A$$

defines a trivial fibration.

(iv) The projection functor

$$A_{/t} \xrightarrow{\sim} A$$

whose domain is the slice of A over t is a trivial fibration.

(v) Any sphere in A whose final vertex is t admits a filler:

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \\ & \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

*When these conditions hold, t defines a **terminal element** of A.*

Proof. Unpacking (i), all that is required to define an adjunction $! \dashv t$ is to define a unit natural transformation $\eta\colon \mathrm{id}_A \Rightarrow t!$ so that the component $\eta t = \mathrm{id}_t$ is an identity; see Lemma 2.2.2. But it suffices, as claimed by (ii), to require only that ηt is an isomorphism. The proof is a specialization of Lemma B.4.2. The vertical composite $(\eta t) \cdot (\eta t)$ is computed by the pasting diagram:

$$\begin{tikzcd} 1 \arrow[r, "t"] & A \arrow[rr, equal] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[rr, equal] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\eta"] & A \\ & & 1 \arrow[ur, "t"'] & & 1 \arrow[ur, "t"'] & \end{tikzcd}$$

By middle-four interchange, we can evaluating this composite by first whiskering the left-hand η with $!$, which yields $!\eta = \mathrm{id}_!$ since the quasi-category 1 is 2-terminal.[1] Hence $(\eta t) \cdot (\eta t) = (\eta t)$, so ηt is an idempotent isomorphism, and hence, by cancelation, an identity. This proves the equivalence of (i) and (ii).

Proposition 4.3.10 establishes the equivalences of (i) and (iii) in any ∞-cosmos.

By Proposition D.6.4, for any vertex t in a quasi-category, we have an equivalence

$$\begin{tikzcd} \mathrm{Hom}_A(A,t) \arrow[rr, "\sim"] \arrow[dr, two heads, "p_0"'] & & A_{/t} \arrow[dl, two heads] \\ & A & \end{tikzcd}$$

between the domain-projection isofibration and the canonical projection from the slice construction of Proposition 4.2.5. Consequently, by the two-of-three property, one isofibration is an equivalence if and only if the other is, proving the equivalence of (iii) and (iv).

By Definition 1.1.24, the projection $A_{/t} \twoheadrightarrow A$ is a trivial fibration if and only if the following right lifting property holds for all $n \geq 0$

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & A_{/t} \arrow[d, two heads] \\ \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed, "w"] & A \end{tikzcd}$$

Via the adjunction

$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "-\star\Delta[0]"] \arrow[r, phantom, "\perp"] & {}^{\Delta[0]/}s\mathcal{S}et \arrow[l, bend left, "-/-"] \end{tikzcd}$$

the sphere $u\colon \partial\Delta[n] \to A_{/t}$ transposes into a map $\Lambda^{n+1}[n+1] \to A$ with final vertex A, with the simplex $v\colon \Delta[n] \to A$ providing a filler for the open face of the horn. Thus, together the maps u and v transpose to define a sphere $\partial\Delta[n+1] \to A$ with final vertex t. The desired lift $w\colon \Delta[n] \to A_{/t}$ exists

[1]See the discussion in the paragraph preceding Proposition 1.4.5.

just when this transposed sphere admits a filler. In this way, we see that the right lifting properties

$$\forall n \geq 0 \qquad \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & A_{/t} \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \\ & \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \forall n \geq 1$$

are transposes, proving the equivalence of (iv) and (v). □

There is a relative extension of Joyal's characterization (v) used in the proof of Proposition 9.3.2, which is involved with the construction of a homotopy coherent adjunction.

F.1.2. Lemma. *Suppose E and B are quasi-categories which possess a terminal element and $p\colon E \twoheadrightarrow B$ is an isofibration which preserves them: if t is terminal in E then pt is terminal in B. Then any lifting problem of the following form has a solution*

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ & \Delta[n] \arrow[r, "v"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Proof. Using the universal property of the terminal object t in E and Proposition F.1.1(v), we we may extend the sphere u to a map $w\colon \mathbf{\Delta}[n] \to E$. This defines two maps $pw, v\colon \mathbf{\Delta}[n] \rightrightarrows B$ with a common boundary $pu\colon \partial\mathbf{\Delta}[n] \to B$, which we may use to define a sphere $h\colon \partial\mathbf{\Delta}[n+1] \to B$ with $h\delta^{n+1} = pw$ and $h\delta^n = v$ by starting with the degenerate simplex $pw\sigma^n\colon \mathbf{\Delta}[n+1] \to B$, restricting to its boundary, and then replacing the n^{th} face in this sphere with $v\colon \mathbf{\Delta}[n] \to B$. By construction, h maps the object $[n+1]$ to the object pt which is terminal in B, so it follows that we may fill this sphere to define a simplex $k\colon \mathbf{\Delta}[n+1] \to B$.

We may also construct a map $g\colon \Lambda^n[n+1] \to E$ by restriction from the degenerate simplex $w\sigma^n\colon \mathbf{\Delta}[n+1] \to E$, which then defines a factorization of the commutative square of the statement:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "u"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r, "v"'] & B \end{tikzcd} = \begin{tikzcd} \partial\mathbf{\Delta}[n] \arrow[r, hook, "\delta^n"] \arrow[d, hook] & \Lambda^n[n+1] \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \mathbf{\Delta}[n] \arrow[r, hook, "\delta^n"'] & \mathbf{\Delta}[n+1] \arrow[r, "k"'] \arrow[ur, dashed, "\ell"'] & B \end{tikzcd}$$

Since the central vertical of this commutative rectangle is an inner horn inclusion and its right hand vertical is an isofibration of quasi-categories, it follows that the lifting problem on the right has a solution $\ell\colon \mathbf{\Delta}[n+1] \to E$ as marked, and now it is clear that the map $\ell\delta^n\colon \mathbf{\Delta}[n] \to E$ provides a solution to the original lifting problem. □

Exercises.

F.1.i. Exercise. Prove that if A and B are quasi-categories which possess a terminal element and $f\colon A \to B$ is a functor which preserves terminal elements, then given any lifting problem as below-left

in which t is terminal in A

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ & \Delta[n] \arrow[r] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] \arrow[r, phantom, shift right=3, "\Downarrow\cong"] & A \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift as above-right so that the upper-left triangle commutes up to natural isomorphism and the bottom-right triangle commutes on the nose.

F.2. Limits and colimits

In this section, we expand Proposition 4.3.2 to prove that the synthetic definition of a limit of a diagram indexed by a simplicial set and taking values in a quasi-category coincides with the analytic definition first introduced by Joyal [**48**, 4.5]. In the following result, we prove the equivalence between four synthetic definitions of a limit cone—(i), the original Definition 2.3.7, (ii), appearing in Proposition 13.5.1, (iii), appearing in Proposition 4.3.4, and (iv), from Proposition 4.3.2—and one analytic one (v), which is Joyal's. In this case, by the results just cited and Proposition F.1.1, there is nothing left to do but state the result and provide references for its components.

