A solution to the stable marriage problem

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Stable marriages

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The stable marriage problem

Find a stable matching for any dating pool.

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description via a metaphor

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- Day $n+1$: (a.m.) each man rejected on day n proposes to his top remaining choice
  (p.m.) each woman rejects all but her top suitor
- When each man is engaged, the algorithm terminates.

An example

Ken	Bev	Cat	Ada
Leo	Ada	Cat	Bev
Max	Ada	Bev	Cat

Ada	Ken	Leo	Max
Bev	Leo	Max	Ken
Cat	Max	Leo	Ken

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Day 1:

- Leo & Max propose to Ada. Ken proposes to Bev.

An example

Ken	[Bev]	Cat	Ada
Leo	[Ada]	Cat	Bev
Max	[~~Ada~~]	Bev	Cat

Ada	Ken	[Leo]	[~~Max~~]
Bev	Leo	Max	[Ken]
Cat	Max	Leo	Ken

Day 1:

- Leo & Max propose to Ada. Ken proposes to Bev.
- Ada rejects Max.
- Ada & Leo and Bev & Ken are engaged.

An example

Ken	Bev	Cat	Ada
Leo	Ada	Cat	Bev
Max	~~Ada~~	Bev	Cat

Ada	Ken	Leo	Max
Bev	Leo	Max	Ken
Cat	Max	Leo	Ken

Day 2:

- Max proposes to Bev.

An example

Ken	~~Bev~~	Cat	Ada
Leo	[Ada]	Cat	Bev
Max	~~Ada~~	[Bev]	Cat

Ada	Ken	[Leo]	Max
Bev	Leo	[Max]	~~Ken~~
Cat	Max	Leo	Ken

Day 2:

- Max proposes to Bev.
- Bev rejects Ken.
- Ada & Leo and Bev & Max are engaged.

An example

Ken	~~Bev~~	Cat	Ada
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Day 2:

- Max proposes to Bev.
- Bev rejects Ken.
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Day 3:

- Ken proposes to Cat.

An example

Ken	~~Bev~~	Cat	Ada
Leo	Ada	Cat	Bev
Max	~~Ada~~	Bev	Cat

Ada	Ken	Leo	Max
Bev	Leo	Max	~~Ken~~
Cat	Max	Leo	Ken

Day 2:

- Max proposes to Bev.
- Bev rejects Ken.
- Ada & Leo and Bev & Max are engaged.

Day 3:

- Ken proposes to Cat.
- It's a match: Ada & Leo, Bev & Max, Cat & Ken.

A solution to the stable marriage problem

Theorem

The deferred-acceptance algorithm arranges stable marriages.

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Proof:

Each of the women that a given man prefers to his wife rejected him in favor of a suitor she preferred.

Heteronormativity

Heteronormativity is an essential feature of the algorithm: the "stable roommates" problem might have no solution!

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Example

Given "roommate" preferences:

Ada	Bet	-	Dot
Bet	Cat	-	Dot
Cat	Ada	-	Dot
Dot	-	-	-

then whomever is paired with Dot would rather swap to be with the suitor who ranks her first.

Comparing stable matchings

Say a man and a woman are possible for each other if some stable matching marries them.

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In the solution found by the deferred-acceptance algorithm, every man gets his best possible match!

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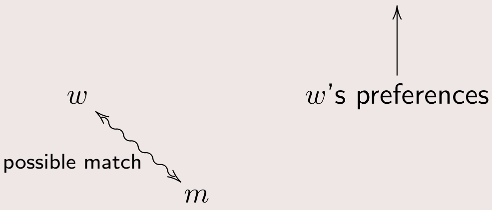
Proof:

We show (by induction) that no man is rejected by a possible wife:



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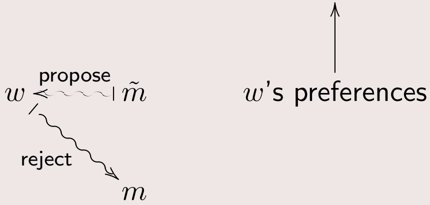

