Homotopy coherent adjunctions

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Thursday, January 16th
AMS Special Session on Homotopy Theory
2014 Joint Mathematics Meetings, Baltimore

Joint with Dominic Verity.

Plan

1. Quasi-categories and adjunctions
2. Homotopy coherent adjunctions
3. Homotopy coherent monads and the quasi-category of algebras
4. The monadicity theorem

Quasi-categories

A **quasi-category** is a simplicial set A in which any inner horn

$$\begin{array}{ccc} \Lambda^{n,k} & \longrightarrow & A \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \qquad 0 < k < n$$

has a filler.

The **homotopy category** hA has

- objects = vertices
- morphisms = homotopy classes of 1-simplices

Via the adjunction

$$\underline{\mathrm{Cat}} \underset{\hookrightarrow}{\overset{h}{\underset{\perp}{\longleftarrow}}} \underline{\mathrm{qCat}}$$

quasi-category theory extends category theory.

Adjunctions of quasi-categories

$\underline{\mathrm{qCat}}_\infty$:= the **simplicial category of quasi-categories** with hom-spaces B^A

$\underline{\mathrm{qCat}}_2$:= the **2-category of quasi-categories** with hom-categories $h(B^A)$

An **adjunction** of quasi-categories is an adjunction in $\underline{\mathrm{qCat}}_2$.

$$A \underset{u}{\overset{f}{\underset{\bot}{\leftrightarrows}}} B \qquad \eta\colon \mathrm{id}_B \Rightarrow uf \qquad \epsilon\colon fu \Rightarrow \mathrm{id}_A$$

Some theorems and examples

Theorems.

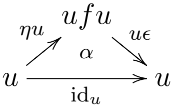
- $f \dashv u$ induces adjunctions $f^X \dashv u^X$ and $C^u \dashv C^f$ for any simplicial set X and quasi-category C.
- Any equivalence can be promoted to an adjoint equivalence.
- Right adjoints preserve limits.
- $f \colon B \to A$ has a left adjoint iff $f \downarrow a$ has a terminal object for each $a \in A$.

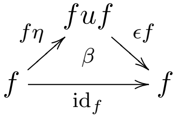
Examples.

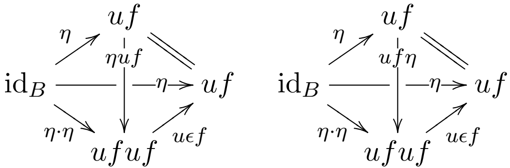
- ordinary adjunctions, topological adjunctions
- simplicial Quillen adjunctions
- colim $\dashv$ const $\dashv$ lim
- loops–suspension

A coherence question

Given $A \underset{u}{\overset{f}{\underset{\perp}{\leftrightarrows}}} B$ in $\underline{\mathrm{qCat}}_2$, what adjunction data exists in $\underline{\mathrm{qCat}}_\infty$?

- $\mathrm{id}_B \xrightarrow{\eta} uf$ in B^B $\qquad fu \xrightarrow{\epsilon} \mathrm{id}_A$ in A^A
- 

in B^A


in A^B
- 

filling $\Lambda^{3,1} \to B^B$

But do there exist fillers with the same bottom face?

The free adjunction

$\underline{\mathrm{Adj}}$:= the **free adjunction**, a 2-category with

- objects $+$ and $-$
- $\underline{\mathrm{Adj}}(+,+) = \underline{\mathrm{Adj}}(-,-)^{\mathrm{op}} := \mathbb{\Delta}_+$
- $\underline{\mathrm{Adj}}(-,+) = \underline{\mathrm{Adj}}(+,-)^{\mathrm{op}} := \mathbb{\Delta}_\infty$

Theorem (Schanuel-Street). 2-functors $\underline{\mathrm{Adj}} \to \underline{\mathrm{qCat}_2}$ correspond to adjunctions in $\underline{\mathrm{qCat}_2}$.

