MADE-TO-ORDER WEAK FACTORIZATION SYSTEMS

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The aim of this note is to briefly summarize techniques for building weak factorization systems whose right class is characterized by a particular lifting property.

Weak factorization systems

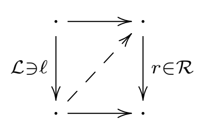
Weak factorization systems are of paramount importance to homotopical algebra. This connection is best illustrated by the following definition, due to Joyal and Tierney [JT07].

Definition 1. A **Quillen model structure** on a category $\mathbf{M}$, with a class of maps $\mathcal{W}$ called **weak equivalences** satisfying the 2-of-3 property, consists of two classes of maps $\mathcal{C}$ and $\mathcal{F}$ so that $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ and $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ are weak factorization systems.

Definition 2. A **weak factorization system** $(\mathcal{L},\mathcal{R})$ on a category $\mathbf{M}$ consists of two classes of maps so that

- Any map $f \in \mathbf{M}$ can be factored as $f = r \cdot \ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$.
- Any **lifting problem**, i.e., any commutative square

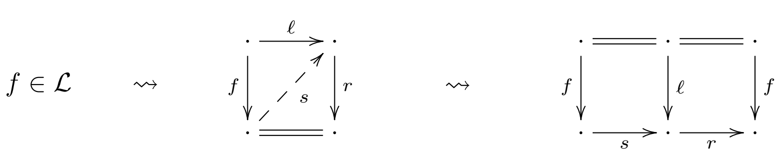
(1)



  has a **solution**, i.e., a diagonal arrow making both triangles commute.
- The classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

Remark 3.

(1) By the **retract argument**:



the class $\mathcal{L}$ consists of retracts of maps appearing as left factors.

(2) Consequently, the left and right classes determine each other. More precisely:

$$\mathcal{L} = {}^{\boxslash}\mathcal{R} \qquad \text{and} \qquad \mathcal{R} = \mathcal{L}^{\boxslash}$$

meaning $\mathcal{L}$ and $\mathcal{R}$ are maximal with the lifting property (1).

I would like to thank Steve Awodey for sparking my interest in homotopy type theory and encouraging me to keep in touch with recent progress.

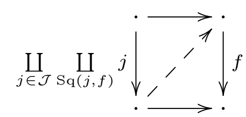
(3) Any class of maps of the form ${}^{\boxslash}\mathcal{R}$ is closed under retracts, coproducts, pushouts, transfinite composition, and contains the isomorphisms. Such classes are called **weakly saturated**. A class $\mathcal{L}^{\boxslash}$ is closed under the dual limit constructions.

(4) Given any set of maps $\mathcal{J}$, there is a **cofibrantly generated** weak factorization system $({}^{\boxslash}(\mathcal{J}^{\boxslash}), \mathcal{J}^{\boxslash})$—provided that it is possible to construct factorizations. This is accomplished by means of the **small object argument**.[1]

The algebraic small object argument

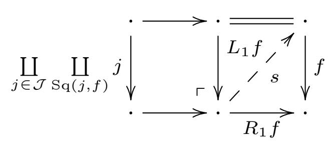
Assuming the category $\mathbf{M}$ is cocomplete and satisfies a certain "smallness" condition (such as being locally presentable), the **algebraic small object argument** defines the functorial factorization necessary for a "made-to-order" weak factorization system with $\mathcal{R} = \mathcal{J}^{\boxslash}$. For now, $\mathcal{J}$ is an arbitrary set of morphisms of $\mathbf{M}$ but later we will use this notation to represent something more sophisticated.

Generic lifting problems. The small object argument begins by defining a **generic lifting problem**, a single lifting problem that characterizes the desired right class:

(2) $f \in \mathcal{J}^{\boxslash} \quad \rightsquigarrow$ 

The diagonal map defines a solution to any lifting problem between $\mathcal{J}$ and f.

Step-one functorial factorization. Taking a pushout transforms the generic lifting problem (2) into the **step-one functorial factorization**, another generic lifting problem that also factors f.

(3) $f \in \mathcal{J}^{\boxslash} \quad \rightsquigarrow$ 

The step-one functorial factorization defines a pointed endofunctor $R_1 \colon \mathbf{M^2} \to \mathbf{M^2}$ of the arrow category. An R_1-**algebra** is a pair (f, s) satisfying (3). A map admits the structure of an R_1-algebra if and only if it is in the class $\mathcal{J}^{\boxslash}$.