F.2.1. Proposition. *Consider a diagram $d\colon J \to A$, where J is a simplicial set and A is a quasi-category. The following are equivalent*

(i) There exists an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

(ii) There exists an absolute right lifting diagram

$$\begin{tikzcd} & A^{J^{\triangleleft}} \arrow[d, two heads, "\mathrm{res}"] \\ 1 \arrow[ur, "\ulcorner\lambda\urcorner"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow\cong"] & A^J \end{tikzcd}$$

(iii) There exists a pointwise right extension diagram

$$\begin{tikzcd} J \arrow[r, "d"] \arrow[d, "!"'] \arrow[dr, phantom, shift left=2, "\Uparrow\lambda"] & A \\ 1 \arrow[ur, "\ell"'] & \end{tikzcd}$$

(iv) The quasi-category of cones $\mathsf{Hom}_{A^J}(\Delta, d)$ *admits a terminal element* $\ulcorner\lambda\urcorner$*, representing a cone*

$$\begin{tikzcd} & 1 \arrow[d, "\ulcorner\lambda\urcorner"] & & & & 1 \arrow[dr, "\ell"] \\ & \mathsf{Hom}_{A^J}(\Delta, d) \arrow[dl] \arrow[dr, "p_0"] & & = & 1 \arrow[ur, equal] \arrow[dr, "d"'] & \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & A \arrow[dl, "\Delta"] \\ 1 \arrow[dr, "d"'] & \overset{\phi}{\Leftarrow} & A \arrow[dl, "\Delta"] & & & A^J \\ & A^J \end{tikzcd}$$

(v) The quasi-category $A_{/d}$ *admits a terminal element* $\ulcorner\lambda\urcorner \colon 1 \to A_{/d}$*, transposing to define an extension of* d *to a diagram*

$$\begin{tikzcd} & J \arrow[dl, hook'] \arrow[dr, "d"] \\ J^{\triangleleft} \arrow[rr, "\ulcorner\lambda\urcorner"'] & & A \end{tikzcd}$$

When these conditions hold, the data variously labeled (ℓ, λ) *or* $\ulcorner\lambda\urcorner$ *defines the* ***limit cone*** *over* d*.*

Proof. The equivalence of (i) and (ii) is proven in Proposition 4.3.4 for any ∞-category A and simplicial set J; the simplicial set $J^{\triangleleft} \coloneqq \Delta[0] \star J$ is the join from Definition 4.2.4. The equivalence of (i) and (iii) is proven in Proposition 4.3.4 for any cartesian closed ∞-cosmos. Proposition 4.3.2 proves the equivalence between (i) and (iv) for any diagram valued in any ∞-cosmos.

Finally, the equivalence between (iv) and (v) is a consequence of the equivalence of quasi-categories $\mathsf{Hom}_{A^J}(\Delta, d) \simeq A_{/d}$ over A of Proposition D.6.5, which provides two models for the quasi-category of cones over d. The final ingredient Lemma 2.2.6, which that if one of these quasi-categories has a terminal element, they both do, as terminal elements are preserved by the equivalence. ☐

F.3. Right adjoint right inverse adjunctions

To our knowledge, right adjoint right inverse adjunctions between quasi-categories have not been given much attention. Nonetheless, we pause to establish a useful analytic characterization of such adjunctions, which will help us compare various other synthetic and analytic definitions.

F.3.1. Lemma. *An isofibration* $f \colon B \twoheadrightarrow A$ *of quasi-categories admits a right adjoint right inverse if and only if for every element* $a \colon 1 \to A$*, there exists an element* $ua \colon 1 \to B$ *with* $fua = a$ *that has the property that that any lifting problem with* $n \geq 1$

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ua"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & B \arrow[d, two heads, "f"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \tag{F.3.2}$$

has a solution.

PROOF. If u is the right adjoint right inverse, then $\epsilon\colon fu = \mathrm{id}_A$ is the identity, and the induced fibered equivalence $\ulcorner\epsilon\circ f(-)\urcorner\colon \mathrm{Hom}_B(B,u) \xrightarrow{\sim} \mathrm{Hom}_A(f,A)$ of Proposition 4.1.1 is represented by the map induced by f between the comma ∞-categories defined in Proposition 3.4.5.

$$\begin{tikzcd}
& \mathrm{Hom}_B(B,u) \arrow[rr] \arrow[dl, dashed, "\ulcorner\epsilon\circ f(-)\urcorner"'] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & B^2 \arrow[dl, two heads, "f^2"] \arrow[dd, two heads] \\
\mathrm{Hom}_A(f,A) \arrow[rr, crossing over] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & A^2 & \\
& A\times B \arrow[rr, "u\times B" near start] \arrow[dl, equal] & & B\times B \arrow[dl, two heads, "f\times f"] \\
A\times B \arrow[rr, "f\times A"'] & & A\times A \arrow[from=uu, two heads, crossing over] &
\end{tikzcd}$$

By that result—or alternatively, by Proposition C.1.12, on which its proof relies—we see that that the induced map between comma ∞-categories is also an isofibration. Combining these facts, we see that $\ulcorner\epsilon\circ f(-)\urcorner\colon \mathrm{Hom}_B(B,u) \overset{\sim}{\twoheadrightarrow} \mathrm{Hom}_A(f,A)$ is a trivial fibration over $A\times B$. This trivial fibration pulls back over any vertex $a\colon 1 \to A$ to define a trivial fibration $\mathrm{Hom}_B(B,ua) \overset{\sim}{\twoheadrightarrow} \mathrm{Hom}_A(f,a)$ over B. By Corollary D.6.6, the domain and codomain are equivalent to Joyal's slices, so the isofibration $f\colon B_{/ua} \twoheadrightarrow f_{/a}$ induced by f is also a trivial fibration between quasi-categories. The defining lifting property of Definition 1.1.24

$$\begin{tikzcd}
\partial\Delta[n-1] \arrow[r] \arrow[d, hook] & B_{/ua} \arrow[d, two heads, "f", "\wr"'] \\
\Delta[n-1] \arrow[r] \arrow[ur, dashed] & f_{/a}
\end{tikzcd}$$

for $n \geq 1$ transposes to the lifting property of (F.3.2).

Conversely, the lifting property (F.3.2) can be used to inductively define a section $u\colon A \to B$ of f extending the choices of elements $ua\colon 1 \to B$ lifting each $a\colon 1 \to A$. The inclusion $\mathrm{sk}_0 A \hookrightarrow A$ can be expressed as a countable composite of pushouts of coproducts of maps $\partial\mathbf{\Delta}[n] \hookrightarrow \mathbf{\Delta}[n]$ with $n \geq 1$, and each intermediate lifting problem required to define a lift

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[rr, bend left, "ua"] & \mathrm{sk}_0 A \arrow[r, "u"] \arrow[d, hook] & B \arrow[d, two heads, "f"] \\
& A \arrow[r, equal] \arrow[ur, dashed, "u"] & A
\end{tikzcd}$$

will have the form of (F.3.2). To show that u is a right adjoint right inverse to f, it suffices, by Lemma B.4.2 to define a 2-cell $\eta\colon \mathrm{id}_B \Rightarrow uf$ that whiskers with u and with f to isomorphisms. We construct a representative for η by solving the lifting problem

$$\begin{tikzcd}
B \sqcup B \arrow[rr, "\mathrm{id}_B \sqcup uf"] \arrow[d, hook] & & B \arrow[d, two heads, "f"] \\
B\times\Delta[1] \arrow[r, two heads, "\pi"'] \arrow[urr, dashed, "\eta"'] & B \arrow[r, "f"'] & A
\end{tikzcd}$$

By construction $f\eta = \mathrm{id}_f$ is certainly invertible.

