Chp 4.5 Gradient system

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\beta^{-1}}\, dW_t \ ; \quad X_0 = x \qquad (*)$$

V: potential

<u>Example</u> $V(x) = \frac{1}{2}|x|^2$; $\quad dX_t = -X_t dt + \sigma dW_t$

$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, $x \in \mathbb{R}$: $\quad dX_t = (-X_t + X_t^3)dt + \sigma dW_t \quad$ double-well potential.

<u>Def (confining potential)</u> V is a confining potential if $\lim_{|x|\to\infty} V(x) = +\infty$, $e^{-\beta V(x)} \in L^1(\mathbb{R}^d)$, $\forall \beta > 0$.

<u>Proposition 4.2</u>. The SDE $(*)$ w/ a confining potential V is ergodic w/ stationary density

$$\boxed{\rho_\beta(x) = \frac{1}{Z} e^{-\beta V(x)}, \qquad Z = \int e^{-\beta V(x)} dx.}$$

Gibbs distribution

Recall: definition of an ergodic process:

- physics: $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_s)\,ds = \int f(x)\,\mu(dx)$, $\forall f \in C_b$. μ is the ergodic meas.

  Long time average = phase-space avg.

- Markov process: $\partial_t \rho = L^* \rho \Rightarrow \rho(t,\cdot) = e^{L^* t}\rho(0,\cdot)$ $\qquad L^*\mu = 0 \Leftrightarrow$ invariant meas.

  $\lim_{t\to\infty} \rho(t,\cdot) = \mu$, $\quad \forall \rho(0,\cdot)$; $\quad \Leftrightarrow \mu$ ergodic.

- The Markov process w/ generator $\boxed{\mathcal{L} = -\nabla V\cdot\nabla + \beta^{-1}\Delta}$ $\xleftrightarrow{\text{weak!}}$ (X_t)

$$\begin{cases} \partial_t u = Lu\,; \\ u(0,x) = f(x) \end{cases} \qquad f(X_t) = f(x) + \int_0^t Lf(X_s)\,ds + mG.$$

$$u(t,x) = \mathbb{E}^x[f(X_t)]$$

- The Fokker-Planck equation: $\boxed{L^* = \nabla\cdot(\nabla V\cdot) + \beta^{-1}\Delta} = \nabla\cdot(\nabla V\cdot + \beta^{-1}\nabla\cdot)$

  $\partial_t \rho = L^*\rho$ $\qquad$ (Smoluchowski Equ.)

$$\nabla\rho_\beta = \rho_\beta(-\beta\nabla V), \quad \Delta\rho_\beta = \rho_\beta[-\beta^2|\nabla V|^2 - \beta\Delta V]$$

"Proof" Outline:

1> ρ_β is a stationary density: $0 = L^*\rho_\beta = \nabla\cdot(\nabla V\rho_\beta) + \beta^{-1}\Delta\rho_\beta = \Delta V\rho_\beta + \rho_\beta(-\beta|\nabla V|^2) + \beta^{-1}\Delta\rho_\beta = 0$

2> ρ_β is a unique solu. to $0 = L^*\rho_\beta$ s.t. $\int \rho_\beta(x)dx = 1$, $\rho_\beta(x) \geq 0$.

spectral gap.

3> Convergence: $\rho(t,\cdot) \xrightarrow{t\uparrow\infty} \rho_\beta(\cdot)$.

$\boxed{\rho(t,x) = h(t,x)\,\rho_\beta(x),\ \forall x}$ $\quad \partial_t h = \mathcal{L}h$ (backward Kolmogorov).

Step 2 follows from the next proposition. $\quad \mathcal{L} = \mathcal{L}^*$ on $L^2(\rho_\beta)$: $\quad \mathcal{L}h = 0 \Rightarrow h = \text{const}$.

Let $L^2(\rho_\beta) = \{ f : \int |f|^2(x)\rho_\beta(x)dx < \infty \}$ with inner product $\langle f, h\rangle_\rho$.

Proposition 4.3 Let V be a smooth confining potential. Then, the operator $\mathcal{L} = -\nabla V\cdot\nabla + \beta^{-1}\Delta$ is self-adjoint in $L^2(\rho_\beta)$. Furthermore, it is nonpositive and its kernel consists of constants.

