FACTORIZATION SYSTEMS

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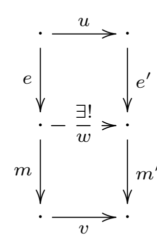
Abstract. These notes were written to accompany a talk given in the Algebraic Topology and Category Theory Proseminar in Fall 2008 at the University of Chicago. We first introduce orthogonal factorization systems, give a few examples, and prove some basic theorems. Next, we turn to weak factorization systems, which play an important role in the theory of model categories, a connection which we make explicit. We discuss what it means for a weak factorization system to be functorial and observe that functoriality does not guarantee the existence of natural lifts. This leads us, naturally one might say, to the definition of a natural weak factorization system, which is where we conclude these notes. The reader is assumed to have some familiarity with category theory — functors, limits and colimits, naturality, monads and comonads, comonoids, 2-categories, and some basic categorical terminology; [9] is a good reference for any concepts that may be unfamiliar.

1. Orthogonal Factorization Systems

Definition 1.1. An *orthogonal factorization system* in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of distinguished class of morphisms such that

(I) $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms and are closed under composition.
(II) Every $f \in \operatorname{mor} \mathcal{K}$ can be factored as $f = me$ with $e \in \mathcal{L}$ and $m \in \mathcal{R}$.
(III) This factorization is *functorial*, i.e., given the solid diagram

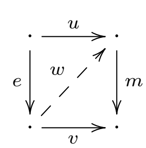
(1.2)



there exists a unique arrow w such that both squares commute.

Definition 1.3. Given classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ we say that $\mathcal{E}$ is *orthogonal* to $\mathcal{M}$ and write $\mathcal{E} \perp \mathcal{M}$ if for all commutative squares

(1.4)



with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there exists a unique arrow w making both triangles commute. We will refer to such squares as *lifting problems* and the arrow w as a

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solution. In this language, we write $\mathcal{E} \perp \mathcal{M}$ if every lifting problem with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ has a unique solution.

Remark 1.5. Let $\mathcal{K}$ be locally small and let $e : A \to B$ and $m : X \to Y$. The condition $e \perp m$ means that the square

$$\begin{array}{ccc} \mathrm{Hom}(B, X) & \xrightarrow{e^*} & \mathrm{Hom}(A, X) \\ {\scriptstyle m_*}\downarrow & & \downarrow{\scriptstyle m_*} \\ \mathrm{Hom}(B, Y) & \xrightarrow[e^*]{} & \mathrm{Hom}(A, Y) \end{array} \tag{1.6}$$

is cartesian (i.e., is a pullback in **Set**).

Conditions (I) and (III) above imply that for any orthogonal factorization system $(\mathcal{L}, \mathcal{R})$:

(IV) $\mathcal{L} \perp \mathcal{R}$.

It follows easily from (IV) that the factorizations guaranteed by (II) are unique up to unique isomorphism.

Definition 1.7. For any class of morphisms $\mathcal{E}$, let

$$\mathcal{E}^{\perp} = \{m \in \mathrm{mor}\,\mathcal{K} \mid \mathcal{E} \perp m\}.$$

Dually, let

$${}^{\perp}\mathcal{E} = \{m \in \mathrm{mor}\,\mathcal{K} \mid m \perp \mathcal{E}\}.$$

Note that

$$\mathcal{E} \subset {}^{\perp}\mathcal{M} \iff \mathcal{E} \perp \mathcal{M} \iff \mathcal{E}^{\perp} \supset \mathcal{M}.$$

Lemma 1.8. *A pair of distinguished classes of morphisms $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorization system if and only if the following conditions are satisfied:*

(II) *Every $f \in mor\mathcal{K}$ can be factored as $f = me$ with $e \in \mathcal{L}$ and $m \in \mathcal{R}$.*

(V) $\mathcal{L} = {}^{\perp}\mathcal{R}$ *and* $\mathcal{R} = \mathcal{L}^{\perp}$.

Proof. (I), (II), (III) $\Rightarrow$ (V): We already remarked that (I) and (III) imply (IV), so $\mathcal{R} \subset \mathcal{L}^{\perp}$. Suppose $f \in \mathcal{L}^{\perp}$ and use (II) to write $f = me$ with $e \in \mathcal{L}$ and $m \in \mathcal{R}$. The lifting problem

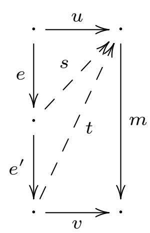
$$\begin{array}{ccc} \cdot & = & \cdot \\ {\scriptstyle e}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[m]{} & \cdot \end{array}$$

has a solution s such that $se = 1$. It follows that es and 1 are both solutions to the lifting problem

$$\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle m} \\ \cdot & \xrightarrow[m]{} & \cdot \end{array}$$

Uniqueness implies that these solutions are equal. So e is an isomorphism and by (I), $e \in \mathcal{R}$; hence, $f = me \in \mathcal{R}$ because (I) also says that $\mathcal{R}$ is closed under composition. The proof that $\mathcal{L} \subset {}^{\perp}\mathcal{R}$ follows dually.

(II), (V) $\Rightarrow$ (I), (III): The statement (V) is quite strong and implies (I) and (III) by itself. (V) $\Rightarrow$ (III) is obvious because a lifting problem (1.2) give rise to a lifting problem of the form (1.4) by composing u and e' and m and v. As there is always a unique solution to a lifting problem if either the left or right arrows are invertible, (V) clearly implies that $\mathcal{L}$ and $\mathcal{R}$ contain the isomorphisms. To show that $\mathcal{L}$ is closed under composition, suppose $e, e' \in \mathcal{L}$ and consider a lifting problem



As $e \in \mathcal{L} = {}^{\perp}\mathcal{R}$, there is a unique lift s as shown, which gives rise to a lifting problem with e' and m. This problem has a unique solution t, which clearly makes the outer triangles commute. Suppose t' were another solution to the lifting problem with $e'e$ and m. Then $t'e'$ and te' both solve the lifting problem between e and m, so we must have $t'e' = s = te'$. But then t and t' are both solutions to the lifting problem between e' and m, and thus orthogonality implies that $t = t'$. So $e'e \perp m$, and (V) implies that $e'e \in \mathcal{L}$, which is therefore closed under composition. The proof for $\mathcal{R}$ is dual. □

This lemma gives an alternative definition of an orthogonal factorization system, which we record now for easy reference. An obvious corollary is that each class of an orthogonal factorization system determines the other.

