THE ALGEBRA AND GEOMETRY OF ∞-CATEGORIES

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Abstract. ∞-categories are a sophisticated tool for the study of mathematical structures with higher homotopical information. This note, directed at the Friends of Harvard Mathematics, introduces this notion from first principles.

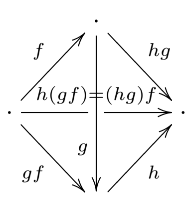
1. The philosophy of category theory

The fundamental philosophy of category theory is that anything one would want to know about a mathematical object is determined by the maps (a.k.a. functions or transformations, sometimes simply "arrows") to or from it. For instance, functions from a singleton set to a set X classify its elements; functions from X to the set $\{0, 1\}$ identify subsets. Linear maps from $\mathbb{R}$ to a real vector space V correspond bijectively to its vectors. More refined analysis can be used to decode the entire vector space structure. Continuous functions from the singleton space to a topological space T identify its points. Maps from T to the Sierpinski space classify open sets. And so forth.

Mathematical objects of a fixed type assemble into a *category.* A category consists of objects $X, Y, Z, \ldots$ and maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z, \ldots$ including specified identities $X \xrightarrow{1_X} X$. This data is subject to the following axiom: to any pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ there must be some specified composite map

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & = & \searrow^{g} \\ X & \xrightarrow[gf]{} & Z \end{array}$$

and furthermore this composition law must be both *associative* and *unital.* The first of these conditions says that a single arrow is the specified composite $(hg)f$ and $h(gf)$ whenever these compositions are defined. The latter says that composition with an identity has no effect.



$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & = & \searrow^{1_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} & X & \\ {}^{1_X}\nearrow & = & \searrow^{f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

For instance, there is a category of sets and functions; of vector spaces and linear transformations; and of topological spaces and continuous maps, to name just a few. One might say that the objects of a category are the "nouns" and the maps the "verbs" in the language appropriate to the mathematical theory of

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interest [Maz07]. The title of this beautiful essay asks "When is one thing equal to some other thing?" — a question which we now address.

Definition. A map $X \xrightarrow{f} Y$ is an *isomorphism* if there exists a map $Y \xrightarrow{g} X$ such that the displayed composites are identities.

$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & = & \searrow^{g} \\ X & \xrightarrow[1_X]{} & X \end{array} \qquad \begin{array}{ccc} & X & \\ {}^{g}\nearrow & = & \searrow^{f} \\ Y & \xrightarrow[1_Y]{} & Y \end{array}$$

The term derives from the Greek: "iso" = "same" "morphic" = "shape". This notion was called an *equivalence* in the foundational paper [EM45] in which categories are introduced as a formalism with which to describe *natural* comparisons between parallel mathematical constructions.

An easy argument shows that the map g is also an isomorphism and furthermore uniquely determined. We say two objects are *isomorphic* if there exists an isomorphism between them. The following lemma says that if two objects are isomorphic, then they are identical from the vantage point of our philosophy.

Lemma. *If X and Y are isomorphic then there is a bijection between the collections of maps*

$$\{Z \to X\} \qquad \leftrightsquigarrow \qquad \{Z \to Y\}$$

Proof. Composition with the map $X \xrightarrow{f} Y$ defines a function from the collection of maps $\{Z \to X\}$ to the collection $\{Z \to Y\}$; composition with $Y \xrightarrow{g} X$ defines a function in the other direction. To show that these functions define a bijection, we prove that they are inverses. For this, suppose we are given a map $Z \xrightarrow{h} X$. Its image under the composite function is

$$g(fh) = (gf)h = 1_X h = h$$

by the associativity and unitality axioms. □

The first major theorem in category theory is that the converse holds as well. It is not possible to overstate the importance of this result which says that a mathematical object is determined up to isomorphism by its *universal property* or, equivalently, by the (set-valued) functor that it *represents*.

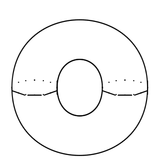
Lemma (Yoneda lemma)**.** *If there exists a natural bijection*

$$\{Z \to X\} \qquad \leftrightsquigarrow \qquad \{Z \to Y\}$$

for all objects Z in the category, then X and Y are canonically isomorphic.