To show that ηu is an isomorphism it suffices, by Corollary D.4.17, to check that each of its components $\eta u(a)\colon ua \to ufua = ua$ are isomorphisms in A. Inverse isomorphisms can be found by elementary applications of the lifting property (F.3.2), whose details we leave to the reader. $\square$

F.4. Cartesian and cocartesian fibrations

The aim in this section is to establish the equivalence between synthetic and analytic characterizations of when an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories defines a cartesian or cocartesian fibration. The following result compares the three synthetic definitions proven equivalent in Theorem 5.1.11 with two analytic definitions due to Lurie [**61**, §2.4.1]. After we give its proof we comment on a few non-obvious aspects of the comparison.

F.4.1. Proposition. *For an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the following are equivalent and define what it means for p to be a* ***cartesisan fibration****:*

(i) Every natural transformation $\beta\colon b \Rightarrow pe$ as below-left admits a lift $\chi\colon e' \Rightarrow e$ as below-right:

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\Uparrow\beta"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

with the property that:

- ***induction****: Given any functor $x\colon Y \to X$ and natural transformations $\begin{tikzcd} Y \arrow[r, bend left, "e''"] \arrow[r, bend right, "ex"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} Y \arrow[r, bend left, "pe''"] \arrow[r, bend right, "pe'x"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ so that $p\tau = p\chi x \cdot \gamma$, there exists a lift $\begin{tikzcd} Y \arrow[r, bend left, "e''"] \arrow[r, bend right, "e'x"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}"] & E \end{tikzcd}$ of γ so that $\tau = \chi x \cdot \bar{\gamma}$.*
- ***conservativity****: Any fibered endomorphism of a restriction of χ is invertible: if $\begin{tikzcd} Y \arrow[r, bend left, "e'x"] \arrow[r, bend right, "e'x"'] \arrow[r, phantom, "\Downarrow\zeta"] & E \end{tikzcd}$ is any natural transformation so that $\chi x \cdot \zeta = \chi x$ and $p\zeta = \mathrm{id}_{pe'}$ then ζ is invertible.*

(ii) The functor $\ulcorner \mathrm{id}_p \urcorner\colon E \to \mathsf{Hom}_B(B, p)$ admits a right adjoint over B.

(iii) The functor $k := \widehat{\{\delta^0, p\}}\colon E^2 \twoheadrightarrow \mathsf{Hom}_B(B, p)$ admits a right adjoint right inverse.

(iv) Any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 1$

$$\begin{tikzcd} \Delta[1] \arrow[r, "{[n]\times\mathrm{id}}"'] \arrow[rrr, bend left, "\chi"] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[rr] \arrow[d, hook] & & E \arrow[d, two heads, "p"] \\ & \Delta[n]\times\Delta[1] \arrow[rr] \arrow[urr, dashed] & & B \end{tikzcd} \tag{F.4.2}$$

has a solution.

(v) *Any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 2$*

$$\begin{tikzcd} \Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \tag{F.4.3}$$

has a solution.

Condition (v) appears to be mildly stronger than [**61**, 2.4.2.1], which only requires that p is an inner fibration with the lifting property (F.4.3), but it follows easily that any such p must be an isofibration; see Exercise F.4.i.

Proof. Condition (i) is a repackaging of the original Definition 5.1.6 with the induction and conservativity properties of the p-cartesian natural transformations of Definition 5.1.1 strengthened to include the observation of Lemma 5.1.5 and the global requirement that these properties be inherited by restrictions of p-cartesian natural transformations. Modulo this translation, the equivalence of (i), (ii), and (iii) is proven in Theorem 5.1.11.

It remains to verify the equivalence between any of these synthetic conditions and the corresponding analytic ones. We'll demonstrate that (iii)⇔(iv) and (iv)⇔(v).

By Lemma F.3.1, the isofibration k admits a right adjoint right inverse if and only if any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E with the lifting property

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & E^2 \arrow[d, two heads, "k"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_B(B,p) \end{tikzcd} \tag{F.4.4}$$

for $n \geq 1$. Since the isofibration k is the Leibniz cotensor of p with the inclusion $1\colon \mathbf{\Delta}[0] \hookrightarrow \Delta[1]$, this lifting property is equivalent to the transposed lifting property

$$\begin{tikzcd} \Delta[1] \arrow[r, "{[n]\times \mathrm{id}}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ & \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

again for $n \geq 1$, proving the equivalence between (iii) and (iv).

Now we'll show that the lifting property (F.4.3) assumed in (v) suffices to solve this lifting problem. Our task is to find lifts along p for each of the $n+1$ shuffles of $\Delta[n]\times\Delta[1]$. We number these shuffles $0, \ldots, n$ starting from the closed end of the cylinder. Proceeding inductively for $k < n$, we choose a lift for the kth shuffle by filling a $\Lambda^{k+1}[n+1]$ horn, which can be done since p is an isofibration between quasi-categories. To lift the nth shuffle, we're required to fill a $\Lambda^{n+1}[n+1]$ horn whose final $\{n, n+1\}$ edge is χ, which can be done with the lifting property (F.4.3). This proves that (v)⇒(iv).

The converse implication holds on account of the retract diagram

$$\begin{tikzcd}
\Delta[1] \arrow[r, "\{n,n+1\}"'] \arrow[rr, bend left, "{[n]\times\mathrm{id}}"] & \Lambda^{n+1}[n+1] \arrow[r, hook] \arrow[d, hook] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & \Lambda^{n+1}[n+1] \arrow[d, hook] \\
& \Delta[n+1] \arrow[r, hook, "\iota"] & \Delta[n]\times\Delta[1] \arrow[r, "\rho"] & \Delta[n+1]
\end{tikzcd} \tag{F.4.5}$$

in which $\iota\colon \mathbf{\Delta}[n+1] \hookrightarrow \mathbf{\Delta}[n]\times\Delta[1]$ is the inclusion of the last shuffle at the open end of the cylinder and $\rho\colon \mathbf{\Delta}[n]\times\Delta[1] \to \mathbf{\Delta}[n+1]$ is the projection that collapses the closed end of the cylinder to a point. In coordinates:

$$\iota(i) \coloneqq \begin{cases} (i,0) & 0\le i\le n \\ (n,1) & i=n+1 \end{cases} \qquad \text{and} \qquad \rho(i,j) \coloneqq \begin{cases} i & j=0 \\ n+1 & j=1. \end{cases}$$

By Lemma C.2.3 it's now clear that the lifting property (F.4.2) implies the lifting property (F.4.3). ☐

The proof of Proposition F.4.1 demonstrated that the lifting properties of (iv) and (v) define equivalent analytic characterizations of what it means for a 1-simplex in the domain of an isofibration $p\colon E \twoheadrightarrow B$ to be "p-cartesian," see also [**61**, 2.4.1.8]. A priori this analytic definition of what it means for an edge in the domain of an isofibration p between quasi-categories to be p-cartesian appears to be significantly stronger than our synthetic Definition 5.1.1 and indeed this is the case—unless p is a cartesian fibration, in which case this weaker universal property, miraculously, turns out to be strong enough.