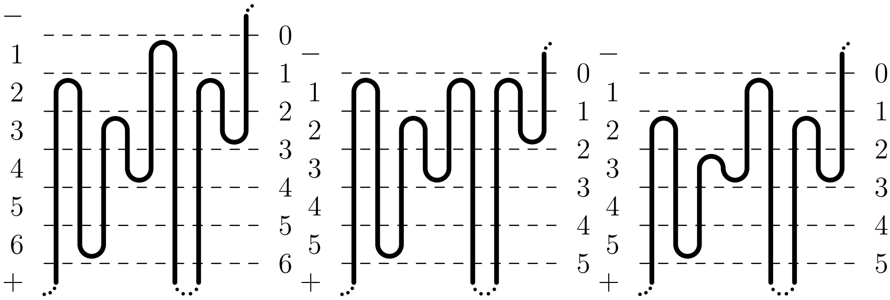
$$\mathrm{id} \xrightarrow{\;\eta\;} uf \;\begin{array}{c}\xrightarrow{\;\eta\;}\\ \xleftarrow{\;u\epsilon\;}\\ \xrightarrow{\;uf\eta\;}\end{array}\; ufuf \;\begin{array}{c}\xrightarrow{\;\eta\;}\\ \xleftarrow{\;u\epsilon\;}\\ \xrightarrow{\;uf\eta\;}\\ \xleftarrow{\;ufu\epsilon\;}\\ \xrightarrow{\;ufuf\eta\;}\end{array}\; ufufuf \cdots$$

$$u \;\begin{array}{c}\xrightarrow{\;\eta\;}\\ \xleftarrow{\;u\epsilon\;}\end{array}\; ufu \;\begin{array}{c}\xrightarrow{\;\eta\;}\\ \xleftarrow{\;u\epsilon\;}\\ \xrightarrow{\;uf\eta\;}\\ \xleftarrow{\;ufu\epsilon\;}\end{array}\; ufufu \;\begin{array}{c}\xrightarrow{\;\eta\;}\\ \xleftarrow{\;u\epsilon\;}\\ \xrightarrow{\;uf\eta\;}\\ \xleftarrow{\;ufu\epsilon\;}\\ \xrightarrow{\;ufuf\eta\;}\\ \xleftarrow{\;ufufu\epsilon\;}\end{array}\; ufufufu \cdots$$

The free homotopy coherent adjunction

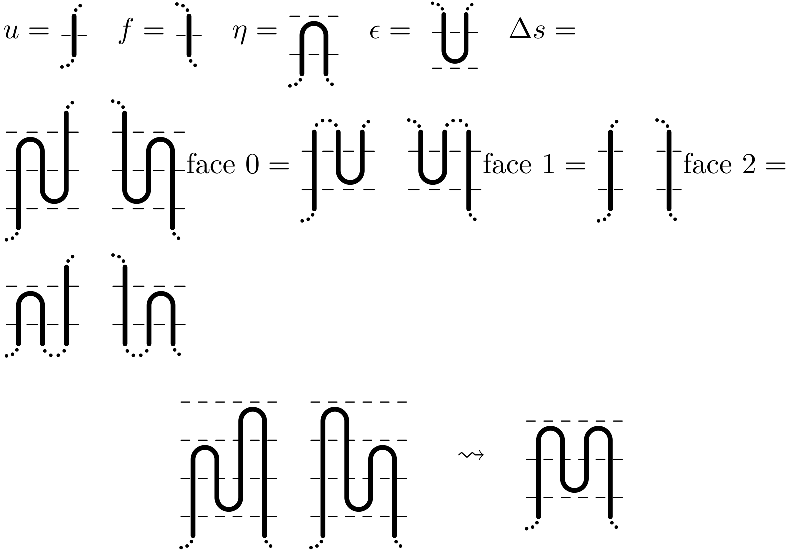
Conjecture.Definition. The **free homotopy coherent adjunction** is $\underline{\mathrm{Adj}}$, regarded as a simplicial category under $2\text{-}\underline{\mathrm{Cat}} \hookrightarrow \underline{\mathrm{sSet}}\text{-}\underline{\mathrm{Cat}}$.

n-arrows are **strictly undulating squiggles** on $n+1$ lines



Proposition. $\underline{\mathrm{Adj}}$ is a simplicial computad (i.e., cofibrant).

Homotopy coherent adjunctions



Homotopy coherent monads

$\underline{\mathrm{Mnd}} :=$ full subcategory of $\underline{\mathrm{Adj}}$ on $+$.

