Higher Dimensional Categories: Model Categories and Weak Factorisation Systems

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1 Introduction

Loosely speaking, "homotopy theory" is a perspective which treats objects as equivalent if they have the same "shape" which, for a category theorist, occurs when there exists a certain class $\mathcal{W}$ of morphisms that one would like to invert, but which are not in fact isomorphisms. Model categories provide a setting in which one can do "abstract homotopy theory" in subjects far removed from the original context of topological spaces. Given a model category, one can form its *homotopy category*, in which the weak equivalences $\mathcal{W}$ become isomorphisms, but it is the additional structure provided by two other distinguished classes of morphisms - cofibrations and fibrations - that enables one to understand the morphisms that result from formally inverting the weak equivalences, in effect allowing one to "do homotopy theory."

The study of higher dimensional categories, which are a weak notion in their most useful form, can benefit immensely from homotopy theory. Hence, it is worthwhile to first gain a thorough understanding of model categories and their properties, which in turn make use of 2-categorical notions. This is the object of this paper. In Section 2, we begin by introducing a few useful concepts from 2-category theory. Then, in Section 3, we define a model category, which will be one of two central topics in this paper. In Sections 3.1 and 3.2, we develop some of the basic theory of model categories and of chain complexes, which will provide one of the main examples. Sections 3.3 and 3.4 give a thorough discussion of two algebraic examples of model categories: $\mathbf{Ch}_R$ and $\mathbf{Cat}$. Section 3.5 gives Quillen's well-known small object argument, completing the discussion of model categories.

In Section 4, we change perspectives somewhat to discuss weak factorisation systems in general, and in the sections that follow we prove some results connecting factorisations to limits and colimits. Notably, we define a stronger *natural* weak factorisation system in Section 4.5, which applies to our two example model categories, providing additional algebraic structure. We conclude with a few suggestions for further study and our acknowledgments.

This paper endeavors to be as self-contained as possible, and includes the basic exposition, rather than referring the reader to outside sources. The material in Section 2 is based on [8] and [7]. The material in Section 3 draws from [6] and [4], though the proofs in those sources are often omitted or incomplete. Section 4 is based upon [5], [11], and [9].

2 2-Categories

The definition of a 2-category is originally due to C. Ehresmann.

Definition 2.1. A 2-*category* $\mathcal{K}$ consists of *objects* or 0-*cells* A, B etc., *arrows* or *morphisms* or 1-*cells*, $f, g : A \to B$ etc., and 2-*cells* $\alpha : f \Rightarrow g$:

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

such that identities exist and subject to axioms that say that the various composites one might form are unambiguous.

2-cells can be composed in two distinct ways. When two objects A and B are fixed, $\mathcal{K}(A, B)$ and the 2-cells between these arrows form a category under the operation of *vertical composition*:

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "g"'{name=D}] & B \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\beta"] \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta\cdot\alpha"] \end{tikzcd}$$

The operation of *horizontal composition* of 2-cells is given by

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "g"'{name=D1}] & B \arrow[r, bend left=50, "h"{name=U2}] \arrow[r, bend right=50, "k"'{name=D2}] & C \arrow[from=U1, to=D1, Rightarrow, "\alpha"] \arrow[from=U2, to=D2, Rightarrow, "\beta"] \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left=50, "hf"{name=U}] \arrow[r, bend right=50, "kg"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta*\alpha"] \end{tikzcd} \tag{1}$$

Finally, a distributive law requires that when

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=U1}] \arrow[r, ""{name=M1}] \arrow[r, bend right=60, "g"'{name=D1}] & B \arrow[r, bend left=60, "h"{name=U2}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, "k"'{name=D2}] & C \arrow[from=U1, to=M1, Rightarrow, "\alpha"] \arrow[from=M1, to=D1, Rightarrow, "\beta"] \arrow[from=U2, to=M2, Rightarrow, "\gamma"] \arrow[from=M2, to=D2, Rightarrow, "\delta"] \end{tikzcd}$$

then the composites

$$(\delta * \beta) \cdot (\gamma * \alpha) = (\delta \cdot \gamma) * (\beta \cdot \alpha) : hf \Rightarrow kg.$$

Notation 2.2. In describing composites, we adopt the usual convention of writing A for 1_A and f for 1_f when the context is clear. For example, if $h = k$ and $\beta = 1_h$ in (1), we write the composite $\beta * \alpha$ as $h * \alpha$.

The objects and arrows of $\mathcal{K}$ form a category called the *underlying category of* $\mathcal{K}$, which may also be denoted $\mathcal{K}$, when the meaning is clear from the context.

The prototypical example of a 2-category is **Cat**, whose objects are small categories, morphisms are functors, and 2-cells are natural transformations. With this example in mind, 2-dimensional category theory can be thought of as "abstract category theory," as its results will apply to the 2-category **Cat** in particular. The study of natural weak factorisation systems in Section 4.5 will require a particular 2-category $\mathbf{Cat}/\mathcal{K}$.

Definition 2.3. Given a category $\mathcal{K}$, the *slice 2-category over* $\mathcal{K}$, $\mathbf{Cat}/\mathcal{K}$, has objects $(\mathcal{A}, V_\mathcal{A})$ consisting of a category $\mathcal{A}$ together with a functor $V_\mathcal{A} : \mathcal{A} \to \mathcal{K}$; arrows $F : (\mathcal{A}, V_\mathcal{A}) \to (\mathcal{B}, V_\mathcal{B})$, which are functors $F : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "V_\mathcal{A}"'] & & \mathcal{B} \arrow[dl, "V_\mathcal{B}"] \\ & \mathcal{K} & \end{tikzcd}$$

commutes; and 2-cells $\alpha : F \Rightarrow G$, which are natural transformations $\alpha : F \Rightarrow G$ such that $V_\mathcal{B} * \alpha = 1_{V_\mathcal{A}}$, i.e.,

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, "V_\mathcal{B}"] & \mathcal{K} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "V_\mathcal{A}"] \arrow[r, bend right=50, "V_\mathcal{A}"'] \arrow[r, phantom, "\Downarrow 1_{V_\mathcal{A}}"] & \mathcal{K} \end{tikzcd}$$

Definition 2.4. A 2-*functor* $F : \mathcal{K} \to \mathcal{P}$ sends objects of $\mathcal{K}$ to objects of $\mathcal{P}$, morphisms of $\mathcal{K}$ to morphisms of $\mathcal{P}$, and 2-cells of $\mathcal{K}$ to 2-cells of $\mathcal{P}$ in such a way that preserves domains, codomains, identity, and composition. A 2-*natural transformation* $\eta : F \Rightarrow G : \mathcal{K} \to \mathcal{P}$ is a map that assigns each object A of $\mathcal{K}$ a map $\eta_A : FA \to GA$ in $\mathcal{P}$ in such a way that η is a natural transformation on the underlying categories of $\mathcal{K}$ and $\mathcal{P}$ but is also 2-natural in the sense that, for each 2-cell $\alpha : f \Rightarrow g : A \to B$ in $\mathcal{K}$, we have:

$$\begin{tikzcd} FA \arrow[r, bend left=50, "Ff"] \arrow[r, bend right=50, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \arrow[r, "\eta_B"] & GB \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, "\eta_A"] & GA \arrow[r, bend left=50, "Gf"] \arrow[r, bend right=50, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GB \end{tikzcd}$$

Many of the familiar concepts in category theory are definable purely in terms of objects, arrows, and 2-cells, allowing one to "do category theory" in a 2-category. For example:

Definition 2.5. A *monad* in a 2-category $\mathcal{K}$ on an object B of $\mathcal{K}$ is an endomorphism $t : B \to B$ together with 2-cells $\eta : 1 \Rightarrow t$ and $\mu : t^2 \Rightarrow t$ called the *unit* and *multiplication* respectively such that the following diagrams commute:

$$
\begin{tikzcd}
t \arrow[r, Rightarrow, "\eta t"] \arrow[dr, Rightarrow, "1_t"'] & t^2 \arrow[d, Rightarrow, "\mu"] & t, \arrow[l, Rightarrow, "t\eta"'] \arrow[dl, Rightarrow, "1_t"] \\
& t &
\end{tikzcd}
\qquad
\begin{tikzcd}
t^3 \arrow[r, Rightarrow, "t\mu"] \arrow[d, Rightarrow, "\mu t"'] & t^2 \arrow[d, Rightarrow, "\mu"] \\
t^2 \arrow[r, Rightarrow, "\mu"'] & t.
\end{tikzcd}
$$

A *comonad* in a 2-category is defined dually.

Example 2.6. A monad on **Cat** is just a usual monad. For any category $\mathcal{K}$, a monad on the 2-category $\mathbf{Cat}/\mathcal{K}$ consists of an object $V_{\mathcal{A}} : \mathcal{A} \to \mathcal{K}$, a functor $T : \mathcal{A} \to \mathcal{A}$ satisfying $V_{\mathcal{A}}T = V_{\mathcal{A}}$, and 2-cells $\eta : 1 \Rightarrow t$, $\mu : t^2 \to t$ such that $V_{\mathcal{A}} * \eta = 1_{V_{\mathcal{A}}} = V_{\mathcal{A}} * \mu$ and such that (T, η, μ) is a monad on $\mathcal{A}$ in **Cat**.

Remark 2.7. The notion of a *comonad* on $\mathbf{Cat}/\mathcal{K}$ can be defined dually: a comonad on an object $V_{\mathcal{A}} : \mathcal{A} \to \mathcal{K}$ of $\mathbf{Cat}/\mathcal{K}$ is simply a comonad (G, ϵ, δ) on $\mathcal{A}$ such that G is a morphism of $\mathbf{Cat}/\mathcal{K}$ and ϵ and δ are 2-cells.

Remark 2.8. We say (G, ϵ, δ) is a *comonad on dom* when (G, ϵ, δ) is a comonad on the object $\mathrm{dom} : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ in the 2-category $\mathbf{Cat}/\mathcal{K}$. Dually, (T, η, μ) is a *monad on cod* when (T, η, μ) is a monad on the object cod :$\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ in $\mathbf{Cat}/\mathcal{K}$. These objects will appear again in Section 4.5.

There are many notions of 2-categorical limits. One general type are *indexed limits*, defined with respect to a 2-functor $F : \mathcal{P} \to \mathbf{Cat}$, called the *indexing type*. A general definition is given in [7]; however, we present only the special case that we will need in Section 3.4. Let $\mathbf{1}$ be the category with a single object and identity morphism, let

$$\mathbf{2} = \bullet \longrightarrow \bullet$$

be the category consisting exactly two objects with one morphism between them, and let

$$\mathbf{I} = \bullet \rightleftarrows \bullet$$

be the category with two objects and one isomorphism between them.

Definition 2.9. Let $F, G : \mathbf{2} \to \mathbf{Cat}$ be 2-functors such that $F\mathbf{2} = \mathbf{1} \to \mathbf{2}$ and $G\mathbf{2} = A \xrightarrow{f} B$. Then the *pseudo-limit* of f is the limit F, G indexed by F. Specifically, the limit is the universal diagram of the form

$$
\begin{tikzcd}
& \{F, G\} \arrow[dl, "u"'] \arrow[dr, "v"] & \\
A \arrow[rr, "f"'] \arrow[rr, phantom, bend left=20, "\Nearrow \eta"] & & B
\end{tikzcd}
$$

where η is invertible.

3 Model Categories

Definition 3.1. A *model structure* on a category $\mathcal{K}$ consists of three distinguished classes of morphisms - *fibrations*, *cofibrations*, and *weak equivalences* - subject to the following axioms:

1. Each distinguished class of morphisms is closed under retracts, where a morphism $f : A \to B$ is a *retract* of $g : X \to Y$ if there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "="] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "="'] & Y \arrow[r] & B \end{tikzcd}$$

2. (2 of 3) If $f, g \in \operatorname{mor} \mathcal{K}$ are such that two out of three of f, g, or fg is a weak equivalence, so is the third.

3. (Lifting) Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

   where j is a cofibration, q is a fibration, and one of j or q is also a weak equivalence has a solution so that both diagrams commute.

4. (Factorisation) Any $f : A \to B$ can be factored in two ways:
   (i) $A \xrightarrow{i} X \xrightarrow{q} B$, where i is a cofibration and q is a fibration and a weak equivalence.
   (ii) $A \xrightarrow{j} Z \xrightarrow{p} B$, where j is a cofibration and a weak equivalence and p is a fibration.

A *model category* is a complete and cocomplete category (i.e., a category $\mathcal{K}$ with all small limits and colimits) with a model structure.

A cofibration that is also a weak equivalence is called *trivial* and similarly for fibrations. Maps j and q that satisfy Axiom 3 are said to have the *left lifting property* (respectively the *right lifting property*) with respect to each other. Hence, Axiom 3 asserts that the trivial cofibrations have the left lifting property with respect to the fibrations and the cofibrations have the left lifting property with respect to the trivial fibrations.

Following Hovey [6], we amend the usual definition of a model structure on a category slightly to require that the factorisations in Axiom 4 are *functorial.* Given a category $\mathcal{K}$, we can form a category $\mathcal{K}^{\mathbf{2}}$ whose objects are morphisms in $\mathcal{K}$ and whose arrows are commutative squares. This category is also called $\mathbf{Arr}(\mathcal{K})$ or $\mathbf{Map}(\mathcal{K})$ and is isomorphic to the category $\mathcal{K}^{\mathbf{2}}$ of functors $\mathbf{2} \to \mathcal{K}$, hence the notation used here. Let dom: $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ and cod: $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ denote the functors that send an arrow to its domain and codomain, respectively. A *functorial factorisation* a pair of functors $L, R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ so that

$$\mathrm{dom}L = \mathrm{dom}, \qquad \mathrm{cod}R = \mathrm{cod}, \qquad \mathrm{cod}L = \mathrm{dom}R,$$

and $f = R(f)L(f)$ for every $f \in \operatorname{mor}\mathcal{K}$.

We modify the definition of a model category to stipulate that both factorisations given by Axiom 4 are functorial. Let α, β be the pair of functors that sends a morphism f to an acyclic cofibration followed by a fibration, and let γ, δ be the functors that send f to a cofibration followed by an trivial fibration. The functorial hypothesis has the following nice consequence: a commutative square

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \text{yields} \quad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\alpha(f)"'] & \cdot \arrow[d, "\alpha(g)"] \\ \cdot \arrow[r] \arrow[d, "\beta(f)"'] & \cdot \arrow[d, "\beta(g)"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

where the unlabled central arrows and objects are determined by the functors that specify this factorisation.

Hovey asserts that in practise the factorisations in all model categories can be made functorial, though it is still not common to define model categories with this requirement. The advantage to doing so, besides for convenience, is that it will allow us to show that various constructions are natural with respect to maps of model categories. For example, in the presence of a functorial model category structure, the definition of a derived functor can be made to depend only on the model category $\mathcal{K}$ and not on a particular choice of cofibrant replacement (see [6, pp 16] and the definition below).

3.1 Basic Properties

Remark 3.2. The axioms for a model category are self-dual. Specifically, if $\mathcal{K}$ is a model category then $\mathcal{K}^{\mathrm{op}}$ is as well, where the cofibrations of $\mathcal{K}^{\mathrm{op}}$ are the fibrations of $\mathcal{K}$, the fibrations of $\mathcal{K}^{\mathrm{op}}$ are the cofibrations of $\mathcal{K}$, and the weak equivalences are the same. The functor α of $\mathcal{K}^{\mathrm{op}}$ is the opposite of the functor δ of $\mathcal{K}$ and so on.

Remark 3.3. Axiom 4 implies that for every object in $\mathcal{K}$ there exists a weak equivalence with that object as its domain. Hence, it follows from Axiom 2 that every identity morphism is a weak equivalence. Using closure under retracts, any isomorphism f with $gf = 1_A$ and $fg = 1_B$ is a weak equivalence as well, as shown by the diagram:

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "f"'] & A \arrow[r, "1"] \arrow[d, "1"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

We will show furthermore that isomorphisms are also fibrations and cofibrations. Let $1_A = kh$ be a factorisation into a cofibration followed by a fibration (the previous argument implying that both are trivial). The diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "g"'] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "h"'] & C \arrow[r, "k"'] & A \end{tikzcd}$$

shows that g is a cofibration as well, while

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[r, "k"] \arrow[d, "k"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & A \arrow[r, "f"'] & B \end{tikzcd}$$

shows that f is a fibration. Repeating this argument with a factorisation of 1_B gives that every isomorphism in a model category is both an trivial fibration and an trivial cofibration.

