Consider the Sturm-Liouville eigenvalue problem:
\[ (p(x)q')' + q(x) = -\lambda \sigma(x) q, \quad \text{subject to } \begin{align*}
\beta_1 q(a) + \beta_2 q'(a) &= 0, \\
\beta_3 q(b) + \beta_4 q'(b) &= 0.
\end{align*} \]

1. All the eigenvalues \( \lambda \) are real.
2. There exist an infinite number of eigenvalues:
   \[ \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots \]
   a. There is a smallest eigenvalue, usually denoted \( \lambda_1 \).
   b. There is not a largest eigenvalue and \( \lambda_n \to \infty \) as \( n \to \infty \).
3. Corresponding to each eigenvalue \( \lambda_n \), there is an eigenfunction, denoted \( \phi_n(x) \) (which is unique to within an arbitrary multiplicative constant). \( \phi_n(x) \) has exactly \( n - 1 \) zeros for \( a < x < b \).
4. The eigenfunctions \( \phi_n(x) \) form a "complete" set, meaning that any piecewise smooth function \( f(x) \) can be represented by a generalized Fourier series of the eigenfunctions:
   \[ f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x). \]

Furthermore, this infinite series converges to \( (f(x^+) + f(x^-))/2 \) for \( a < x < b \) (if the coefficients \( a_n \) are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function \( \sigma(x) \). In other words,
   \[ \int_a^b \phi_n(x)\phi_m(x)\sigma(x) \, dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m. \]
6. Any eigenvalue can be related to its eigenfunction by the Rayleigh quotient:
   \[ \lambda = \frac{\int_a^b \phi' \sigma \phi \, dx}{\int_a^b \phi \sigma \phi \, dx} = \frac{\langle \phi, \phi \rangle_{\sigma}}{\langle \phi \rangle_{\sigma}}, \]
   where the boundary conditions may somewhat simplify this expression.

To do: prove 1 & 5.

Denote \( L(\phi) = (p\phi')' + q\phi \). \( \lambda \) is a linear operator \( L(\psi_1, \psi_2) = \lambda_1 \psi_1 + \lambda_2 \psi_2 \).

Then the SLE becomes \( L(\phi) = -\lambda \phi \).

Let's prove 5. First: let \( (\lambda_n, \phi_n), (\lambda_m, \phi_m) \) be two solutions to SLE:

Then \( L(\phi_n) = -\lambda_n \phi_n \Rightarrow \int_a^b L(\phi_n) \phi_m \, dx = -\lambda_n \int_a^b \phi_n \phi_m \, dx \).

\[ \Rightarrow \int_a^b \left[ L(\phi_n) \phi_m - L(\phi_m) \phi_n \right] \, dx = -\lambda_n \int_a^b \phi_n \phi_m \, dx \]
A more general equality: for any two functions \(u, v \in C^2\)

\[
U_L(v) - V_L(u) = \left( p \left( u v' - v u' \right) \right)'
\]

Logrange's identity.

**Proof.**

\[
u L(v) = u \left[ (p v') y + q v \right] = u \left( p v' + p v u' \right) + q u v
\]

\[
v L(u) = v \left[ (p u') y + q u \right] = v \left( p u' + p u v' \right) + q u v
\]

\[
u L(v) - v L(u) = p' \left( u v' - v u' \right) + p \left( u v'' - v u' \right)
\]

Note that \(p\left( u v' - v u' \right) = p' \left( u v' - v u' \right) + p \frac{d}{dt} \left( u v' - v u' \right)\).

\[
\Rightarrow \text{Green's formula: } \int_a^b \left[ U_L(v) - V_L(u) \right] dx = p \left( u v' - v u' \right) \bigg|_a^b
\]

Then we have

\[
\int_a^b \left[ L(p a) - L(p b) \right] dx = p \left( q p_a^b - p q p_a^b \right) = 0
\]

Note that B.C.\( \begin{pmatrix} q_n(a) & q_n(b) \\ \frac{q_n'(a)}{\alpha} & \frac{q_n'(b)}{\beta} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \)

has a sol. \( (\alpha, \beta) \) s.t. \( \alpha^2 + \beta^2 = 0 \).

\(q_n \text{ & } p_n \text{ satisfy Eq. (a)}\)

Equivalently, \( A x = 0 \) has a non-zero soln.