Definition 1.9. An *orthogonal factorization system* in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of distinguished class of morphisms such that

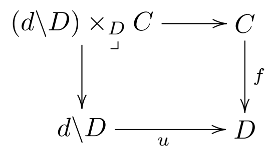
(II) Every $f \in \operatorname{mor} \mathcal{K}$ can be factored as $f = me$ with $e \in \mathcal{L}$ and $m \in \mathcal{R}$.
(V) $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.

By now some examples are perhaps overdue.

Example 1.10. The classes $\mathcal{L}$ of epimorphisms and $\mathcal{R}$ of monomorphisms form an orthogonal factorization system in **Set** and also in other categories. In **Top**, we take $\mathcal{L}$ to be the quotient maps and $\mathcal{R}$ to be continuous inclusions.

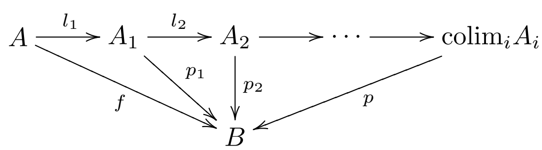
The classes of an orthogonal factorization system are often denoted by $(\mathcal{E}, \mathcal{M})$ in the literature, which I suspect is due to a recognition of this example. Orthogonal factorization systems are somtimes called E-M factorization systems, a term which in [7] serves as an abbreviation for *Eilenberg-Moore* factorization systems. These are defined as functorial weak factorization systems with a particular condition about the factorization of trivial squares; the authors prove that this definition is equivalent to that of an orthogonal factorization system, correcting any potential historical discrepancy. We will say more about this later.

Example 1.11. A functor $p : E \to B$ is a *discrete fibration* if for each $e \in \operatorname{ob} E$ and $g \in \operatorname{mor} B$ with codomain $p(e)$, there is a unique $f \in \operatorname{mor} E$ with codomain e such that $p(f) = g$. A functor $f : C \to D$ is *final* if for all $d \in D$, the category defined by the pullback



is connected. **Cat** has an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is the class of final functors and $\mathcal{R}$ is the class of discrete fibrations.

Example 1.12. (Due to [6].) **Cat** has another orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ where $\mathcal{R}$ is the class of conservative functors, i.e., functors which reflect isomorphisms. The class $\mathcal{L}$ consists of *iterated strict localizations*. For any set $S \subset \operatorname{mor} A$, where A is a small category, there is a functor $A \to S^{-1}A$ that inverts S universally, which we call a *strict localization* of A at S. Given any $f : A \to B$, there is a factorization $f = p_1 l_1$ where l_1 is a strict localization of the morphisms inverted by f. The functor p_1 may not be conservative, but when we iterate this process and take the colimit, the functor p induced by the resulting cone from the colimit to B *is* conservative. The functor from A to the colimit is an iterated strict localization, and this construction gives the functorial factorization of f.



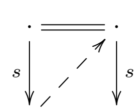
Lemma 1.13. *If* $(\mathcal{L}, \mathcal{R})$ *is an orthogonal factorization system, then* $\mathcal{L} \cap \mathcal{R}$ *is the class of isomorphisms of* $\mathcal{K}$*. Furthermore,* $\mathcal{L}$ *has the* right cancellation property*:*

$$gf \in \mathcal{L} \text{ and } f \in \mathcal{L} \Rightarrow g \in \mathcal{L}.$$

Dually, $\mathcal{R}$ *has the* left cancellation property*:*

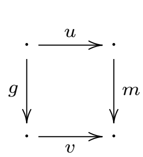
$$gf \in \mathcal{R} \text{ and } g \in \mathcal{R} \Rightarrow f \in \mathcal{R}.$$

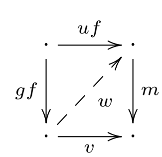
Proof. If $s \in \mathcal{L} \cap \mathcal{R}$ then the lifting problem



defines its inverse.

Suppose $gf \in \mathcal{L}$ and $f \in \mathcal{L}$ and let $m \in \mathcal{R}$. A lifting problem

(1.14) 

gives rise to a lifting problem 

which has a unique solution w. Then wg and u are both solutions to the lifting problem

$$\begin{array}{ccc} \cdot & \xrightarrow{uf} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m} \\ \cdot & \xrightarrow[vg]{} & \cdot \end{array}$$

so these must be equal and w is a solution to (1.14). Thus $g \in {}^{\perp}\mathcal{R} = \mathcal{L}$. The proof that $\mathcal{R}$ has the left cancellation property is dual. □

Let $\mathbf{2}$ be the category consisting of two objects and one non-identity arrow:

$$\bullet \to \bullet$$

The functor category $\mathcal{K}^{\mathbf{2}}$ has morphisms of $\mathcal{K}$ as objects; arrows $(u, v) : f \to g$ are commutative squares

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \tag{1.15}$$

This is sometimes called the *arrow category* of $\mathcal{K}$, for obvious reasons. This category is equipped with functors

$$\mathrm{dom} : \mathcal{K}^{\mathbf{2}} \to \mathcal{K} \quad \text{and} \quad \mathrm{cod} : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$$

that project to the domain and codomain respectively. There is a natural transformation $\kappa : \mathrm{dom} \to \mathrm{cod}$ such that $\kappa_f = f$ for all $f \in \mathrm{mor}\,\mathcal{K}$.

The category $\mathbf{2}$ is a comonoid $(\mathbf{2}, \delta, \epsilon)$ in $\mathbf{CAT}$ with respect to the cartesian closed structure (as is any small category). The functor $\delta : \mathbf{2} \to \mathbf{2} \times \mathbf{2}$ is the diagonal and $\epsilon : \mathbf{2} \to \mathbf{1}$ is the unique functor to the terminal object. The internal-hom of $\mathbf{CAT}$ is a 2-functor $\mathbf{CAT}^{\mathrm{op}} \times \mathbf{CAT} \to \mathbf{CAT}$, which gives rise to a 2-functor

$$\Psi : \mathbf{CAT}^{\mathrm{op}} \to [\mathbf{CAT}, \mathbf{CAT}]$$

because $\mathbf{CAT}$ is cartesian closed. We write $(-)^A$ for $\Psi(A)$, if A is a small category; this is consistent with our notation for the arrow category above. Together, the 2-functor Ψ and the comonoid $(\mathbf{2}, \delta, \epsilon)$ give rise to a 2-monad

$$(\Phi, \mu, \eta) = (\Psi(\mathbf{2}), \Psi(\delta), \Psi(\epsilon))$$

on $\mathbf{CAT}$ called the *squaring monad* because the functor Φ takes a category A to the category $A^{\mathbf{2}}$.

