Ex.1. Consider the system
\[ \frac{d\tilde{x}_i(t)}{dt} = \frac{1}{N} \sum_{j=1}^{N} \varphi(\tilde{x}_j(t) - \tilde{x}_i(t)) \quad i = 1, \ldots, N. \]
\[ \tilde{x}_i(0) = \tilde{x}_i \in \mathbb{R}^d \]
Show that global soln. exists for the following type of \( \varphi: \mathbb{R}^2 \to \mathbb{R}^d \):

(a) \( \varphi \) is piecewise \( \mathcal{L} \) Lipschitz; \( \mathcal{L} > 0 \).

(b) The Lennard-Jones kernel, \( \varphi(r) = \varphi(t) = 4\varepsilon \left[ (\frac{\sigma}{r})^2 - (\frac{\sigma}{r})^6 \right] \)

Proof (sketch): issue to address: \( \varphi \) may be discontinuous; e.g. piecewise constant

1. If \( \varphi \) is local Lipschitz, then local existence + A priori bd. \( \Rightarrow \) global soln.

2. (uniqueness: \( \langle \varphi(\tilde{x}), \tilde{\varphi}(\tilde{x}) \rangle = \tfrac{1}{2} \sum \varphi(\tilde{x}_j) \tilde{\varphi}(\tilde{x}_j) \leq 0 \Rightarrow \tilde{x}(t) = \tilde{x}(0) \).)

\( \mathbf{\Rightarrow} \) global \( \mathbf{\Rightarrow} \) local Lipschitz.

2. If \( \varphi \) is piecewise \( \mathcal{L} \) Lipschitz, let \( \varphi_k \to \varphi \) pointwise, \( \varphi_k \) Lipschitz.

\[ \tilde{x}_k(t) = \tilde{x}(0) + \int_0^t \varphi_k(\tilde{x}_k(u)) \, du, \text{ global solns.} \]

1. \( \| \tilde{x}_k \|_{L^1([0,T])} \leq C + T \| \varphi_k \|_{L^1([0,T])} \leq C (1 + T \mathcal{L}_k) \)

2. \( \tilde{x}_k \) is uniformy bd. \( \mathbf{\Rightarrow} \) global soln.

\[ \int_0^t \| \varphi_k(\tilde{x}_k(u)) \| \, du \leq C(t,T) \]

(5) \( \tilde{x}(t) = \lim \tilde{x}_k(t) \quad \text{in } L^1([0,T]) \)

(6) \( \int_0^T |\varphi_k(\tilde{x}_k(t)) - \varphi(\tilde{x}(t))| \, dt \to 0 \)

(7) \( \tilde{x}(t) \to \tilde{x}(0) \quad \text{in } \mathbb{R}^d \)

1b) \( \tilde{x}_i(0) \in \mathbb{R}^d \quad \text{if } \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_d) \in \mathbb{R}^d \quad \text{and} \quad \tilde{x}_i(0) \neq \tilde{x}_j(0) \quad \text{for } i \neq j \quad \text{(otherwise, not defined)}.

1c) Global existence: need to show a priori bd. We already have \( |\tilde{x}(t)| \leq |\tilde{x}(0)| \) but here we need \( \tilde{x}(t) \in I \), i.e., no collision.

Ex.2: Similar to 1(b).