L5. §2.3 Modeling w/ 1st order DEs

§2.4 Linear vs. Nonlinear Equations

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Mathematical modeling

1. Construct a model
2. Analysis of the model
3. Comparison w/ Experiment or Observation

Recent research trend: infer a model from data based on analysis

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Eve 8. A young person w/ no initial capital invests $k$ $\text{per year}$ at an annual return rate of $r$. Assume that investments are made continuously & the return is compounded continuously.

(a) Der. the sum $S(t)$ at any time $t$.

(b) If $r = 7.5\%$, determine $k$ so that $1m$ will be achieved in $40$ years.

(c) If $r = 2$ $\text{K/year}$, determine the return rate $k$ that must be obtained to have $1m$ available in $40$ years.

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Solu.

(a) \[ \frac{dS}{dt} = rS + k \quad \Rightarrow \quad S(0) = \frac{-k}{r} + (S_0 + \frac{k}{r}) e^{rt} \]

(b) $r = 0.075$,

\[ S(40) = \frac{k}{0.075} (e^{0.075 \times 40} - 1) > 10^6 \Rightarrow k > \frac{10^6 \times r}{e^{rt} - 1} = \frac{7.5 \times 10^4}{e^{0.075 \times 40} - 1} \approx 3930. \]

(c) $S(40) = \frac{k}{r} (e^{10x} - 1) > 10^6$

\[ e^{10x} > 10^6 \Rightarrow x \geq \log_{10} 10^6 \]

\[ x > 6 \]

\[ r > \log_{10} 1.0977. \]
Section 4. Linear vs. Nonlinear Equ.

Existence & Uniqueness (∃!)

Theorem 2.4.1. If \( p(t) \) and \( g(t) \) are continuous on an open interval \( I = (a, b) \)
and the initial time to \( \in (a, b) \), then \( \exists ! \) solution \( y = y(t) \) s.t.

\[
\begin{align*}
y'(t) + p(t)y &= g(t), & t \in I, \\
y(t_0) &= y_0, & \text{where } y_0 \text{ is an arbitrary value}
\end{align*}
\]

Proof: Following the method of Integrating Factor, we have that

\[
y(t) = \frac{1}{\mu(t)} \left[ \int_{t_0}^{t} \mu(s) g(s) \, ds + \psi(t) \right].
\]

\[
\mu(t) = e^{\int_{t_0}^{t} p(s) \, ds}
\]

satisfies the IVP. \( \Rightarrow \) Existence of solution.

Furthermore, any solution \( y(t) \) can be written as above \( \Rightarrow ! \) #

Theorem 2.4.2 (Nonlinear) Let \( f(t, y) \), \( \frac{df}{dy} (y) \) be continuous in a rectangle \( (a, b) \times (Y, S) \)
containing \( (t_0, y_0) \). Then \( \exists! \) an interval \( (t_h - h, t_h + h) \subseteq (a, b) \)
so there is a unique soln. to

\[
\begin{align*}
y' &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

1. Rank about prof. by Implicit function theorem.
2. Rank on the The. Solution exists in a neighborhood of \( (t_0, y_0) \). Not everywhere. In the \( (a, b) \times (Y, S) \)
   - a sufficient but NOT necessary condition.
Example 1: Determine (without solving the problem) an interval on which \( \exists ! \) soln to the IVP

\[
\begin{align*}
    ty' + 2y &= 4t^2 \\
    y(1) &= 2
\end{align*}
\]

Solution:

\[
\begin{align*}
    y' + \frac{2}{t}y &= 4t \\
    y(1) &= 2
\end{align*}
\]

By theorem 2.4.1, soln \( \exists ! \) in either \((-\infty, 0)\) or \((0, \infty)\).

Since \( y(1) = 2 \), \( t = 1 \), the interval should be \((0, \infty)\).

Example 2: Solve the IVP and determine how the interval the soln exists depends on the critical value \( y_0 \):

\[
\begin{align*}
    y' &= 4t/y; \quad y(0) = y_0 \quad \frac{1}{2}y^2 = -2t^2 + c \\
    y_0 &< 0: \text{No}
\end{align*}
\]

Example 3:

\[
\begin{align*}
    y' &= y^{1/3}, \quad t \geq 0 \\
    y(0) &= 0
\end{align*}
\]

Solution:

\[
y^{-1/3} \, dy = dt \quad \Rightarrow \quad \frac{3}{2} y^{\frac{2}{3}} = t + C
\]

\[
y = \left( \frac{2}{3} (t + C) \right)^{3/2}
\]

\( y(0) = 0 \quad \Rightarrow \quad C = 0 \quad \Rightarrow \quad y(t) = \left( \frac{3}{2} t \right)^{3/2}, \quad t \geq 0.
\]

Other solutions:

\[
y = -\left( \frac{3}{2} t \right)^{3/2}
\]

(\( \infty \)-many solutions)

\[
y \equiv 0
\]

\[
y = 1, \quad \text{if } 0 \leq t < t_1
\]

\[
y = -\left( \frac{3}{2} (t - t_0) \right)^{3/2}, \quad t \geq t_0
\]

If \( y(t_0) = y_0 \), \( t_0 > 0 \), \( y(t) \rightarrow \infty \)

Near \((t_0, y_0)\), \( \exists ! \) soln.