(B1) \( \alpha q_n(a) + \beta q_n(b) = 0 \) \( \Rightarrow \text{det}(B) = 0 \)

(B2) \( \beta q_n'(a) + \beta q_n'(b) = 0 \)

Similarly:

\( q_n(b) p_n'(b) - q_n'(b) p_n(b) = 0 \)

Remark: Greens formula w/ B.C.:

\[
\int_a^b \left[ U_L(v) - V_L(u) \right] dx = 0
\]

Integration by parts:

\[
\int_a^b \left[ \frac{d}{dx} \left( u v \right) \right] dx = \int_a^b u \frac{d}{dx} v dx
\]

\[
\langle v, u' \rangle = \langle u, v' \rangle \Rightarrow L^* = L \text{ self-adjoint operator.}
\]

\( \text{matrix: } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is Hermitian.} \Rightarrow \langle u, Av \rangle = \langle Au, v \rangle = \langle Av, u \rangle \)
Next, we prove 1. the eigenvalues are real.

Let \((\lambda, \phi)\) be a solution. Then
\[
L \phi = -\lambda \phi(\alpha) \phi
\]
Take complex conjugate:
\[
\bar{L} \phi = -\lambda \bar{\phi}(\alpha) \bar{\phi}
\]
Note that
\[
\frac{L \phi}{\phi(\alpha)} = \frac{\bar{L} \bar{\phi}}{\bar{\phi}(\alpha)} = \frac{L \phi}{\phi(\alpha)}
\]

If \(\lambda \) is not real, then \(\lambda = \lambda + \bar{\lambda} \), then by (3), \(\phi \bar{\phi} \) must be orthogonal.
Then
\[
\int_0^\beta |\phi(\alpha)|^2 d\alpha = 0
\]
\[
= \int_0^\beta |\phi(\alpha)|^2 d\alpha \text{ which } \neq 0 \text{ unless } |\phi| = 0.
\]
Therefore, \(\lambda = \lambda \). \(\Rightarrow\) \(\lambda \) is real.

To prove (6), let \((\lambda, \phi)\) be a solution to the SLE. \(\Rightarrow\)
\[
L \phi = -\lambda \phi(\alpha) \phi
\]
\[
\langle \phi, \phi \rangle = \int_0^\beta \phi(\alpha) \bar{\phi}(\alpha) d\alpha
\]
\[
= -\lambda \langle \phi, \phi \rangle
\]
\[
\lambda = -\frac{\langle \phi, \phi \rangle}{\langle \phi, \phi \rangle}
\]
\[
\lambda = -\frac{-\lambda |\phi|^2}{2|\phi|^2}
\]

Question: for each eigenvalue, is the eigenfunction unique?

\[
\begin{bmatrix}
L \phi_1 = -\lambda_1 \phi_1 \\
L \phi_2 = -\lambda_2 \phi_2
\end{bmatrix}
\]

Ans: suppose we have two eigenfunctions \(\phi_1 \) & \(\phi_2\) then
\[
\langle \phi_2, L \phi_1 \rangle = -\lambda \langle \phi_2, \phi_1 \rangle
\]
\[
\langle \phi_1, L \phi_2 \rangle = -\lambda \langle \phi_1, \phi_2 \rangle
\]
\[
\Rightarrow \langle \phi_2, L \phi_1 \rangle = \langle \phi_1, L \phi_2 \rangle = 0
\]

By Lagrange's Identity: \((P(\phi_1, \phi_2)) = 0 \Rightarrow P(\phi_1, \phi_2) = \text{constant.}
\]

[(P1)]
If \((BC)\) is regular, then 
\[
\left( \phi_1 \phi_2' - \phi_2 \phi_1' \right)(x) = 0.
\]
Then \(\text{const} = 0\), we must have 
\[
\left( \phi_1 \phi_2' - \phi_2 \phi_1' \right)(x) = 0, \quad \forall x \in [a,b].
\]

\[
\phi_1, \left( \frac{\phi_1'}{\phi_1} \right) = \phi_1 \left( \frac{\phi_2' - \phi_2 \phi_1'}{\phi_1^2} \right) = 0
\]

\[
\Rightarrow \left( \frac{\phi_2'}{\phi_2} \right)' = 0 \Rightarrow \frac{\phi_2'}{\phi_2} = c \Rightarrow \phi_2 = c \phi_1
\]

linearly dependent.