The free monad construction. By the closure properties enumerated in Remark 3(3), $L_1 f \in \mathcal{L} = {}^{\boxslash}(\mathcal{J}^{\boxslash})$. However, there is no reason to expect that $R_1 f \in \mathcal{J}^{\boxslash}$: maps in the image of R_1 need not be R_1-algebras—unless R_1 is a monad. The idea of the algebraic small object argument, due to Garner [Gar09], is to freely replace the pointed endofunctor R_1 by a monad.[2]

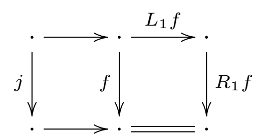
Following Kelly [Kel80], assuming certain "smallness" or "boundedness" conditions, it is possible to construct the free monad $\mathbb{R}$ on a pointed endofunctor R_1 in such a way that the categories of algebras are isomorphic. Garner shows that with sufficient care, Kelly's construction can be performed in a way that preserves the

[1] Dual "fibrantly generated" weak factorization systems are much rarer because the technical conditions necessary to "permit the small object argument" are satisfied by **Set** but not by $\mathbf{Set}^{\mathrm{op}}$.

[2] When all maps in the left class are monomorphisms, the free monad is defined by "iteratively attaching non-redundant cells" until this process converges.

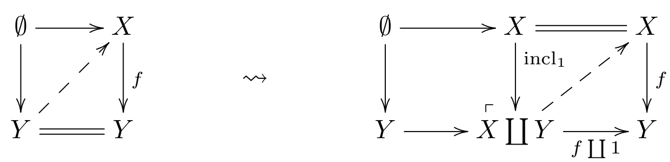
fact that the endofunctor R_1 is the right factor of a functorial factorization whose left factor L_1 is already a comonad. In this way, the algebraic small object produces a functorial factorization $f = \mathbb{R}f \cdot \mathbb{L}f$ in which $\mathbb{L}$ is a comonad, $\mathbb{R}$ is a monad, and $\mathbb{R}\text{-}\mathbf{Alg} \cong R_1\text{-}\mathbf{Alg} \cong \mathcal{J}^{\boxslash}$.

Remark 4. When the weak saturation of the generating set $\mathcal{J}$ is contained within the monomorphisms, the algebraic small object argument has the following simplified description: Iteratively apply the construction of the step-one functorial factorization to the right factor constructed in the previous stage, but avoid attaching solutions to redundant lifting problems. This means that any lifting problem whose "attaching map" factors through a previous stage



should be omitted from the coproduct defining the left-hand side of (2).

Example 5. Consider $\{\emptyset \to *\}$ on the category **Set**. The algebraic small object argument produces the generic lifting problem displayed on the left and the step-one functorial factorization displayed on the right:



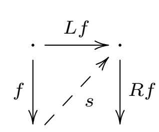
Every lifting problem after step one is redundant. Indeed, $\mathbb{R}f = f \coprod 1$ is already a monad and the algebraic small object argument converges in one step to define the functorial factorization $(\mathbb{L} = \mathrm{incl}_1, \mathbb{R})$.

Example 6. Consider $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ on the category of simplicial sets. Here we may consider lifting problems against a single generator at a time inductively by dimension. The step-one factorization of $X \to Y$ attaches the 0-skeleton of Y to X. There are no non-redundant lifting problems involving the generator $\emptyset \hookrightarrow \Delta^0$, so we move up a dimension. The step-two factorization of $X \to Y$ now attaches 1-simplices of Y to all possible boundaries in $X \cup \mathrm{sk}_0 Y$. After doing so, there are no non-redundant lifting problems involving $\partial\Delta^1 \hookrightarrow \Delta^1$. The construction converges at step ω.

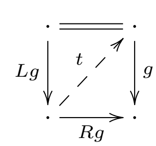
Algebraic weak factorization systems

Functorial factorizations and lifting properties. A functorial factorization $f = Rf \cdot Lf$ for a weak factorization system $(\mathcal{L}, \mathcal{R})$ characterizes that weak factorization system:

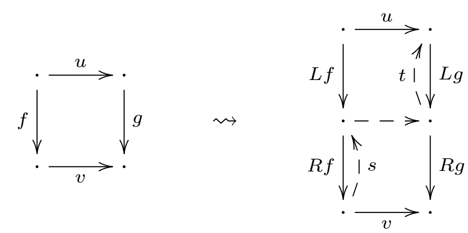
$f \in \mathcal{L} \quad \leftrightsquigarrow$



$g \in \mathcal{R} \quad \leftrightsquigarrow$

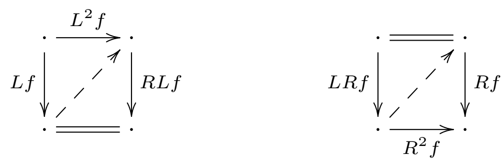


because the specified lifts assemble into a canonical solution to any lifting problem



A natural question is whether any functorial factorization defines a weak factorization system. In order to have $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ we must have maps

(4)



These need not exist in general but do when the pointed endofunctors L and R underlie a comonad and monad respectively.[3] So, in particular, the functorial factorization produced by the algebraic small object argument determines its weak factorization system.

