In case $\psi_j (v_i, s) = \frac{1}{2} (v_i - v_j)^2,$ \((3.7)\) e.g. C-S $\psi(t) = (1 + t)^{-\beta}$

Prop 6.5 under the assumption in Thm 6.3, assume \((3.7)\) and that $\psi$ is bdd.

Then $L^{2,2}$-swarming implies $L^{2,2}$-flocking

**Proof:** Recall $L^{2,2}$-swarming $\iff$ sup $\sup_{t \geq 0}$ $E[|\psi(t)|^2]$ $< \infty$ $\&$ lim $\sup_{t \to \infty}$ $E[|\psi(t)|^2] < \infty$

Lemma 3.3 Assume $\psi_j (v_i, s) = \frac{1}{2} (v_i - v_j)^2,$ where $\psi$ is a bdd function, and

1. $\exists$ a unique soln. $(v_i, t)_{t \geq 0}$, and that $\psi_j (v_i, t)$ is a $L^{1}$ semi-mix.
2. $\psi_j$, $s \mapsto E[\psi_j (v_i, t) - \psi_j (v_i, s)]$ is well defined & $C^0$
3. The system is $L^{2,2}$-strong swarming

Then the system is $L^{2,2}$-flocking $\iff$ $E[\psi(t)] < \infty$ $\&$ $E[|\psi(t)|^2] < \infty$

(1) $\langle v_i (t), v_j (t) \rangle$ is an $L^{2}$-semi-mix b.c.

$\frac{d\bar{x}_i}{dt} = u_i$ $\frac{d\bar{v}_i}{dt} = -\frac{1}{m_i} \frac{\partial v_i}{\partial v_i} (v_i - v_j) \frac{d\bar{v}_i}{dt} + \bar{w}_i$ $d\psi_j = \bar{d} w_i = \bar{d} w_j = \bar{d} w_i + \bar{d} w_j$

$$\Rightarrow \quad \frac{d\bar{w}_i}{dt} = \left( \langle \bar{w}_i, d\bar{w}_i \rangle + \int d\bar{w}_i \bar{d} w_i \right)$$

$\Rightarrow \quad \langle \bar{w}_i (t), \bar{w}_i (s) \rangle = \langle \bar{w}_i (s), \bar{w}_i (s) \rangle + \int_s^t \langle \bar{w}_i (r), \bar{w}_i (r) \rangle \quad \forall t, s \geq 0$

$$\Rightarrow \quad \langle \bar{w}_i (t), \bar{w}_i (s) \rangle = \langle \bar{w}_i (s), \bar{w}_i (s) \rangle + \frac{1}{2} \int_s^t \langle \bar{w}_i (r), \bar{w}_i (r) \rangle \quad \forall t, s \geq 0$$

$\int_s^t \bar{d} w_i \bar{d} w_i = \int_s^t X_{\bar{w}_i} [\bar{w}_i (r) - \bar{w}_i (r)] \bar{d} w_i = \int_s^t X_{\bar{w}_i} [\bar{w}_i (r) - \bar{w}_i (r)] \bar{d} w_i \quad \forall s \leq t \leq \infty$

(2) $E[|\psi(t)|^2] \leq m C_6 b.c.$ $\Rightarrow$ $\psi(t) = \psi(t) < C_6$ b.c.

$\psi(t) = 2 \langle v_i, \bar{w}_i \rangle + \frac{1}{2} \langle \bar{d} w_i \bar{d} w_i \rangle$ $\psi(t) = \psi(t) < C_6$ b.c.

$|\psi(t)|^2 = \psi(t) \psi(t) + \frac{1}{2} \langle \bar{d} w_i \bar{d} w_i \rangle$ $\psi(t) = \psi(t) < C_6$ b.c.

$E[|\psi(t)|^2] = E[|\psi(t)|^2] + \frac{1}{2} E[\psi(t) \psi(t) + \psi(t) \psi(t) + \psi(t) \psi(t)]$ $\psi(t) = \psi(t) < C_6$, where boldness follows from $\psi$ is bdd.