A model category has all small limits, so in particular it contains both an initial and a terminal object (as a colimit and limit, respectively, for an empty diagram). Hence, the following definition makes sense:

Definition 3.4. An object X in a model category $\mathcal{K}$ is *cofibrant* if the unique morphism from the initial object to X is a cofibration. More generally, a *cofibrant replacement* for an object Z is a weak equivalence $X \to Z$ with X cofibrant. A cofibrant replacement exists for any object Z by the factorisation axiom. The terms *fibrant* and *fibrant replacement* are defined dually.

By applying the functors γ and δ to the map from the initial object to Z, we get a functor $Z \mapsto QZ$ such that QZ is cofibrant called the *cofibrant replacement functor* of $\mathcal{K}$. This functor comes equipped with a natural transformation $q : Q \to 1_{\mathcal{K}}$ such that the maps $q_Z : QZ \to Z$ are trivial fibrations. Dually, there is a *fibrant replacement functor* $RZ \to Z$ equipped with a natural transformation $r : 1_{\mathcal{K}} \to R$, with each $r_Z : Z \to RZ$ an trivial cofibration.

Remark 3.5. The axioms for a model structure on a category are overdetermined, as is shown by the following lemma.

Lemma 3.6 (The Retract Argument)**.** *Let $f = qj$ in a category $\mathcal{K}$ and suppose that f has the left lifting property with respect to q. Then f is a retract of j. Dually, if f has the right lifting property with respect to j, then f is a retract of q.*

Proof. Suppose $f : A \to B$ has the left lifting property with respect to $q : X \to B$. Then we have a lift $h : B \to X$ in the following diagram.

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & X \arrow[d, "q"] \\ B \arrow[r, equal] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

Hence, the diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "j"'] & A \arrow[d, "f"] \\
B \arrow[r, "h"'] & X \arrow[r, "q"'] & B
\end{tikzcd}$$

commutes, and the identity $qh = 1_B$ shows that f is a retract of j. The dual statement follows similarly. $\square$

Lemma 3.7. *A morphism in a model category $\mathcal{K}$ is a cofibration (trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a morphism is a fibration (trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations).*

Proof. We prove that a morphism is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations; the other proofs are similar. If f is a cofibration, then it has the requisite lifting property by Axiom 3, so there is nothing to prove. Conversely, assume f has the left lifting property with respect to all trivial fibrations. Factorise $f = qj$ into a product of a cofibration followed by an trivial fibration. By assumption f has the left lifting property with respect to q and so Lemma 3.6 implies that f is a retract of j, and hence a cofibration. $\square$

Remark 3.8. This gives another proof that every isomorphism in a model category is both an trivial cofibration and an trivial fibration.

Corollary 3.9. *The classes $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ are closed under* cobase change*, meaning that if $i \in \mathcal{C}$ and j is the morphism given by a pushout diagram*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "i"'] & \cdot \arrow[d, dashed, "j"] \\
\cdot \arrow[r, dashed] & \cdot
\end{tikzcd}$$

then $j \in \mathcal{C}$ and likewise for $\mathcal{C} \cap \mathcal{W}$. Dually, the classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under base change*.*

Proof. We show that $\mathcal{C}$ is closed under cobase change using Lemma 3.7; the other proofs are similar. Let $i \in \mathcal{C}$ and let j be a cobase change of i. For any $f \in \mathcal{F} \cap \mathcal{W}$, a lifting problem $(u, v) : j \to f$ gives a lifting problem

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "i"'] & \cdot \arrow[r, "u"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[urr, dashed, "w"] & \cdot \arrow[r, "v"'] & \cdot
\end{tikzcd} \tag{2}$$

by composing with the pushout diagram, which has a solution w by Axiom 3. Because j is part of a pushout diagram, the map w and the top row of (2) induce a lift h from the codomain of j to the domain of f. By definition, $h \cdot j = u$. Similarly, the universal mapping property of a pushout square implies that $f \cdot h = v$, as the map from the codomain of j to the codomain of f such that (2) commutes must be unique. $\square$

Remark 3.10. In addition to being closed under retracts, the class $\mathcal{C}$ of cofibrations is closed under various colimits. Specifically $\mathcal{C}$ is closed under coproducts (which will be proven in Section 4.3), cobase change (by Corollary 3.9), and sequential colimits (which will be discussed in Section 3.5). The trivial cofibrations $\mathcal{C} \cap \mathcal{W}$ are closed under these colimits as well, and fibrations and trivial fibrations are closed under products and sequential limits, dually. However, in general the classes $\mathcal{C}$ and $\mathcal{F}$ fail to be closed under all colimits and limits. This will be discussed further in Section 4.3.

Before proceeding further, we introduce two algebraic examples of model categories: $\mathbf{Ch}_R$, the category of non-negatively indexed chain complexes of modules over a commutative ring with identity, and **Cat**, the category of small categories. **Cat** has two well-known model structures, and we choose to describe the "categorical" structure. Readers interested in the "topological" model structure on **Cat** are referred to [13]. Before describing the model structure on $\mathbf{Ch}_R$, we given some algebraic preliminaries.

3.2 $\mathbf{Ch}_R$, homology, and projective resolutions

The object of this and the following section will be to show first that chain complexes of R-modules form a model category. This result is well-known, though because our study of weak factorisation systems makes repeated use of their functoriality, we prove explicitly that the two factorisations are functorial, something that is not often done. But first, we provide some preliminaries, in consultation with [14]:

Let R be a commutative ring with identity; the category of R-modules will be denoted $\mathbf{Mod}_R$. This is a complete and cocomplete abelian category. An R-module P is *projective* if and only if for every surjection $f : M \twoheadrightarrow N$ and every map $g : P \to N$ there exists an $h : P \to M$ such that the following commutes:

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & N \end{tikzcd}$$

Clearly any free module is projective, for the map h can be constructed by lifting the image of the basis vectors. Similarly, direct summands of free modules are projective, and these encompass all examples of projective modules.

Proposition 3.11. *An R-module is projective if and only if it is a direct summand of a free module.*

Proof. Let P be a projective module, and let $F(P)$ denote the free module on the set underlying P. There is a canonical surjection $\pi : F(P) \twoheadrightarrow P$; hence, there must exist a lift i such that the diagram

$$\begin{tikzcd} & F(P) \arrow[d, two heads, "\pi"] \\ P \arrow[ur, dashed, "i"] \arrow[r, "1"'] & P \end{tikzcd}$$

commutes. As $\pi i = 1$, i must be an inclusion $P \hookrightarrow F(P)$. Hence, P is a direct summand of the free module $F(P)$. □

Over fields or division rings, the only projectives are free modules, but this is not always the case. For example, if $R = R_1 \times R_2$ then $P_1 = R_1 \times 0$ and $P_2 = 0 \times R_2$ are projective but not free, since $(0, 1)P_1 = (1, 0)P_2 = 0$.

We say that an abelian category $\mathcal{A}$ *has enough projectives* if for every $A \in \operatorname{ob} \mathcal{A}$ there exists a projective P and surjection $P \twoheadrightarrow A$. We show that $\mathbf{Mod}_R$ has enough projectives by using the fact that the module R is projective and the isomorphism of sets $\mathbf{Mod}_R(R, M) = M$ given by evaluation at $1 \in R$. For $M \in \operatorname{ob} \mathbf{Mod}_R$ define

$$P(M) = \oplus_{f:R\to M} R,$$

where the sum runs over all morphisms $f : R \to M$ in $\mathbf{Mod}_R$. Then define $e : P(M) \to M$ to be the unique morphism given by the following diagram:

$$\begin{tikzcd}
R \arrow[dr, "i_R"] \arrow[ddr, "f"'] & \cdots & R \arrow[dl, "i_R"'] \arrow[ddl, "f'"] \\
& P(M) \arrow[d, dashed, "e"] & \\
& M &
\end{tikzcd}$$

The set isomorphism $\mathbf{Mod}_R(R, M) = M$ implies that this map is surjective, and $P(M)$ is a projective module because it is free. Note that $P : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is a functor. Given $h : M \to N$, the map $P(h) : P(M) \to P(N)$ is induced by the maps $R \to P(N)$ that include the component of $P(M)$ corresponding to $f : R \to M$ into the component of $P(N)$ corresponding to $hf : R \to N$. This fact will be useful later.

Definition 3.12. A *chain complex* of R-modules

$$C_\bullet = \cdots \to C_3 \to C_2 \to C_1 \to C_0$$

is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules together with R-module homomorphisms $\partial = \partial_n : C_n \to C_{n-1}$ such that $\partial^2 : C_n \to C_{n-2}$ is zero.

Definition 3.13. Let $\mathbf{Ch}_R$ denote the category of non-negatively indexed chain complexes of R-modules. A morphism $f : M_\bullet \to N_\bullet$ of chain complexes is a family of module homomorphisms $f_n : M_n \to N_n$ such that

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"'] & M_n \arrow[r, "\partial"] \arrow[d, "f_n"'] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r] & N_{n+1} \arrow[r, "\partial"'] & N_n \arrow[r, "\partial"'] & N_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes for all n.

$\mathbf{Ch}_R$ is an abelian category with addition of chain maps defined in each degree by the modules $\mathrm{Hom}_R(M_n, N_n)$. It has all small limits and colimits, taken degreewise, as these exist in $\mathbf{Mod}_R$. In this paper, any chain complex is assumed to be an object in $\mathbf{Ch}_R$; in particular, we adopt the convention that $M_n = 0$ for all $n < 0$ for any chain complex $M_\bullet$.

Definition 3.14. Let $C_\bullet$ be a chain complex. The kernel

$$Z_n(C_\bullet) := \ker(C_n \xrightarrow{\partial_n} C_{n-1})$$

is a R-module called the *n-cycles* of $C_\bullet$, while the image

$$B_n(C_\bullet) := \mathrm{im}\,(C_{n+1} \xrightarrow{\partial_{n+1}} C_n)$$

is an R-module called the *n-boundaries*. Because $\partial^2 = 0$, $B_n \subset Z_n \subset C_n$ for all n, so we may form the quotient

$$H_n(C_\bullet) := Z_n(C_\bullet)/B_n(C_\bullet),$$

called the n-th *homology module* of $C_\bullet$.

It is easy to check that a morphism $f : M_\bullet \to N_\bullet$ of chain complexes maps cycles to cycles and boundaries to boundaries, hence inducing morphisms $f_n : H_n(M_\bullet) \to H_n(N_\bullet)$ of homology modules. Indeed, homology is a covariant functor $\mathbf{Ch}_R \to \mathbf{Mod}_R$ for each $n \in \mathbb{Z}$.

Definition 3.15. A *projective resolution* of an R-module M is a chain complex $P_\bullet$ together with a map $e : P_0 \to M$ such that each P_i is projective and the sequence

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{e} M \to 0$$

is exact. Equivalently, if we regard M as a chain complex concentrated in degree zero, a projective resolution is a chain map $e : P_\bullet \to M$ which induces an isomorphism on homology.

Lemma 3.16. *Every R-module M has a projective resolution.*

Proof. Choose a projective P_0 and surjection $e_0 : P_0 \twoheadrightarrow M$, and let M_0 be the kernel of this map. Similarly, choose a projective P_1 and surjection $e_1 : P_1 \twoheadrightarrow M_0$. Define $\partial : P_1 \to P_0$ to be the composite of e_1 with the inclusion $M_0 \hookrightarrow P_0$. Proceed inductively. Let M_1 be the kernel of e_1 and choose a projective P_2 and surjection $e_2 : P_2 \twoheadrightarrow M_1$, and so forth. This construction produces the following diagram, with diagonals and horizontal sequence exact.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots P_3 \arrow[rr, "\partial"] \arrow[dr, "e_3"] & & P_2 \arrow[rr, "\partial"] \arrow[ur, "e_2"] & & P_1 \arrow[rr, "\partial"] \arrow[dr, "e_1"] & & P_0 \arrow[r, "e_0"] & M \arrow[r] & 0 \\
 & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The map $e : P_\bullet \to M$ is the desired projective resolution. $\square$

Note Lemma 3.16 is also true for any abelian category $\mathcal{A}$ with enough projectives.

Proposition 3.17. *Let $e : P_\bullet \to M$ be a projective resolution and $g : M \to N$ a module homomorphism. Then for any resolution $Q_\bullet \to N$ (not necessarily with the Q_n projective), there exists a chain map $f : P_\bullet \to Q_\bullet$.*

Proof. Because $Q_\bullet \to N$ is a resolution, the map $h : Q_0 \to N$ is a surjection, so the fact that P_0 is projective gives a map $f_0 : P_0 \to Q_0$ such that

$$\begin{tikzcd} P_0 \arrow[r, two heads, "e"] \arrow[d, "f_0"'] & M \arrow[d, "g"] \\ Q_0 \arrow[r, two heads, "h"'] & N \end{tikzcd}$$

commutes. We construct the remainder of the chain map $f : P_\bullet \to Q_\bullet$ by induction. Assume that $f_k : P_k \to Q_k$ exists for $k = 0, \ldots, n$ and is a chain map so far as it is defined. By definition $Q_{n+1} \twoheadrightarrow B_n(Q_\bullet)$; similarly the image of $\partial : P_{n+1} \to P_n$ is $B_n(P_\bullet)$, so $f_n \mid_{B_n} \circ \partial$ gives a map $Q_{n+1} \to B_n(Q_\bullet)$. Because P_{n+1} is projective, this induces a map $f_{n+1} : P_{n+1} \to Q_{n+1}$ such that $\partial \circ f_{n+1} = f_n \circ \partial$; hence, this construction gives a chain map $f : P_\bullet \to Q_\bullet$. $\square$

3.3 Model structure on $\mathbf{Ch}_R$

The object of this section is to prove the following theorem.

Theorem 3.18. *The category $\mathbf{Ch}_R$ has a model structure with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$, and fibrations $\mathcal{F}$ given by:*
$\mathcal{W} = \{f : M \to N \mid H_n f \text{ is an isomorphism for all } n \geq 0\}$
$\mathcal{C} = \{f \mid M_n \to N_n \text{ is an injection with projective cokernel for } n \geq 0\}$
$\mathcal{F} = \{f \mid M_n \to N_n \text{ is surjective for } n \geq 1\}$.

The substance of this theorem will be in proving that these classes satisfy Axioms 3 and 4. Indeed, we prove the other two axioms now. A morphism f of chain complexes is a weak equivalence if and only if its image under each of the homology maps $H_n : \mathbf{Ch}_R \to \mathbf{Mod}_R$ is an isomorphism. The class $\mathcal{W}$ satisfies the 2 of 3 Axiom precisely because homology is *functorial*. Similarly, the functoriality of homology shows that $\mathcal{W}$ is closed under retracts. Apply the homology functor H_n to a retract diagram

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, "g"'] & M_\bullet \arrow[r] \arrow[d, "f"'] & A_\bullet \arrow[d, "g"] \\ B_\bullet \arrow[r] & N_\bullet \arrow[r] & B_\bullet \end{tikzcd}$$

with $f \in \mathcal{W}$. Because $H_n f$ is an iso, there exists a lift $h : H_n(B_\bullet) \to H_n(A_\bullet)$ such that the upper-left and lower-right triangles commute. This map h is an inverse for $H_n g$, so $g \in \mathcal{W}$ as desired.

We will show in Section 4.1 that any class of morphisms, which is defined to be precisely those maps that have the left lifting property (or right lifting property) with respect to another class of morphisms, is closed under retracts, completing the proof of Axiom 1.

Note that $\mathbf{Ch}_R$ has all small limits and colimits, which are taken degreewise, so Theorem 3.18 implies that $\mathbf{Ch}_R$ is a model category. The proofs of Axioms 3 and 4 will require some preliminary work.

Recall that a *pullback* of a diagram

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} A \times_C B \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

such that this diagram commutes and $A \times_C B$ is universal in the sense that for any object Z and maps x, y such that

$$\begin{tikzcd} Z \arrow[dr, dashed, "(x,y)"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\ & A \times_C B \arrow[r] \arrow[d] & B \arrow[d] \\ & A \arrow[r] & C \end{tikzcd}$$

commutes, there exists a unique map $Z \to A \times_C B$ that gives a factorisation of x and y. The category $\mathbf{Ch}_R$ has pullbacks as it clearly has products and equalisers.