F.4.6. Proposition. *Fix $p\colon E \twoheadrightarrow B$ an isofibration of quasi-categories and consider a 1-simplex $\chi\colon e' \to e$ in E. The following are equivalent and characterize when χ is p-**cartesian**:*

(i) The induced map to the pullback in the square

$$\begin{tikzcd}
E_{/\chi} \arrow[r, two heads] \arrow[d, two heads, "p"'] & E_{/e} \arrow[d, two heads, "p"] \\
B_{/p\chi} \arrow[r, two heads] & B_{/pe}
\end{tikzcd}$$

is a trivial fibration.

(ii) Any lifting problem for $n\ge 2$

$$\begin{tikzcd}
\Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\
& \Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a solution.

(iii) Any lifting problem for $n \geq 1$

$$\begin{tikzcd}
\Delta[1] \arrow[r, "{[n]\times \mathrm{id}}"'] \arrow[rr, bend left, "\chi"] & \partial\Delta[n]\times\Delta[1] \underset{\partial\Delta[n]\times\{1\}}{\cup} \Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\
& \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a solution.

If $p\colon E \twoheadrightarrow B$ *is known to be a cartesian fibration, then the following criterian also characterizes the* p*-cartesian edges* $\chi\colon e' \to e$*.*

(iv)
- ***induction****: Given any 1-simplices* $\tau\colon e'' \to e'$ *in* E *and* $\gamma\colon pe'' \to pe'$ *in* B *that bound a 2-simplex*

$$\begin{tikzcd}
pe'' \arrow[rr, "p\tau"] \arrow[dr, "\gamma"'] & & pe \\
& pe' \arrow[ur, "p\chi"'] &
\end{tikzcd}$$

in B*, there exists a lift* $\bar{\gamma}\colon e'' \to e'$ *of* γ *that bounds a 2-simplex*

$$\begin{tikzcd}
e'' \arrow[rr, "\tau"] \arrow[dr, "\bar{\gamma}"'] & & e \\
& e' \arrow[ur, "\chi"'] &
\end{tikzcd}$$

in E*.*

$$\begin{tikzcd}
e'' \arrow[rr, "\tau"] \arrow[dr, dotted, "\bar{\gamma}"'] & & e & & \\
& e' \arrow[ur, "\chi"'] \arrow[d, mapsto] & & \in & \mathsf{h}E \arrow[dd, two heads, "p_*"] \\
& {} & & & \\
pe'' \arrow[rr, "p\tau"] \arrow[dr, "\gamma"'] & & pe & & \\
& pe' \arrow[ur, "p\chi"'] & & \in & \mathsf{h}B
\end{tikzcd}$$

- ***conservativity****: Any fibered endomorphism of* χ *is invertible: if* $\zeta\colon e' \to e'$ *is a 1-simplex in* E *so that there exists a 2-simplex*

$$\begin{tikzcd}
e' \arrow[rr, "\chi"] \arrow[dr, "\zeta"'] & & e \\
& e' \arrow[ur, "\chi"'] &
\end{tikzcd}$$

in E *and* $p\zeta$ *is an isomorphism, then* ζ *is an isomorphism.*

Proof. By the adjunction of Proposition 4.2.5, the lifting property that characterizes the trivial fibration of (i)

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, hook] & E_{/\chi} \arrow[d, two heads] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B_{/p\chi} \underset{B_{/pe}}{\times} E_{/e}
\end{tikzcd}$$

for $n \geq 0$ transposes to the lifting property of (ii), as argued in the proof of Proposition F.1.1. The proof of Proposition F.4.1(iv)⇔(v) demonstrates that (ii)⇔(iii).

Finally, in the case where $p\colon E \twoheadrightarrow B$ is known to be a cartesian fibration, Lemma 5.3.5 proves that the synthetically p-cartesian natural transformations, satisfying the weak universal property of (iv), coincide with the class of **p-cartesian 1-arrows** introduced in Definition 5.3.4, namely those vertices in the essential image of the right adjoint right inverse $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E^{\mathbb{2}}$ to $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathsf{Hom}_B(B,p)$. By Lemma F.3.1, these vertices have a lifting property (F.4.4), which we've just seen transposes to the lifting property of (iii). Thus, in the case where p is a cartesian fibration, (iv) is as strong as the a priori higher-dimensional lifting properties (i), (ii), and (iii). ☐

We now extend the notion of cartesian edge from Definition 5.3.4 to define a **cartesian cylinder** to describe a variant of the lifting properties appearing in Proposition F.4.6.

F.4.7. Definition (cartesian cylinders). Suppose that $p\colon E \twoheadrightarrow B$ is a cartesian fibration of quasi-categories and that X is any simplicial set. We say that a cylinder $e\colon X \times \Delta[1] \to E$ is **pointwise p-cartesian** if and only if for each 0-simplex $x \in X$ it maps the 1-simplex $(x\cdot\sigma^0, \mathrm{id}_{[1]})\colon (x,0) \to (x,1)$ to a p-cartesian arrow in E.

F.4.8. Lemma. *Let $p\colon E \twoheadrightarrow B$ be a cartesian fibration of quasi-categories. A cylinder $e\colon X \times \Delta[1] \to E$ is pointwise p-cartesian if and only if its dual $e\colon \Delta[1] \to E^X$ defines a p^X-cartesian arrow for the cartesian fibration $p^X\colon E^X \twoheadrightarrow B^X$.*

Proof. First note that Corollary 5.1.16 implies that $p^X\colon E^X \twoheadrightarrow B^X$ is a cartesian fibration. To prove the stated equivalence, note that a vertex is in the essential image of $\bar{r}^X\colon \mathsf{Hom}_{B^X}(B^X,p^X) \to E^{X\times\mathbb{2}}$ if and only if it transposes to a diagram $X \to E^{\mathbb{2}}$ whose vertices land in the essential image of $\bar{r}\colon \mathsf{Hom}_B(B,p) \to E$. ☐

Thus, the exponentiated functor $p^X\colon E^X \twoheadrightarrow B^X$ is a cartesian fibration whose cartesian arrows are the pointwise p-cartesian cylinders. It follows that for all $f\colon X \to Y$ restriction along f defines a cartesian functor of cartesian fibrations:

$$\begin{tikzcd} E^Y \arrow[r, "E^f"] \arrow[d, two heads, "p^Y"'] & E^X \arrow[d, two heads, "p^X"] \\ B^Y \arrow[r, "B^f"'] & B^X \end{tikzcd}$$

We give one further lifting property characterization of the p-cartesian edges for an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories that will be used in the body of the text.