#
To consider the C-S kernel, \( \Rightarrow \) the communication rate decay, w/ distance.
we need to remove the condition \( N_{\min} \rightarrow \inf \) \( \forall (r) > 0 \).

Define:
\[
\Phi(r) = \min_{\nu \in \Omega} \Phi(\nu) \quad T_r = \inf \{ s > 0; \max_{\xi_{1,2}} | y(\xi_{1,2}) - y(\xi_{1,2}) | > r \}
\]
we would like to show that \( T_r = +\infty \) w/ a high prob.

**Rmk 6.6** (Back to the deterministic model). Assume \( \delta = 0 \).

\[ z(t) = \frac{1}{\lambda_{in}} \sum_{k=0}^{\infty} \left\{ y(t) \right\}
\]

1. \( dz(t) \leq 0 \Rightarrow z(t) \leq z(0) \Rightarrow \sup_{t} | y(t) - y(t) | < \infty \), \( \forall t \).

So, the process is\( f \)-locking as soon as \( \sup_{t} | y(t) - y(t) | < \infty \), \( \forall t \).

But \( \sup_{t} | y(t) | \leq \left\{ y(t) \right\} + \int_{0}^{t} | y(s) | ds \leq \int_{0}^{t} \frac{z(u)^{\frac{1}{2}}}{\frac{1}{\lambda_{in}}} \left( 1 - e^{-\lambda_{in} t} \right) \]

\[ \Rightarrow \sup_{t} | y(t) | \leq \left\{ y(t) \right\} + \frac{\lambda_{in}^{\frac{1}{2}}}{\lambda_{in}} \int_{0}^{t} \frac{e^{-\lambda_{in} t}}{\lambda_{in}} \]

In particular, \( \frac{\text{RHS} \leq \gamma}{=\gamma} \), then \( T_r = +\infty \), and the system is\( f \)-locking.

\[ r = \sup_{t} \left\{ y(t) \right\} \quad r_{0} + \frac{\lambda_{in}^{\frac{1}{2}}}{\lambda_{in}} \left( \frac{\lambda_{in}^{\frac{1}{2}}}{\lambda_{in}} \right)^{\gamma} \leq \gamma \]

\( \gamma = C r_{0} \quad C = \frac{1}{\lambda_{in}^{\frac{1}{2}}} \left( 1 - \lambda_{in}^{\frac{1}{2}} \right) \left( \lambda_{in}^{\frac{1}{2}} \right)^{\gamma} \quad \infty \)

Thus, a sufficient condition for unconditional\( f \)-locking (for all ICs) is that,

for any \( (z(t), r_{0}) \), there exists \( C^{-1} t \)-satisfy true, or

\[ \lim_{C \to \infty} \lambda_{in}(C^{-1}) \frac{r_{0}^{1/2}}{C r_{0}^{1/2}} = \infty \]

\[ = \frac{\lambda_{in}(C^{-1})}{(C r_{0}^{1/2})^{2}} \left( \frac{1}{(C r_{0}^{1/2})^{2}} \frac{y_{0}^{1/2}}{(C r_{0}^{1/2})^{2}} \right) \]

\[ \text{for some } 0 < \delta \leq \frac{1}{2} \left( \frac{y_{0}^{1/2}}{(C r_{0}^{1/2})^{2}} = 1 \text{ of } C=r \right) \]
In stochastic case, much to show $T_r = \infty$ (w. a certain prob.)

For $t < T_r$, we have

$$h(t, \xi(t)) = \Theta(\xi(t), t)$$

Then, if $\Theta(\xi(t), t) \to 0$ a.s., $Z(t) \to z_0$ as $T_r \to \infty$.

To understand the behavior of $T_r$, write as above

$$|Y_g(t, T_r)| \leq \left| Y_g(z_0) \right| + \int_{t_0}^{T_r} |Y_g(s)| ds$$

Note: If $T_r = \infty$, we have, as earlier, $\sup_{t \geq 0} |Y_g(t)| < \infty$.

But we need to control the a.s. behavior of $N(t) = e^{M_t - t M(\tau)}$, so as to have an estimate like the deterministic case.