The proof of this theorem relies in several instances on the following lemma, which is stated but not proven in [4].

Lemma 3.19. *Let $f : M_\bullet \to N_\bullet$ be a morphism of chain complexes. Then the following statements are equivalent:*

(1) f is a trivial fibration

(2) The induced map

$$M_n \to Z_{n-1}M \times_{Z_{n-1}N} N_n$$

is a surjection for $n \geq 0$.

Proof. For simplicity of notation, let $X = Z_{n-1}M \times_{Z_{n-1}N} N_n$. We construct X as the equaliser of the maps $\partial\pi_N, f_{n-1}\pi_M : Z_{n-1}M \times N_n \to Z_{n-1}N$. We treat X as a subobject of $Z_{n-1}M \times N_n$ whose elements are precisely those ordered pairs $(a, b) \in Z_{n-1}M \times N_n$ such that $f_{n-1}(a) = \partial(b)$.

((1) $\to$ (2)). For $n = 0$ the induced map is simply $M_0 \to N_0$, and the fact that $H_0 f$ is an isomorphism implies that this is surjective. For $n > 0$, the induced map $M_n \to X$ is surjective if and only if for every ordered pair $(a, b) \in Z_{n-1}M \times N_n$ such that $f_{n-1}(a) = \partial(b)$ there exists some $m \in M_n$ such that $a = \partial(m)$ and $b = f_n(m)$, as is clear from the following diagram.

$$\begin{tikzcd}
M_n \arrow[dr] \arrow[rrrdd, dashed, "f_n"] \arrow[rrddd, bend right, "\partial"'] & & & \\
& X \arrow[dr, hook] & & \\
& & Z_{n-1}M \times N_n \arrow[r, "\pi_N"] \arrow[d, "\pi_M"'] & N_n \arrow[d, "\partial"] \\
& & Z_{n-1}M \arrow[r, two heads, "f_{n-1}"] & Z_{n-1}N
\end{tikzcd}$$

Let $(a,b) \in X$. By definition, $f_{n-1}(a) = \partial(b) \in B_{n-1}N$, and because $H_{n-1}f$ is an isomorphism, this implies that $a \in B_{n-1}M$. The coset $\partial^{-1}(a) = z + Z_nM$ for some $z \in M_n$, and its image under f_n is $f_n(\partial^{-1}(a)) = f_n(z) + Z_nN$ as f_n induces a surjection on the cycles. But $\partial(f_n(z)) = \partial(b)$, so this coset is the fibre of ∂ that contains b, which implies that there is some $m \in z + Z_nM$ such that $\partial(m) = a$ and $f_n(m) = b$, and the induced map $M_n \to X$ is surjective.

((2) $\to$ (1)). We now assume that for every $(a,b) \in Z_{n-1}M \times N_n$ such that $f_{n-1}(a) = \partial(b)$ there exists some $m \in M_n$ such that $\partial(m) = a$ and $f_n(m) = b$, which for the case $n = 0$ tells us that f_0 is surjective. Let $n > 0$. Given $a \in Z_{n-1}M$ such that $f_{n-1}(a) \in B_{n-1}N$, then by definition $f_{n-1}(a)$ has a lift to N_n under ∂. Thus, the surjection $M_n \twoheadrightarrow X$ gives us $m \in M_n$ such that $\partial(m) = a$, which implies that $a \in B_{n-1}M$. So H_nf is injective for all $n \geq 0$. We observe that the ordered pair $(0,b)$ satisfies $f_{n-1}(0) = \partial b$ for any $b \in Z_nN$, so the surjection $M_n \twoheadrightarrow X$ gives us $m \in Z_nM$ such that $f_n(m) = b$. Hence, f_n restricts to a surjection $Z_nM \twoheadrightarrow Z_nN$ for $n > 0$, which tells us that H_nf is an isomorphism for all $n \geq 0$. Finally, any $b \in N_n$ gives $\partial(b) \in Z_{n-1}N$, which we now know can be lifted to some $a \in Z_{n-1}M$ under f_{n-1}. Once again, surjectivity of the induced map $M_n \twoheadrightarrow X$ supplies an $m \in M_n$ such that $f_n(m) = b$, so $f_n : M_n \twoheadrightarrow N_n$ is surjective as well. $\square$

Proof of Theorem 3.18. It remains to show that the three classes of morphisms satisfy the lifting and factorisation axioms. We begin with the trivial cofibration - fibration half of the factorisation axiom.

For $k > 0$, let D^k denote the chain complex where $D^k_k = D^k_{k-1} = R$ and $D^k_n = 0$ for $n \neq k, k-1$, with $\partial : D^k_k \to D^k_{k-1}$ the identity. For any chain complex $N_\bullet$, the set $\mathbf{Ch}_R(D^k, N_\bullet) = N_k$ as each morphism picks out an element of N_k as the image of 1 in R. Analogously to our construction in $\mathbf{Mod}_R$, define

$$P(N_\bullet) = \oplus_{f:D^k \to N} D^k,$$

which gives a natural map $e : P(N_\bullet) \to N_\bullet$ with the component functions $e_n : P(N_\bullet)_n \to N_n$ surjective for $n \geq 1$. The map $N_\bullet \mapsto P(N_\bullet)$ is functorial and each component of the resulting chain complex is projective. Note also that $P(N_\bullet)$ is a direct sum of exact sequences, so it is also exact.

Now let $f : M_\bullet \to N_\bullet$ be any morphism of chain complexes. Then f map be factorised as

$$\begin{tikzcd} M_\bullet \arrow[rr, "f"] \arrow[dr, "i_{M_\bullet}"'] & & N_\bullet \\ & M_\bullet \oplus P(N_\bullet) \arrow[ur, "{[f,e]}"'] & \end{tikzcd}$$

where the first factor is injective with projective cokernel and the second is surjective in positive degrees. Furthermore, $i_{M_\bullet} : M_\bullet \to M_\bullet \oplus P(N_\bullet)$ induces an isomorphism on homology because $P(N_\bullet)$ is exact. Therefore, this gives a fibration into a trivial cofibration followed by a fibration. This factorisation is clearly functorial, as required.

The cofibration-trivial fibration factorisation is constructed inductively. Let $f : M_\bullet \to N_\bullet$ be a chain map. At $n > 0$, we assume that there exists a factorisation up to that point; namely, assume for $k = 0, \ldots, n-1$ that there exist R-modules Q_k and morphisms $\partial : Q_k \to Q_{k-1}$, $i : M_k \to Q_k$, $q : Q_k \to N_k$ such that $f = qi$, $\partial^2 = 0$, and i and q are chain maps as far as they are defined. So that the resulting factorisation will have the desired form we assume also that i is injective with projective cokernel and that the map $Q_k \to Z_{k-1}Q \times_{Z_{k-1}N} N_k$ induced by ∂ and q is surjective. Lemma 3.19 then implies that $H_k q$ is an isomorphism and q is surjective. Hence, if we can complete the induction step, this construction will give a factorisation into a cofibration followed by a trivial fibration.

For $n = 0$, $Z_{-1}Q \times_{Z_{-1}N} N_0 = 0 \times_0 N_0 \simeq N_0$ and the factorisation $M_0 \xrightarrow{i_{M_0}} M_0 \oplus P(N_0) \xrightarrow{[f_0,e]} N_0$ suffices. For the inductive step, we wish to construct a factorisation to complete the top row of the diagram

$$\begin{tikzcd} M_n \arrow[d, "\partial"'] & & N_n \arrow[d, "\partial"'] \\ M_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\partial"'] & Q_{n-1} \arrow[r, "q_{n-1}"] \arrow[d, "\partial"'] & N_{n-1} \arrow[d, "\partial"'] \\ M_{n-2} \arrow[r, "i_{n-2}"] \arrow[d] & Q_{n-2} \arrow[r, "q_{n-2}"] \arrow[d] & N_{n-2} \arrow[d] \\ \vdots & \vdots & \vdots \end{tikzcd}$$

Note that N_n maps into $Z_{n-1}N$ and the commutativity of the right hand square implies that $Z_{n-1}Q$ does as well. Form the pullback $Z_{n-1}Q \times_{Z_{n-1}N} N_n$. Because $\partial^2 : M_n \to M_{n-2}$ is zero, the composite $i_{n-1}\partial$ defines a map from M_n to $Z_{n-1}Q$. This, together with $f_n : M_n \to N_n$

induces a map $j : M_n \to Z_{n-1}Q \times_{Z_{n-1}N} N_n$ factorising f_n as seen from the diagram:

$$\begin{tikzcd}
M_n \arrow[dr, dashed, "j"] \arrow[drr, bend left, "f_n"] \arrow[ddr, bend right, "i_{n-1}\partial"'] & & \\
& Z_{n-1}Q \times_{Z_{n-1}N} N_n \arrow[r] \arrow[d] & N_n \arrow[d, "\partial"] \\
& Z_{n-1}Q \arrow[r, "q_{n-1}"] & Z_{n-1}N
\end{tikzcd}$$

Because $\mathbf{Mod}_R$ has enough projectives, we may choose a projective P such that there is a factorisation of j as

$$M_n \xrightarrow{i} M_n \oplus P \longrightarrow Z_{n-1}Q \times_{Z_{n-1}N} N_n$$

with the second map a surjection. We set $Q_n = M_n \oplus P$ and call the first map of this factorisation i and the composite of the second with the map to N_n given by the pullback q. The chain maps $i : M_\bullet \to Q_\bullet$ and $q : Q_\bullet \to N_\bullet$ thus constructed are a cofibration and a trivial fibration, respectively, as desired.

It is less obvious that this map is functorial, but this is true. Let $f : M_\bullet \to N_\bullet$ and $g : A_\bullet \to B_\bullet$ be two chain maps with factorisations $M_\bullet \to Q_\bullet \to N_\bullet$ and $A_\bullet \to C_\bullet \to B_\bullet$ respectively, and let $(u, v) : f \to g$ be a morphism in $(\mathbf{Ch}_R)^{\mathbf{2}}$. This factorisation is functorial if there exists a morphism $h : Q_\bullet \to C_\bullet$ such that

$$\begin{tikzcd}
M_\bullet \arrow[rr, "f"] \arrow[dr] \arrow[dd, "u"'] & & N_\bullet \arrow[dd, "v"] \\
& Q_\bullet \arrow[ur] \arrow[dd, dashed, "h"' near end] & \\
A_\bullet \arrow[rr, "g" near end] \arrow[dr] & & B_\bullet \\
& C_\bullet \arrow[ur] &
\end{tikzcd}$$

commutes. This is true in degree zero because P is a functor on $\mathbf{Mod}_R$. Hence, we may assume that such a map exists for $k = 0, \ldots, n-1$ and is a chain map as far as it is defined. For the inductive step, we must construct a morphism $h_n : Q_n \to C_n$ such that

$$\begin{tikzcd}
M_n \arrow[r] \arrow[d] \arrow[ddr] & Q_n \arrow[r] \arrow[d] \arrow[ddr, dashed] & N_n \arrow[d] \arrow[ddr] & \\
M_{n-1} \arrow[r, dashed] \arrow[ddr] & Q_{n-1} \arrow[r, dashed] \arrow[ddr] & N_{n-1} \arrow[ddr] & \\
& A_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & B_n \arrow[d] \\
& A_{n-1} \arrow[r] & C_{n-1} \arrow[r] & B_{n-1}
\end{tikzcd}$$

commutes. By the inductive hypothesis, we are given that the solid-arrow diagram commutes, so there exists a map $k : Q_n \to Z_{n-1}C \times_{Z_{n-1}B} C_n$. By construction, $C_n = A_n \oplus P_C$, where P_C is a projective module mapping surjectively onto this pullback. Because $Q_n = M_n \oplus P_Q$ with P_Q projective, the map k induces a lift $e : P_Q \to P_C$. It follows that

$$Q_n = M_n \oplus P_Q \xrightarrow{[i_{A_n}u, i_{P_C}e]} A_n \oplus P_C = C_n$$

is the desired map h_n, which shows that this factorisation is indeed functorial.

The following two propositions demonstrate that the three classes have the required lifting properties, completing the proof that $\mathbf{Ch}_R$ is a model category. □

Proposition 3.20. *In $\mathbf{Ch}_R$, every cofibration satisfies the left lifting property with respect to every trivial fibration.*

Proof. Consider the following lifting problem

$$\begin{tikzcd} A_\bullet \arrow[r, "u"] \arrow[d, "i"'] & M_\bullet \arrow[d, "q"] \\ B_\bullet \arrow[r, "v"'] & N_\bullet \end{tikzcd}$$

We construct a lift inductively. As we saw in the proof of Lemma 3.19, the fact that a fibration is also a weak equivalence implies that the map $q_0 : M_0 \to N_0$ is a surjection as well. Because $i_0 : A_0 \to B_0$ is injective with projective cokernel, it is isomorphic to a morphism of the form $A_0 \to A_0 \oplus P_0$ where P_0 is projective. Hence, the lifting property of projective modules allows us to construct a lift $k_0 : P_0 \to M_0$ such that

$$\begin{tikzcd} A_0 \arrow[r, equal] \arrow[d, "i_{A_0}"'] & A_0 \arrow[r, "u"] \arrow[d, "i_0"'] & M_0 \arrow[d, "q_0"] \\ A_0 \oplus P_0 \arrow[r, shift left, "\sim"] \arrow[urr, dashed, "{[i_0,k_0]}"] & B_0 \arrow[l, shift left] \arrow[r, "v"'] & N_0 \end{tikzcd}$$

commutes, giving the required lift. For the induction step, we assume that a lift exists for all $k < n$ and that it is a chain map as far as it is defined. By Lemma 3.19, the map $M_n \to Z_{n-1}M \times_{Z_{n-1}N} N_n$ is a surjection, so the fact that $i_n : A_n \to B_n$ is injective with projective cokernel, means that we can solve the lifting problem

$$\begin{tikzcd} A_n \arrow[r] \arrow[d, "i_n"'] & M_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed, "g_n"] & Z_{n-1}M \times_{Z_{n-1}N} N_n \end{tikzcd}$$

as in the base case. The use of the pullback guarantees that the morphisms g_n constructed in this matter form a chain map $g : B_\bullet \to M_\bullet$, which is the desired lift. □

Lemma 3.21. *A chain map $q : Q_\bullet \to N_\bullet$ is a fibration if and only if q has the right lifting property with respect to the maps $0 \to D^n$ for all $n > 0$.*

Proof. For any chain map $f : D^n \to N_\bullet$ with $n > 0$, there exists a lifting problem

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Q_\bullet \arrow[d, "q"] \\ D^n \arrow[r, "f"'] & N_\bullet \end{tikzcd}$$

The morphism $f : D^n \to N_\bullet$ is determined uniquely by the element $f_n(1) \in N_n$, where 1 is the multiplicative unit of $R = D^n_n$. Assume that q is a fibration. Then a lift g_n exists in degree n because $D^n_n = R$ is projective and $q_n : Q_n \to N_n$ is surjective. The map $g_{n-1} : D^n_{n-1} \to Q_{n-1}$ is then determined by the image under $\partial : Q_n \to Q_{n-1}$ of $f_n(1)$, so the resulting $g : D^n \to Q_\bullet$ is a chain map and gives the desired lift.

Conversely, assume that q satisfies the right lifting property with respect to all morphisms $0 \to D^n$ for all $n > 0$. For any $x \in N_n$, there exists some $f' : R \to N_n$ such that $f'(1) = x$ and this can be extended to a chain map $f : D^n \to N_\bullet$. By hypothesis, there exists a lift $g : D^n \to Q_\bullet$ such that $qg = f$; in the n-th degree, this implies that x is in the image of $q_n g_n$ and hence q_n. So q_n is surjective for all $n > 0$. Hence, q is a fibration. □

Proposition 3.22. *In $\mathbf{Ch}_R$, every trivial cofibration satisfies the left lifting property with respect to every fibration.*

Proof. It follows from Lemma 3.21 that any fibration $p : M_\bullet \to N_\bullet$ has the right lifting property with respect to maps $0 \to P(B_\bullet)$ for any chain complex $B_\bullet$, and hence also for maps $A_\bullet \to A_\bullet \oplus P(B_\bullet)$. Let $j : A_\bullet \to B_\bullet$ be any trivial cofibration. The trivial cofibration-fibration factorisation given in Theorem 3.18 gives rise to a lifting problem

$$\begin{tikzcd} A_\bullet \arrow[r, "i_{A_\bullet}"] \arrow[d, "j"'] & A_\bullet \oplus P(B_\bullet) \arrow[d, "{[j,e]}"] \\ B_\bullet \arrow[r, equal] & B_\bullet \end{tikzcd}$$

and the 2-out-of-3 property gives that the fibration is a weak equivalence as well. By Proposition 3.20, j has the left lifting property with respect to $[j, e]$, so Lemma 3.6 implies that j is a retract of $i_{A_\bullet}$. Because $i_{A_\bullet}$ has the left lifting property with respect to any fibration, j must as well, completing the proof. □

3.4 Model structure on Cat

Let **Cat** be the category of all small categories with functors as morphisms.