F.4.9. Lemma. *Let $X \hookrightarrow Y$ be a simplicial subset of a simplicial set Y.*

(i) Any lifting problem for $n \geq 0$

$$\begin{tikzcd} X \times \Delta[1] \cup Y \times \{1\} \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ Y \times \Delta[1] \arrow[r, "b"'] \arrow[ur, dashed, "\bar{e}"'] & B \end{tikzcd}$$

with the property that the cylinder $X \times \Delta[1] \subseteq X \times \Delta[1] \cup Y \times \{1\} \xrightarrow{e} E$ is pointwise p-cartesian admits a solution $\bar{e}$ which is also pointwise p-cartesian.

(ii) Any lifting problem for $n \geq 1$

$$\begin{tikzcd} X \times \boldsymbol{\Delta}[n] \cup Y \times \Lambda^n[n] \arrow[r, "e"] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ Y \times \boldsymbol{\Delta}[n] \arrow[r, "b"'] \arrow[ur, dashed, "\bar{e}"'] & B \end{tikzcd}$$

in which the cylinder $Y \times \boldsymbol{\Delta}^{\{n-1,n\}} \subseteq X \times \boldsymbol{\Delta}[n] \cup Y \times \Lambda^n[n] \xrightarrow{e} E$ is pointwise p-cartesian admits a solution $\bar{e}$.

Proof. The Leibniz tensor with $X \hookrightarrow Y$ defines a functor $s\mathcal{S}et^{\mathbb{2}} \to s\mathcal{S}et^{\mathbb{2}}$ that preserves the retract diagram (F.4.5), so the lifting property (ii) follows from (i). In turn, the lifting property (i) follows inductively from Proposition F.4.6(iii) combined with the fact that any monomorphism $X \hookrightarrow Y$ can be decomposed as a sequential composite of pushouts of coproducts of inclusions $i_n \colon \partial\Delta[n] \hookrightarrow \Delta[n]$, and by Proposition C.2.9(vi), the pushout product with $\{1\} \hookrightarrow \Delta[1]$ is then similarly a sequential composite of pushouts of coproducts of the pushout products $\partial\boldsymbol{\Delta}[n] \times \Delta[1] \cup \boldsymbol{\Delta}[n] \times \{1\} \hookrightarrow \boldsymbol{\Delta}[n] \times \Delta[1]$. $\square$

A similar argument proves the following result.

F.4.10. Lemma. *A cocartesian fibration $q \colon E \twoheadrightarrow A$ of quasi-categories admits a right adjoint right inverse $t \colon A \to E$ if and only if for each object $a \in A$ the fiber E_a over that object has a terminal object.*

Proof. To prove necessity, consider the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r] \arrow[d, hook] & E_a \arrow[r, hook] \arrow[d, two heads, "q"] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & 1 \arrow[r, "a"'] & A \end{tikzcd}$$

Under our assumption that q has a right adjoint right inverse t we may apply the lifting condition depicted in equation F.3.2 of Lemma F.3.1 to show that the outer composite square has a lifting (the dotted arrow) and then apply the pullback property of the right hand square to obtain a lifting for the left hand square (the dashed arrow). This lifting property of the left hand square shows that ta is a terminal object in E_a.

To establish sufficiency, we start by taking the object $ta \in E$ to be the terminal object in the fiber E_a for each object $a \in A$. Now by Lemma F.3.1 our desired result follows if we can show that each lifiting problem

$$\begin{tikzcd} 1 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "ta"] & \partial\Delta[n] \arrow[r, "y"'] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ & \Delta[n] \arrow[r, "x"'] & A \end{tikzcd} \tag{F.4.11}$$

has a solution. Consider the order preserving function $k \colon [n] \times [1] \to [n]$ defined by $k(i, 0) \coloneqq i$ and $k(i, 1) \coloneqq n$. Taking nerves, this gives rise to a simplicial map $k \colon \boldsymbol{\Delta}[n] \times \Delta[1] \to \boldsymbol{\Delta}[n]$ and it is easy

to check that this restricts to a simplicial map $k\colon \mathbf{\Delta}[n] \times \Delta[1] \to \mathbf{\Delta}[n]$ which we may compose with $x\colon \mathbf{\Delta}[n] \to A$ to give (a representative of) a 2-cell

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "y"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\kappa"] & E \arrow[d, two heads, "q"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd}$$

with the property that $\kappa[n]$ is the identity 2-cell on a. Now we may take a cocartesian lift $\chi\colon y \Rightarrow u$ of κ and by construction the image of $u\colon \mathbf{\Delta}[n] \to E$ is contained entirely in the fiber $E_a \subseteq E$. What is more, we know that $\chi[n]$ is an isomorphism since we have, by pre-composition stability of cocartesian 2-cells, that it is a cocartesian lift of the identity 2-cell on a. Consequently, we see that $u\{n\}$ is a terminal object of E_a, since it is isomorphic to ta, and it follows that we may apply its universal property to extend $u\colon \partial\mathbf{\Delta}[n] \to E_a$ to a simplex $v\colon \mathbf{\Delta}[n] \to E_a$.

We may combine (a representative of) the 2-cell χ with v to assemble the upper horizontal map in the following commutative square:

$$\begin{tikzcd} \partial\mathbf{\Delta}[n] \times \Delta[1] \cup \mathbf{\Delta}[n] \times \{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "q"] \\ \mathbf{\Delta}[n] \times \Delta[1] \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

Now we can construct a solution for this lifting problem by successively picking fillers for each of the non-degenerate $(n+1)$-simplices in $\mathbf{\Delta}[n]\times\Delta[1]$. These are guaranteed to exist for the first $n-1$ of those because q is an isofibrations and they entail the filling of an inner horn. To obtain a filler for the last one we need to fill an outer horn, but observe that its final edge maps to an isomorphism of E, since it is the image of $\chi[n]$, and so it too has a filler. Finally on restricting the resulting map $\ell\colon \mathbf{\Delta}[n]\times\Delta[1] \to E$ to the initial end of the cylinder that is its domain, we obtain an n-simplex $\mathbf{\Delta}[n] \to E$ which is easily seen to be a solution to the original lifting problem in (F.4.11) as required. □

A final result demonstrates that discrete cocartesian and discrete cartesian fibrations between quasi-categories coincide with the classes of **left fibrations** and **right fibrations** introduced by Joyal [**48**].

