Chp 2. 1st order DEs

§2.1. Linear Eq. Method of IF

$y' = f(t, y)$

Solution:

(1) $y' = ay + b \rightarrow y(t) = \frac{b}{a} + \left(y_0 + \frac{b}{a}\right)e^{at}$

(2) $y' = ay + g(t) \rightarrow ?$

(3) $y' = -p(t)y + g(t) \rightarrow ?$

General: $p(t)y' + q(t)y = g(t) \rightarrow ?$

To find a solution, we would like to integrate the equation.

Sometimes we can:

$$(4 + t^2) \frac{dy}{dt} + 2ty = 4t$$

$$\frac{d}{dt}[(4 + t^2)y] = 4t$$

$$(4 + t^2)y = 2t^2 + C$$

$$y = \frac{2t^2 + C}{4 + t^2}$$

In general, we cannot. (e.g. (3))

(Leibniz) Integrating factor method.

Multiply the equation by a factor $\mu(t)$ to make the equation exact that can be integrated.

Remark: we get (1) by integrating the equation

$$y' = a(y + \frac{b}{a})$$

$$\frac{dy}{dt} = a$$

$$\frac{d}{dt}ln|y + \frac{b}{a}| = a$$

$$ln|y + \frac{b}{a}| = a + C$$
\[
\frac{dy}{dt} + ay = g(t)
\]

(\star)

\[
\mu(t) \frac{dy}{dt} + a \mu(t) y = \mu(t) g(t)
\]

To choose \( \mu(t) \) s.t.

\[
11 \quad \frac{d}{dt}[\mu(t)y] = \mu(t) \frac{dy}{dt} + y \frac{d\mu}{dt}
\]

\[
\implies \quad \frac{d\mu}{dt} = a \mu
\]

\[
\mu(t) = e^{at}
\]

Then the equation (\star) becomes.

\[
\frac{d}{dt}(e^{at}y) = e^{at}g(t)
\]

\[
e^{at}y(t) = \int_0^t e^{as}g(s)ds + C
\]

\[
y(t) = Ce^{-at} + \int_0^t e^{as}g(s)ds + C
\]

Example 2: Find the solution of \( y' - 2y = 4 - t \), and plot the graphs of several solutions. Discuss the behavior of the solution as \( t \to \infty \).

Solution: In form of \( y' + ay = f(t) \): \( a = 2 \), \( f(t) = 4 - t \)

Therefore, the integrating factor is \( \mu(t) = e^{-2t} \).

Multiplying \( \mu(t) \) on both side of the eqn., we have

\[
\frac{d}{dt}(e^{-2t}y) = (4-t)e^{-2t} = 4e^{-2t} - te^{-2t}
\]

\[
e^{-2t}y = \int_0^t (4e^{-2s} - te^{-2s})ds + C
\]

\[
y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}
\]

If \( C = 0 \), \( y(t) = -\frac{7}{4} + \frac{1}{2}t \), increase linearly as \( t \to \infty \)

If \( C \neq 0 \), decrease exponentially as \( t \to \infty \)

\[
y = \int_0^t e^{as}g(s)ds + C
\]
General first order linear Eq: \[ y' + p(t) y = g(t) \]

To determine an appropriate integrating factor:

\[ \mu y' + \mu p(t) y = (\mu y)' = \frac{\mu'}{\mu} y + \mu g(t) \]

\[ \mu' = \mu p(t) \]

\[ \frac{\mu'}{\mu} = p(t) \Rightarrow \ln \mu(t) = \int p(t) \, dt + C \]

Multiplying the original Eq by \( \mu(t) \):

\[ (\mu y)' = g(t) \mu(t) \]

\[ \Rightarrow \mu(t)y = \int g(t) \mu(t) + C \]

\[ \int g(t) \mu(t) + C \]

Example 4: Solve the IVP \[ ty' + 2y = 4t^2 \quad t > 1 \]

\[ y(1) = 2 \]

Multiply \( t \) to the Eq:

\[ y' + \frac{2}{t} y = 4t \]

Solution:

Integrating factor: \( \mu(t) = e^{\int \frac{2}{t} \, dt} = e^{2 \ln t} = t^2 \)

\[ (t^2 y)' = 4t \cdot t^2 = 4t^3 \]

\[ t^2 y = t^4 + C \]

\[ y = t^2 + \frac{C}{t^2} \]

\[ z = y(1) = 1 + C \Rightarrow C = 1 \]

\[ y(t) = t^2 + \frac{1}{t^2}, \quad t > 1 \]