Definition 3.23. A functor $P : \mathcal{E} \to \mathcal{B}$ in **Cat** is a *categorical fibration* if for every map $f : b' \to b$ in $\mathcal{B}$ and object e lying over b there exists an object e' and a map $g : e' \to e$ in

$\mathcal{E}$ lying over f and universal in the following sense: given $h : e'' \to e$ such that Ph factors through f in $\mathcal{B}$, there exists a unique $k : e'' \to e'$ lying above this factorisation such that $gk = h$. We say P is a *fibration for isos* if this is only required for all isomorphisms $b' \xrightarrow{\sim} b$.

It is easy to see that if P is a fibration for isos, the lift of an isomorphism $f : b' \xrightarrow{\sim} b$ is also an isomorphism in $\mathcal{E}$.

Theorem 3.24. *The category* **Cat** *has a model structure where:*
- *weak equivalences $\mathcal{W}$ are categorical equivalences, that is, functors which are full, faithful, and essentially surjective on objects*
- *cofibrations $\mathcal{C}$ are functors which are injective on objects*
- *fibrations $\mathcal{F}$ are functors which are fibrations for isos.*

Once again, the substance of this theorem is the proof of the lifting and factorisation axioms. Categorical equivalences clearly satisfy the 2 of 3 property. The following Lemma will show that the three classes $\mathcal{C}, \mathcal{F}$, and $\mathcal{W}$ are closed under retracts.

Lemma 3.25. *Let $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{W}$ be classes of morphisms defined as in Theorem 3.24. Then each class is closed under retracts.*

Proof. Consider a retract diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "G"'] \arrow[dr, phantom, "\boxed{1}"] & \mathcal{E} \arrow[r] \arrow[d, "F"] \arrow[dr, phantom, "\boxed{2}"] & \mathcal{A} \arrow[d, "G"] \\ \mathcal{B} \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{B} \end{tikzcd}$$

Because the horizontal composites are the identity, the upper and lower arrows of $\boxed{1}$ and $\boxed{2}$ must be injective and surjective on both objects and arrows, respectively. Hence, if $F \in \mathcal{C}$, then from $\boxed{1}$, $G \in \mathcal{C}$ as well. If $F \in \mathcal{W}$, then $\boxed{1}$ implies that G is faithful, and $\boxed{2}$ implies that G is full and essentially surjective on objects, so $G \in \mathcal{W}$. Finally, if $F \in \mathcal{F}$, we use surjectivity of $\boxed{2}$ to lift an iso $b' \xrightarrow{\sim} b$ in $\mathcal{B}$ to $\mathcal{K}$, then lift it to a morphism in $\mathcal{E}$ satisfying the required universal mapping property, using the fact that F is a fibration for isos. Finally, we map this lift forward to $\mathcal{A}$ using the top edge of $\boxed{2}$, to show that G is a fibration for isos as well. Hence, all three classes are closed under retracts, as required. □

Proof of Theorem 3.24. We begin by showing that these classes satisfy the required lifting properties. Consider a lifting problem

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{C} \arrow[d, "P"] \\ \mathcal{B} \arrow[r, "G"'] \arrow[ur, dashed, "H"] & \mathcal{D} \end{tikzcd}$$

where I is a cofibration and P is a fibration. Objects in the image of I lift uniquely to $\mathcal{A}$ as I is injective on objects, so H can be defined on these objects simply by taking the image under F of their lifts. Indeed, this definition is required to make the upper triangle commute. It remains to define the rest of the map.

Assume that I is also a weak equivalence. Then every $b \in \operatorname{ob}\mathcal{B}$ is isomorphic to some Ia for $a \in A$. The image of the isomorphism $b \xrightarrow{\sim} Ia$ under the functor G is also an isomorphism with codomain $GIa = PFa$. Because P is a fibration, this map lifts to an isomorphism $c \xrightarrow{\sim} Fa$ in $\mathcal{C}$, and we define $Hb = c$. To define H on morphisms of $\mathcal{B}$, consider $h : b' \to b$ with $b \simeq Ia$ and $b' \simeq Ia'$. Composing h with these isomorphisms gives a map $g : Ia' \to Ia$ in $\mathcal{B}$, which lifts to a map $f : a' \to a$ in $\mathcal{A}$ as I is full. Hence, $Ff : Fa' \to Fa$ is a morphism in $\mathcal{C}$ and composition with the isos $Hb' \xrightarrow{\sim} Fa'$ and gives a map $k : Hb' \to Fa$ whose image under P factors through $Gb \xrightarrow{\sim} PFa$.

$$\begin{tikzcd}
Hb' \arrow[r, "\sim"] \arrow[drr, dashed, "k"] & Fa' \arrow[dr, "Ff"] & & \\
& Hb \arrow[r, "\sim"'] & Fa & \mathcal{C} \\
Gb' \arrow[r, "\sim"] \arrow[drr, dashed, "Pk"] \arrow[dr, "Gh"'] & PFa' \arrow[dr, "Gg"] & & \\
& Gb \arrow[r, "\sim"'] & PFa & \mathcal{D}
\end{tikzcd}$$

Hence, it must have a factorisation $Hb' \to Hb$ in $\mathcal{C}$, which is defined to be the image of h under H. This defines the functor H in such a way that both triangles commute, so the lifting property is satisfied in this case.

Now assume that P is a weak equivalence and drop this assumption for I. Again take $b \in \operatorname{ob}\mathcal{B}$, not in the image of I. Because P is essentially surjective, the image $Gb \xrightarrow{\sim} Pc$ for some $c \in \operatorname{ob}\mathcal{C}$. Since P is a fibration for isos, this map lifts to some $\tilde{c} \xrightarrow{\sim} c$ in $\mathcal{C}$. Define $Hb = \tilde{c}$. Given a morphism $b' \to b$, its image under G, composed with the isomorphisms from Gb' and Gb, gives a morphism $Pc' \to Pc$. Because P is full, this lifts to a morphism $c' \to c$, and as before the fact that P is a fibration for isos gives a map $Hb' = \tilde{c}' \to \tilde{c} = Hb$, which we define to be the image of this map under H.

$$\begin{tikzcd}
\tilde{c}' \arrow[r, "\sim"] & c' \arrow[dr] & & \\
& \tilde{c} \arrow[r, "\sim"] & c & \mathcal{C} \\
Gb' \arrow[r, "\sim"] \arrow[dr] & Pc' & & \\
& Gb \arrow[r, "\sim"] & Pc & \mathcal{D}
\end{tikzcd}$$

To define the functorial factorisations of **Cat**, we construct pseudo-limits and colimits in the sense of Section 2. We begin with the trivial cofibration-fibration factorisation. For reasons that will become clear in Section 4, we would like the functor $F : \mathbf{Cat}^{\mathbf{2}} \to \mathbf{Cat}$, which sends a functor $H : \mathcal{A} \to \mathcal{B}$ to the category $\mathcal{K}$ through which it factorises, to absorb some of the "choosing" that took place in constructing lifting maps. For the trivial cofibration-fibration lifting problem, the principal choice was that of an isomorphism $b \xrightarrow{\sim} Ha$. Hence, we let $\mathcal{K}$ be the category whose objects are triples $(a, b, b \xrightarrow{\sim} Ha)$ with $a \in \operatorname{ob}\mathcal{A}$, $b \in \operatorname{ob}\mathcal{B}$, and $b \xrightarrow{\sim} Ha$ an isomorphism in $\mathcal{B}$. Morphisms are pairs $a \to a'$ and $b \to b'$ that commute with the specified isomorphisms. Note that the commuting requirement implies that the pair is completely determined by the morphism $a \to a'$. This category comes equipped with some obvious forgetful functors:

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "U_{\mathcal{A}}"'] \arrow[dr, "U_{\mathcal{B}}"] \arrow[d, phantom, "\nearrow\alpha"] & \\ \mathcal{A} \arrow[rr, "H"'] & & \mathcal{B} \end{tikzcd} \tag{3}$$

and we define the 2-cell α by using the isomorphisms given with each object of $\mathcal{K}$. It is easy to check that (3) is universal, so that $\mathcal{K}$ is the pseudo-limit of H; hence, F is indeed a functor. It remains to show that this construction yields a factorisation.

Define a functor $I : \mathcal{A} \to \mathcal{K}$ by

$$a \mapsto (a, Ha, 1_{Ha} : Ha \xrightarrow{\sim} Ha).$$

Then $H = U_{\mathcal{B}}I$. The functor I is clearly injective on objects, full, and faithful. It is also essentially surjective: given $(a, b, b \xrightarrow{\sim} Ha)$, the pair $1_a : a \to a$ and $b \xrightarrow{\sim} Ha$ gives an isomorphism to $(a, a, 1_{Ha}) : Ha \xrightarrow{\sim} Ha) = Ia$. So I is a trivial cofibration. Furthermore, the forgetful functor $U_{\mathcal{B}}$ is a fibration for isos. Given $b' \xrightarrow{\sim} b$, with $b = U_{\mathcal{B}}(a, b, b \xrightarrow{\sim} Ha)$ then b' lifts to $(a, b', b' \xrightarrow{\sim} b \xrightarrow{\sim} Ha)$. If

$$\begin{tikzcd} (a'', \arrow[d] & b'', \arrow[d] & b'' \arrow[r, "\sim"] \arrow[d] & Ha'') \arrow[d] \\ (a, & b, & b \arrow[r, "\sim"] & Ha) \end{tikzcd}$$

is a map in $\mathcal{K}$ such that $b'' \to b$ factors through $b' \xrightarrow{\sim} b$, then

$$\begin{tikzcd} b'' \arrow[r, "\sim"] \arrow[d] & Ha'' \arrow[d] \\ b' \arrow[r, "\sim"] & Ha \end{tikzcd}$$

commutes, by definition of the bottom map, so the lift factorises as well. Hence, $U_{\mathcal{B}}$ is a fibration, and we have exhibited one half of the factorisation axiom.

For the cofibration-trivial fibration factorisation, we simply dualise this construction. However, the pseudo-colimit $\mathcal{E}$ of $H : \mathcal{A} \to \mathcal{B}$ is less natural to describe. We define a category $\mathcal{E}$ with

$$\operatorname{ob} \mathcal{E} = \{(a, Ha) \mid a \in \operatorname{ob} \mathcal{A}\} \cup \{(*, b) \mid b \in \operatorname{ob} \mathcal{B}\},$$

where the $*$ is just a formal symbol. Morphisms in $\mathcal{E}$ are simply morphisms in $\mathcal{B}$ between the objects in the second coordinate. There exist two obvious functors: $V_{\mathcal{A}} : \mathcal{A} \to \mathcal{K}$ with

$a \mapsto (a, Ha)$ and $V_{\mathcal{B}} : \mathcal{B} \to \mathcal{K}$ with $b \mapsto (*, b)$. These functors define the pseudo-colimit:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "H"] \arrow[dr, "V_{\mathcal{A}}"'] & \arrow[d, phantom, "\swarrow\beta"] & \mathcal{B} \arrow[dl, "V_{\mathcal{B}}"] \\ & \mathcal{E} & \end{tikzcd} \tag{4}$$

where β is an invertible 2-cell whose components are the identity maps $1_{Ha} : (*, Ha) \to (a, Ha)$. The universal mapping property of (4) follows easily; given functors $M : \mathcal{A} \to \mathcal{D}$ and $N : \mathcal{B} \to \mathcal{D}$ to some category $\mathcal{D}$, the map $\mathcal{E} \to \mathcal{D}$ is the obvious one. Furthermore, $V_{\mathcal{A}}$ is clearly injective on objects and hence a cofibration in **Cat**.

Define a functor $P : \mathcal{E} \to \mathcal{B}$ to be the forgetful functor that projects to the second coordinate, and note that $H = PV_{\mathcal{A}}$. By definition, $\mathcal{E}(y, z) = \mathcal{B}(Py, Pz)$ for any $y, z \in \mathrm{ob}\,\mathcal{E}$, so P is automatically full and faithful, and by the construction of $\mathrm{ob}\,\mathcal{E}$, P is surjective. Hence P is a weak equivalence. Furthermore, P is a fibration for isos. Because P is surjective and full, any morphism in $\mathcal{B}$ has a lift. Because B is faithful, once lifts of objects are chosen, the lifts of the morphisms between them must be unique, which gives the required universal property. Hence, P is a trivial fibration, so our factorisation, and hence the proof, is complete. □

3.5 Quillen's Small Object Argument

Recall that an *ordinal* is the well-ordered set of all smaller ordinals. Formally, the ordinals can be constructed using a transfinite recurrence as follows:

- $\emptyset$ is an ordinal denoted by 0.
- If α is an ordinal then $\alpha + 1 = \alpha \cup \{\alpha\}$ is its *successor ordinal*.
- If A is a set of ordinals, then $\cup_{\alpha \in A} \alpha$ is an ordinal.

It is clear from this definition that every ordinal has a successor, but not every ordinal has an immediate predecessor. An ordinal that is neither 0 nor a successor ordinal is called a limit ordinal. The first of these is ω, the limit of the set of ordinals denoted by natural numbers. The collection of all ordinals is denoted ON and is not itself a set, for if this were a case then it would also be an ordinal, by the above definition; in this case, it would be equal to a member of itself, contradicting the axiom of regularity.

An ordinal can be thought of as a category with a unique map $\alpha \to \beta$ if and only if $\alpha \leq \beta$. Given an ordinal γ and a category $\mathcal{K}$ with all small colimits, a *γ-sequence* is a functor $X : \gamma \to \mathcal{K}$ that is colimit preserving. Since X preserves colimits, the induced map

$$\mathrm{colim}_{\alpha<\beta} X\alpha \to X\beta$$

is an isomorphism for all ordinals $\beta < \gamma$. Note that an ω-sequence is simply a sequence.

Given a set A, its *cardinality* is defined to be the smallest ordinal γ such that there is a bijection $\gamma \to A$. This ordinal is denoted $|A|$; it is called an *initial ordinal*, for it is the smallest ordinal having a given cardinality. Infinite initial ordinals are denoted ω_α and have cardinality denoted $\aleph_\alpha$. A *cardinal* is an ordinal κ such that $\kappa = |\kappa|$. For finite sets, the notions of cardinals and ordinals coincide, but this is not true for infinite sets (e.g., the ordinal ω defined above an its successor $\omega + 1$ both have cardinality $\aleph_0$).

Definition 3.26. Let $\mathcal{K}$ be a category and $\mathcal{F} \subset \operatorname{mor} \mathcal{K}$ be a class of morphisms. Then an object A of $\mathcal{K}$ is $\mathcal{F}$-*small* if whenever

$$X_0 \xrightarrow{e_0} X_1 \xrightarrow{e_1} X_2 \xrightarrow{e_2} \cdots$$

is a ω-sequence of morphisms in $\mathcal{F}$, then the map of sets

$$\operatorname{colim} \mathcal{K}(A, X_n) \to \mathcal{K}(A, \operatorname{colim} X_n)$$

is an isomorphism. An object is *small* if it is small relative to the entire class $\operatorname{mor} \mathcal{K}$.