F.4.12. Proposition. *For an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories, the following are equivalent and define what it means for p to be a **discrete cartesian fibration**:*

(i) The map $p\colon E \twoheadrightarrow B$ is a cartesian fibration whose fibers are Kan complexes.
(ii) Every 2-cell $\beta\colon b \Rightarrow pe$ in the homotopy 2-category of quasi-categories has an essentially unique lift: given $\chi\colon e' \Rightarrow e$ and $\psi\colon e'' \Rightarrow e$ so that $p\chi = p\psi = \beta$, then there exists an isomorphism $\gamma\colon e'' \Rightarrow e'$ with $\chi \cdot \gamma = \psi$ and $p\gamma = \mathrm{id}$.
(iii) The induced functor $k\colon E^{\mathbf{2}} \to \mathsf{Hom}_B(B, p)$ is a trivial fibration.
(iv) For $n \geq 0$ and $0 < k \leq n$ any lifting problem

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution.

PROOF. The first characterization (i) is a reinterpretation of the original Definition 5.4.2 using Proposition 16.2.3, which says that an isofibration $p\colon E \twoheadrightarrow B$ defines a discrete object in $\mathcal{QC}at_{/B}$ if and only if its fibers are Kan complexes. The equivalence of (i) with (ii) is proven in Proposition 5.4.3, while the equivalence of (i) with (iii) is proven in Proposition 5.4.6.

We conclude by demonstrating the equivalence of (iii) and (iv). Recall from Remark 5.1.13, that the canonical functors $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ can be constructed as the Leibniz cotensor of the monomorphism $1\colon \mathbb{1} \hookrightarrow \mathbb{2}$ with the trivial fibration $p\colon E \overset{\sim}{\twoheadrightarrow} B$. By adjunction, k is a trivial fibration if and only if the lifting problem below-right has a solution

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & E^{\mathbb{2}} \arrow[d, two heads, "k", "\wr"'] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_B(B,p) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta[n]\times\Delta[1]\cup\Delta[n]\times\{1\} \arrow[r] \arrow[d, hook] & E \arrow[d, two heads, "p"] \\ \Delta[n]\times\Delta[1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

By Proposition D.3.8, if $p\colon E \twoheadrightarrow B$ is a right fibration, satisfying condition (iv), then the lifting problem above-right admits a solution and hence $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ is a trivial fibration proving (iii). Conversely, for $0 < k \leq n$ the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ is a retract

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, hook, "\Lambda^k[n]\times\{0\}"] \arrow[d, hook] & \Lambda^k[n]\times\Delta[1]\cup\Delta[n]\times\{1\} \arrow[r, two heads, "r"] \arrow[d, hook] & \Lambda^k[n] \arrow[d, hook] \\ \Delta[n] \arrow[r, hook, "\Delta[n]\times\{0\}"] & \Delta[n]\times\Delta[1] \arrow[r, two heads, "r"] & \Delta[n] \end{tikzcd} \quad \text{where} \quad \begin{aligned} r(i,0) &= i \\ r(i,1) &= \begin{cases} i & i \neq k-1 \\ k & i = k-1 \end{cases} \end{aligned}$$

Thus, to solve the lifting problem postulated by (iv), it suffices to show that p lifts against the pushouts products $(\Lambda^k[n] \hookrightarrow \Delta[n]) \hat{\times} (\{1\} \hookrightarrow \Delta[1])$, which transposes to a lifting problem between the monomorphism $\Lambda^k[n] \hookrightarrow \Delta[n]$ and $k\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$. If (iii) holds and k is a trivial fibration, then this constructs the desired lift. ☐

Exercises.

F.4.i. EXERCISE. Suppose $p\colon E \to B$ is an inner fibration between quasi-categories so that any 1-simplex $\beta\colon b \to pe$ in B admits a lift $\chi\colon e' \to e$ in E so that any lifting problem for $n \geq 2$

$$\begin{tikzcd} \Delta[1] \arrow[r, "\{n-1,n\}"'] \arrow[rr, bend left, "\chi"] & \Lambda^n[n] \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\ & \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a solution. Show that p is an isofibration.

F.5. Adjunctions

The comparison between the analytic and synthetic definitions of adjunction between quasi-categories is somewhat more subtle as these are typically presented with different data. The synthetic definition, originally due to Joyal [**49**], of an adjunction involves two specified functors $f\colon B \to A$ and $u\colon A \to B$, together with specified maps $\eta\colon B\times\Delta[1] \to B$ and $\eta\colon A\times\Delta[1] \to A$ up to homotopy in

B^B or A^A, together with 2-simplices in the quasi-categories B^A and A^B that witness the triangle equalities. By contrast, the analytic notion, due to Lurie [**61**, 5.2.2.1], is defined to be an isofibration[2] that is both a cartesian fibration and a cocartesian fibration equipped with specified equivalences $M_0 \simeq B$ and $M_1 \simeq A$.

Since Proposition 2.1.12 demonstrates that the synthetic notion of adjunction is equivalence-invariant, we simplify our notation somewhat and let B and A denote the fibers over 0 and 1 respectively of the isofibration $M \twoheadrightarrow \Delta[1]$. Our aim in this section is to show that from a cocartesian and cartesian fibration $M \twoheadrightarrow \Delta[1]$, one can extract an adjunction between B and A, with the adjoint functors determined uniquely up to isomorphism, and conversely from a 2-categorical adjunction, one can construct a corresponding **correspondence** $M \twoheadrightarrow \Delta[1]$, which is unique up to fibered equivalence.

F.5.1. Proposition. *Let M be a quasi-category equipped with a map $M \twoheadrightarrow \Delta[1]$ that is both a cocartesian fibration and cartesian fibration. Then*

- *the fibers $B := M_0$ and $A := M_1$ and*
- *the functors $f\colon B \to A$ and $u\colon A \to B$ defined by the cocartesian and cartesian lift, respectively, of the generic arrow in $\Delta[1]$*

define an adjunction

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$$

Proof. This is a special case of Proposition 5.2.5. We recall the construction of f and u and leave the rest of the details to that result and Remark 5.2.6. Let χ denote a cocartesian lift of the generic arrow in $\Delta[1]$ whose domain is the inclusion $B \hookrightarrow M$ of the fiber over 0. The codomain of this lifted arrow lands in the fiber over 1 and thus factors uniquely through the inclusion $A \hookrightarrow M$ of that fiber:

$$\begin{tikzcd} B \arrow[rr, hook, bend left] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner", very near start] & & M \arrow[d, two heads] \\ 1 \arrow[rr, bend left, "0"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow"] & & \Delta[1] \end{tikzcd} = \begin{tikzcd} B \arrow[rr, hook, bend left] \arrow[rr, phantom, "\Downarrow\chi"] \arrow[rr, dotted, bend right] \arrow[dr, dashed, "f"] \arrow[dd, two heads] & & M \arrow[dd, two heads] \\ & A \arrow[ur, hook, bend right] \arrow[dl] & \\ 1 \arrow[rr, bend right, "1"'] & & \Delta[1] \end{tikzcd}$$

This factorization defines the functor $f\colon B \to A$. Since cocartesian lifts of a fixed arrow are unique up to isomorphism, this proves that the left adjoint f is unique up to isomorphism as claimed. The construction of the right adjoint $u\colon A \to B$ is dual, involving a cartesian lift of the generic arrow in $\Delta[1]$. □

The converse makes use of something we call the **quasi-categorical collage construction**.