Korostenski and Tholen prove the following theorem in [7].

Theorem 1.16. *Orthogonal factorization systems in small categories are equivalently described by the normal pseudo-algebras with respect to the squaring 2-monad* (Φ, μ, η) *on* $\mathbf{CAT}$.

The adjective pseudo means that the associativity diagrams are only required to commute up to isomorphism, while normal means that the unit axiom holds strictly.

Theorem 1.17. *The class* ${}^{\perp}\mathcal{M}$ *is closed under colimits (taken in* $\mathcal{K}^{\mathbf{2}}$*) for any class of maps* $\mathcal{M}$ *in a category* $\mathcal{K}$*. Hence, the left class of an orthogonal factorization system is closed under colimits and dually the right class is closed under limits.*

Proof. It is possible to verify directly that ${}^{\perp}\mathcal{M}$ is closed under coequalizers and arbitrary coproducts; this would be a good exercise. A proof of the latter is given below for a class of maps that satisfy a weaker left lifting property. By contrast, the proof that ${}^{\perp}\mathcal{M}$ is closed under coequalizers depends explicitly on the natural choice of lifts (1.4).

Here we give a slick proof due to [6] of the dual statement (when $\mathcal{K}$ is locally small): that $\mathcal{M}^{\perp}$ is closed under limits. For any arrows $e : X \to Y$ and $m : Z \to W$, there is a commutive square (1.6). This yields a functor

$$S : (\mathcal{K}^{\mathrm{op}})^{\mathbf{2}} \times \mathcal{K}^{\mathbf{2}} \to \mathbf{Set}^{\mathbf{2}\times\mathbf{2}}$$

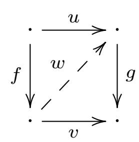
that is continuous in each variable. An arrow m belongs to $\mathcal{M}^{\perp}$ if and only if the square $S(e, m)$ is cartesian for each $e \in \mathcal{M}$. The full subcategory of $\mathbf{Set}^{\mathbf{2}\times\mathbf{2}}$ spanned by the cartesian squares is closed under limits, completing the proof. □

This is part of what is meant when one says that the left and right classes of an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ have *good stability properties*. These classes have other stability properties as well; these hold more generally for *weak* factorization systems, which we now introduce.

2. Weak Factorization Systems

Let $\mathcal{K}$ be a category. A morphism f has the *left lifting property* with respect to a morphism g, or equivalently, g has the *right lifting property* with respect to f if every lifting problem

(2.1)



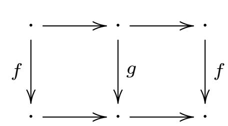
has a solution w, not necessarily unique, such that both triangles commute. We denote this by $f \boxslash g$.

As before, let $\mathcal{E}^{\boxslash}$ be the class of arrows with the right lifting property with respect to each $e \in \mathcal{E}$ and let ${}^{\boxslash}\mathcal{E}$ be the class of arrows with the left lifting property with respect to each $e \in \mathcal{E}$. Write $\mathcal{E} \boxslash \mathcal{M}$ to indicate that $\mathcal{E} \subset {}^{\boxslash}\mathcal{M}$ or equivalently that $\mathcal{M} \subset \mathcal{E}^{\boxslash}$.

Definition 2.2. A *weak factorization system* in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of distinguished classes of morphisms such that

(1) Every $h \in \operatorname{mor} \mathcal{K}$ can be factored as $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.
(2) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Recall that a morphism f is a *retract* of g if there is a commutative diagram

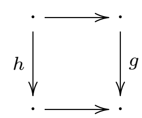


with horizontal composites the identity.

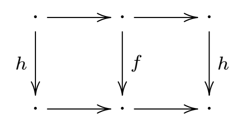
Condition (2) implies:

(3) $\mathcal{L} \boxslash \mathcal{R}$.
(4) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

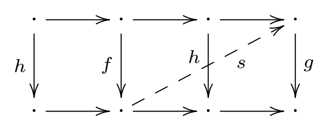
The first statement is obvious. For the second, let h be a retract of $f \in \mathcal{L}$ and let $g \in \mathcal{R}$. Given a lifting problem



and a retract diagram



with horizontal composites the identity. Combining these diagrams, we obtain an expanded picture of the same lifting problem with an obvious solution:

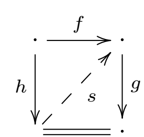


In the above, s is the lift provided by the fact that f has the left lifting property with respect to g. Thus, h has the left lifting property with respect to g, and so $h \in \mathcal{L} = {}^{\boxslash}\mathcal{R}$, proving that $\mathcal{L}$ is closed under retracts. The proof for $\mathcal{R}$ is similar.

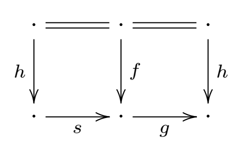
Conversely, (2) follows from (1), (3), and (4) by what is called the *retract argument*, giving an alternative definition of a weak factorization system.

Lemma 2.3. *If* $(\mathcal{L}, \mathcal{R})$ *is a pair of classes of morphisms satisfying* (1), (3), *and* (4), *then* $(\mathcal{L}, \mathcal{R})$ *is a weak factorization system.*

Proof. By (3), $\mathcal{L} \subset {}^{\boxslash}\mathcal{R}$, so it remains to show the reverse inclusion. Given $h \in {}^{\boxslash}\mathcal{R}$ use (1) to write $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$. The lifting problem



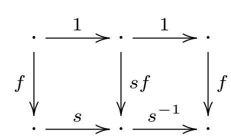
has a solution s, which we use to express h as a retract of f



By (4), $h \in \mathcal{L}$. The dual proof shows that $\mathcal{L}^{\boxslash} = \mathcal{R}$. □

Condition (4) implies:

(5) If s is a split monic and $sf \in \mathcal{L}$, then $f \in \mathcal{L}$. This follows from the retract diagram:



Dually if t is a split epic and $gt \in \mathcal{R}$, then $g \in \mathcal{R}$.

Condition (5) is the best analog of the cancellation properties of Lemma 1.13 for weak factorization systems.