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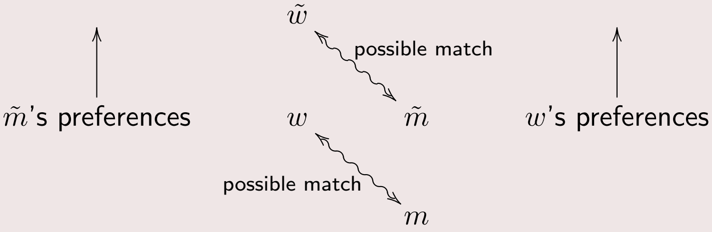

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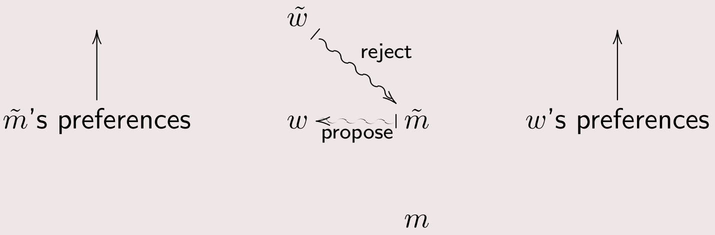

Partial order on stable matchings

Define $\geq_M$ to mean preferred by every man and
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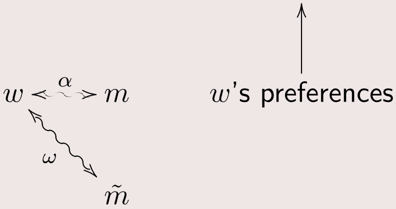
Proof:

Suppose $\alpha \geq_M \omega$ but some woman w prefers α.



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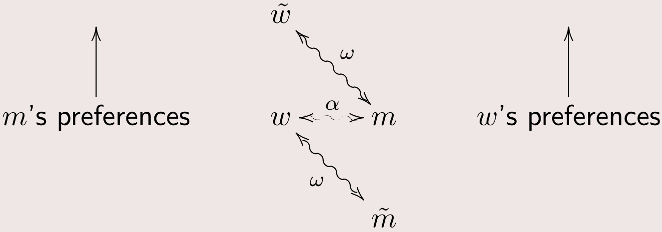

Stability of ω implies that ω matches m to some woman $\tilde{w}$ he prefers, but then $\alpha \not\geq_M \omega$.

The complete lattice of stable matchings

Theorem (Conway)

The set of stable matchings for any fixed dating pool is a complete lattice with partial order $\geq_M$.

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The set of stable matchings for any fixed dating pool is a complete lattice with partial order $\geq_M$.

Proof: construction of $\alpha \vee \omega$

Suppose everyone joins right hands with their α-match and left hands with their ω-match (forming disjoint circles with men facing in and women facing out).

If all drop hands and point at their preferred partner, in each circle, everyone will point in the same direction (so that the men and women have opposite preferences).

Sexism in the male-proposing algorithm

Corollary

In the stable matching found by the male-proposing algorithm, every woman gets her worst possible match!

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Of course it's a different matter if the women propose...

Upshot: waiting to receive proposals is a bad strategy.

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- The women steal preference lists and compute their optimal matches.
- Each woman truncates her preference list below her best match.
- Male proposals will be rejected until the result is women-optimal.

Men shouldn't lie.

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No man or consortium of men can improve their results in the male-proposing algorithm by submitting false preferences.

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Real-world consequences

At least on one side, the deferred-acceptance algorithm is strategy-proof.

National Resident Matching Program: History

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The medical match

- In 1952, after a few false starts, the National Intern Matching Program (NRMP) found a stable solution:
- ... the deferred-acceptance algorithm!

NRMP: Who proposes?

Failure to communicate (Gale-Sotomayor 1985)

"The question of course then arises as to whether these results can be applied 'in practice'. [Gale-Shapley '62] had expressed some reservations on this point,—and then came another surprise. Not only could the method be applied, it had been more than ten years earlier!"

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But who proposes?

- Prior to the mid-1990s the hospitals acted as the proposers.
- After a review by Roth et al., the students propose.

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Theorem (Ronn)

The couples match is NP-complete.