The complicial sets model of higher $\infty$-categories

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A prehistory of higher categorical physics

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The idea of a higher $\infty$-category

An $\infty$-category, a nickname for an $(\infty, 1)$-category, has:

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

A higher $\infty$-category, meaning an $(\infty, n)$-category for $0 \leq n \leq \infty$, has:

- objects
- 1-arrows between these objects
- 2-arrows between these 1-arrows
- $\vdots$
- $n$-arrows between these $n-1$-arrows
- plus higher invertible arrows witnessing composition, units, associativity, and coherence all the way up
Fully extended topological quantum field theories

The $(\infty, n)$-category $\text{Bord}_n$ has

- objects = compact 0-manifolds
- $k$-arrows = $k$-manifolds with corners, for $1 \leq k \leq n$
- $n+1$-arrows = diffeomorphisms of $n$-manifolds rel boundary
- $n+m+1$-arrows = $m$-fold isotopies of diffeomorphisms, $m \geq 1$

often with extra structure (e.g., framing).

A fully extended topological quantum field theory is a homomorphism with domain $\text{Bord}_n$, preserving the monoidal structure and all compositions. The cobordism hypothesis classifies fully extended TQFTs of framed bordisms by the value taken by the positively oriented point.

Dan Freed

On the unicity of the theory of higher $\infty$-categories

The schematic idea of an $(\infty, n)$-category is made rigorous by various models: $\theta_n$-spaces, iterated complete Segal spaces, Segal $n$-categories, $n$-quasi-categories, $n$-relative categories, …

Theorem (Barwick–Schommer-Pries, et al). All of the above models of $(\infty, n)$-categories are equivalent.

Clark Barwick and Christopher Schommer-Pries

- On the Unicity of the Homotopy Theory of Higher Categories
  arXiv:1112.0040

But the theory of higher $\infty$-categories has not yet been sufficiently developed in any model, so there is “analytic” work still to be done.
Goal: introduce a user-friendly model of higher $\infty$-categories

1. A simplicial model of $(\infty, 1)$-categories

2. Towards a simplicial model of $(\infty, 2)$-categories

3. The complicial sets model of $(\infty, n)$-categories

4. Complicial sets in the wild
A simplicial model of $(\infty, 1)$-categories
The idea of a 1-category

A 1-category has:

- objects: ★
- 1-arrows: ★ ➔ ★
- composition: ★ ➔ ★ ➔ ★
- identity 1-arrows: ★ = ★
- identity axioms:
  - $\text{id} \circ f = f$
  - $f \circ \text{id} = f$
- associativity axioms:
  - $h \circ (g \circ f) = (h \circ g) \circ f$
  - $g \circ f$
  - $h$
From 1-categories to $(\infty, 1)$-categories

In an $(\infty, 1)$-category, the composition operation and associativity and unit axioms become higher data.

An $(\infty, 1)$-category has:

- objects $\bullet$; 1-arrows $\bullet \rightarrow \bullet$; identity 1-arrows $\bullet = \bullet$

- composition $f \Rightarrow g$ witnessed by invertible 2-arrows

- identity composition witnesses

- invertible 3-arrows witnessing associativity
A model for $(\infty, 1)$-categories

In a quasi-category, one popular model for an $(\infty, 1)$-category, this data is structured as a simplicial set with:

- 0-simplices = objects
- 1-simplices = 1-arrows
- 2-simplices = binary composites
- 3-simplices = ternary composites
- $n$-simplices = $n$-ary composites

with degenerate simplices used to encode identity arrows and identity composition witnesses.
A model for $(\infty, 1)$-categories

A quasi-category is a “simplicial set with composition”: a simplicial set in which every inner hom can be filled to a simplex.

Low dimensional horn filling:

An inner hom is the subcomplex of an $n$-simplex missing the top cell and the face opposite the vertex $\bullet_k$ for $0 < k < n$.

Corollary: In a quasi-category, all $n$-arrows with $n > 1$ are equivalences.
A quasi-category is a model of an infinite-dimensional category structured as a simplicial set.

- Basic data is given by low dimensional simplices:
  - 0-simplices = objects
  - 1-simplices = 1-arrows
- Axioms are witnessed by higher simplices:
  - 2-simplices witness binary composites
  - 3-simplices witness associativity of ternary composition
- Higher simplices also regarded as arrows: $n$-simplices = $n$-arrows
- Axioms imply that $n$-arrows are equivalences for $n > 1$.