Proof: $\forall f, h \in C_0^2(\mathbb{R}^d)\cap L^2(\rho_\beta)$: (further discussion on domain of $\mathcal{L}$ and $\mathcal{L}^*$)

$$\langle \mathcal{L}f, h\rangle_{\rho_\beta} = \int (\cancel{-\nabla V\cdot\nabla f} + \beta^{-1}\Delta f) h \rho_\beta dx = \int \cdots$$

$$= -\beta^{-1}\int \nabla f\cdot\nabla h\, \rho_\beta\, dx = \langle f, \mathcal{L}h\rangle_\rho .$$

$$\beta^{-1}\int \Delta f\, h\rho_\beta = -\beta^{-1}\int \nabla f\cdot \underbrace{\nabla(h\rho_\beta)}_{=\nabla h\,\rho_\beta + h\nabla\rho_\beta}, \quad \nabla\rho_\beta = -\beta\nabla V\rho_\beta$$

$$= -\beta^{-1}\int \nabla f\cdot\nabla h\,\rho_\beta \underbrace{- \beta^{-1}\int \nabla f\cdot h(-\beta)\nabla V\rho_\beta}_{=\cancel{+\int\nabla f\cdot\nabla V h\rho_\beta}}$$

It is nonpositive: $\underline{\langle \mathcal{L}f, f\rangle_{\rho_\beta} = -\beta^{-1}\|\nabla f\|^2_{L^2(\rho_\beta)}} \le 0$.

Its kernel: $\mathcal{L}f = 0 \Rightarrow \langle\mathcal{L}f, f\rangle = 0 \Rightarrow \|\nabla f\|^2_{L^2(\rho_\beta)} = 0 \Rightarrow f \equiv$ constant. #.

Thm 4.4 (Convergence to equilibrium). Let $\rho(t,\cdot)$ be the sln. to $\partial_t\rho = \mathcal{L}^*\rho$ with initial $\rho(0,\cdot)\in L^2(\mathbb{R}^d, \rho_\beta^{-1})$.

Assume that the potential V satisfies a Poincare inequality w/ constant λ:

$\mu(dx) = e^{-V(x)}dx$: $\quad \lambda\|f\|^2_{L^2(\mu)} \le \|\nabla f\|^2_{L^2(\mu)}$, $\forall f\in C^1(\mathbb{R}^d)\cap L^2(\mu)$, s.t. $\mu(f) = 0$.

Then, $\rho(t,\cdot)$ converges to ρ_β exponentially fast:

$$\boxed{\|\rho(t,\cdot) - \rho_\beta\|_{L^2(\rho_\beta^{-1})} \le e^{-\lambda\beta^{-1}t}\|\rho(0,\cdot) - \rho_\beta\|_{L^2(\rho_\beta^{-1})}}$$

Rmk0. $\|\rho(t,\cdot) - \rho_\beta(\cdot)\|^2_{L^2(\rho_\beta^{-1})} = \int |\rho(t,x) - \rho_\beta(x)|^2\rho_\beta^{-1}(x)dx = \int \left|\frac{\rho(t,x)}{\rho_\beta(x)} - 1\right|^2\rho_\beta(x)dx = \int |h-1|^2 d\rho_\beta$.

$$\underline{\underline{0 < \lambda \le \lambda_{min}}} = \min_f \frac{\langle -\mathcal{L}f, f\rangle}{\langle f, f\rangle}$$

Rmk1. - the Poincare inequality is for V, or for μ: $\mu(dx) = e^{-V(x)}dx$, $\Leftrightarrow \rho_\beta(x) = e^{-\beta V(x)}$, $\forall \beta > 0$.

- $\lambda \operatorname{Var}(f) \le \|\nabla f\|^2_{L^2(\mu)} = \beta\, \mathcal{D}_{\mathcal{L}}(f) = \langle -\mathcal{L}f, f\rangle_\rho$ Dirichlet form. $\boxed{\text{spectral Gap of } \mathcal{L}.}$

Thm 4.3 (Poincare Ineq.) Let $V\in C^2(\mathbb{R}^d)$, $\mu \triangleq \frac{1}{Z}e^{-V(x)}dx$. If $\lim_{|x|\to\infty}\left(\frac{1}{2}|\nabla V(x)|^2 - \Delta V(x)\right) = +\infty$. Then

P.I.: $\exists \lambda > 0$; $\lambda\|f\|^2_{L^2(\mu)} \le \|\nabla f\|_{L^2(\mu)}$ $\quad \forall f\in C^1(\mathbb{R}^d)\cap L^2(\mu)$, $\mu(f) = 0$

Rmk2. Example of V: $V(x) = \frac{1}{2}x^Tx$ on $\mathbb{R}^d$, or $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ for $\mathbb{R}^1$.