The colimit $\operatorname{colim} \mathcal{K}(A, X_n)$ is taken in **Set**, where it can be constructed as a coproduct followed by a coequaliser. More specifically, form the coproduct

$$\coprod_{n \geq 0} \mathcal{K}(A, X_n),$$

which is simply the set of maps $A \to X_k$ for some k, equipped with inclusions

$$\mathcal{K}(A, X_k) \xrightarrow{i_k} \textstyle\coprod_{n \geq 0} \mathcal{K}(A, X_n).$$

Next, form a coequaliser for each pair of maps i_j, $i_k e_{k-1} \cdots e_j$ for all $j < k$. This construction produces $\operatorname{colim} \mathcal{K}(A, X_n)$ as a quotient of the coproduct by relations of the form $i_j(f) = i_k e_{j,k}(f)$ for each $f \in \mathcal{K}(A, X_j)$ and each $j < k$, where $e_{j,k}$ is the composite $e_{k-1} \cdots e_j$. By transitivity, this means that distinct morphisms $f, g : A \to X_j$ are identified if for some $k > j$, $e_{j,k} f = e_{j,k} g$. This shows that the map $\operatorname{colim} \mathcal{K}(A, X_n) \to \mathcal{K}(A, \operatorname{colim} X_n)$ is well-defined since maps $f : A \to X_j$, $g : A \to X_k$ represent the same element in the colimit if there is some m such that $e_{j,m} f = e_{k,m} g$, in which case their image is

$$A \xrightarrow{e_{j,m} f} X_m \longrightarrow \operatorname{colim} X_n.$$

If we mimicked this construction for a finite sequence $X_0 \to \cdots \to X_m$, then $\operatorname{colim} \mathcal{K}(A, X_n)$ is precisely the set $\mathcal{K}(A, X_m) = \mathcal{K}(A, \operatorname{colim} X_n)$ as X_m is the colimit of this diagram. Returning our attention to ω-sequences, we see that $\operatorname{colim} \mathcal{K}(A, X_n) \to \mathcal{K}(A, \operatorname{colim} X_n)$ is surjective if and only if every map $A \to \operatorname{colim} X_n$ factorises through some X_k. Injectivity says that given $f, g : A \to X_k$ such that $e_{k,m} f \neq e_{k,m} g$ for all $m \geq k$, then the composites of these maps with maps to the colimit are also distinct. But, by the nature of the colimit construction in **Set**, this will always be the case for certainly

$$i_m e_{k,m} f \neq i_m e_{k,m} g : A \to \coprod_{n \geq 0} X_n,$$

which means that the coequaliser will not project these maps onto the same element.

Hence, this discussion can be summed up by saying that an object A of a category $\mathcal{K}$ is small if a map from it to the colimit of any ω-sequence factors through some stage of that sequence.

Example 3.27. In **Set**, every finite set A is small. Given an ω-sequence

$$X_0 \to X_1 \to X_2 \to \cdots$$

let $f : A \to \operatorname{colim} X_n$ be any map. For each $a \in A$, $f(a)$ is in the image of some $X_{k(a)}$, because a colimit in **Set** is constructed a coequaliser of a coproduct, which is simply a disjoint union. Let

$$k = \sup\{k(a) \mid a \in A\}.$$

Then f factors through a map $g : A \to X_k$, which shows that the map $\operatorname{colim} \mathcal{K}(A, X_n) \to \mathcal{K}(A, \operatorname{colim} X_n)$ is surjective as required.

Example 3.28. Similarly, any bounded chain complex of finitely presented R-modules is small in $\mathbf{Ch}_R$.

Definition 3.29. A model category $\mathcal{K}$ is *cofibrantly generated* if there are sets of morphisms I and J such that

(1) The domain of every morphism in I is $\mathcal{C}$-small where $\mathcal{C}$ is the class of all cofibrations and the trivial fibrations are exactly those maps that have the right lifting property with respect to all of I.

(2) The domain of every morphism in J is $\mathcal{C} \cap \mathcal{W}$-small where $\mathcal{C} \cap \mathcal{W}$ is the class of all trivial cofibrations and the fibrations are exactly those maps that have the right lifting property with respect to all of J.

Remark 3.30. The name comes from the fact that the sets I and J generate the cofibrations and the trivial cofibrations, respectively, in the sense that the cofibrations are the smallest class of maps that contains I and is closed under coproducts, cobase change, sequential colimits, and retracts, and likewise for the trivial cofibrations and J. By Lemma 3.7, it is clear that $\mathcal{C} \supset I$ and $\mathcal{C} \cap \mathcal{W} \supset J$, and Remark 3.10 shows that these classes are closed under the required limits. Conversely, Garner makes the more specific claim that the cofibrations are realisable as "a retract of a transfinite composition of pushouts of coproducts of generating cofibrations" [3, pp 47].

The conclusion in the following theorem repeats a fact that we have already asserted in requiring that the factorisations given by a model structure on a category are functorial. However, the proof of this result sheds considerably light into why the constructions given in Sections 3.3 and 3.4 worked. The argument is called *Quillen's small object argument* and first appeared in [10].

Theorem 3.31 (Quillen's small object argument)**.** *If $\mathcal{K}$ is a cofibrantly generated category with a model structure, then the factorisations given by this structure can be chosen to be functorial.*

Proof. We prove this for the cofibration-trivial fibration factorisation. The proof for the other half is obtained by replacing I with J, "cofibration" with "trivial cofibration," and "trivial fibration" with "fibration." Let $f : Y \to Z$ be any map, and let $X_0 = Y$. Let S_0 be the set of commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "f"] \\ B \arrow[r] & Z \end{tikzcd}$$

with $i \in I$. Define X_1 to be the pushout given by the diagram

$$\begin{tikzcd} \oplus_{S_0} A \arrow[r] \arrow[d] & X_0 \arrow[d, "j_1"] \\ \oplus_{S_0} B \arrow[r] & X_1 \end{tikzcd}$$

which exists because $\mathcal{K}$ is cocomplete. The maps $f = q_0 : X_0 \to Z$ and $\oplus_{S_0} B \to Z$ induce a map $q_1 : X_1 \to Z$. Inductively, let S_n be the set of commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_n \arrow[d, "f"] \\ B \arrow[r] & Z \end{tikzcd}$$

with $i \in I$, and define X_{n+1} to be the pushout

$$\begin{tikzcd} \oplus_{S_n} A \arrow[r] \arrow[d] & X_n \arrow[d, "j_{n+1}"] \\ \oplus_{S_n} B \arrow[r] & X_{n+1} \end{tikzcd} \tag{5}$$

inducing a map $q_{n+1} : X_{n+1} \to Z$. This construction yields a cocone Z over an ω-sequence

$$\begin{tikzcd} Y = X_0 \arrow[r, "j_1"] \arrow[dr, "f=q_0"'] & X_1 \arrow[r, "j_2"] \arrow[d, "q_1"] & X_2 \arrow[r] \arrow[dl, "q_2"] & \cdots \\ & Z & & \end{tikzcd}$$

Let $X = \operatorname{colim} X_n$, and let $j : Y \to X$ and $q : X \to Z$ be the induced maps to and from the colimit. Closure under coproducts implies that the left hand side of (5) is a cofibration, so closure under cobase change implies that each j_n is as well. Finally, closure under sequential colimits implies that the map j is a cofibration. Hence, it remains only to show that q is a trivial fibration, which we do by showing that q has the right lifting property with respect to any $i \in I$. Given a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r, "v"'] & Z \end{tikzcd}$$

with $i \in I$, A is small, so there exists a factorisation

$$\begin{tikzcd} A \arrow[r, "\tilde{u}"] \arrow[dr, "u"'] & X_n \arrow[d] \\ & X \end{tikzcd}$$

for some n. This gives a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{u}"] \arrow[d, "i"'] & X_n \arrow[r] \arrow[d, "q_n"] & X \arrow[dl] \\ B \arrow[r, "v"'] & Z \end{tikzcd}$$

Hence, the construction of X_{n+1} as a pushout over coproducts of such diagrams defines maps $\tilde{v}$ and w satisfying

$$\begin{tikzcd} A \arrow[r, "\tilde{u}"] \arrow[d, "i"'] & X_n \arrow[d, "j_{n+1}"] \\ B \arrow[r, "\tilde{v}"] \arrow[dr, "w"'] & X_{n+1} \arrow[d] \arrow[dr, "q_{n+1}"] \\ & X \arrow[r, "q"'] & Z \end{tikzcd} \tag{6}$$

We claim that $w : B \to X$ is the desired lift. The composite of the arrows from A to X in a clockwise direction in (6) equals u, so the upper triangle commutes. By definition of the pushout X_{n+1}, $v = q_{n+1}\tilde{v}$, so the parallelogram at the bottom of (6) shows that the lower triangle commutes as well, completing the proof. □

In light of Theorem 3.31, it should come as no surprise that $\mathbf{Ch}_R$ is cofibrantly generated. The proof of this fact will illuminate connections between the small object argument and the factorisations we had previously established.

Recall that D^n denotes the chain complex where $D^n_n = D^n_{n-1} = R$ and $D^n_k = 0$ for $k \neq n, n-1$, with $\partial : D^n_n \to D^n_{n-1}$ the identity. Let S^n be the chain complex with $S^n_n = R$ and $S^n_k = 0$ for $k \neq n$.

Proposition 3.32. $\mathbf{Ch}_R$ *is cofibrantly generated with* $I = \{S^{n-1} \to D^n \mid n \geq 1\} \cup \{0 \to S^0\}$ *and* $J = \{0 \to D^n \mid n \geq 1\}$.

Proof. Clearly, the maps in I and J are cofibrations and the maps in J are also weak equivalences. Lemma 3.21 shows that the fibrations are precisely those maps that satisfy the right lifting property for J. It remains to show that a map is a trivial fibration if and only if it satisfies the right lifting property for all maps in I. Let $q : Q_\bullet \to N_\bullet$ be a trivial fibration. Then q_0 is surjective, which implies that q has the right lifting property with respect to

$0 \to S^0$. Thus, we need only consider lifting problems of the form

$$
\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d] & Q_\bullet \arrow[d, "q"] \\
D^n \arrow[r, "v"'] & N_\bullet
\end{tikzcd}
\tag{7}
$$

A lift $w : D^n \to Q_\bullet$ is trivial in all but degrees n and $n-1$, so we need only construct maps w_n and w_{n-1} such that

$$
\begin{tikzcd}
0 \arrow[rr] \arrow[dr] \arrow[dd] & & R \arrow[dr, "u_{n-1}"] \arrow[dd] & \\
& Q_n \arrow[rr, "\partial_Q"] \arrow[dd, "q_n"] & & Q_{n-1} \arrow[dd, "q_{n-1}"] \\
R \arrow[rr] \arrow[ur, dashed, "w_n"] \arrow[dr, "v_n"'] & & R \arrow[ur, dashed, "w_{n-1}"'] \arrow[dr] & \\
& N_n \arrow[rr, "\partial_N"'] & & N_{n-1}
\end{tikzcd}
\tag{8}
$$

commutes. In order for the upper triangle on the right face of the cube to commute, we need $w_{n-1}(1) = u_{n-1}(1) = z \in Q_{n-1}$. Extending this diagram to degree $n-2$, $1 \in S^{n-1}_{n-1}$ gets mapped to 0, so z is a cycle. Furthermore, $q_{n-1}(z) = \partial_N v_n(1)$, so $q_{n-1}(z) \in B_{n-1}N$, which means that $z \in B_{n-1}Q$, since q is a homology isomorphism. Hence z is in the image of ∂_Q. Pick $x \in Q_n$ such that $\partial_Q(x) = z$. Then $\partial_N q_n(x) = q_{n-1}(z) = \partial_N v_n(1)$, so $v_n(1) - q_n(x) \in Z_nN$. Because q_n is a homology isomorphism, there is some $\tilde{x} \in Z_nQ$ such that $q_n(\tilde{x}) = v_n(1) - q_n(x)$. Then $\tilde{x} + x$ maps to $v_n(1)$ under q_n and to z under ∂_Q. Take $w_n(1) = \tilde{x} + x$ and this defines a lift such that the entire diagram commutes.

Conversely, we assume that $q : Q_\bullet \to N_\bullet$ has the right lifting property with respect to every element of J. The right lifting property of q with respect to $0 \to S^0$ implies that q_0 is surjective. We prove that q_n is surjective by induction. Assuming this is true for q_{n-1}, we can construct a lifting problem (7) for every chain map $v : D^n \to N_\bullet$. More specifically, there is a chain map v that sends $1 \in R = D^n_n$ to any element $x \in N_n$, and by surjectivity of q_{n-1}, this can extended to a lifting problem (7). The existence of a lift w_n in (8) implies that q_n is surjective. Hence, q is a fibration.

More precisely, the set of diagrams of the form (7) are in bijection with the set

$$\{(x, y) \in N_n \times Z_{n-1}Q \mid q_{n-1}y = \partial_N x\}.$$

A lift (8) corresponds to an element $z \in Q_n$ such that $\partial_Q z = y$ and $q_n z = x$. This perspective will make it clear that that q induces an isomorphism on homology. Let $x \in Z_nN$. Then $(x, 0)$ is a lifting problem with solution $z \in Q_n$ such that $\partial_Q z = 0$. Hence, $z \in Z_nQ$ and H_nq is surjective for $n > 0$ (when $n = 0$, surjectivity is immediate because $Z_0Q = Q_0$ and $Z_0N = N_0$). For injectivity, let $y \in Z_{n-1}Q$ be such that $q_{n-1}y = \partial_N x$ for any x. Then (x, y) is a lifting problem with solution z such that $\partial_Q z = y$. Hence, $y \in B_nQ$ and H_nq is injective for all n. □

4 Weak Factorization Systems

The trivial cofibration-fibration and cofibration-trivial fibration factorisations given by a model structure provide examples of a more general categorical notion of a *weak factorisation system*. To better understand the relationship between these factorisations and the lifting property of Axiom 3, we now study such systems in general.

4.1 Definitions

Let $\mathcal{K}$ be a category. A morphism f has the *left lifting property* with respect to a morphism g, or equivalently, g has the *right lifting property* with respect to f if every lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot \end{tikzcd} \tag{9}$$

has a solution w such that both triangles commute, and denote this by $f \leq g$.

Definition 4.1. A *weak factorisation system* in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of distinguished classes of morphisms such that

(1) Every $h \in \operatorname{mor} \mathcal{K}$ can be factorised as $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

(2) $\mathcal{R} = \mathcal{L}^{\leq}$ is precisely the class of morphisms that have the right lifting property with respect to each $f \in \mathcal{L}$.

(3) $\mathcal{L} = {}^{\leq}\mathcal{R}$ is precisely the class of morphisms that have the left lifting property with respect to each $g \in \mathcal{R}$.

Recall that a morphism f is a *retract* of g if there is a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

with horizontal composites the identity.

Conditions 2 and 3 imply:

(2') $f \leq g$ for every $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

(3') $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

The first statement is obvious. For the second, let h be a retract of $f \in \mathcal{L}$ and let $g \in \mathcal{R}$. Given a lifting problem

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "h"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad \text{and a retract diagram} \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "h"'] & \cdot \arrow[r] \arrow[d, "f"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

with horizontal composites the identity. Combining these diagrams, we obtain an expanded picture of the same lifting problem with an obvious solution:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "h"'] & \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[r] \arrow[d, "h"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r] & \cdot \arrow[r] \arrow[urr, dashed, "s"] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

In the above, s is the lift provided by the fact that f has the left lifting property with respect to g. Thus, h has the left lifting property with respect to g, and so $h \in \mathcal{L}$, proving that $\mathcal{L}$ is closed under retracts. The proof for $\mathcal{R}$ is similar.

Condition 3' implies:

(3") If s is a split monic and $s \cdot f \in \mathcal{L}$, then $f \in \mathcal{L}$. Dually if t is a split epic and $g \cdot t \in \mathcal{R}$, then $g \in \mathcal{R}$. This follows from the following retract diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "f"'] & \cdot \arrow[r, "1"] \arrow[d, "sf"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "s"] & \cdot \arrow[r, "s^{-1}"] & \cdot
\end{tikzcd}$$

Proposition 4.2. *If* $(\mathcal{L}, \mathcal{R})$ *is a pair of classes of morphisms satisfying* (1),(2'), *and* (3"), *then* $(\mathcal{L}, \mathcal{R})$ *is a weak factorisation system.*

Proof. We must show that $\mathcal{L} = {}^{\leq}\mathcal{R}$; $\mathcal{R} = \mathcal{L}^{\leq}$ will follows dually. Clearly $\mathcal{L} \subset {}^{\leq}\mathcal{R}$. Given $f \in {}^{\leq}\mathcal{R}$, factorize f as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$. By the lifting property for f, there exists a lift s in the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "l"] \arrow[d, "f"'] & \cdot \arrow[d, "r"] \\
\cdot \arrow[r, "1"'] \arrow[ur, dashed, "s"] & \cdot
\end{tikzcd}$$

and by commutativity of the lower triangle, s is split monic. Hence (3") implies that $f \in \mathcal{L}$ as desired. $\square$

Given a category $\mathcal{K}$, we can form a category $\mathcal{K}^{\mathbf{2}}$ whose objects are morphisms in $\mathcal{K}$ and whose arrows $(u, v) : f \to g$ are commutative squares

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd} \tag{10}$$

Let dom: $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ and cod: $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ denote the functors that send an arrow to its domain and codomain, respectively, and let $\kappa : \mathrm{dom} \to \mathrm{cod}$ be the natural transformation given by $\kappa_f = f$ for all morphisms f in $\mathcal{K}$.