Lemma 2.4. *If* $(\mathcal{L}, \mathcal{R})$ *is a pair of classes of morphisms satisfying* (1), (3), *and* (5), *then* $(\mathcal{L}, \mathcal{R})$ *is a weak factorization system.*

Proof. Again, we must show that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$; $\mathcal{R} = \mathcal{L}^{\boxslash}$ follows dually. By (3), $\mathcal{L} \subset {}^{\boxslash}\mathcal{R}$. Given $h \in {}^{\boxslash}\mathcal{R}$, write $h = gf$ as in the proof of (2.3) and use the same lifting problem to define s. Commutativity of the upper triangle tells us that $sh = f \in \mathcal{L}$ and commutativity of the lower triangle says that s is split monic. So $h \in \mathcal{L}$ by (5). □

For convenience, we record the results of this discussion in the following definition.

Definition 2.5. A *weak factorization system* in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of distinguished classes of morphisms such that

(1) Every $h \in \operatorname{mor} \mathcal{K}$ can be factored as $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.
(3) $\mathcal{L} \boxslash \mathcal{R}$.

and either of the following conditions hold

(4) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.
(5) If s is split monic and $sf \in \mathcal{L}$ then $f \in \mathcal{L}$, and if t is split epic and $gt \in \mathcal{R}$ then $g \in \mathcal{R}$.

Model categories provide numerous examples of weak factorization systems, though we defer the task of making this connection explicit until the next section. For now, we give a simple example that provides a nice comparison with Example 1.10.

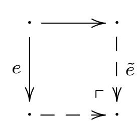
Example 2.6. The classes $\mathcal{L}$ of monomorphisms and $\mathcal{R}$ of epimorphisms form a weak factorization system in **Set**. This is not an orthogonal factorization system because solutions to the lifting problem (2.1) are no longer unique when the positions of the epimorphism and the monomorphism are reversed.

Example 2.7. Any orthogonal factorization system is a weak factorization system, as is most easily seen by comparing Definition 1.9 with Definition 2.2.

The classes $\mathcal{L}$ and $\mathcal{R}$ of a weak factorization system still have several nice closure properties, which are also shared by orthogonal factorization systems.

Definition 2.8. Let $\mathcal{K}$ be a cocomplete category. A class of morphisms $\mathcal{E}$ in $\mathcal{K}$ is called *saturated* if

(i) $\mathcal{E}$ is closed under pushouts; i.e., if



is a pushout square with $e \in \mathcal{E}$, then $\tilde{e} \in \mathcal{E}$.
(ii) $\mathcal{E}$ is closed under retracts.
(iii) $\mathcal{E}$ is closed under *transfinite composition*; i.e., if λ is a non-empty ordinal regarded as a category and $X : \lambda \to \mathcal{K}$ is a colimit preserving functor such that for all $\beta < \lambda$ the natural map

$$\operatorname{colim}_{\alpha<\beta} X^\alpha \to X^\beta$$

is in $\mathcal{E}$[1], then the induced map

$$X^0 \to \operatorname{colim}_{\alpha<\lambda} X^\alpha$$

is in $\mathcal{E}$. This morphism is called the *transfinite composite* of X.

[1]Equivalently, such that $X^\beta \to X^{\beta+1}$ is in $\mathcal{E}$ for all $\beta + 1 < \lambda$.

Remark 2.9. Let $\mathcal{E}$ be saturated. Then $\mathcal{E}$ necessarily contains all isomorphisms by property (iii) applied to the ordinal $1 = \{0\}$. $\mathcal{E}$ is also closed under composition, by property (iii) applied to the ordinal $3 = \{0, 1, 2\}$.

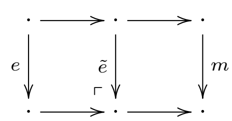
Properties (i) and (iii) together imply that $\mathcal{E}$ is closed under arbitrary coproducts, taken in $\mathcal{K}^{\mathbf{2}}$. Suppose we have a collection of morphisms $e_\alpha : x_\alpha \to y_\alpha$ in $\mathcal{E}$ indexed by the ordinals $\alpha < \lambda$ for some ordinal λ. For each $\beta < \lambda$ define e'_β to be the pushout

$$\begin{array}{ccc} x_\beta & \longrightarrow & (\sqcup_{\alpha<\beta} y_\alpha) \sqcup (\sqcup_{\beta\le\alpha<\lambda} x_\alpha) \\ {\scriptstyle e_\beta}\downarrow & & \downarrow{\scriptstyle e'_\beta} \\ y_\beta & \longrightarrow & (\sqcup_{\alpha\le\beta} y_\alpha) \sqcup (\sqcup_{\beta<\alpha<\lambda} x_\alpha) \end{array}$$

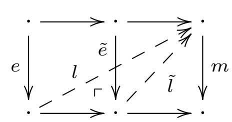
along the obvious inclusion. Then $\sqcup_{\alpha<\lambda} e_\alpha$ is the transfinite composite of the e'_α, and hence lies in $\mathcal{E}$.

Theorem 2.10. *Let $\mathcal{K}$ be a cocomplete category and let $\mathcal{M}$ be a fixed class of maps in $\mathcal{K}$. Let $\mathcal{E} = {}^{\boxslash}\mathcal{M}$. Then $\mathcal{E}$ is saturated.*

Proof. To show that $\mathcal{E}$ satisfies (i), let $\tilde{e}$ be a pushout of $e \in \mathcal{E}$ and consider a lifting problem

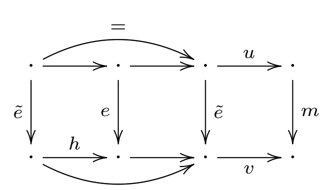


with $m \in \mathcal{M}$. Because $e \boxslash m$, there is a lift l, which induces the solution $\tilde{l}$ to our lifting problem by the universal property of the pushout

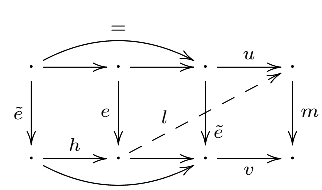


Thus $\tilde{e} \in \mathcal{E} = {}^{\boxslash}\mathcal{M}$.

