§ 2.2 Separable Equations

Recall: \[ \frac{dy}{dt} = ay + b \]
\[ \frac{1}{y + b/a} \, dy = a \, dt \]
\[ \ln |y + b/a| = at + c \]
\[ y = \frac{-b}{a} + \frac{e^c}{a} e^{at} + c \]

Separable equations:
\[ M(x) + N(y) \frac{dy}{dx} = 0 \]
\[ M(x) \, dx + N(y) \, dy = 0 \] (in differential form)
\[ N(y) \, dy = -M(x) \, dx \]
\[ \int N(y) \, dy = -\int M(x) \, dx + C \]

If \( H_0(x) = M(x), \quad H_1(y) = N(y) \). Then
\[ H_2(y) = H_1(x) + C \]

Rule: Any function \( y = q(x) \) that satisfies the above equation is a solution to (1).

- The solution is defined implicitly, rather than explicitly.
- If we given initial condition \( y(x_0) = y_0 \), then from (4), we get \( C = H_2(y_0) - H_1(x_0) \).

Note that
\[ H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s) \, ds, \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s) \, ds \]

We obtain from (1) that
\[ \int_{y_0}^{y} N(y) \, dy = -\int_{x_0}^{x} M(s) \, ds \]

[We can also get this equation directly from (2) by integrating \( \int_{y}^{y_0} \) left = \( \int_{x}^{x_0} \) right.]
Example 1
\[ \frac{dy}{dx} = \frac{x^2}{1-y^2} \quad (5) \]

Solution:
\[(1-y^2)\,dy = x^2\,dx \]
\[y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C \quad (6)\]
This is an equation of integral curves of the eqn.(5). Any differentiable function \(y = f(x)\) that satisfies (5) is a solution to (5).

Example 2
Solve the IVP:
\[ \frac{dy}{dx} = \frac{3x^2(x+2)}{2(y+1)} \quad , \quad y(0) = -1 \]
\[Z(y+1)\,dy = (x^2+4x+2)\,dx \]
\[y^2 - 2y = x^3 + 2x^2 + 2 + C \]
\[-1 = y(0) \Rightarrow 1 - Z(1) = 0 + 0 + 0 + C \Rightarrow C = 3 \]
\[y = 1 + \sqrt{x^2 + 2x^2 + 2x^4} \]
\[-1 = y(0) \Rightarrow y = 1 - \sqrt{x^2 + 2x^2 + 2x^4} \]

Example 3
Solve
\[ \frac{dy}{dx} = \frac{4x^3 - x^3}{4+y^3} \]
and find the solution that passes through the point \((0,1)\) and determine its interval of validity.

Solution:
\[4+y^3 = (4x^3-x^3)\,dx \]
\[y^4 + 16y = 8x^2 - x^4 + C \]
\[x=0, y=1 \Rightarrow C = 16 - 0 = 17 \]
\[y^4 + 16y = 8x^2 - x^4 + 17 \]

The solution is valid until the equation does not hold: \(4+y^2 = 0 \quad y = \pm \sqrt{-4} \rightarrow x \in \pm 3.5 \)
\[\frac{dy}{dx} = \frac{8x^2 - x^4 + 17}{y^2 + 16} \quad \text{valid} \quad \Rightarrow \text{the solution is valid on } (-3.5, 3.5).
Note 1. Sometimes the equation \( \frac{dy}{dx} = f(x, y) \) may have a constant solution \( y = y_0 \).

For example: \( \frac{dy}{dx} = \frac{(y-3) \cos x}{1+2y^2} \) has a constant solution \( y = y_0 \).

Note 2. In Example 2, the solution is solved explicitly for \( y \) as a function of \( x \). However, this solution of an explicit solution is exceptional. In general it is better to leave the solution in implicit form.

\[
\frac{dy}{dx} \cdot \frac{dx}{dy} = F(x, y) \implies \frac{dy}{dx} = F(x, y) \quad \text{(see chp 9)}
\]

Exercise 30: **Homogeneous Equation**

If \( \frac{dy}{dx} = f(x, y) \) can be expressed as a function of \( \frac{y}{x} \) only, then the equation is said to be homogeneous.

Such equations can be transformed into separate equations.

Consider the equation: \( \frac{dy}{dx} = \frac{y + x}{y - x} = \frac{y/x + 1}{1 - y/x} \), homogeneous.

Let \( v = y/x \), \( y = xv(x) \). Then \( \frac{dv}{dx} = v(1) + x \frac{dv}{dx} = \frac{v + 1}{1 - v} \)

\[
x \frac{dv}{dx} = \frac{v + 1}{1 - v} - v = \frac{v + 1 - v(1 - v)}{1 - v} = \frac{v^2}{1 - v}
\]

If \( v 
eq 2 \): \( \frac{1}{v^2} dv = \frac{1}{x} dx \implies \ln (v^2 + 1) = \ln |v| + C \implies C = \frac{1}{(1 + 2)} = \frac{1}{(1 + 2)^2}
\]

\[
\left[ = \frac{1}{v^2} dv = \frac{1}{v^2} dv - \frac{1}{v^2} dv \right] = d\left( \ln |v| - \ln |v^2 + 1| - \ln |v^2 - 1| \right)
\]

If \( v = 2 \): \( y = 2x \).

If \( v = -2 \): \( y = -2x \). 

Hw: 2, 3, 5, 14, 22, 25, 29

For pilot: 32, 33, 32