- A sufficient condition: $\operatorname{Hess}(V) \ge \lambda I$ uniformly convex. [Bakry-Emery].
- General: $V\in C^2(\mathbb{R}^d)$, $\lim_{|x|\to\infty}\left(\frac{|\nabla V|^2}{2} - \Delta V(x)\right) = +\infty$.

Proof: Let $\rho(t,x) = h(t,x)\rho_\beta(x)$. Then $\partial_t h = Lh$.

Also, $\int h(t,x)\rho_\beta(x)dx = 1$. Thus, $h-1$ has mean zero, and $\partial_t(h-1) = L(h-1)$.

Further, $h(0,\cdot) \in L^2(\rho_\beta)$ by definition. $\Rightarrow h(0,\cdot) - 1 \in L^2(\rho)$.

Thus, $\xrightarrow{\text{Poincare Ineq.:}}$ $\frac{1}{2}\partial_t \|h-1\|^2_{L^2(\rho)} = \langle L(h-1), h-1\rangle_\rho = -\mathcal{D}_L(h-1,h-1) \le -\beta^{-1}\lambda\|h-1\|^2_{L^2(\rho)}$

$$\Rightarrow \quad \underbrace{\|h-1\|^2_{L^2(\rho)}}_{\|\rho(t,\cdot)-\rho_\beta\|^2_{L^2(\rho_\beta^{-1})}} \le e^{-2\lambda\beta^{-1}t}\underbrace{\|h(0,\cdot)-1\|^2_\rho}_{\|\rho(0,\cdot)-\rho_\beta\|^2_{L^2(\rho_\beta^{-1})}} \to 0 \text{ as } t\uparrow\infty.$$

#.

Rmk $\|h-1\|^2_{L^2(\rho)}$ is a "Lyapunov function" of the backward Kolmogorov eq: $\partial_t h = Lh$.

General "Lyapunov function" of the diffusion process X_t?

Lyapunov Fn. $U \triangleq$:

(i) $U(x) \ge 0 \quad \forall x \in \mathbb{R}^d$

(ii) $\lim_{|x|\uparrow\infty} U(x) = +\infty$

(iii) $\exists\, a \,\&\, b$ s.t. $U(x) \le a e^{b|x|}$, $|\nabla U(x)| \le a e^{b|x|}$

- $\mathcal{L}U(x) \le -\alpha U(x) + \beta$

Proposition 4.4 Assume that V is a confining potential & $h(t,\cdot)$: $\partial_t h = Lh$.

Then $\frac{d}{dt} H(h(t,\cdot)) \le 0$,

for any $H(h) \triangleq \int \varphi(h)\, d\rho_\beta$ if $\varphi \in C^2(\mathbb{R})$ is convex.

Proof: $\frac{d}{dt}H(h(t,\cdot)) = \int \varphi'(h)\, \partial_t h\, \rho_\beta dx$

$$= \langle \varphi'(h), Lh\rangle = -\beta^{-1}\int \nabla\varphi'(h)\nabla h\,\rho_\beta dx = -\beta^{-1}\int \underbrace{\varphi''(h)}_{\varphi \text{ convex}} |\nabla h|^2 \rho_\beta dx \le 0. \ \#$$

- If $\varphi(h) = h\ln h - h + 1$, $\rightarrow$ free energy functional:

$$H(h) = \int (h\ln h - h + 1)\rho_\beta dx \overset{h=\rho/\rho_\beta}{=} \int \left(\frac{\rho}{\cancel{\rho_\beta}}\ln\frac{\rho}{\rho_\beta} - \frac{\rho}{\cancel{\rho_\beta}} + 1\right)\cancel{\rho_\beta}\, dx$$

$$= \int \rho \ln\frac{\rho}{\rho_\beta}\, dx \qquad \text{Entropy}$$

$\rho_\beta = \frac{1}{Z}e^{-\beta V}$

$$= \int \rho\ln\rho + \beta\int V\rho + \ln Z$$

$$= \beta F(\rho)$$

$$\underline{F(\rho) = \int V\rho\, dx + \beta^{-1}\int \rho\ln\rho\, dx + \beta^{-1}\ln Z}. \qquad \text{Free energy}$$

chp6 The Langevin Equ. (Pav14)

Motion of a particle that is subject to friction:

$$(\text{LE}) \quad \ddot{q} = -\nabla V(q) \underbrace{-\nu \dot{q}}_{} + \underbrace{\sqrt{2\nu\beta^{-1}}\,\dot{W}}_{} \to \text{stochastic forcing}: \quad \beta^{-1} = K_B T \to \text{Absolute temperature}$$

linear dissipation: ν = friction coefficient. Boltzmann's constant.