For (ii), now suppose that $\tilde{e}$ is a retract of e and consider the following lifting problem

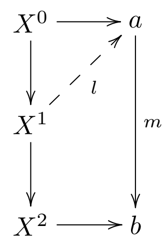


with $m \in \mathcal{M}$ as above. Because $e \boxslash m$, there is a lift l as shown

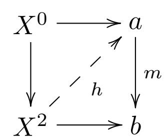


and the composite lh is the solution to our lifting problem given by u and v. Thus $\mathcal{E}$ satisfies (ii).[2]

For (iii), we use transfinite induction. The base case for the ordinal 1 is trivial because isomorphisms satisfy the left lifting property with respect to any map, but it is instructive to see how this works for the ordinal 2, which we regard as the category $\mathbf{2}$ with two objects and a unique non-identity arrow: $0 \to 1$. Suppose $X : \mathbf{2} \to \mathcal{K}$ is a functor such that the map $X^0 \to X^1$ is in $\mathcal{E}$. Let X^2 be the colimit of $X^0 \to X^1$; we wish to show that the composite $X^0 \to X^2$ has the left lifting property with respect to $\mathcal{M}$. Given $m \in \mathcal{M}$ and a lifting problem as shown, we factor $X^0 \to X^2$ and use the lifting property of $X^0 \to X^1$ with respect to m to define the map l:

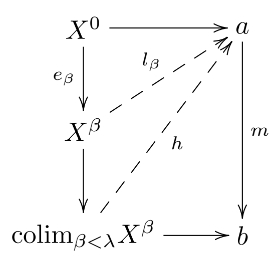


Now a is the summit of a cone under $X^0 \to X^1$, so there is a map $h : X^2 \to a$ defined by the universal property of the colimit X^2 such that the upper triangle of



commutes. The lower triangle commutes by the uniqueness of this universal property, since ml and $X^1 \to X^2 \to b$ define the same cone with summit b. Thus h is the desired lift and $X^0 \to X^2$ is in $\mathcal{E}$.

For the inductive hypothesis, suppose we have shown that $\mathcal{E}$ is closed under β-composition for all ordinals $\beta < \lambda$. Let $X : \lambda \to \mathcal{K}$ be a colimit preserving functor with each morphism $\mathrm{colim}_{\alpha<\beta} X^\alpha \to X^\beta$ in $\mathcal{E}$. By hypothesis, the composites $e_\beta : X^0 \to X^\beta$ are in $\mathcal{E}$ so there exist lifts $l_\beta : X^\beta \to a$ as shown



which define a cone under the diagram X with summit a. Hence, there is a map h induced by the universal property of the colimit; the proof that h is the desired lift is exactly analogous to the case where $\lambda = 2$. Thus, $\mathcal{E}$ satisfies (iii), which shows

[2]This is the same proof that was given at the beginning of this section. It is repeated here to emphasize that we only need that $\mathcal{E} = {}^{\boxslash}\mathcal{M}$ and not that $\mathcal{E}$ is the left class of a weak factorization system

that any class of maps defined by a left lifting property in a cocomplete category is saturated. ☐

The corollory of course is that if $(\mathcal{L}, \mathcal{R})$ is a weak factorization system, then $\mathcal{L}$ is saturated and $\mathcal{R}$ is closed under retracts, products, pullbacks, and composition. Note that as in the case of orthogonal factorization systems, these closure properties have nothing to do with factorization; the proofs rely only on the lifting properties.

Now that we know that the classes $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms of $\mathcal{K}$, the argument given for the first part of Lemma 1.13 proves the following:

Lemma 2.11. *If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system in a category $\mathcal{K}$, then $\mathcal{L} \cap \mathcal{R}$ is the class of isomorphisms in $\mathcal{K}$.*

3. Functorial Weak Factorization Systems

Many weak factorization systems are functorial, though in the absence of unique lifts, we need to be slightly more careful about what this means. It is instructive to compare the following definition with condition (III) for orthogonal factorization systems.

Definition 3.1. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is *functorial* if there is a pair of functors $L, R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$ such that

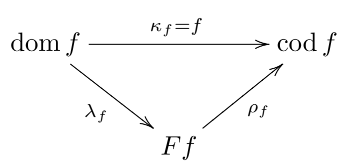
$$\mathrm{dom}L = \mathrm{dom}, \qquad \mathrm{cod}R = \mathrm{cod}, \qquad \mathrm{cod}L = \mathrm{dom}R,$$

and $f = Rf \cdot Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for every $f \in \mathrm{mor}\,\mathcal{K}$.

Remark 3.2. The functors L and R can be equivalently described by the functor

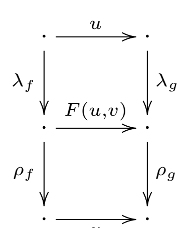
$$F = \mathrm{cod}L = \mathrm{dom}R : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$$

and natural transformations $\lambda : \mathrm{dom} \to F$ and $\rho : F \to \mathrm{cod}$ such that $\kappa = \rho \cdot \lambda$ and $\lambda_f \in \mathcal{L}$ and $\rho_f \in \mathcal{R}$ for all morphisms f; explicitly, the factorization of $f \in \mathrm{mor}\,\mathcal{K}$ is given by



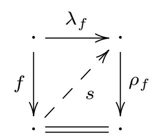
We call the triple (F, λ, ρ) a *functorial realization* for the weak factorization $(\mathcal{L}, \mathcal{R})$. Naturality of λ and ρ implies that for any commutative square (1.15), the diagram

(3.3)

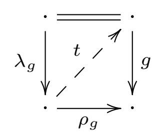


commutes.

A functorial realization of a weak factorization system determines the system itself. Suppose $f \in \mathcal{L}$. The factorization of any $f \in \mathcal{L}$ gives rise to a lifting problem



which necessarily has a solution s. Dually, if $g \in \mathcal{R}$, its factorization gives rise to a lifting problem

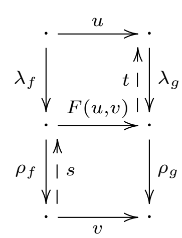


which has a solution t. Accordingly, given a functorial realization (F, λ, ρ) of a weak factorization system $(\mathcal{L}, \mathcal{R})$, define

$$\begin{aligned} \mathcal{L}_F :=& \{f \mid \exists s : \lambda_f = s \cdot f, \rho_f \cdot s = 1\}, \\ \mathcal{R}_F :=& \{g \mid \exists t : \rho_g = g \cdot t, t \cdot \lambda_f = 1\}. \end{aligned} \tag{3.4}$$

Given $f \in \mathcal{L}_F$ and $g \in \mathcal{R}_F$ the lifts s and t allow one to construct a solution to the lifting problem (2.1)

(3.5)



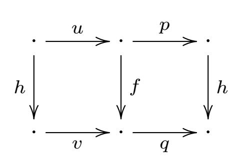
by taking $w = t \cdot F(u, v) \cdot s$. The equations defining the lifts s and t guarantee that the required triangles commute.

