MATH 727: CATEGORY THEORY IN CONTEXT

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Course description: An introduction to categories, functors, natural transformations, the Yoneda lemma, limits and colimits, adjunctions, monads, and other topics as time permits with the aim of revisiting a broad range of mathematical examples from the categorical perspective.

Overview

Atiyah described mathematics as the "science of analogy"; in this vein, the purview of category theory is *mathematical analogy*. Specifically, category theory provides a mathematical language that can be deployed to describe phenomena in any mathematical context. Perhaps surprisingly given this level of generality, these concepts are neither meaningless and nor in many cases so clearly visible prior to their advent. In part, this is accomplished by a subtle shift in perspective. Rather than characterize mathematical objects directly, the categorical approach emphasizes the morphisms, which give comparisons between objects of the same type. Structures associated to particular objects can frequently be characterized by their *universal properties*, i.e., by the existence of certain canonical morphisms to or from other objects of a similar form.

A great variety of constructions can be described in this way: products, kernels, and quotients for instance are all *limits* or *colimits* of a particular shape, a characterization that emphasizes the universal property associated to each construction. Tensor products, free objects, and localizations are also uniquely characterized by universal properties in appropriate categories. Important technical differences between particular sorts of mathematical objects can be described by the distinctive properties of their categories: that rings have all limits and colimits while fields have few, that certain classes maps are *monomorphisms* or *epimorphisms*. Constructions that take one type of mathematical object to objects of another type are often morphisms between categories, called *functors*. In contrast with earlier numerical invariants in topology, functorial invariants (the fundamental group, homology) tend both to be more easily computable and also provide more precise information. Functors can then be said to *preserve* particular categorical structures, or not. Of particular interest is when a functor describes an *equivalence* of categories, which means that objects of the one sort can be translated into and reconstructed from objects of another sort.

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Category theory also contributes new proof techniques, such as *diagram chasing* or duality; Steenrod called these methods "abstract nonsense."[1] The aim of this course is to introduce the language, philosophy, and basic theorems of category theory. A complementary objective is to put this theory into practice: studying functoriality in algebraic topology, naturality in group theory, and universal properties in algebra.

Practitioners often assert that the hard part of category theory is to state the correct definitions. Once these are established and the categorical style of argument is sufficiently internalized, proving the theorems tends to be relatively easy.[2] The relative simplicity of the proofs of major theorems occasionally leads detractors to assert that there are no theorems in category theory. This is not at all the case! Counterexamples abound in the topics that we will discuss. A list of further major theorems, beyond the scope of this course, appears as an epilogue to the lecture notes.

So why study category theory?

• It's fun, and elegant.

• It provides a useful organizing principle, which can make new mathematical ideas easier to learn and familiar concepts easier to remember:

> The aim of theory really is, to a great extent, that of systematically organizing past experience in such a way that the next generation, our students and their students and so on, will be able to absorb the essential aspects in as painless a way as possible, and this is the only way in which you can go on cumulatively building up any kind of scientific activity without eventually coming to a dead end. — Atiyah "How research is carried out"

• It will give us a chance to explore some really deep ideas, e.g., of *representability*, that are nonetheless relatively accessible. My hope is that you will leave this course feeling more in command of the mathematics that you already know and in a position to more easily absorb the mathematics you will soon learn.

Course logistics

Lectures.

- MW 9-10:15am, Krieger 312

Course website.

- http://www.math.jhu.edu/~eriehl/727

Contact.

- eriehl@math.jhu.edu, Krieger 312

[1] Contrary to popular belief, this was not intended as an epithet.

[2] A famous exercise in Serge Lang's *Algebra* asks the reader to "Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book." Homological algebra is the subject whose development induced Eilenberg and Mac Lane to introduce the general notions of category, functor, and natural transformation.

Office hours.

- Monday 3-4, immediately following each lecture, or by appointment.

Textbook.

- Detailed lecture notes can be found on the course website. These form the primary text. For supplementary reading, I recommend *Basic Category Theory* by Tom Leinster or *Categories for the Working Mathematician* by Saunders Mac Lane.

Assessment

Exercises. Students are expected to work through exercises, which can be found at the end of each section in the course lecture notes. Collaboration is encouraged.

Oral assessment. Roughly every other week, each student will be asked to give a short oral exposition of a selection of these exercises. The specific choice of which exercises to present will be left to the student, but they must be drawn from the relevant sections of the lecture notes. Oral presentations will take place at the chalkboard and will ordinarily require no more than 10 minutes. The grading scheme will be pass (full credit) or fail (no credit). Passing signifies that the student has succeeded in communicating a reasonable understanding of the assigned material. Failed oral examinations may be retaken as many times as necessary within one week of the original assignment.

Extra credit. Any student who brings me an example of a mathematical application of a categorical idea that (a) is not already described in the course lecture notes and (b) has not yet been told to me by anyone else will be rewarded with extra credit. This exercise may be repeated as often as desired.

Exams. There will be three written one-hour exams, to be scheduled after each third of the course, with the aim of facilitating internalization and long-term understanding. Students will be asked to state definitions presented in class and concisely solve a handful of short-answer questions. The focus will be on the main ideas, the material most worth remembering, not on technicalities.

Course grades. A numerical grade will be assigned based on the following formula: 40% oral assignments and 20% for each exam. I predict that everyone who consistently attends the lectures will do very well.

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