$$\mathbb{E}[\xi_t \xi_s] = \nu \beta^{-1} \delta(t-s) \quad \text{(Fluctuation–dissipation Thm)}$$

§6.1 The Fokker-Planck equ. in phase space

$$(\text{LE}) \begin{cases} \dot{q} = p \\ \dot{p} = -\nabla V(q) - \nu p + \sqrt{2\nu\beta^{-1}}\,\dot{W} \end{cases}$$

Generator Here $b = \begin{pmatrix} p \\ -\nabla V(q) - \nu p \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 \\ c \end{pmatrix}$, $\sigma\sigma^T = \begin{pmatrix} 0 & 0 \\ 0 & cc^T \end{pmatrix}$

$$L = p \cdot \nabla_q + (-\nabla V(q) - \nu p)\nabla_p + \nu\beta^{-1}\Delta_p$$

$$= \underbrace{p\cdot\nabla_q - \nabla V(q)\nabla_p}_{=B} + \nu\underbrace{(-p\nabla_p + \beta^{-1}\Delta p)}_{=S}$$

$$L^* \rho = [-p\cdot\nabla_q + \nabla V(q)\cdot\nabla_p]\rho + \nu\nabla_p(p\rho) + \nu\beta^{-1}\Delta_p\rho$$

$$\partial_t \rho = L^* \rho$$

$dX_t = b(X_t)dt + \sigma dB_t$; $X_0 = x$;

generator: $L = b\cdot\nabla + \frac{1}{2}\sigma\sigma^T : \text{Hess}$

$f(X_t) = f(x) + \int_0^t Lf(X_s)ds + m.G$, $\forall f \in C_0^\infty$

$u(t,x) = \mathbb{E}[f(X_t^x)] = f(x) + \int_0^t \mathbb{E}[Lf(X_s)]ds$,

$\partial_t u = Lu$ $\qquad X_t \sim \rho(t,\cdot)$

$\int f(y)\rho(t,y)dy = f(x) + \int_0^t Lf(y)\rho(s,y)dy\,ds$

$\partial_t \rho = L^*\rho = -\nabla(b\rho) + \frac{1}{2}\sigma\sigma^T : \text{Hess}(\rho)$ $\quad \sigma = \text{const.}$

$= \nabla\cdot[-b\rho + \frac{1}{2}\nabla\cdot(\Sigma\rho)] = \nabla\cdot J(\rho)$

Rmk1: L^* is NOT uniformly elliptic; (Existence of sln. ρ ? $\to$ Hypoelliptic)

Rmk2: (Hamiltonian system) The drift part:

$$\boxed{H(p,q) = \tfrac{1}{2}|p|^2 + V(q)} \to \begin{cases} \dot{q} = p & = \partial_p H \\ \dot{p} = -\nabla V(q) & = -\partial_q H \end{cases}$$

- conservation of energy: $H(p(t),q(t)) \equiv H(p(0),q(0))$, $\forall t \geq 0$.

$$\frac{d}{dt}H(p(t),q(t)) = \partial_p H\dot{p} + \partial_q H\dot{q} = \partial_p H(-\partial_q H) + \partial_q H\partial_p H \equiv 0, \quad \forall t.$$

- Liouville operator. $B = p\cdot\nabla_q - \nabla V\cdot\nabla_p$

$B(H) = +p\partial_q H - \nabla V\partial_p H = p\cdot\nabla V(q) - \nabla V\cdot p = 0$

$B(f(H)) = f' BH = 0$, $\forall f \in C^1$

$u(t,p_0,q_0) = f(p(t),q(t))$

$\partial_t u = Bu$.

$B^* = -p\cdot\nabla_q + \nabla V\cdot\nabla_p = -B$

- Many Invariant measures. If random IC $(p(0),q(0)) \to \rho(t,\cdot)$: $\partial_t\rho = B^*\rho$

$\partial_t\rho = 0 \Leftrightarrow B^*\rho = 0 = -B\rho \Rightarrow$ any ρ s.t. $\rho = f(H) \geq 0$, $\int\rho = 1$. eg. $\frac{1}{Z_\beta}e^{-\beta H}$;

Liouville equ. $\dot{x} = b(x)$; $X(0)=x$; $b \in C^1$

e.g. $\frac{1}{Z} I_{\{H \equiv const\}}$

$u(x,t) = f(X(t,x))$, $f \in C^1 \Rightarrow \boxed{\partial_t u = Lu = (b \cdot \nabla) u}$

Proof. $\partial_t u = [\nabla f \cdot b](X(t,x)) \overset{(*)}{=} \nabla f(X(t,x)) \cdot \nabla_x X(t,x) \cdot b(x) = (b \cdot \nabla) u$ #.

