Homework 1
Due date: Feb. 6th

1. Consider a first-order system

\[
\frac{d}{dt} x_i(t) = \frac{1}{N} \sum_{i' = 1}^{N} \phi(\|x_j(t) - x_i(t)\|)(x_j(t) - x_i(t)), \quad \text{for } i = 1, \ldots, N, \tag{1}
\]

where \(x_i(t) \in \mathbb{R}^d\) represents the position of agent \(i\) at time \(t\).

(a) Let \(\phi \in L^\infty(\mathbb{R}^+)\) be a piecewise global Lipschitz function. Show that there exists a unique solution for any initial condition \(x(0) \in \mathbb{R}^{Nd}\).

(b) [optional] What if \(\phi(r) = \Phi(r)\) with \(\Phi(r) = c_1 (r^{-12} - c_2 r^{-6})\), the Lennard-Jone potential? This is an interaction kernel with short range repulsion and long-range attraction.

2. Consider a second-order system

\[
\frac{d}{dt} x_i(t) = v_i(t), \quad \text{for } i = 1, \ldots, N,
\]

\[
\frac{d}{dt} v_i(t) = \frac{1}{N} \sum_{i' = 1}^{N} \phi(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)), \tag{2}
\]

Let \(\phi \in L^\infty(\mathbb{R}^+)\) be a piecewise global Lipschitz function. Show that there exists a unique solution for any initial condition \((x(0), v(0)) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\).