2. A motivating example from homotopy theory

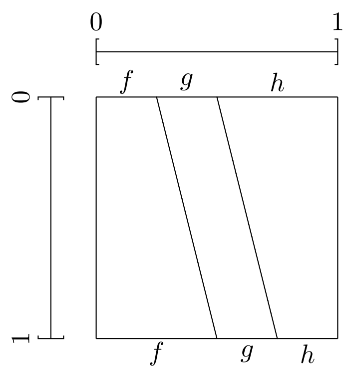
Let us now bring this discussion into the realm of homotopy theory. Fix a topological space T—for instance a surface, perhaps a torus, on which, we might imagine, lives a very small bug.



Returning to the philosophy introduced above, we might try to investigate the space T by considering continuous maps from simple geometric objects into T. In particular, we might choose maps that describe possible trajectories for the bug, whose positions are represented by points in T and whose meanderings are described by means of paths, i.e., continuous functions from the interval $[0,1]$ to T.

These considerations naturally suggest a category whose objects are points and whose morphisms are paths. Paths that start and end at a common point can be composed by "traveling twice as fast." The composition of a path f from position x to position y with a path g from position y to position z is the path where the bug traverses f over the course of the interval $[0, \frac{1}{2}]$ and traverses g over the course of the interval $[\frac{1}{2}, 1]$. But this composition law fails to be associative: a bug traveling along the path $h(gf)$ spends half its time on h and a quarter each along g and f while a bug traveling along $(hg)f$ spends half its time along f and only a quarter each on g and h.

While these paths aren't identical, they are *homotopic*. The homotopy takes the form of a continuous function from the product $[0,1] \times [0,1]$ to T as depicted by the following schematic picture.

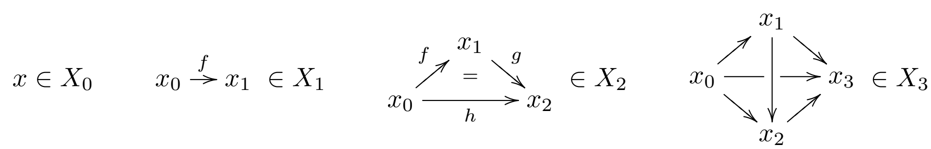


The construction suggested above does yield a category $\Pi_1 T$ provided we instead define maps to be *homotopy classes of paths*. Incidentally, the reason we suggested that T might be a surface, such as a torus, with non-zero genus is so that this category will have more than one map between any two given points.

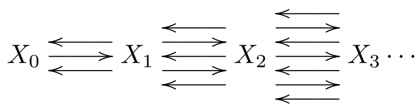
Assuming the space T is *path connected*, the category $\Pi_1 T$ is equivalent (as a category) to the *fundamental group* of T, "one of the most celebrated invariants in algebraic topology" [Lur08]. Nonetheless, this construction is somewhat unsatisfying, because the "homotopy classes" lose the information provided by the explicit homotopies. The forgetting involved with this truncation was necessary because categories, as classically defined, are only 1-dimensional. The "higher homotopical information" of the space T instead organizes into an ∞-*category*.

3. Simplicial sets

An ∞-category is a particular sort of *simplicial set*. A simplicial set X consists of sets $X_0, X_1, X_2, \ldots$ of *simplices* in varying dimension which we might visualize in the following manner



The data of a simplicial set also consists of functions

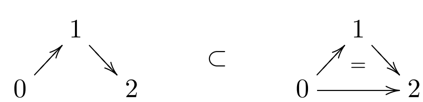


that specify which lower-dimensional simplices are faces of higher-dimensional simplices and which higher-dimensional simplices represent degenerate copies of lower-dimensional simplices. These functions satisfy certain relations that are evident from their geometric description.

Example. To any category $\mathcal{C}$ there is an associated simplicial set $N\mathcal{C}$ with 0-simplices the objects of $\mathcal{C}$, 1-simplices the maps, 2-simplices composable pairs of maps, 3-simplices composable triples, and so on. In this simplicial set, a simplex is uniquely determined by its *spine*, the sequence of edges connecting the 0th vertex to the 1st to the 2nd to the 3rd and so on. This simplicial set is used to define the *classifying space* of a group.

Example. To any topological space T there is an associated simplicial set ST with 0-simplices the points of T, 1-simplices the paths, 2-simplices continuous functions from the topological 2-simplex into T, 3-simplices continuous functions from the topological 3-simplex to T, and so on. This simplicial set is used to define the *homology* of a topological space.