Theorem 3.6. (a) *For every weak factorization system* $(\mathcal{L}, \mathcal{R})$ *with functorial realization* (F, λ, ρ)*, the classes* $\mathcal{L} = \mathcal{L}_F$ *and* $\mathcal{R} = \mathcal{R}_F$*.*

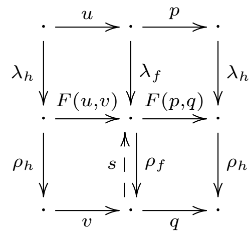
(b) *For any triple* $(F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}, \lambda : \mathrm{dom} \to F, \rho : F \to \mathrm{cod})$ *with* $\kappa = \rho \cdot \lambda$ *and such that* $\lambda_f \in \mathcal{L}_F$ *and* $\rho_f \in \mathcal{R}_F$ *for all* $f \in mor\mathcal{K}$*, the pair* $(\mathcal{L}_F, \mathcal{R}_F)$ *is a weak factorization system with functorial realization* (F, λ, ρ)*.*

Proof. (a) By definition $f \in \mathcal{L}_F$ satisfies $s \cdot f = \lambda_f \in \mathcal{L}$ with s a split monic, so $\mathcal{L}_F \subset \mathcal{L}$ by (5) and similarly $\mathcal{R}_F \subset \mathcal{R}$. Conversely, if $f \in \mathcal{L}$, factor $f = \rho_f \lambda_f$ and use $f \boxslash \rho_f$ as in the proof of Lemma 2.3 to find a lift s such that $\lambda_f = s \cdot f$ and $\rho_f \cdot s = 1$. Hence, $\mathcal{L} \subset \mathcal{L}_F$ and dually $\mathcal{R} \subset \mathcal{R}_F$.

(b) The equality $\kappa = \rho \cdot \lambda$ gives factorization and (3.5) shows that $\mathcal{L}_F \boxslash \mathcal{R}_F$. We show that $\mathcal{L}_F$ and $\mathcal{R}_F$ are closed under retracts. Given a retract diagram

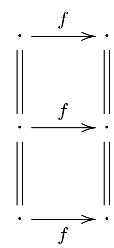


factor f and h to obtain



If $f \in \mathcal{L}_F$ there exists a lift s such that $\lambda_f = s \cdot f$ and $\rho_f \cdot s = 1$. Define $j = F(p,q) \cdot s \cdot v$. Then using that the outer horizontal composites are the identity, j satisfies $\lambda_h = j \cdot h$ and $\rho_h \cdot j = 1$. Hence, $h \in \mathcal{L}_F$ and a dual argument shows $\mathcal{R}_F$ is closed under retracts as well. This implies that $(\mathcal{L}_F, \mathcal{R}_F)$ satisfies (4). Hence, this pair is a weak factorization system with functoral realization (F, λ, ρ). □

Remark 3.7. In a functorial weak factorization system, the middle arrow in (3.3) depends functorially on u and v, but it is not in general the *unique* choice such that this diagram commutes. Surprisingly, to guarantee uniqueness it suffices to impose a condition on the functorial factorization of trivial arrows $(f,f) : 1_{\mathrm{dom} f} \to 1_{\mathrm{cod} f}$ in $\mathcal{K}^{\mathbf{2}}$. Specifically, one requires that the factorization of this square is



or at least something isomorphic to it. A functorial weak factorization system satisfying this condition is called an Eilenberg-Moore factorization system by [7], which gives a proof that these are equivalent to orthogonal factorization systems (i.e., that functorial factorizations plus an identity condition implies unique choice). On the other hand, there are many functorial weak factorization systems that exist in nature (see Theorem 4.4 below) that are not orthogonal factorization systems, which suggests that this identity condition is unreasonable to expect.

Rosicky and Tholen prove that many functorial weak factorization systems arise as lax algebras for the squaring monad described in Theorem 1.16. These include all cofibrantly generated weak factorization systems in a locally presentable category. See [10] for details.

4. Weak Factorization Systems and Model Categories

The language of weak factorization systems allows us to give a particularly concise definition of a model category. There are many good references for this subject such as [1] or [5].

Definition 4.1. A *model category* $\mathcal{K}$ is a complete and cocomplete category together with three distinguished classes of morphisms — the *cofibrations* $\mathcal{C}$, the *fibrations* $\mathcal{F}$, and the *weak equivalences* $\mathcal{W}$ — satisfying the following axioms:

(M1) The class $\mathcal{W}$ is closed under retracts and satisfies the 2 of 3 property: if

$f, g \in \operatorname{mor} \mathcal{K}$ are such that any two of f, g, or fg is a weak equivalence, so is the third.

(M2) The pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems on $\mathcal{K}$.

Model categories provide a particularly abundant source of weak factorization systems. Conversely, any weak factorization system $(\mathcal{L}, \mathcal{R})$ in a complete and cocomplete category $\mathcal{K}$ can be extended to a model structure with $\mathcal{C} = \mathcal{L}$, $\mathcal{F} = \mathcal{R}$, and $\mathcal{W} = \operatorname{mor} \mathcal{K}$. This structure is not particularly interesting.

Hovey [5] notes that the weak factorization systems in most familiar model structures are in fact functorial. Furthermore, many of them are *cofibrantly generated*, which means that the right class of each weak factorization system is defined to be those maps with the right lifting property with respect to some *set* of maps. The left class is then defined to be those maps with the left lifting property with respect to the right class.

The main tool for constructing cofibrantly generated weak factorization systems is Quillen's small object argument, which takes a set of maps $\mathcal{J}$ with a certain condition about their domains and produces a functorial weak factorization system $({}^{\boxslash}(\mathcal{J}^{\boxslash}), \mathcal{J}^{\boxslash})$. When this argument can be applied, the left class ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ is the smallest saturated class of morphisms containing $\mathcal{J}$, called the *saturation* of $\mathcal{J}$.

We won't attempt to reproduce the theory of cofibrantly generated model categories (see [5] for a good exposition), but we do give a few definitions leading up to a statement of Quillen's small object argument, which produces the cofibrantly generated functorial weak factorization systems.