F.5.2. Definition (the quasi-categorical collage construction). For any cospan $f\colon A \to C$ and $g\colon B \to C$ of quasi-categories, define a simplicial set $\mathrm{col}(f,g)$ by declaring that

$$\mathrm{col}(f,g)_n = \left\{ \left(\boldsymbol{\Delta}[i] \xrightarrow{a} A, \boldsymbol{\Delta}[j] \xrightarrow{b} B, \boldsymbol{\Delta}[n] \xrightarrow{c} C \right) \middle| \begin{array}{ll} c|_{\{0,\ldots,i\}} = f(a), & i,j \geq -1, \\ c|_{\{n-j,\ldots,n\}} = g(b), & i+j = n-1. \end{array} \right\}$$

[2]The nitpicker might note that Lurie only requires an inner fibration, but any inner fibration over $\Delta[1]$ is automatically an isofibration. In fact, so long as M is a quasi-category, any simplicial map $M \to \Delta[1]$ is automatically an isofibration.

with the convention that conditions indexed by $\mathbf{\Delta}[-1]$ are empty (or that each simplicial set is terminally augmented). There are simplicial maps

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{col}(f,g) \arrow[d, "\rho"] & A \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] \\ \{1\} \arrow[r, hook] & \Delta[1] & \{0\} \arrow[l, hook'] \end{tikzcd}$$

the top ones being the evident inclusions. The map ρ is defined to send an n-simplex $(a\colon \mathbf{\Delta}[i] \to A, b\colon \mathbf{\Delta}[j] \to B, c\colon \mathbf{\Delta}[n] \to C)$ to the n-simplex $[n] \to [1]$ that carries $0, \ldots, i$ to 0 and $i+1, \ldots, n$ to 1. Note that the fiber of ρ over 0 is isomorphic to A while the fiber of ρ over 1 is isomorphic to B

As was our custom for two-sided fibrations and modules, we customarily write $B + A \hookrightarrow \mathrm{col}(f,g)$ for the inclusions of the fibers over 1 and 0 — with the fiber over 1 on the left and the fiber over 0 on the right. This positions the covariantly-acting quasi-category on the "left" and the contravariantly-acting quasi-category on the "right."

F.5.3. Lemma. *The map $\rho\colon \mathrm{col}(f,g) \to \Delta[1]$ is an inner fibration. In particular, the simplicial set $\mathrm{col}(f,g)$ is a quasi-category.*

Proof. Since the fibers of ρ over 0 and 1 are the quasi-categories A and B, it suffices to consider inner horns

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{col}(f,g) \arrow[d, "\rho"] \\ \mathbf{\Delta}[n] \arrow[r, "\alpha"'] \arrow[ur, dashed] & \Delta[1] \end{tikzcd}$$

for which $\alpha\colon [n] \to [1]$ is a surjection. Suppose α carries $0, \ldots, i$ to 0 and $i+1, \ldots, n$ to 1. Note that for any $0 < k < n$, the faces $\{0, \ldots, i\}$ and $\{i+1, \ldots, n\}$ of $\mathbf{\Delta}[n]$ belong to the horn $\Lambda^k[n]$. In particular, the map $\Lambda^k[n] \to \mathrm{col}(f,g)$ identifies simplices $a\colon \mathbf{\Delta}[i] \to A$ and $\mathbf{\Delta}[n-i-1] \to B$ together with a horn $\Lambda^k[n] \to C$ whose initial and final faces are the images of these simplices under $f\colon A \to C$ and $g\colon B \to C$. Since C is a quasi-category this horn admits a filler $c\colon \mathbf{\Delta}[n] \to C$ and the triple (a,b,c) defines an n-simplex in $\mathrm{col}(f,g)$ solving the lifting problem. $\square$

We write $\mathrm{col}(f,B)$ for the collage of $f\colon A \to B$ with the identity on B.

F.5.4. Lemma. *For any $f\colon A \to B$, the map $\rho\colon \mathrm{col}(f,B) \to \Delta[1]$ is a cocartesian fibration.*

Proof. To prove the claim, we need only specify cocartesian lifts of the non-degenerate 1-simplex of $\Delta[1]$ and demonstrate that these edges have the corresponding universal property. To that end, for any vertex $a \in A_0$, let $\chi_a\colon \Delta[1] \to \mathrm{col}(f,B)$ be the 1-simplex

$$\chi_a \coloneqq (a\colon \mathbf{\Delta}[0] \to A, fa\colon \mathbf{\Delta}[0] \to B, fa \cdot \sigma^0\colon \Delta[1] \to B),$$

defined by the copy degenerate edge at $fa \in B_0$ lying over the 1-simplex in $\Delta[1]$. To show that χ_a is ρ-cocartesian, we must construct fillers for any left horn

$$\begin{tikzcd} \Delta[1] \arrow[r, "\{0,1\}"'] \arrow[rr, bend left, "\chi_a"] & \Lambda^0[n] \arrow[r] \arrow[d] & \mathrm{col}(f,B) \arrow[d, "\rho"] \\ & \mathbf{\Delta}[n] \arrow[r, "\beta"'] \arrow[ur, dashed] & \Delta[1] \end{tikzcd}$$

whose initial edge is χ_a. Note that this condition implies that the bottom map $\beta\colon [n] \to [1]$ carries 0 to 0 and the remaining vertices to 1. The map $\Lambda^0[n] \to \mathrm{col}(f,B)$ defines a horn $\Lambda^0[n] \to B$ in the quasi-category B whose first edge is degenerate. By Proposition 1.1.14, this "special outer horn" admits a filler $b\colon \mathbf{\Delta}[n] \to B$ and the triple

$$(a\colon \mathbf{\Delta}[0] \to A, b \cdot \delta^0\colon \mathbf{\Delta}^{n-1} \to B, b\colon \mathbf{\Delta}[n] \to B)$$

defines an n-simplex in $\mathrm{col}(f,B)$ that solves the lifting problem. ☐

F.5.5. Proposition. *For any $f\colon A \to B$ between quasi-categories, the collage $\mathrm{col}(f,B)$ defines the **oplax colimit**[3] of f in $\mathcal{QC}at$. That is $\mathrm{col}(f,B)$ defines a cone under the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\mathrm{id}\times\delta^0"'] & B \arrow[d, hook] \arrow[ddr, bend left] & \\ A \times \Delta[1] \arrow[r] \arrow[drr, bend right, "h"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "k"] & \\ & & \mathrm{col}(f,B) \end{tikzcd}$$

so that the induced map k is inner anodyne, and in particular an equivalence in the Joyal model structure.