Simplicial sets themselves form a category. Indeed, returning to the philosophy expressed above, a simplicial set X is entirely described by maps to it. In particular, the n-simplices of X correspond bijectively to maps $\Delta^n \to X$ whose domain is the *standard n-simplex*. Here Δ^n is the simplicial set with a single non-degenerate n-simplex, together with its faces and their degeneracies. There is a sub simplicial set $\Lambda^n_k \subset \Delta^n$ for each $0 \leq k \leq n$ called the (n,k)-*horn* which is the simplicial set formed by the throwing away the non-degenerate n-simplex and its k-th $(n-1)$-dimensional face. For example, the $(2,1)$-horn is the simplicial set depicted below



A map $\Lambda^2_1 \to X$ specifies a pair of 1-simplices in X so that the target vertex of one is the source vertex of the other.

4. ∞-categories

We are now prepared to define an ∞-category.

Definition. An ∞-*category* is a simplicial set X so that for each $n \geq 2$, $0 \leq k \leq n$ any (n, k)-horn in X extends to an n-simplex.

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

We have already seen two classes of examples.

Example. For any category $\mathcal{C}$, $N\mathcal{C}$ is an ∞-category. Each $(2, 1)$-horn admits an extension to a 2-simplex because composable maps have a specified composite. Each $(3, 1)$- and $(3, 2)$-horn admits an extension because this composition law is associative. Indeed, each (n, k)-horn admits a unique extension to an n-simplex because the simplices in $N\mathcal{C}$ are uniquely characterized by their spine, which is visible in the horn.

Example. For any topological space T, ST is an ∞-category. A map $\Lambda^n_k \to ST$ corresponds to a continuous function from a topological realization of the (n, k)-horn into the space T. This topological horn includes into the topological n-simplex and this inclusion is a *deformation retract*, meaning there is a continuous retraction of the n-simplex onto the (n, k)-horn. This retraction may be used to define the desired extension. Indeed, note that this argument works equally well for $(n, 0)$- and (n, n)-horns in ST.

In general, we think of an ∞-category as a weak category with maps in each dimension. The 0-simplices are the objects and the 1-simplices are the maps. For any composable pair of maps $x \xrightarrow{f} y, y \xrightarrow{g} z$, the extension condition in the case $n = 2$ guarantees that there exists a 2-simplex

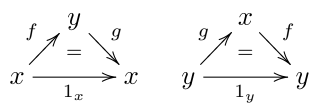
$$\begin{array}{ccccc} & & y & & \\ & {}^{f}\nearrow & = & \searrow^{g} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

which we think of as a homotopy witnessing that h is a composite of f and g. The 3-dimensional extension condition implies, among other things, that any two composites are themselves homotopic. The 4-dimensional extension condition implies, among other things, that any two parallel homotopies have a "higher homotopy" comparing them, and so on. There is a sense in which these (higher) homotopies may themselves be composed; with respect to this composition law, all (higher) homotopies are invertible. One way to express this property is to say that between any two vertices in an ∞-category there is a topological space of maps, though this space is only well-defined up to *homotopy type*.

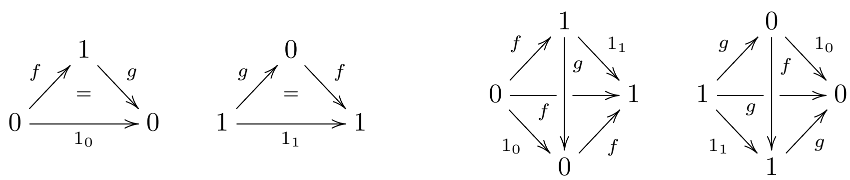
5. Equivalences in ∞-categories

Finally, we return to a question considered at the beginning: What should it mean for two objects of an ∞-category to be equivalent? A quick definition is that objects of an ∞-category are equivalent just when they are isomorphic in the associated homotopy category. This leads to the following concrete definition.

Definition. A 1-simplex $x \xrightarrow{f} y$ in a ∞-category is an *equivalence* if there exists a 1-simplex $y \xrightarrow{g} x$ and 2-simplices



Part of the magic of ∞-categories is visible in the following result, which says that equivalences in an ∞-category are automatically "infinite-dimensional." Let J be the simplicial set with two 0-simplices $0, 1$; two non-degenerate 1-simplices $0 \xrightarrow{f} 1$, $1 \xrightarrow{g} 0$; and indeed two non-degenerate simplices in each dimension continuing in the pattern depicted



Theorem. *Equivalences in an ∞-category correspond to maps $J \to X$. More precisely, a 1-simplex in X is an equivalence if and only if it can be extended to a map whose domain is J.*

The proof is reasonably elementary though it requires some clever combinatorics. The reader might enjoy proving a special case: that the data described above can be extended to dimension three.

References

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