When κ is a regular cardinal, we say that an ordinal λ is κ*-filtered* if every subset of cardinal less than κ has an upper bound in λ.

Definition 4.2. Let $\mathcal{K}$ be a locally small, cocomplete category, let $\mathcal{E}$ be a class of maps in $\mathcal{K}$, and let κ be a regular cardinal. An object A of $\mathcal{K}$ is κ*-small relative to* $\mathcal{E}$ if for every κ-filtered ordinal λ, and every colimit preserving functor $X : \lambda \to \mathcal{K}$ such that $e_\beta : \operatorname{colim}_{\alpha<\beta} X^\alpha \to X^\beta$ is in $\mathcal{E}$ for all $\beta < \lambda$, the canonical map

$$\psi : \operatorname{colim}_{\beta<\lambda} \mathcal{K}(A, X^\beta) \longrightarrow \mathcal{K}(A, \operatorname{colim}_{\beta<\lambda} X^\beta)$$

is a bijection. We say A is *small relative to* $\mathcal{E}$ if A is κ-small for some regular cardinal κ, and say that A is *small* if it is small relative to $\operatorname{mor} \mathcal{K}$.

The colimit on the left is a quotient of

$$\sqcup_\beta \mathcal{K}(A, X^\beta)$$

by the relation $f \sim g$ iff there is some $\gamma < \lambda$ such that the composites

$$A \xrightarrow{f} X^\alpha \to X^\gamma \quad \text{and} \quad A \xrightarrow{g} X^\beta \to X^\gamma$$

are equal. Thus, ψ is well-defined by composing any representative f by the appropriate map in the colimiting cone. In the examples, ψ is usually injective for all objects A. The main point of this definition comes from surjectivity, which says that if A is a small object, then any map $A \to \operatorname{colim}_{\beta<\lambda} X^\beta$ factors through some object in the colimit diagram.

Example 4.3. Any set is small. More precisely, a set A is $|A|$-small where $|A|$ denotes the cardinality of A.

Theorem 4.4 (Small Object Argument)**.** *Let $\mathcal{K}$ be a locally small, cocomplete category, and let $\mathcal{J}$ be a set of maps that is small relative to transfinite compositions of pushouts of elements of $\mathcal{J}$. Then $({}^{\boxslash}(\mathcal{J}^{\boxslash}), \mathcal{J}^{\boxslash})$ is a functorial weak factorization system.*

Proof. It follows from the definitions that ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ and $\mathcal{J}^{\boxslash}$ satisfy the desired lifting properties. (Note there *is* something to show here; namely, that $({}^{\boxslash}(\mathcal{J}^{\boxslash}))^{\boxslash} = \mathcal{J}^{\boxslash}$, which is true because $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ form a Galois connection on subsets of mor $\mathcal{K}$.) The small object argument will provide a functorial factorization, provided we choose all pushouts and colimits of all ordinal diagrams in $\mathcal{K}$ in advance.

Let κ be a sufficiently large cardinal so that all domains of morphisms in $\mathcal{J}$ are κ-small and let λ be a κ-filtered ordinal. Let $f : X \to Y$ be any morphism in $\mathcal{K}$, and let $X = Z^0$. Let S_0 be the set of commutative squares

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

with $j \in \mathcal{J}$. Define Z^1 to be the pushout given by the diagram

$$\begin{array}{ccc} \sqcup_{S_0} A & \longrightarrow & Z^0 \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ \sqcup_{S_0} B & \longrightarrow & Z^1 \end{array}$$

which exists because $\mathcal{K}$ is cocomplete. The maps $q_0 = f : Z^0 \to Y$ and $\sqcup_{S_0} B \to Y$ induce a map $q_1 : Z^1 \to Y$. Inductively, let S_β be the set of commutative squares

$$\begin{array}{ccc} A & \longrightarrow & Z^\beta \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q_n} \\ B & \longrightarrow & Y \end{array}$$

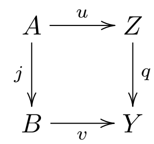
with $j \in \mathcal{J}$, and define $Z^{\beta+1}$ to be the pushout

$$\begin{array}{ccc} \sqcup_{S_n} A & \longrightarrow & Z^\beta \\ \downarrow & & \downarrow{\scriptstyle i_{n+1}} \\ \sqcup_{S_n} B & \longrightarrow & Z^{\beta+1} \end{array} \tag{4.5}$$

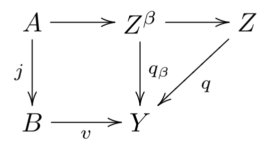
The cone $q_\beta : Z^\beta \to Y$ and $\sqcup_{S_\beta} B \to Y$ induces a map $q_{\beta+1} : Z^{\beta+1} \to Y$. For limit ordinals α define $Z^\alpha = \operatorname{colim}_{\beta<\alpha} Z^\beta$ and define $q_\alpha : Z^\alpha \to Y$ to be the map induced by the q_β.

This construction yields a λ-sequence $Z : \lambda \to \mathcal{K}$ and a cone under this sequence with summit Y. Let $Z = \operatorname{colim}_{\beta<\lambda} Z^\beta$, and let $i : X \to Z$ and $q : Z \to Y$ be the induced maps to and from the colimit. This gives a factorization $f = qi$ that is functorial because all of the colimits were chosen in advance.

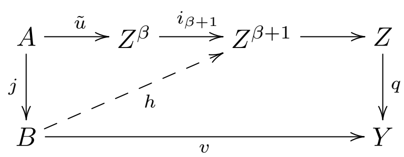
It remains to show that $i \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$ and $q \in \mathcal{J}^{\boxslash}$. The left hand factor i was constructed as a transfinite composite of pushouts of coproducts of elements of $\mathcal{J}$ and thus $i \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$ by Remark 2.9 and Theorem 2.10. Hence, it remains only to show that $q \in \mathcal{J}^{\boxslash}$. Given a commutative square



with $j \in \mathcal{J}$, A is κ-small and λ is κ-filtered, so there exists a factorization of u through some Z^β with $\beta < \lambda$. This gives a commutative diagram



The object $Z^{\beta+1}$ was defined as the pushout over coproducts of such diagrams, so we have a map $h : B \to Z^{\beta+1}$ such that



commutes. The top composite in this rectangle is u, so it is clear that this gives the desired lift. $\square$

5. Natural Weak Factorization Systems

In the context of a weak factorization system, *functoriality* of the factorization does not imply *naturality* of the lifts s and t of (3.5). To begin with, we do not know in general that $F(u, v)$ is the unique horizontal morphism that makes (3.5) commute. Surprisingly, this follows from a simple condition concerning the functorial factorization of trivial squares, which was described in Remark 3.7. Under this additional hypothesis, the arrow $F(u, v)$ is unique, and it follows from Definition 1.1 that the result is an orthogonal factorization system.