Thus a quasi-category is an $(\infty, 1)$-category, with all $n$-arrows with $n > 1$ weakly invertible.
Towards a simplicial model of $(\infty, 2)$-categories
Towards a simplicial model of an \((\infty, 2)\)-category

How might a simplicial set model an \((\infty, 2)\)-category?

- 0-simplices = \( \bullet \) = objects
- 1-simplices = \( \bullet \xrightarrow{f} \bullet \) = 1-arrows
- 2-simplices = \( \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \) = 2-arrows

Problem: the 2-simplices must play a dual role, in which they are
- interpreted as inhabited by possibly non-invertible 2-cells
- while also serving as witnesses for composition of 1-simplices
  in which case it does not make sense to think of their inhabitants as non-invertible.

Idea: “mark” the 2-simplex witnesses for composition and demand that these marked 2-simplices behave as 2-dimensional equivalences.
Towards a simplicial model of an \((\infty, 2)\)-category

- 2-simplices = \[ f \xrightarrow{g} \quad \xrightarrow{\uparrow} \quad \xrightarrow{h} g \]
  = 2-arrows

- marked 2-simplices witness 1-arrow composition

Now 3-simplices witness composition of 2-arrows:

given

\[
\begin{array}{c}
\bullet_0 \xrightarrow{f} \bullet_1 \xrightarrow{\alpha} \bullet_2 \\
\downarrow h \quad \downarrow \beta \quad \downarrow k
\end{array}
\]

fill

\[
\begin{array}{c}
\bullet_2 \xrightarrow{g} \bullet_3 \\
\downarrow \gamma \quad \downarrow k \circ g
\end{array}
\]

then fill

\[
\begin{array}{c}
\bullet_0 \xrightarrow{0} \bullet_1 \\
\downarrow f \\
\bullet_3
\end{array}
\]
3

The complicial sets model of \((\infty, n)\)-categories
Marked simplicial sets

For a simplicial set to model a higher $\infty$-category with non-invertible arrows in each dimension:

- It should have a distinguished set of “marked” $n$-simplices witnessing composition of $n - 1$-simplices.
- Identity arrows, encoded by the degenerate simplices, should be marked.
- Marked simplices should behave like equivalences.
- In particular, 1-simplices that witness an equivalence between objects should also be marked.

This motivates the following definition:

A marked simplicial set is a simplicial set with a designated subset of marked simplices that includes all degenerate simplices.

The symbol “$\simeq$” is used to decorate marked simplices.
Complicial sets

Recall:

A quasi-category is a “simplicial set with composition”: a simplicial set in which every inner hom can be filled to a simplex.

A complicial set is a “marked simplicial set with composition”: a simplicial set in which every admissible horn can be filled to a simplex and in which composites of marked simplices are marked.

Low dimensional admissible horn filling:

and if $f$ and $g$ are marked so is $g \circ f$. 
A complicial set is a “marked simplicial set with composition”: a simplicial set in which every admissible horn can be filled to a simplex and in which composites of marked simplices are marked.

Low dimensional admissible horn filling:

and if $\alpha$ and $\beta$ are marked so is $\alpha * \beta$. 
An $n$-simplex in a marked simplicial set is $k$-admissible — “its $k$th face is the composite of its $k - 1$ and $k + 1$-faces” — if every face that contains all of the vertices $\bullet_{k-1}, \bullet_k, \bullet_{k+1}$ is marked.

Marked faces include:

- the $n$-simplex
- all codimension-1 faces except the $(k - 1)$th, $k$th, and $(k + 1)$th
- the 2-simplex spanned by $\{\bullet_{k-1}, \bullet_k, \bullet_{k+1}\}$ when $0 < k < n$
- the edge spanned by $\{\bullet_{0}, \bullet_{1}\}$ when $k = 0$ or $\{\bullet_{n-1}, \bullet_{n}\}$ when $k = n$.

An $k$-admissible $n$-horn is the subcomplex of the $k$-admissible $n$-simplex that is missing the $n$-simplex and its $k$-th face.
Strict $\omega$-categories as strict complicial sets

A strict complicial set is a complicial set in which every admissible horn can be filled uniquely, a “marked simplicial set with unique composition.”

Any strict $\omega$-category $\mathcal{C}$ defines a strict complicial set $\mathcal{N}\mathcal{C}$ whose $n$-simplices are strict $\omega$-functors

$$\mathcal{O}_n \to \mathcal{C},$$

where

- $\mathcal{O}_n$ is the free strict $n$-category generated by the $n$-simplex and
- an $n$-simplex is marked in $\mathcal{N}\mathcal{C}$ just when the $\omega$-functor $\mathcal{O}_n \to \mathcal{C}$ carries the top-dimensional $n$-arrow in $\mathcal{O}_n$ to an identity in $\mathcal{C}$.