Lemma. Let $(M_t)$ be a m.b. satisfying $\langle M \rangle_t \in C_t$. Then for

$$S(a, b) = \inf \{ t \geq 0 : M_t - b \langle M \rangle_t > a \}$$

we have

$$p(S(a, b) < \infty) \leq e^{-2ab}$$

Proof. Under the assumption, $e^{\int \theta M_t - \frac{1}{2} \gamma^2 \langle M \rangle_t} \leq e^{a}$ a.s. for

$$\int \gamma^2 \langle M \rangle_t \leq \gamma^2 \langle M \rangle_t$$

Thus

$$IE \left[ e^{\int \gamma^2 \langle M \rangle_t} \right] = 1, \quad \forall t, \gamma > 0$$

Therefore

$$IE \left[ e^{b M_t - \frac{1}{2} b^2 \langle M \rangle_t} \right] = 1, \quad \forall t, b > 0$$

$$\Rightarrow \quad IE \left[ \int_{t_0}^{T_r} e^{b M_s - \frac{1}{2} b^2 \langle M \rangle_s} \right] \leq 1$$

$$\Rightarrow \quad p(S < \infty) \leq e^{-2ab}$$

Remark: If $M = \text{standard Brown}$, the equality holds.
Now, on \( S(a,b) = \omega_{\frac{1}{2}} \), whose probability is larger than \( 1 - e^{-2ab} \),
\[
M_r - b < M^2 \leq a, \quad \forall \xi.
\]
\[
\Rightarrow \quad |x_g(t)\sum_{r=0} \leq \sqrt{\lambda e^t} \int_{e^{-t}} \frac{1}{\lambda \mathcal{K}(r^2)} e^{\frac{1}{2} \lambda \mathcal{K}(r^2)} e^{\lambda \mathcal{K}(r^2) - (2)k^2a^2 - 2k^2(b + \frac{1}{2})} d\lambda
\]
\[
\Rightarrow \quad \lim_{T \to \infty} P(T < 2a, S(a,b) = \omega_{\frac{1}{2}}) \geq 1 - e^{-2ab}.
\]

\( \Rightarrow \quad \textbf{THEOREM 6.9.} \quad \text{In the situation of Theorem 6.3, assume in addition that (3.7) is in force. Let } r > 0. \text{ Let } a, b > 0. \text{ Assume that:}
\]
- \( \lambda \psi_1(r^2) > 2K^2(d^2 + (b - \frac{1}{2})) \) where \( \psi_1 \) is defined in (6.6),
- the initial condition satisfies, for all \( (i, j) \),
\[
|x_i(0) - x_j(0)| + \frac{z(0)e^{\frac{1}{2} r^2}}{\lambda \psi_1(r^2) - 2K^2(d^2 + (b - \frac{1}{2}))} < r,
\]
where \( z(0) = \sum_{k=1}^{N} |v_k(0) - \bar{v}(0)|^2 \).

Then the system (6.2) is flocking with a probability larger than \( 1 - e^{-2ab} \).

\( \text{REMARK 6.10.} \quad \text{Note that, as in Theorem 6.3, when } \sigma \text{ is diagonal, we may replace condition } \lambda \psi_1(r^2) > 2K^2(d^2 + (b - \frac{1}{2})) \text{ by } \lambda \psi_1(r^2) > 2K^2(d^2 + (b - \frac{1}{2})). \text{ If in addition the diagonal term } \sigma^{k,k}(v) = \sigma^{k,k}(u^k), \text{ we may replace it by } \lambda \psi_1(r^2) > 2K^2(1 + (b - \frac{1}{2})). \)

\( \text{REMARK 6.11.} \quad \text{Remark that when } K = 0 \text{ corresponding to a constant } \sigma, \text{ we may take any } b \text{ going to infinity and } a \text{ going to } 0 \text{ so that } ba \text{ goes to infinity. We thus obtain almost sure flocking under the same initial conditions than for the deterministic result [in particular for any initial condition if } r\psi_1(r^2) \to +\infty \text{ as } r \to +\infty]. \text{ This is not surprising since the microscopic variables satisfy the deterministic system of differential equations. Only the center of mass is driven by some Brownian motion.} \)