Algebraic weak factorization systems. Put another way, the algebraic small object argument produces an **algebraic weak factorization system** $(\mathbb{L}, \mathbb{R})$, a functorial factorization that underlies a comonad $\mathbb{L}$ and a monad $\mathbb{R}$ and in which the natural transformation defined using the canonical lifts (4) defines a distributive law $LR \Rightarrow RL$ [GT06]. In fact, either of the categories $\mathbb{L}$-**coalg** or $\mathbb{R}$-**alg** determine the other; see [Rie11, §2.5], [Rie13c, §5.1].

Example 7. In Example 5, $\mathbb{L}$-**coalg** is the category of monomorphisms and pullback squares, while $\mathbb{R}$-**alg** $\cong \{\emptyset \to *\}^{\boxslash}$ is the category of split epimorphisms and commutative squares preserving the splittings.

Example 8. In Example 6, $\mathbb{L}$-**coalg** is the category of monomorphisms and commutative squares that induce pullback squares between the relative latching maps, while $\mathbb{R}$-**alg** $\cong \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}^{\boxslash}$ is the category of algebraic acyclic Kan fibrations and maps preserving the chosen liftings. This is the Reedy algebraic weak factorization system defined with respect to the algebraic weak factorization system of Example 7 [Rie13d].

Generalizations of the algebraic small object argument

The construction of the generic lifting problem admits a more categorical description which makes it evident that it can be generalized in a number of ways, expanding the class of weak factorization systems whose functorial factorizations can be "made-to-order."

[3]This is why $\mathbb{L}f \in \mathcal{L} = {}^{\boxslash}(\mathcal{J}^{\boxslash})$.

Coherence. Step zero of the algebraic small object argument forms the **density comonad**, i.e., the left Kan extension along itself, of the inclusion of the generating set of arrows:

$$\begin{array}{ccc} & \mathbf{M^2} & \\ \nearrow & \Uparrow & \searrow^{\text{Lan}=L_0} \\ \mathcal{J} & \longrightarrow & \mathbf{M^2} \end{array} \qquad\qquad L_0 f = \int^{j\in\mathcal{J}} \coprod_{\text{Sq}(j,f)} j$$

When $\mathbf{M}$ is cocomplete, this construction makes sense for any small *category* of arrows $\mathcal{J}$. The counit of the density comonad defines the generic lifting problem (2), admitting a solution if and only if $f \in \mathcal{J}^{\boxslash}$—but now $\mathcal{J}^{\boxslash}$ denotes the category in which an object is a map f together with a choice of solution to any lifting problem against $\mathcal{J}$ that is coherent with respect to (i.e., commutes with) morphisms in $\mathcal{J}$. Proceeding as before, the algebraic small object argument produces an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ so that $\mathbb{R}$-**alg** $\cong \mathcal{J}^{\boxslash}$ over $\mathbf{M^2}$ and $\mathbb{L}$-coalgebras lift against $\mathbb{R}$-algebras.

Example 9. In the category of cubical sets, let $\sqcap, \sqsupset, \sqcup, \sqsubset$ suggestively denote four subfunctors of the 2-dimensional representable $\square^2$. For $n > 2$ and $J \subset \{1, \ldots, n\}$ with $|J| = n - 2$, define $\sqcap^J \subset \square^n$ to be $\sqcap \otimes \square^J$ and similarly for the three other shapes. Consider the category whose objects are the inclusions $\sqcap^J \hookrightarrow \square^n$ for each shape and whose morphisms are generated by

- the projections $\begin{array}{ccc} \sqcap^J & \longrightarrow & \sqcap^{J\setminus\{j\}} \\ \downarrow & & \downarrow \\ \square^n & \xrightarrow[\pi_j]{} & \square^{n-1} \end{array}$ for each $j \in J$, and
- the inclusions $\begin{array}{ccc} \sqcap^J & \longrightarrow & \sqcap^{J\cup\{i\}} \\ \downarrow & & \downarrow \\ \square^n & \longrightarrow & \square^{n+1} \end{array}$ embedding $\square^n$ as the face $i = 0$ or $i = 1$.