Definition 4.3. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ is *functorial* if there is a pair of functors $L, R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ so that

$$\mathrm{dom}L = \mathrm{dom}, \qquad \mathrm{cod}R = \mathrm{cod}, \qquad \mathrm{cod}L = \mathrm{dom}R,$$

and $f = Rf \cdot Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for every $f \in \mathrm{mor}\,\mathcal{K}$.

The functors L and R can be equivalently described by the functor

$$F = \mathrm{cod}L = \mathrm{dom}R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$$

and natural transformations $\lambda : \mathrm{dom} \to F$ and $\rho : F \to \mathrm{cod}$ such that $\kappa = \rho \cdot \lambda$ and $\lambda_f \in \mathcal{L}$ and $\rho_f \in \mathcal{R}$ for all morphisms f. Hence, the factorisation of a morphism f in $\mathcal{K}$ is given by

$$\begin{tikzcd} \mathrm{dom}\, f \arrow[rr, "\kappa_f = f"] \arrow[dr, "\lambda_f"'] & & \mathrm{cod}\, f \\ & Ff \arrow[ur, "\rho_f"'] & \end{tikzcd}$$

We call the triple (F, λ, ρ) a *functorial realisation* for the weak factorisation $(\mathcal{L}, \mathcal{R})$. Naturality of λ and ρ implies that for any commutative square (10), the diagram

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\lambda_f"'] & \cdot \arrow[d, "\lambda_g"] \\ \cdot \arrow[r, "F(u,v)"] \arrow[d, "\rho_f"'] & \cdot \arrow[d, "\rho_g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{11}$$

commutes.

Surprisingly, a functorial realisation of a weak factorisation system determines the system itself. Given such a triple (F, λ, ρ), define

$$\mathcal{L}_F := \{f \mid \exists s : \lambda_f = s \cdot f, \rho_f \cdot s = 1\}, \tag{12}$$
$$\mathcal{R}_F := \{g \mid \exists t : \rho_g = g \cdot t, t \cdot \lambda_f = 1\}. \tag{13}$$

Given $f \in \mathcal{L}_F$ and $g \in \mathcal{R}_F$ the lifts s and t allow one to construct a solution to the lifting problem (9)

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\lambda_f"'] & \cdot \arrow[d, "\lambda_g"] \\ \cdot \arrow[r, "F(u,v)"] \arrow[d, "\rho_f"'] \arrow[ur, phantom, ""] & \cdot \arrow[u, dashed, bend left, "t"] \\ \cdot \arrow[r, "v"'] \arrow[u, dashed, bend right, "s"'] & \cdot \arrow[u, phantom] \end{tikzcd} \tag{14}$$

by taking $w = t \cdot F(u, v) \cdot s$. The equations defining the lifts s and t guarantee that the required triangles commute.

Theorem 4.4. (1) *For every weak factorisation system* $(\mathcal{L}, \mathcal{R})$ *with functorial realisation* (F, λ, ρ)*, the classes* $\mathcal{L} = \mathcal{L}_F$ *and* $\mathcal{R} = \mathcal{R}_F$*.*

(2) *For any triple* $(F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}, \lambda : \mathrm{dom} \to F, \rho : F \to \mathrm{cod})$ *with* $\kappa = \rho \cdot \lambda$ *and such that* $\lambda_f \in \mathcal{L}_F$ *and* $\rho_f \in \mathcal{R}_F$ *for every morphism* f *in* $\mathcal{K}$*, then* $(\mathcal{L}_F, \mathcal{R}_F)$ *is a weak factorisation system with functorial realisation* (F, λ, ρ)*.*

Proof. (1) By definition $f \in \mathcal{L}_F$ satisfies $s \cdot f = \lambda_f \in \mathcal{L}$ with s a split monic, so $\mathcal{L}_F \subset \mathcal{L}$ by (3") and similarly $\mathcal{R}_F \subset \mathcal{R}$. Conversely, if $f \in \mathcal{L}$, factorise $f = \rho_f \lambda_f$ and use $f \leq \rho_f$ as in the proof of Proposition 4.2 to find a lift s such that $\lambda_f = s \cdot f$ and $\rho_f \cdot s = 1$. Hence, $\mathcal{L} \subset \mathcal{L}_F$ and dually $\mathcal{R} \subset \mathcal{R}_F$.

(2) The equality $\kappa = \rho \cdot \lambda$ gives factorisation and (14) shows that $\mathcal{L}_F \leq \mathcal{R}_F$. We show that $\mathcal{L}_F$ and $\mathcal{R}_F$ are closed under retracts. Given a retract diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "h"'] & \cdot \arrow[r, "p"] \arrow[d, "f"] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "v"'] & \cdot \arrow[r, "q"'] & \cdot
\end{tikzcd}$$

factorise f and h to obtain

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "\lambda_h"'] & \cdot \arrow[r, "p"] \arrow[d, "\lambda_f"] & \cdot \arrow[d, "\lambda_h"] \\
\cdot \arrow[r, "F(u,v)"] \arrow[d, "\rho_h"'] & \cdot \arrow[r, "F(p,q)"] \arrow[d, "\rho_f"] & \cdot \arrow[d, "\rho_h"] \\
\cdot \arrow[r, "v"'] & \cdot \arrow[r, "q"'] \arrow[u, dashed, bend left, "s"] & \cdot
\end{tikzcd}$$

If $f \in \mathcal{L}_F$ there exists a lift s such that $\lambda_f = s \cdot f$ and $\rho_f \cdot s = 1$. Define $j = F(p, q) \cdot s \cdot v$. Then using that the outer horizontal composites are the identity, j satisfies $\lambda_h = j \cdot h$ and $\rho_h \cdot j = 1$. Hence, $h \in \mathcal{L}_F$ and a dual argument shows $\mathcal{R}_F$ is closed under retracts as well. This implies that $(\mathcal{L}_F, \mathcal{R}_F)$ satisfies (3"). Hence, this pair is a weak factorisation system with functorial realisation (F, λ, ρ). □

Definition 4.5. An *orthogonal factorisation system* is a weak factorisation system such that every lifting problem (9) has a *unique* solution w such that both triangles commute.

Example 4.6. In **Set**, $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorisation system where $\mathcal{E}$ is the class of epis and $\mathcal{M}$ is the class of monos. Given a lifting problem (9) with f epic and g monic, it is easy to construct a unique lift. For each object x in the codomain of f, its preimage under f must be mapped to a single element of the domain of g, because g is monic. Define $w(x)$ to be this element; it is clear that both triangles commute.

Example 4.7. Similarly, in **Top**, orthogonal factorisation systems are given by either $\mathcal{E}_0 =$ surjections and $\mathcal{M}_0 =$ embeddings or $\mathcal{E}_1 =$ quotients and $\mathcal{M}_1 =$ injections.

It is well known that in an orthogonal factorisation system, the classes $\mathcal{L}$ and $\mathcal{R}$ enjoy many good stability properties. For example, $\mathcal{L}$ is closed under all colimits, $\mathcal{R}$ is closed under all limits, and both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition. But these properties are not enjoyed by all weak factorisation systems. Various restrictions of weak factorisation systems have been proposed to recover the closure properties enjoyed by orthogonal systems, including most recently the notion of *natural* weak factorisation systems in [5]. We show that comonad-monad structure of this system is not necessary to achieve closure of the classes $\mathcal{L}$ and $\mathcal{R}$ under colimits and limits. Indeed, it is not even necessary to assume that the lifts in (9) are *natural*, as the authors suggest. Instead, a simple assumption about the behavior of the factorisation with regard to identity morphisms suffices.

In Section 4.2, we describe this assumption and prove a key theorem showing that the morphism $F(u, v)$ is unique in a certain sense. In Section 4.3, we show that this uniqueness implies that the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under colimits and limits, respectively. In Section 4.4, we give a generalisation of these results. Finally, in Section 4.5, we define the natural weak factorisation systems of [5], an algebraisation of the notion of a weak factorisation system that will describe other circumstances under which the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under all (co)limits.

The author was introduced to this topic through [5], [9], and [11], and in discussions with J.M.E. Hyland. The preceding section owes a great debt to [11] and the last to [5]. The lemma and theorem from Section 4.2 are proven in a slightly different form in [9].

4.2 Uniqueness Theorem

Let $E : \mathcal{K} \to \mathcal{K}^{\mathbf{2}}$ be the embedding functor $A \mapsto 1_A$ and $f \mapsto (f, f) : 1 \to 1$ for all $A \in \operatorname{ob} \mathcal{K}$ and $f \in \operatorname{mor} \mathcal{K}$. We will prove the following theorem.

Theorem 4.8. *If $(\mathcal{L}, \mathcal{F})$ is a weak factorisation system with functorial realisation (F, λ, ρ) such that*

$$(F, \lambda, \rho)E = 1, \tag{15}$$

then for any commutative square (10), $F(u, v)$ is the unique morphism k such that both squares of

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\lambda_f"'] & \cdot \arrow[d, "\lambda_g"] \\ \cdot \arrow[r, "k"] \arrow[d, "\rho_f"'] & \cdot \arrow[d, "\rho_g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \tag{16}$$

commute.

By $(F, \lambda, \rho)E = 1$, we mean that $FE = 1_{\mathcal{K}}$ and $\lambda_E, \rho_E = 1 : 1_{\mathcal{K}} \to 1_{\mathcal{K}}$. Specifically, this means that $F(1_A) = A$ and $\lambda_{1_A} = \rho_{1_A} = 1_A$ for all $A \in \operatorname{ob} \mathcal{K}$, and $F((f, f) : 1 \to 1) = f$ for all $f \in \operatorname{mor} \mathcal{K}$. Hence, (15) is equivalent to the assertion that the factorisation of a commutative square $(f, f) : 1 \to 1$ determined by (F, λ, ρ) is

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "1"'] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "f"] \arrow[d, "1"'] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}$$

The implications of this theorem on the question of colimits and limits in $\mathcal{L}$ and $\mathcal{R}$ will be explored in Section 4.3.

Recall that $L, R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ are functors sending a morphism f to λ_f and ρ_f, respectively. For future notational convenience, define natural transformations $\alpha : L \to 1_{\mathcal{K}^{\mathbf{2}}}$ and $\beta : 1_{\mathcal{K}^{\mathbf{2}}} \to R$ by

$$\alpha_f = (1, \rho_f) : \lambda_f \to f \qquad \text{and} \qquad \beta_f = (\lambda_f, 1) : f \to \rho_f.$$

For any functorial weak factorisation system, $\lambda_f \in \mathcal{L}_F$ and $\rho_f \in \mathcal{R}_F$ implies that there exists s and t such that $\rho_{\lambda_f} \cdot s = 1$ and $t \cdot \lambda_{\rho_f} = 1$. So for any morphism f, $\lambda_{\rho_f} = \lambda_{Rf}$ is always monic and $\rho_{\lambda_f} = \rho_{Lf}$ is always epic.

Lemma 4.9. *Let $(\mathcal{L}, \mathcal{F})$ be a functorial weak factorisation system satisfying (15). Then $\lambda_R = F\beta$ and $\rho_L = F\alpha$.*

Proof. We must show that $\lambda_{Rf} = F\beta_f$ and $\rho_{Lf} = F\alpha_f$ for all $f \in \operatorname{mor} \mathcal{K}$. The diagram

$$\begin{tikzcd}
\cdot \arrow[r, "\lambda_f"] \arrow[d, "\lambda_f"'] & \cdot \arrow[d, "\lambda_{Rf}"] \\
\cdot \arrow[r, "z"] \arrow[d, "\rho_f"'] & \cdot \arrow[d, "\rho_{Rf}"] \\
\cdot \arrow[r, "1"'] & \cdot
\end{tikzcd}$$

commutes for $z = F\beta_f$ and $z = \lambda_{Rf}$. Hence, the morphism $(\lambda_{Rf}\lambda_f, \rho_f) : \lambda_f \to \rho_{Rf}$ factorises as

$$\begin{tikzcd}
\cdot \arrow[r, "\lambda_f"] \arrow[d, "\lambda_f"'] & \cdot \arrow[r, "z"] \arrow[d, "1"] & \cdot \arrow[d, "\rho_{Rf}"] \\
\cdot \arrow[r, "1"'] & \cdot \arrow[r, "\rho_f"'] & \cdot
\end{tikzcd} \tag{17}$$

for both choices of z. Because $\rho_E = 1$, the map $F(\lambda_f, 1)$ must satisfy $1 \cdot F(\lambda_f, 1) = 1 \cdot \rho_{Lf}$, that is, $F(\lambda_f, 1) = \rho_{Lf}$. Applying the functor F to the factorisation (17) with each value of z yields

$$F(F\beta_f, \rho_f) \cdot \rho_{Lf} = F(\lambda_{Rf}, \rho_f) \cdot \rho_{Lf},$$

and since ρ_{Lf} is epic it can be canceled from this equation. Hence, $\overline{z} = F(z, \rho_f)$ is independent of the choice of the value for z. For either value of z, the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "z"] \arrow[d, "1"'] & \cdot \arrow[d, "\lambda_{RRf}"] \\
\cdot \arrow[r, "\overline{z}"] \arrow[d, "1"'] & \cdot \arrow[d, "\rho_{RRf}"] \\
\cdot \arrow[r, "\rho_f"'] & \cdot
\end{tikzcd}$$

commutes. Hence, $\lambda_{RRf} F\beta_f = \lambda_{RRf}\lambda_{Rf}$, but λ_{RRf} is monic, so $F\beta_f = \lambda_{Rf}$. Dually, $\rho_L = F\alpha$. □

Proof of Theorem 4.8. Let k be a morphism such that (16) commutes. Then the diagram

$$\begin{tikzcd}
& \cdot \arrow[r, "u"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \arrow[dr, "\lambda_g"] & \\
\cdot \arrow[ur, "1"] \arrow[d, "\lambda_f"'] \arrow[dr, "\lambda_f"'] & \cdot \arrow[r, "v"'] & \cdot \arrow[dr, "1"] & \cdot \arrow[d, "\rho_g"] \\
\cdot \arrow[ur, "\rho_f"] \arrow[dr, "1"'] & \cdot \arrow[r, "k"] \arrow[d, "1"] & \cdot \arrow[d, "1"] \arrow[ur, "1"] & \cdot \\
& \cdot \arrow[r, "k"'] & \cdot \arrow[ur, "\rho_g"'] &
\end{tikzcd}$$

commutes. The first and last $\mathcal{K}^{\mathbf{2}}$-morphisms on the back of the hexagon are α_f and β_g. Recall that $\rho_E = 1$ implies that $F(\lambda_f, 1) = \rho_{Lf}$; dually, $\lambda_E = 1$ implies that $F(1, \rho_g) = \lambda_{Rg}$. Apply the functor F to the hexagon to get

$$\lambda_{Rg} \cdot F(k, k) \cdot \rho_{Lf} = F\beta_g \cdot F(u, v) \cdot F\alpha_f.$$

From Lemma 4.9 and $FE = 1$, this is equivalent to

$$\lambda_{Rg} \cdot k \cdot \rho_{Lf} = \lambda_{Rg} \cdot F(u, v) \cdot \rho_{Lf}.$$

Because λ_{Rg} is monic and ρ_{Lf} is epic, it follows that $F(u, v) = k$, as desired. □

We conclude this section with a corollary to Theorem 4.8 that will prove useful in the discussion to follow, but first, we clarify some terminology. By a *lift* of a morphism in $\mathcal{L}$ or $\mathcal{R}$, we mean the map s or t specified by (12) or (13). By contrast, a lift of a commutative square (9) means the map w such that both triangles commute.