Proof. The map k is a quotient of the map h, which has the following explicit description. For each n-simplex $(a,\alpha)\colon \mathbf{\Delta}[n] \to A \times \Delta[1]$ define $i := |\alpha^{-1}(0)| - 1$, so that $-1 \le i \le n$. Then h carries (a,α) to the n-simplex of $\mathrm{col}(f,B)$ corresponding to the triple

$$(a|_{\{0,\ldots,i\}}\colon \mathbf{\Delta}[i] \to A, fa|_{\{i+1,\ldots,n\}}\colon \mathbf{\Delta}[n-i-1] \to B, fa\colon \mathbf{\Delta}[n] \to B).$$

Note that the composite $\rho h\colon A \times \Delta[1] \to \Delta[1]$ is the projection.

It remains to present k as a sequential composite of pushouts of coproducts of inner horn inclusions. To do so, first note that

$$\mathrm{col}(f,B)_n = A_n \coprod A_{n-1} \times_{B_{n-1}} B_n \coprod \cdots \coprod A_0 \times_{B_0} B_n \coprod B_n$$

where each map $B_n \to B_i$ is the initial face map corresponding to $\{0,\ldots,i\} \hookrightarrow \mathbf{\Delta}[n]$. From the perspective of this decomposition, P_n is the subset containing the sets A_n and B_n and the subset of $A_i \times_{B_i} B_n$ whose component in B_n is in the image of f. The n-simplices of $\mathrm{col}(f,B)$ that remain to be attached correspond to elements of $A_i \times_{B_i} B_n$, for $0 \le i < n$, that are not in the image of f in the sense just discussed. Note in particular that $k\colon P_0 \hookrightarrow \mathrm{col}(f,B)_0$ is an isomorphism and $k\colon P_n \hookrightarrow \mathrm{col}(f,B)_n$ is an injection for all $n \ge 1$.

[3]See Exercise 17.1.i for an explication of the "oplax" appearing here.

To enumerate our attaching maps, we start with the collection of non-degenerate n-simplices of $\mathrm{col}(f, B)$ for $n \geq 1$ that are not in the image of f and remove also those elements of $A_i \times_{B_i} B_n$ whose components $b \in B_n$ are in the image of the degeneracy map $\sigma_i \colon B_{n-1} \to B_n$. Partially order this set of simplices first in the order of increasing n and the in order of increasing index i; that is we lexicographically order the collection of pairs (n, i) with $n \geq 1$ and $0 \leq i < n$. We will filter the inclusion $P \hookrightarrow \mathrm{col}(f, B)$ as

$$P \hookrightarrow P_{<(1,0)} \hookrightarrow P_{<(2,0)} \hookrightarrow P_{<(2,1)} \hookrightarrow P_{<(3,0)} \hookrightarrow \cdots \hookrightarrow P_{<(n,i)} \hookrightarrow \cdots \hookrightarrow \mathrm{colim} \cong \mathrm{col}(f, B)$$

where the simplicial set $P_{<(n,i)}$ is built from the previous one by a pushout of a coproduct of inner horns indexed by the set of n-simplices $(a, b) \in A_i \times_{B_i} B_n$ with b not in the image of f or σ_i. The filler for the horn indexed by (a, b) will attach this n simplex to B_n as the missing face of the horn and also the $n+1$ simplex $(a, b\sigma^i) \in A_i \times_{B_i} B_{n+1}$.

Consider a simplex $(a, b) \in A_i \times_{B_i} B_n$ with b not in the image of f or σ_i. Define a horn

$$\begin{tikzcd} \Lambda^{i+1}[n+1] \arrow[r] \arrow[d, hook] & P_{<(n,i)} \arrow[d, hook] \\ \mathbf{\Delta}[n+1] \arrow[r, "(a,b\sigma^i)"'] & \mathrm{col}(f, B) \end{tikzcd}$$

For each $0 \leq j < i+1$, the δ^j-face of the $n+1$ simplex $(a, b\sigma^i)$ is the n-simplex $(a\delta^j, b\sigma^i\delta^j)$, which lies in $P_{<(n,i-1)}$ or in $B \hookrightarrow P$ in the case $i = 0$. For each $i+1 < j \leq n+1$, the δ^j-face of the $n+1$ simplex $(a, b\sigma^i)$ is the n-simplex $(a, b\sigma^i\delta^j) = (a, b\delta^{j-1}\sigma^i) \in A_i \times_{B_i} B_n$, which was previously attached to $P_{<(n-1,i)}$. So the $\Lambda^{i+1}[n+1]$ indeed maps to $P_{<(n,i)}$, permitting an inductive construction of the next simplicial set in this sequence as the pushout

$$\begin{tikzcd} \coprod_{\sim} \Lambda^{i+1}[n+1] \arrow[r] \arrow[d, hook] & P_{<(n,i)} \arrow[d] \\ \coprod_{\sim} \mathbf{\Delta}[n+1] \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet \end{tikzcd}$$

defining $P_{<(n+1,0)}$ in the case $i = n-1$ and $P_{<(n,i+1)}$ otherwise. $\square$

F.5.6. Theorem. *Consider a pair of functors between quasi-categories $f \colon B \to A$ and $u \colon A \to B$. Then f is left adjoint to u if and only if the collages $\mathrm{col}(f, A)$ and $\mathrm{col}(B, u)$ are equivalent under $A + B$ and over $\Delta[1]$, in which case $\mathrm{col}(f, A) \twoheadrightarrow \Delta[1]$ or equivalently $\mathrm{col}(B, u) \twoheadrightarrow \Delta[1]$ defines both a cocartesian and a cartesian fibration.*

Proof. First suppose that $\mathrm{col}(f, A) \simeq \mathrm{col}(B, u)$ under $A + B$ and over $\Delta[1]$. By Lemma F.5.4 and Corollary 5.1.17 this means that the map $\mathrm{col}(f, A) \to \Delta[1]$ is both a cocartesian and a cartesian fibration. By Proposition F.5.1 it follows that the 1-arrow in $\Delta[1]$ from 0 to 1 induces an adjunction between the fibers B and A. By inspection of that proof, the left adjoint functor so-constructed in the case of the bifibration $\mathrm{col}(f, A) \to \Delta[1]$ is f; substituting the equivalent bifibration $\mathrm{col}(B, u) \to \Delta[1]$, we see that the right adjoint is equivalent to u.

For the converse, we work in the opposite ∞-cosmos $\mathcal{QC}at^{\mathrm{op}}$, an ∞-cosmos in which "not all objects are cofibrant," as described in §E.5. In that context, Proposition F.5.5 proves that $\mathrm{col}(f, A)$ and $\mathrm{col}(B, u)$ construct the contravariant and covariant comma objects associated to the functors f

and u. If $f \dashv u$ in $\mathcal{QC}at$ then these functors are also adjoint in $\mathcal{QC}at^{\mathrm{op}}$ and Proposition 4.1.1 then proves that the commas $\mathrm{col}(f, A)$ and $\mathrm{col}(B, u)$ are equivalent under $A + B$. By construction, this equivalence also lies over $\Delta[1]$. $\square$

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