This category generates the fibrant replacement functor described by Simon Huber.

Example 10 ([Rie11, §4.2])**.** Any algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ on $\mathbf{M}$ induces a pointwise-defined algebraic weak factorization system $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ on the category $\mathbf{M}^{\mathbf{A}}$ of diagrams. Moreover, when $(\mathbb{L}, \mathbb{R})$ is generated by $\mathcal{J}$, $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ is generated by the category $\mathbf{A}^{\text{op}} \times \mathcal{J}$, whose objects are tensors of arrows of $\mathcal{J}$ with covariant representables.

Enrichment. Now suppose that $\mathbf{M}$ is tensored, cotensored, and enriched over a closed symmetric monoidal category $\mathbf{V}$. In this context, we may choose to define the generic lifting problem using the $\mathbf{V}$-enriched left Kan extension

$$L_0 f = \int^{j\in\mathcal{J}} \underline{\text{Sq}}(j, f) \otimes j,$$

where $\underline{\text{Sq}}(j, f) \in \mathbf{V}$ is the object of commutative squares from j to f defined by the evident pullback involving the hom-objects of $\mathbf{M}$.

The enriched algebraic small object argument produces an algebraic weak factorization system whose underlying left and right classes satisfy an enriched lifting property, defined internally to $\mathbf{V}$. The classes of an ordinary weak factorization system satisfy this enriched lifting property if and only if tensoring with objects in **V** preserves the morphisms in the left class. See [Rie13a, §13] for more details.

Example 11. Consider $\{0 \to R\}$ in the category of modules over a commutative ring R with identity. In analogy with Example 5, the unenriched algebraic small object argument produces the functorial factorization

$$X \xrightarrow{\text{incl}} X \oplus (\oplus_Y R) \xrightarrow{f \oplus \text{ev}} Y$$

whereas the enriched algebraic small object argument produces the factorization

$$X \xrightarrow{\text{incl}} X \oplus Y \xrightarrow{f \oplus 1} Y.$$

Example 12 ([Rie13b])**.** On the category of unbounded chain complexes of R-modules, consider the sets $\{0 \to D^n\}_{n\in\mathbb{Z}}$ and $\{S^{n-1} \hookrightarrow D^n\}_{n\in\mathbb{Z}}$ where D^n is the chain complex with R in degrees n and $n-1$ and identity differential and S^n has R in degree n and zeros elsewhere. The enriched algebraic small object argument converges at step one in the former case and at step two in the latter case to produce the natural factorizations through the mapping cocylinder and the mapping cylinder respectively [BMR13].

Class cofibrantly generated. The algebraic weak factorization systems constructed in Examples 10 and 12 are not cofibrantly generated (in the usual sense) [CH02, Lac07]. The following examples are not cofibrantly generated even in the expanded sense introduced here.

Example 13 ([BR13])**.** There are two algebraic weak factorization systems on topological spaces whose right class is the class of Hurewicz fibrations. A map is a **Hurewicz fibration** if it has the homotopy lifting property, i.e., solutions to lifting problems

$$(5) \qquad \begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle \text{incl}_0}\downarrow & \nearrow & \downarrow {\scriptstyle f} \\ A \times I & \longrightarrow & Y \end{array}$$

defined for every topological space A. As there is proper class of generators, it is not possible to form the coproduct in (2). However, the functor $\mathbf{Top}^{\text{op}} \to \mathbf{Set}$ that sends A to the set of lifting problems (5) is represented by the mapping cocylinder Nf:

$$\begin{array}{ccc} Nf & \longrightarrow & Y^I \\ \downarrow & \lrcorner & \downarrow {\scriptstyle \text{ev}_0} \\ X & \xrightarrow[f]{} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} Nf & \longrightarrow & X \\ {\scriptstyle \text{incl}_0}\downarrow & & \downarrow {\scriptstyle f} \\ Nf \times I & \longrightarrow & Y \end{array}$$

It follows that any lifting problem (5) factors uniquely through the generic lifting problem displayed on the right. The algebraic small object argument proceeds as usual, though there are some subtleties to the proof that it converges.

There is another algebraic weak factorization system "found in the wild": the factorization through the space of Moore paths. The category of algebras for the Moore paths monad admits the structure of a double category in such a way that the forgetful functor to the arrow category becomes a double functor. A recognition criterion due to Garner implies that this defines an algebraic weak factorization system.

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