Definition. A **homotopy coherent monad** is a simplicial functor $T\colon \underline{\mathrm{Mnd}} \to \underline{\mathrm{qCat}}_\infty$., i.e.,

- $+ \mapsto B \in \underline{\mathrm{qCat}}_\infty$
- $\Delta_+ \xrightarrow{t} B^B =:$ the **monad resolution**

$$\mathrm{id}_B \xrightarrow{\ \eta\ } t \;\begin{array}{c}\xrightarrow{\ \eta\ }\\ \xleftarrow{\ \mu\ }\\ \xrightarrow{\ t\eta\ }\end{array}\; t^2 \;\begin{array}{c}\xrightarrow{\ \eta\ }\\ \xleftarrow{\ \mu\ }\\ \xrightarrow{\ t\eta\ }\\ \xleftarrow{\ t\mu\ }\\ \xrightarrow{\ tt\eta\ }\end{array}\; t^3 \cdots$$

and higher data, e.g.,

$$\begin{array}{ccc} & t^2 & \\ {\scriptstyle \eta t}\nearrow & \sim & \searrow{\scriptstyle \mu} \\ t & \Longrightarrow\!=\!= & t \end{array}$$

The quasi-category of algebras

Fix a homotopy coherent monad $T\colon \underline{\mathrm{Mnd}} \to \underline{\mathrm{qCat}}_\infty$ and define the **quasi-category of algebras** by:

$$B[t] = \mathrm{eq}\left(B^{\mathbb{\Delta}_\infty} \rightrightarrows B^{\mathbb{\Delta}_+ \times \mathbb{\Delta}_\infty}\right).$$

A vertex in $B[t]$ is a map $\mathbb{\Delta}_\infty \to B$ of the form:

$$\begin{array}{ccccccc} b & \begin{matrix}\xrightarrow{\ \eta\ }\\ \xleftarrow{\ \beta\ }\end{matrix} & tb & \begin{matrix}\xrightarrow{\ \eta\ }\\ \xleftarrow{\ \mu\ }\\ \xrightarrow{\ t\eta\ }\\ \xleftarrow{\ t\beta\ }\end{matrix} & t^2b & \begin{matrix}\xrightarrow{\ \eta\ }\\ \xleftarrow{\ \mu\ }\\ \xrightarrow{\ t\eta\ }\\ \xleftarrow{\ t\mu\ }\\ \xrightarrow{\ tt\eta\ }\\ \xleftarrow{\ tt\beta\ }\end{matrix} & t^3b\cdots \end{array}$$

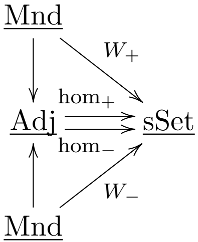
and higher data, e.g.,

$$\begin{array}{ccc} & tb & \\ {}^{\eta}\nearrow & \sim & \searrow^{\beta} \\ b & = & b \end{array}$$

The quasi-category of algebras, continued

More formally, the quasi-category of algebras is defined to be the **weighted limit**

$$B[t] := \{W_-, T\} \qquad\qquad B := \{W_+, T\}.$$



The monadic homotopy coherent adjunction

. . . is all in the weights!

$$\underline{\mathrm{Adj}}^{\mathrm{op}} \xrightarrow{\mathrm{hom}} \underline{\mathrm{sSet}}^{\underline{\mathrm{Adj}}} \xrightarrow{\mathrm{res}} \underline{\mathrm{sSet}}^{\underline{\mathrm{Mnd}}} \xrightarrow{\{-,T\}} \underline{\mathrm{qCat}}_{\infty}^{\mathrm{op}}$$

$$\begin{array}{ccccccc} - & \mapsto & \mathrm{hom}_- & \mapsto & W_- & \mapsto & B[t] \\ f \uparrow \dashv \downarrow u & & \downarrow \vdash \uparrow & & \downarrow \vdash \uparrow & & f^t \uparrow \dashv \downarrow u^t \\ + & \mapsto & \mathrm{hom}_+ & \mapsto & W_+ & \mapsto & B \end{array}$$

Theorem. Given a homotopy coherent adjunction $f \dashv u$ with homotopy coherent monad t, there is a comparison functor

$$\begin{array}{ccc} A & \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} & B[t] \\ f \nwarrow \searrow u & & f^t \nearrow \swarrow u^t \\ & B & \end{array}$$

that is an adjoint equivalence under the expected conditions.

Further reading

"The 2-category theory of quasi-categories" `arXiv:1306.5144`

"Homotopy coherent adjunctions and the formal theory of monads" `arXiv:1310.8279`

"A weighted limits proof of monadicity" on the n-Category Café