As remarked above, a consequence of this result is that the uniqueness condition, while seemingly innocuous, is too much to hope for because many weak factorization systems are not orthogonal. Grandis and Tholen take a different approach in [4], ignoring the question of whether $F(u, v)$ is unique, and asking instead that the lifts s and t be somehow naturally chosen. This leads them to formulate the definition of a *natural weak factorization system*, of which there are surprisingly many examples. Richard Garner has developed the theory of natural weak factorization systems much further in very recent years (see [2] and [3]).

The basic idea is that the factorization functors L and R of a functorial weak factorization system defined in (3.1) can often be replaced by a comonad $\mathbb{L}$ and monad $\mathbb{R}$ respectively, and the resulting extra data can be used to construct natural lifts. A precise definition follows.

Let $\mathbf{CAT}/\mathcal{K}$ be the slice 2-category of small categories over $\mathcal{K}$. A monad (T, μ, η) on $f : \mathcal{A} \to \mathcal{K}$ in $\mathbf{CAT}/\mathcal{K}$ is a monad (T, μ, η) on $\mathcal{A}$ in $\mathbf{CAT}$ such that $fT = f$ and $f\eta = 1 = f\mu$.

Definition 5.1. A *natural weak factorization system* in a category $\mathcal{K}$ is a pair $(\mathbb{L}, \mathbb{R})$ such that

(i) $\mathbb{L} = (L, \Phi, \Sigma)$ is a comonad on dom in $\mathbf{CAT}/\mathcal{K}$.
(ii) $\mathbb{R} = (R, \Lambda, \Pi)$ is a monad on cod in $\mathbf{CAT}/\mathcal{K}$.
(iii) $\mathrm{cod}L = \mathrm{dom}R$, $\mathrm{cod}\Phi = \kappa_R$, $\mathrm{dom}\Lambda = \kappa_L$.

Example 5.2. Let $\mathcal{K}$ be any category with binary products. Then every $h : X \to Y$ has a functorial *graph factorization*

$$X \xrightarrow{(1,h)} X \times Y \xrightarrow{\pi_Y} Y.$$

Dually, if $\mathcal{K}$ has binary coproducts, h has a functorial *cograph factorization*

$$X \xrightarrow{\iota_X} X \cup Y \xrightarrow{h+1} Y.$$

Both of these can be made into natural weak factorization systems. Details are in [4].

When one keeps in mind that the factorizations of an orthogonal factorization system are unique up to unique isomorphism, the following result, also in [4], is not terribly surprising.

Theorem 5.3. *Orthogonal factorization systems of $\mathcal{K}$ are equivalently described as those natural weak factorization systems $(\mathbb{L}, \mathbb{R})$ for when $\mathbb{L}$ and $\mathbb{R}$ are idempotent.*

In [4], the authors assert that any cofibrantly generated weak factorization system in a locally finitely-presentable category is natural, though they do not give a proof. Garner's paper [3] works toward this result.

For a natural weak factorization system $(\mathbb{L}, \mathbb{R})$, we may construct the Eilenberg-Moore categories $\mathcal{L}_{\mathbb{L}}$ and $\mathcal{R}_{\mathbb{R}}$ for the comonad $\mathbb{L}$ and the monad $\mathbb{R}$, respectively. An object of $\mathcal{L}_{\mathbb{L}}$ is a pair $(f, (1, s) : f \to Lf) \in \mathrm{ob}\,\mathcal{K}^{\mathbf{2}} \times \mathrm{mor}\,\mathcal{K}^{\mathbf{2}}$ such that

$$\begin{array}{ccccc} & & f & \xrightarrow{(1,s)} & Lf \\ & \diagup\!\!\diagup & \Big\downarrow{\scriptstyle (1,s)} & & \Big\downarrow{\scriptstyle \Sigma_f} \\ f & \xleftarrow[\Phi_f]{} & Lf & \xrightarrow[L(1,s)]{} & LLf \end{array}$$

The arrow s of $\mathcal{K}$ is a given splitting of $\rho_f = Rf$ as in (3.5) and must be respected by morphisms in $\mathcal{L}_{\mathbb{L}}$. Objects of $\mathcal{R}_{\mathbb{R}}$ are pairs $(g, (t, 1) : Rg \to g) \in \mathrm{ob}\,\mathcal{K}^{\mathbf{2}} \times \mathrm{mor}\,\mathcal{K}^{\mathbf{2}}$ satisfying a dual condition; the arrow t is a given splitting of $\lambda_f = Lf$.

Because the splittings of a natural weak factorization system are naturally chosen, the left and right classes enjoy the same closure properties of orthogonal factorization systems.

Theorem 5.4. *Let $(\mathbb{L}, \mathbb{R})$ be a natural weak factorization system. Then every f factors as $f = Rf \cdot Lf$ with the left factor and its accompanying lift an object of $\mathcal{L}_{\mathbb{L}}$ and the right factor and its accompanying lift an object of $\mathcal{R}_{\mathbb{R}}$. Furthermore, for all $(f, (1, s)) \in \mathcal{L}_{\mathbb{L}}$ and $(g, (t, 1)) \in \mathcal{R}_{\mathbb{R}}$ a lifting problem (1.15) has a natural solution*

$$w = t \cdot F(u, v) \cdot s.$$

The classes $\mathcal{L}_{\mathbb{L}}$ and $\mathcal{R}_{\mathbb{R}}$ are closed under all colimits and limits, respectively, that exist in $\mathcal{K}$, formed as in $\mathcal{K}^{\mathbf{2}}$.

Proof. It is well-known that the forgetful functors $\mathcal{L}_{\mathbb{L}} \to \mathcal{K}^{\mathbf{2}}$ and $\mathcal{R}_{\mathbb{R}} \to \mathcal{K}^{\mathbf{2}}$ create all colimits and limits, respectively, that exist in $\mathcal{K}$. $\square$

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