Corollary 4.10. *Given a functorial weak factorisation system* $(\mathcal{L}, \mathcal{R})$ *satisfying (15), let* $f, g \in \mathcal{R}$ *with lifts* i *and* j *respectively. Then, given a commutative square* (10)*, the diagram*

$$\begin{tikzcd}
\cdot \arrow[r, "u"] & \cdot \\
\cdot \arrow[u, "i"] \arrow[r, "F(u,v)"'] & \cdot \arrow[u, "j"']
\end{tikzcd}$$

commutes. Dually, given $f, g \in \mathcal{L}$ with lifts i and j, the diagram

$$\begin{tikzcd} \cdot \arrow[r, "F(u,v)"] & \cdot \\ \cdot \arrow[u, "i"] \arrow[r, "v"'] & \cdot \arrow[u, "j"'] \end{tikzcd}$$

commutes.

Proof. We prove this lemma for the case $f, g \in \mathcal{R}$. The other statement follows dually. We wish to show that $j \cdot F(u, v) = u \cdot i$. Let

$$k = \lambda_g \cdot j \cdot F(u, v) \cdot \lambda_f \cdot i.$$

We wish to show that k is a morphism such that (16) commutes, so that Theorem 4.8 implies that $k = F(u, v)$. That is, we must show that $k \cdot \lambda_f = \lambda_g \cdot u$ and $\rho_g \cdot k = v \cdot \rho_f$. For the first equation,

$$\begin{aligned} k \cdot \lambda_f &= \lambda_g \cdot j \cdot F(u, v) \cdot \lambda_f \cdot i \cdot \lambda_f \\ &= \lambda_g \cdot j \cdot (F(u, v) \cdot \lambda_f) && \text{as } i \cdot \lambda_f = 1 \\ &= \lambda_g \cdot j \cdot \lambda_g \cdot u && \text{by commutativity of (11)} \\ &= \lambda_g \cdot u && \text{as } j \cdot \lambda_g = 1, \end{aligned}$$

as desired. For the second equation,

$$\begin{aligned} \rho_g \cdot k &= \rho_g \cdot \lambda_g \cdot j \cdot F(u, v) \cdot \lambda_f \cdot i \\ &= \rho_g \cdot F(u, v) \cdot \lambda_f \cdot i && \text{as } g \cdot j = \rho_g \\ &= v \cdot f \cdot i && \text{by commutativity of (11)} \\ &= v \cdot \rho_f && \text{as } f \cdot i = \rho_f, \end{aligned}$$

as desired.

Hence, $F(u, v) = k = \lambda_g \cdot j \cdot F(u, v) \cdot \lambda_f \cdot i$. Composing on the left by j gives

$$j \cdot F(u, v) = j \cdot F(u, v) \cdot \lambda_f \cdot i = j \cdot \lambda_g \cdot u \cdot i = u \cdot i.$$

□

4.3 Limits

If $\mathcal{K}$ has limits of a given type, then $\mathcal{K}^{\mathbf{2}}$ will as well. We illustrate this general fact by giving constructions of products and equalisers in $\mathcal{K}^{\mathbf{2}}$, assuming that they exist in $\mathcal{K}$ and leave the proofs that these objects satisfy the required universal mapping properties as an exercise.

Assume that $\mathcal{K}$ has binary products, and consider two objects $f : A \to B$ and $g : C \to D$ of $\mathcal{K}^{\mathbf{2}}$. Define the product $f \times g : A \times C \to B \times D$ to be the unique morphism given by the following diagram:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times C \arrow[l, "\pi_A"'] \arrow[r, "\pi_C"] \arrow[d, dashed, "! \, f\times g"] & C \arrow[d, "g"] \\ B & B \times D \arrow[l, "\pi_B"'] \arrow[r, "\pi_D"] & D \end{tikzcd}$$

Given any $z : X \to Y$ that maps to both f and g, there exists a unique $\mathcal{K}^{\mathbf{2}}$-morphism $z \to f \times g$, whose domain arrow is given by the unique induced $\mathcal{K}$-morphism $X \to A \times C$ and whose codomain arrow is the unique $\mathcal{K}$-morphism $Y \to B \times D$.

Now assume that $\mathcal{K}$ has equalisers. Given two parallel morphisms $(u, v), (u', v') : f \to g$, form the equalisers e_u and e_v of the pairs u, u' and v, v' respectively. Because $f \cdot e_u$ equalises v and v', there exists a unique $\mathcal{K}$-morphism $e : E_u \to E_v$ such that the left hand side of the following diagram commutes with either square on the right hand side.

$$\begin{tikzcd} E_u \arrow[r, "e_u"] \arrow[d, dashed, "! \, e"'] & A \arrow[r, shift left, "u"] \arrow[r, shift right, squiggly, "u'"'] \arrow[d, "f"] & C \arrow[d, "g"] \\ E_v \arrow[r, "e_v"] & B \arrow[r, shift left, "v"] \arrow[r, shift right, squiggly, "v'"'] & D \end{tikzcd}$$

(Note that the squares formed with exactly one squiggly edge will not commute in general.)

Proposition 4.11. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system. Then $\mathcal{L}$ is closed under coproducts and $\mathcal{R}$ is closed under products.*

Proof. We show that $\mathcal{R}$ is closed under products; closure of $\mathcal{L}$ under coproducts will follow dually. Suppose $f, g \in \mathcal{R}$ and consider a lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "l"'] & \cdot \arrow[d, "f\times g"] \\ \cdot \arrow[r, "q"'] & \cdot \end{tikzcd}$$

for some $l \in \mathcal{L}$. The map $(p, q) : l \to f \times g$ induces maps $\pi_f \cdot (p, q) : l \to f$ and $\pi_g \cdot (p, q) : l \to g$, giving two more lifting problems. As $f, g \in \mathcal{R}$, these have solutions w_f and w_g respectively, and these maps induce a map

$$w : \operatorname{cod} l \to \operatorname{dom} f \times g = \operatorname{dom} f \times \operatorname{dom} g.$$

Commutativity of the two triangles follows from the fact that w_f and w_g are lifts and the universal mapping property of the $\mathcal{K}$-products dom $f \times g$ and cod $f \times g$. Hence $f \times g \in \mathcal{L}^{\leq} = \mathcal{R}$. $\square$

Note that the proof of Proposition 4.11 does not even require that the weak factorisation system be functorial. However, this will not be the case for the next proposition, which requires the use of Corollary 4.10, and hence depends on the condition (15).

Proposition 4.12. *Let $(\mathcal{L}, \mathcal{R})$ be a functorial weak factorisation system satisfying (15). Then $\mathcal{L}$ is closed under coequalisers and $\mathcal{R}$ is closed under equalisers.*

Proof. We show that $\mathcal{R}$ is closed under equalisers; closure of $\mathcal{L}$ under coequalisers will follow dually. Let $f, g \in \mathcal{R}$ with parallel $\mathcal{K}^{\mathbf{2}}$-morphisms $(u, v), (u', v') : f \to g$, and let $(e_u, e_v) : e \to f$ be their equaliser. Given $l \in \mathcal{L}$ and a lifting problem $(x, y) : l \to e$, extend this diagram to the lifting problem

$$\begin{tikzcd}
\cdot \arrow[r, "x"] \arrow[d, "l"'] & \cdot \arrow[r, "e_u"] \arrow[d, "e"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "y"'] \arrow[urr, dashed, "z"] & \cdot \arrow[r, "e_v"'] & \cdot
\end{tikzcd} \tag{18}$$

Let i be a lift of f, j be a lift of g, and k be a lift of l. which has a solution z since $f \in \mathcal{R}$. Then (18) has a solution $z = i \cdot F(e_u \cdot x, e_v \cdot y) \cdot k$. We can use this to construct lifts $u \cdot z$, $u' \cdot z$ of

$$\begin{tikzcd}
\cdot \arrow[r, "x"] \arrow[d, "l"'] & \cdot \arrow[r, "e_u"] \arrow[d, "e"'] & \cdot \arrow[r, "u"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "y"'] \arrow[urr, dashed, "z"] & \cdot \arrow[r, "e_v"'] & \cdot \arrow[r, "v"'] & \cdot
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\cdot \arrow[r, "x"] \arrow[d, "l"'] & \cdot \arrow[r, "e_u"] \arrow[d, "e"'] & \cdot \arrow[r, "u'"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "y"'] \arrow[urr, dashed, "z"] & \cdot \arrow[r, "e_v"'] & \cdot \arrow[r, "v'"'] & \cdot
\end{tikzcd}$$

respectively.

We wish to show that $u \cdot z = u' \cdot z$, because it would follow that z factorises through the equaliser e_u of u and u'. We have

$$\begin{aligned}
u \cdot z &= u \cdot i \cdot F(e_u \cdot x, e_v \cdot y) \cdot k \\
&= j \cdot F(u, v) \cdot F(e_u \cdot x, e_v \cdot y) \cdot k && \text{by Corollary 4.10} \\
&= j \cdot F(u \cdot e_u \cdot x, v \cdot e_v \cdot y) \cdot k && \text{by functoriality of } F \\
&= j \cdot F(u' \cdot e_u \cdot x, v' \cdot e_v \cdot y) \cdot k && \text{since } e \text{ is an equaliser} \\
&= j \cdot F(u', v') \cdot F(e_u \cdot x, e_v \cdot y) \cdot k && \text{by functoriality of } F \\
&= u' \cdot i \cdot F(e_u \cdot x, e_v \cdot y) \cdot k && \text{by Corollary 4.10} \\
&= u' \cdot z.
\end{aligned}$$

Hence, $z = e_u \cdot w$.

$$\begin{tikzcd}
\cdot \arrow[r, "x"] \arrow[d, "l"'] & \cdot \arrow[r, "e_u"] \arrow[d, "e"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "y"'] \arrow[ur, "w"] \arrow[urr, "z"'] & \cdot \arrow[r, "e_v"'] & \cdot
\end{tikzcd}$$

We note that $e_u \cdot x = z \cdot l = e_u \cdot w \cdot l$ and e_u is monic, so $x = w \cdot l$. Similarly, $e_v \cdot y = f \cdot z = f \cdot e_u \cdot w = e_v \cdot e \cdot w$ and e_v is monic, so $y = e \cdot w$. Hence, w is indeed the lift we sought, and $e \in \mathcal{L}^{\leq} = \mathcal{R}$. $\square$

Proposition 4.11 and Proposition 4.12, together with the well-known result that all limits can be constructed from products and equalisers, prove the following theorem.

Theorem 4.13. *If $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system with functorial realisation (F, λ, ρ) such that $(F, \lambda, \rho)E = 1$, then $\mathcal{L}$ is closed under all colimits and $\mathcal{R}$ is closed under all limits.*

4.4 Generalisation

In fact, in Theorems 4.8 and 4.13, the requirement that $(F, \lambda, \rho)E = 1_K$ can be relaxed somewhat. Suppose that the functor FE is not identically 1_K, but instead there exists a natural isomorphism $\gamma : FE \to 1_K$. We define a functor $F' : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ by

$$F'(1_A) = A, \forall A \in \operatorname{ob} \mathcal{K}; \qquad F'(f) = F(f), \forall f \neq 1.$$

Given a $\mathcal{K}^{\mathbf{2}}$-morphism (10), define

$$F'(u, v) = \delta_g \cdot F(u, v) \cdot \delta_f^{-1}$$

where $\delta_{1_A} = \gamma_A, \forall A \in \operatorname{ob} \mathcal{K}$ and $\delta_f = 1_{Ff}, \forall f \neq 1$.

Lemma 4.14. *Let (F, λ, ρ) determine a functorial weak factorisation system $(\mathcal{L}, \mathcal{R})$ and suppose $\gamma : FE \to 1_K$ is a natural isomorphism. Then there exists a functor $F' : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ and natural isomorphism $\delta : F \to F'$ such that $F'E = 1_K$ and $(F', \delta \cdot \lambda, \rho \cdot \delta^{-1})$ determines the functorial weak factorisation system $(\mathcal{L}, \mathcal{R})$.*

Proof. Define F' and δ as above. It is not difficult to check that F' is indeed a functor, and the equation $F'E = 1_K$ is clear from construction. The definition of $F'(u, v)$ gives that

$$\begin{tikzcd} F(f) \arrow[r, "\delta_f"] \arrow[d, "F(u,v)"'] & F'(f) \arrow[d, "F'(u,v)"] \\ F(g) \arrow[r, "\delta_g"'] & F'(g) \end{tikzcd}$$

commutes for any square (10); it follows from the definition of δ that $\delta : F \to F'$ is a natural isomorphism.

Let $\lambda' = \delta \cdot \lambda$ and $\rho' = \rho \cdot \delta^{-1}$. It remains to show that (F', λ', ρ') determines the same weak factorisation system as (F, λ, ρ). By the definition of a functorial realisation of a weak factorisation system, it suffices to show that λ' and ρ' take values in $\mathcal{L}$ and $\mathcal{R}$ respectively. But this follows easily. Given a lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\lambda'_f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

with $g \in \mathcal{R}$, the fact that $\mathcal{L} \leq \mathcal{R}$ gives a lift s for the related problem

$$\begin{tikzcd} \cdot \arrow[rr, "u"] \arrow[d, "\lambda_f"'] & & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "\delta_f"'] \arrow[urr, dashed, "s"] & \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

Hence, $s \cdot \delta_f^{-1}$ is a solution to the original lifting problem, and λ'_f has the left lifting property with respect to all of $\mathcal{R}$ and is thus in $\mathcal{L}$. Similarly, ρ' takes values in $\mathcal{R}$, completing the proof. □

Corollary 4.15. *Let (F, λ, ρ) be a functorial realisation of a weak factorisation system $(\mathcal{L}, \mathcal{R})$ such that there exists a natural isomorphism $\gamma : FE \to 1_{\mathcal{K}}$. Suppose further that $\gamma_A \cdot \lambda_{1_A} = 1_A = \rho_{1_A} \cdot \gamma_A^{-1}$ for all $A \in \mathrm{ob}\,\mathcal{K}$. Then*

(1) $F(u, v)$ is the unique morphism k such that both squares of (16) commute.

(2) $\mathcal{L}$ is closed under all colimits and $\mathcal{R}$ is closed under all limits.

Proof. It suffices to prove the first statement as the proof of Theorem 4.13 will then apply in this case. By Lemma 4.14, we may construct a functor $F' : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$, naturally isomorphic to F, such that $F'E = 1_{\mathcal{K}}$. The extra condition on λ and ρ ensures that $\lambda'_E = \rho'_E = 1_{\mathcal{K}}$ as well. Hence, given a commutative square (10), Theorem 4.8 implies that $F'(u, v)$ is the unique morphism such that

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\lambda'_f"'] & \cdot \arrow[d, "\lambda'_g"] \\ \cdot \arrow[r, "F'(u,v)"] \arrow[d, "\rho'_f"'] & \cdot \arrow[d, "\rho'_g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

commutes. Suppose k is a morphism such that both squares of (16) commute. Then

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\delta_f \cdot \lambda_f = \lambda'_f"'] & \cdot \arrow[d, "\lambda'_g = \delta_g \cdot \lambda_g"] \\ \cdot \arrow[r, "\delta_g \cdot k \cdot \delta_f^{-1}"] \arrow[d, "\rho_f \cdot \delta_f^{-1} = \rho'_f"'] & \cdot \arrow[d, "\rho'_g = \rho_g \cdot \delta_g^{-1}"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

commutes, so by Theorem 4.8,

$$\delta_g \cdot k \cdot \delta_f^{-1} = F'(u, v) = \delta_g \cdot F(u, v) \cdot \delta_f^{-1}.$$

Of course, δ_f and δ_g are isos, so it follows that $F(u, v) = k$ as claimed. □

4.5 Natural Weak Factorisation Systems

While it is reasonable to expect that a generic weak factorisation system gives a well-behaved factorisation of identity morphisms, this is not often the case for model categories, where cofibrations and fibrations are defined with regard to the category's more complicated structure. Of course, well-behaved factorisations of identity morphisms into identity morphisms, which are simultaneously trivial cofibrations and trivial fibrations, always exist, but these are likely not provided by the functorial factorisation, as is required by Theorem 4.8. Indeed, this is one of the reasons why functorial factorisations are not required in the standard definition of a model category; often, "simpler" non-functorial factorisations are preferable.