The strict complicial set $\mathcal{N}\mathcal{C}$ is called the Street nerve of $\mathcal{C}$.

Street-Roberts Conjecture (Verity). The Street nerve defines a fully faithful embedding of strict $\omega$-categories into marked simplicial sets, and the essential image is the category of strict complicial sets.
Strict $\omega$-categories as *weak* complicial sets

Strict $\omega$-categories can also be a source of *weak* rather than *strict* complicial sets, simply by choosing a more expansive marking convention.

Any strict $\omega$-category $\mathcal{C}$ defines a complicial set $N\mathcal{C}$ whose $n$-simplices are strict $\omega$-functors

$$O_n \rightarrow \mathcal{C},$$

where

- $O_n$ is the free strict $n$-category generated by the $n$-simplex and
- an $n$-simplex is marked in $N\mathcal{C}$ just when the $\omega$-functor $O_n \rightarrow \mathcal{C}$ carries the top-dimensional $n$-arrow in $O_n$ to an *equivalence* in $\mathcal{C}$.

Moreover the complicial sets that arise in this way are *saturated*, meaning that every equivalence is marked.
The $n$-complicial sets model of $(\infty, n)$-categories

An $n$-complicial set is a saturated complicial set in which every simplex above dimension $n$ is marked.

For example:

- the nerve of an ordinary 1-groupoid defines a 0-complicial set with everything marked
- the nerve of an ordinary 1-category defines a 1-complicial set with the isomorphisms marked
- the nerve of a strict 2-category defines a 2-complicial set with the 2-arrow isomorphisms and 1-arrow equivalences marked

In fact:

- A 0-complicial set is the same thing as a Kan complex, with everything marked.
- A 1-complicial set is exactly a quasi-category, with the equivalences marked.
A complicial set is a model of an infinite-dimensional category structured as a marked simplicial set.

- Basic data is given by simplices:
  - 0-simplices = objects
  - $n$-simplices = $n$-arrows
- Axioms are witnessed by marked simplices:
  - marked $n$-simplices exhibit binary composites of $(n-1)$-simplices
- Marked simplices define invertible arrows:
  - marked $n$-simplices = $n$-equivalences
- In a saturated complicial set, all equivalences are marked.

An $n$-complicial set, a saturated complicial set in which every simplex above dimension $n$ is marked, is a model of an $(\infty, n)$-category.
Complicial sets in the wild
A $n$-simplicial bordism is a functor from the category of faces of the $n$-simplex to the category of PL-manifolds and regular embeddings satisfying a boundary condition.

- Simplicial bordisms assemble into a semi simplicial set that admits fillers for all horns, constructed by gluing in cylinders.
- By a theorem of Rourke–Sanderson, degenerate simplices exist and make simplicial bordisms into a genuine Kan complex.
The Kan complex of simplicial bordisms can be marked in various ways:

- mark all bordisms as equivalences
- mark only trivial bordisms, which collapse onto their odd faces
- mark the simplicial bordisms that define $h$-cobordisms from their odd to their even faces

**Theorem (Verity).** All three marking conventions turn simplicial bordisms into a complicial set, and the third is the saturation of the second.
Complicial sets defined as homotopy coherent nerves

The homotopy coherent nerve converts a simplicially enriched category into a simplicial set.

Theorem (Cordier–Porter). The homotopy coherent nerve of a Kan complex enriched category is a quasi-category.

Theorem (Cordier–Porter). The homotopy coherent nerve of a 0-complicial set enriched category is a 1-complicial set.

Similarly:

Theorem*(Verity). The homotopy coherent nerve of a $n$-complicial set enriched category is a $n + 1$-complicial set.

In particular, there are a plethora of 2-complicial sets of $\infty$-categories.
References

For more on the complicial sets model of higher $\infty$-categories see:

Dominic Verity

- Complicial sets, characterising the simplicial nerves of strict $\omega$-categories, Mem. Amer. Math. Soc., 2008; arXiv:math/0410412

Emily Riehl

- Complicial sets, an overture, 2016 MATRIX Annals, arXiv:1610.06801

Emily Riehl and Dominic Verity

- Elements of $\infty$-Category Theory, draft book in progress www.math.jhu.edu/~eriehl/elements.pdf (particularly Appendix D: the combinatorics of (marked) simplicial sets)