Nonetheless, some model categories such as $\mathbf{Ch}_R$ and $\mathbf{Cat}$ have a rich algebraic structure that turns the functors α and γ into comonads and β and δ into monads. It turns out that this additional structure provides the algebraic data necessary to make the lifts given by Axiom 3 *natural*, which in turn implies that the classes $\mathcal{C}$, $\mathcal{C} \cap \mathcal{W}$, $\mathcal{F}$, and $\mathcal{F} \cap \mathcal{W}$ are closed under all colimits and limits, respectively. In this section, we prove this fact in the general setting of functorial weak factorisation systems.

Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system with functorial realisation (F, λ, ρ) and consider a $\mathcal{K}^{\mathbf{2}}$-morphism (10). Recall that the natural transformation λ is equivalently described by a functor $L : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ satisfying

$$\mathrm{dom} L = \mathrm{dom}, \qquad \mathrm{cod} L = F, \qquad \kappa L = \lambda.$$

Then ρ can be encoded in a natural transformation $\epsilon : L \to 1_{\mathcal{K}^{\mathbf{2}}}$ given by the following diagram:

$$\begin{tikzcd}
\cdot \arrow[rr, "1"] \arrow[dr, "u"] \arrow[dd, "\lambda_f"'] & & \cdot \arrow[dr, "u"] \arrow[dd, "f"' near start] & \\
& \cdot \arrow[rr, "1" near start] \arrow[dd, "\lambda_g" near start] & & \cdot \arrow[dd, "g"] \\
\cdot \arrow[rr, "\rho_f" near end] \arrow[dr, "F(u,v)"'] & & \cdot \arrow[dr, "v"] & \\
& \cdot \arrow[rr, "\rho_g"'] & & \cdot
\end{tikzcd}$$

The morphisms ϵ_f and ϵ_g are, respectively, the back and front faces of this cube. The natural transformation $\rho : F \to \mathrm{cod}$ is given by the bottom face.

Similarly, we may describe ρ by a functor $R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ satisfying

$$\mathrm{cod} R = \mathrm{cod}, \qquad \mathrm{dom} R = F, \qquad \kappa R = \rho.$$

Then λ is encoded by a natural transformation $\eta : 1_{\mathcal{K}^{\mathbf{2}}} \to R$ given by the following diagram:

$$\begin{tikzcd}
\cdot \arrow[rr, "\lambda_f"] \arrow[dr, "u"] \arrow[dd, "f"'] & & \cdot \arrow[dr, "F(u,v)"] \arrow[dd, "\rho_f"' near start] & \\
& \cdot \arrow[rr, "\lambda_g" near start] \arrow[dd, "g" near start] & & \cdot \arrow[dd, "\rho_g"] \\
\cdot \arrow[rr, "1" near end] \arrow[dr, "v"'] & & \cdot \arrow[dr, "v"] & \\
& \cdot \arrow[rr, "1"'] & & \cdot
\end{tikzcd}$$

The morphisms η_f and η_g are, respectively, the back and front faces of this cube. This time, the natural transformation $\lambda : \mathrm{dom} \to F$ is given by the top face.

In order to construct *natural* solutions to lifting problems between elements of $\mathcal{L}$ and $\mathcal{R}$, we need the splittings s and t of Section 4.1 to be given by natural transformations. More precisely, we require natural transformations $\sigma : F \to FL$ and $\tau : FR \to F$ such that

$$\lambda_L = \sigma\lambda, \qquad \rho_L \sigma = 1_F, \qquad \rho_R = \rho\tau, \qquad \text{and} \qquad \tau\lambda_R = 1_F \tag{19}$$

where these equations ensure that σ and τ satisfy

$$\begin{tikzcd}
& \cdot \arrow[rr, "f"] \arrow[dl, "\lambda_L f"'] \arrow[dr, "\lambda_f"] & & \cdot & \\
\cdot \arrow[rr, shift right, "\rho_L f"'] & & \cdot \arrow[ll, dashed, shift right, "\sigma_f"'] \arrow[ur, "\rho_f"] \arrow[rr, shift right, "\lambda_R f"'] & & \cdot \arrow[ul, "\rho_R f"'] \arrow[ll, dashed, shift right, "\tau_f"']
\end{tikzcd}$$

and are such that (14) commutes.

Analogously, σ can be encoded by a natural transformation $\delta : L \to L^2$ given by the diagram

$$\begin{tikzcd}
\cdot \arrow[rr, "1"] \arrow[dr, "u"] \arrow[dd, "\lambda_f"'] & & \cdot \arrow[dr, "u"] \arrow[dd, "\lambda_{\lambda_f}"' near start] & \\
& \cdot \arrow[rr, "1" near start] \arrow[dd, "\lambda_g"' near start] & & \cdot \arrow[dd, "\lambda_{\lambda_g}"] \\
\cdot \arrow[rr, "\sigma_f"' near end] \arrow[dr, "F(u,v)"'] & & \cdot \arrow[dr, "F(u{,}F(u{,}v))"] & \\
& \cdot \arrow[rr, "\sigma_g"'] & & \cdot
\end{tikzcd}$$

where we assume also that $\epsilon_L\delta = 1_L$ to ensure that $\rho_L\sigma = 1_F$. The morphisms δ_f and δ_g are, respectively, the back and front faces of the cube, and the natural transformation $\sigma : F \to FL$ is given by the bottom face.

Likewise, τ can be encoded by a natural transformation $\mu : R^2 \to R$ given by the diagram

$$\begin{tikzcd}
\cdot \arrow[rr, "\tau_f"] \arrow[dr, "F(F(u{,}v){,}v)"] \arrow[dd, "\rho_{\rho_f}"'] & & \cdot \arrow[dr, "F(u{,}v)"] \arrow[dd, "\rho_f"' near start] & \\
& \cdot \arrow[rr, "\tau_g" near start] \arrow[dd, "\rho_{\rho_g}"' near start] & & \cdot \arrow[dd, "\rho_g"] \\
\cdot \arrow[rr, "1"' near end] \arrow[dr, "v"'] & & \cdot \arrow[dr, "v"] & \\
& \cdot \arrow[rr, "1"'] & & \cdot
\end{tikzcd}$$

where again we also assume that $\mu\eta_R = 1_R$ to ensure that $\tau\lambda_R = 1_F$. The morphisms μ_f and μ_g are, respectively, the back and front faces of the cube, and the natural transformation $\tau : FR \to F$ is given by the top face.

At this point, it seems natural to assume that the triples (L, ϵ, δ) and (R, η, μ) form a comonad and a monad on $\mathcal{K}^{\mathbf{2}}$, respectively. In fact, the particular properties of these triples that describe their behavior with respect to the weak factorisation system $(\mathcal{L}, \mathcal{R})$ are captured by requiring that (L, ϵ, δ) and (R, η, μ) are respectively a comonad on the functor dom and a monad on the functor cod in the 2-category $\mathbf{Cat}/\mathcal{K}$.

This leads to the following definition, originally due to Grandis and Tholen [5].

Definition 4.16. A *natural weak factorisation system* in a category $\mathcal{K}$ is a pair $(\mathbb{L}, \mathbb{R})$ such that
(1) $\mathbb{L} = (L, \epsilon, \delta)$ is a comonad on dom in $\mathbf{Cat}/\mathcal{K}$,
(2) $\mathbb{R} = (R, \eta, \mu)$ is a monad on cod in $\mathbf{Cat}/\mathcal{K}$,
(3) $\operatorname{cod} L = \operatorname{dom} R$, $\operatorname{cod} \epsilon = \kappa_R$, $\operatorname{dom} \eta = \kappa_L$.

Example 4.17. The model categories $\mathbf{Ch}_R$ and $\mathbf{Cat}$ described in Sections 3.3 and 3.4 give rise to natural weak factorisation systems for both types of factorisation. Proving this fact is a useful exercise for the reader interested in developing a greater familiarity with these notions.

In a natural weak factorisation system, the natural transformations σ and τ are given by the algebraic data associated to morphisms in the Eilenberg-Moore categories of $\mathbb{L}$ and $\mathbb{R}$. Recall the following definition.

Definition 4.18. The *Eilenberg-Moore category* $\mathcal{K}^T$ of a monad (T, η, μ) on a category $\mathcal{K}$ has objects (A, α), where $A \in \operatorname{ob} \mathcal{K}$ and $\alpha : TA \to A \in \operatorname{mor} \mathcal{K}$ such that

$$\begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "1"] & \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

commute. Morphisms $h : (A, \alpha) \to (B, \beta)$ in $\mathcal{K}^T$ are arrows $h : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "Th"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

commutes.

Define functors $U : \mathcal{K}^T \to \mathcal{K}$ by $U(A, \alpha) = A$ and $Uh = h$ and $G : \mathcal{K} \to \mathcal{K}^T$ by $GA = (TA, \mu_A)$ and $Gh = Th$. One can show that G is left adjoint to U and this adjunction gives rise to the monad T (see, e.g., [1, pp 229-232]). A dual construction exists for comonads.

Returning to the case at hand, let $\mathcal{K}^{\mathbb{L}}$ and $\mathcal{K}^{\mathbb{R}}$ be the Eilenberg-Moore categories for the comonad $\mathbb{L}$ and the monad $\mathbb{R}$ respectively. An object of $\mathcal{K}^{\mathbb{L}}$ is a pair

$$(f, (i, s) : f \to Lf) \in \operatorname{ob} \mathcal{K}^{\mathbf{2}} \times \operatorname{mor} \mathcal{K}^{\mathbf{2}}.$$

By definition, $\epsilon_f \cdot (i, s) = 1_f$, so i must be the identity on $\operatorname{dom} f$. Hence,

$$\operatorname{ob} \mathcal{K}^{\mathbb{L}} = \{(f, s) \mid \lambda_f = s \cdot f, \rho_f \cdot s = 1, \sigma_f \cdot s = F(1, s) \cdot s\}.$$

A morphism $(u, v) : (f, s) \to (g, p)$ in $\mathcal{K}^{\mathbb{L}}$ is a morphism $(u, v) : f \to g$ in $\mathcal{K}^{\mathbf{2}}$ such that $p \cdot v = F(u, v) \cdot s$; i.e., such that the bottom dotted arrow square of

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "\lambda_f"'] & \cdot \arrow[d, "\lambda_g"] \\
\cdot \arrow[r, "F(u,v)"] \arrow[d, shift right, "\rho_f"'] & \cdot \arrow[d, shift left, "\rho_g"] \\
\cdot \arrow[r, "v"'] \arrow[u, dashed, shift right, "s"'] & \cdot \arrow[u, dashed, shift left, "p"]
\end{tikzcd}$$

commutes. Hence, $(f, s) \in \operatorname{ob} \mathcal{K}^{\mathbb{L}}$ if and only if s is a lift of f in the sense of (12) and if

$$(1, s) : (f, s) \to (\lambda_f, \sigma_f)$$

is a morphism in $\mathcal{K}^{\mathbb{L}}$.

Likewise,

$$\operatorname{ob} \mathcal{K}^{\mathbb{R}} = \{(f, t) \mid \rho_f = f \cdot t, t \cdot \lambda_f = 1, t \cdot \tau_f = t \cdot F(t, 1)\}$$

and morphisms $(u, v) : (f, t) \to (g, q)$ are morphisms $(u, v) : f \to g$ in $\mathcal{K}^{\mathbf{2}}$ such that $u \cdot t = q \cdot F(u, v)$. So, $(f, t) \in \operatorname{ob} \mathcal{K}^{\mathbb{R}}$ if and only if t is a lift of f in the sense of (13) and if $(t, 1) : \rho_f \to f$ is a morphism in $\mathcal{K}^{\mathbb{R}}$.

Suppose we have a functorial weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{K}$ that can be described as a natural weak factorisation system $(\mathbb{L}, \mathbb{R})$. Then, objects of $\mathcal{K}^{\mathbb{L}}$ are morphisms of $\mathcal{L}_F$ that come with a given splitting s, and likewise, objects of $\mathcal{K}^{\mathbb{R}}$ are morphisms of $\mathcal{R}_F$ that come with a given splitting t. With this construction, we have proven the following theorem.

Theorem 4.19. *Let $(\mathbb{L}, \mathbb{R})$ be a natural weak factorisation system of $\mathcal{K}$. Then, in the notation of this section, every morphism f factorises as $f = \rho_f \cdot \lambda_f$ with $(\lambda_f, \sigma_f) \in \mathcal{K}^{\mathbb{L}}$ and $(\rho_f, \tau_f) \in \mathcal{K}^{\mathbb{R}}$. Furthermore, for all $(f, s) \in \mathcal{K}^{\mathbb{L}}$, $(g, t) \in \mathcal{K}^{\mathbb{R}}$, and $(u, v) : f \to g$, there is a naturally chosen diagonal morphism w such that*

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd}$$

commutes; namely, $w = t \cdot F(u, v) \cdot s$.

In a sense, we have not accomplished much because these lifts existed from the beginning and naturality only follows because we have taken the lifting data to be a part of the objects in the categories $\mathcal{K}^{\mathbb{L}}$ and $\mathcal{K}^{\mathbb{R}}$. However, these categories do have nice closure properties regarding limits and isomorphisms.

Theorem 4.20. *If $\mathcal{K}$ has colimits of a given type, then $\mathcal{K}^{\mathbb{L}}$ has them, formed as in $\mathcal{K}^{\mathbf{2}}$. Dually, if $\mathcal{K}$ has limits of a given type, then $\mathcal{K}^{\mathbb{R}}$ has them.*

Proof. The forgetful functors $U_L : \mathcal{K}^{\mathbb{L}} \to \mathcal{K}^{\mathbf{2}}$ and $U_R : \mathcal{K}^{\mathbb{R}} \to \mathcal{K}^{\mathbf{2}}$ create colimits and limits, respectively. More generally, any category of coalgebras for a comonad on $\mathcal{K}^{\mathbf{2}}$ is closed under colimits, and dually any category of algebras for a monad on $\mathcal{K}^{\mathbf{2}}$ is closed under limits. $\square$

Proposition 4.21. *The categories $\mathcal{K}^{\mathbb{L}}$ and $\mathcal{K}^{\mathbb{R}}$ contain the isomorphisms of $\mathcal{K}$, in that every isomorphism f can be equipped with a unique coalgebra and algebra structure.*

Proof. Given an iso f in $\mathcal{K}$, let $s = \lambda_f \cdot f^{-1}$ and $t = f^{-1} \cdot \rho_f$. Then it is easy to check that $(f, s) \in \operatorname{ob} \mathcal{K}^{\mathbb{L}}$ and $(f, t) \in \operatorname{ob} \mathcal{K}^{\mathbb{R}}$. Conversely, if (f, s) is a coalgebra, then the Eilenberg-Moore axioms imply that $s \cdot f = \lambda_f$, so $s = \lambda_f \cdot f^{-1}$. A similar argument shows that t is unique. $\square$

Remark 4.22. Note, however, that unlike weak factorisation systems, it is not clear that $\mathcal{K}^{\mathbb{L}}$ and $\mathcal{K}^{\mathbb{R}}$ are closed under retracts. Suppose $(f, s) \in \mathcal{K}^{\mathbb{L}}$ and that we are given a retract diagram

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "g"'] & \cdot \arrow[r, "p"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \arrow[r, "q"'] & \cdot \end{tikzcd}$$

The only logical lift for g is $r = F(p, q) \cdot s \cdot v$ and indeed this satisfies the lifting conditions of (12). However, there is no reason why $(1, r) : (g, r) \to (\lambda_g, \sigma_g)$ must be a morphism of $\mathcal{K}^{\mathbb{L}}$. Garner explores this problem further, introducing a looser notion of *retract equalisers* to describe the instances when $(1, r)$ is $\mathcal{K}^{\mathbb{L}}$ morphism and hence $(g, r) \in \mathcal{K}^{\mathbb{L}}$ (see [3, 40-41]).

5 Further Study

At the commencement of this project, the author had two primary questions in mind:

(1) Is the full comonad/monad structure of a natural weak factorisation system necessary for the classes $(\mathcal{L}, \mathcal{R})$ of functorial weak factorisation system to be closed under limits?

(2) Given the considerable algebraic structure of certain model categories such as $\mathbf{Ch}_R$ and $\mathbf{Cat}$, is there some relationship between the two factorisations of a weak equivalence given by Axiom 4?

Theorem 4.13 provides an answer to (1) though, unfortunately, not one that is applicable for most model categories. However, (2) remains unanswered, though I remain optimistic that a thorough examination of the examples provided by the model categories $\mathbf{Ch}_R$ and $\mathbf{Cat}$ will provide some insight into this question.

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