$\nabla u = \nabla f(X(t,x)) \nabla_x X(t,x)$

$X(t+s,x) = X(t, X(s,x)) \Rightarrow \frac{d}{ds}: \ b(X(t+s,x)) = \nabla_x X(t, X(s,x)) \cdot b(X(s,x))$, $\forall s$

$\xrightarrow{s \downarrow 0} \quad b(X(t,x)) = \nabla_x X(t,x)\, b(x) \qquad (*)$

Proposition 6.1 Let $V(x)$ be a smooth confining potential (i.e. $\lim_{|x| \to \infty} V(x) = +\infty$, $e^{-\beta V(x)} \in L^1(\mathbb{R}^d)$, $\forall \beta > 0$). Then, the Markov process w/ generator L (i.e, the LE) is ergodic, w/ invariant density;

$$\boxed{\rho_\beta(p,q) = \frac{1}{Z} e^{-\beta H(p,q)}}, \qquad H(p,q) = \frac{1}{2}|p|^2 + V(q)$$

Proof: ① $\partial_t \rho_\beta = L^* \rho_\beta = 0$: $\quad L = B + \gamma S$, $\quad \underline{S = \nabla_p (p \cdot + \beta^{-1} \nabla_p \cdot)}$ [OU].

$B \rho_\beta = 0$; ✓ $\qquad Sf = d \cdot f + p \cdot \nabla_p f + \beta^{-1} \Delta_p f$

$S \rho_\beta = \frac{1}{Z} e^{-\beta H} [\cancel{d - \beta |p|^2 - d + \beta |p|^2}]$

$= 0.$ $\quad \Leftarrow$ $\quad H(p,q) = \frac{1}{2}|p|^2 + V(q)$

$f = e^{-\beta H}$; $\nabla_p f = -\beta e^{-\beta H} \cdot p$

$\Delta_p f = \nabla_p \cdot (-\beta e^{-\beta H} p)$

$= -\beta e^{-\beta H} \cdot d + \beta^2 e^{-\beta H} |p|^2$

$\Rightarrow \rho_\beta$ is an invariant density.

② convergence to ρ_β: Hypo coercivity.

(along with hypoellipticity $\Rightarrow \exists!$ sln. to $\partial_t \rho = L^* \rho$). #.

Rmk1. ρ_β is indpt of $\gamma > 0$. Gibbs / Maxwell-Boltzmann / canonical distribution.

Rmk2 The marginal distribution of q: $\rho_\beta(q) = Z^{-1} e^{-\beta V(q)}$ is the invariant density of the overdamped Langevin dynamics:

$$dq_t = -\nabla V(q_t) dt + \sqrt{2\beta^{-1}}\, dW_t$$

[Reversible stationary process: if $\{X_t\}$ and $\{X_{T-t}\}$ have the same distribution, $\forall T$

A stationary diffusion X_t in $\mathbb{R}^d$ w/ generator L and invariant measure μ is reversible $\Leftrightarrow L = L^*$ in $L^2(\mu)$.

$dX_t = b(X_t) dt + \sigma(X_t) dW_t$. $\quad J(\rho) = -b\rho + \frac{1}{2} \nabla \cdot (\Sigma \rho) = \vec{0}$ detailed balance.

Rmk 3 Operators: $L = B + \nu S = p\cdot\nabla_q - \nabla V(q)\nabla_p + \nu(-p\nabla_p + \beta^{-1}\Delta_p)$.

$$L^* = B^* + \nu S^* = -p\nabla_q + \nabla V(q)\nabla_p + \nu(\nabla_p(p\,\cdot) + \beta^{-1}\Delta_p)$$

Function space: $L^2(\rho_\beta)$. $B^* = -B$, $S^* = S$.

§6.2 Hypoellipticity & Hypocoercivity.

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad \leftrightarrow \quad L = b\cdot\nabla + \tfrac{1}{2}\sigma\sigma^T : \mathrm{Hess}$$

$\sigma \in \mathbb{R}^{n\times m}$, $W_t \in \mathbb{R}^m$

$$= \sum_{j=1}^n b_j \partial_{x_j} + \frac{1}{2}\sum_{i,j} \sigma_{ik}\sigma_{jk}\partial_{x_i}\partial_{x_j}$$

Define the vector fields:

$$\to L = \sum_{i=1}^n A_i^2 + A_0$$

$$A_0(x) = \sum_{j=1}^n b_j(x)\partial_{x_j}$$

$$A_k = \sum_{j=1}^n \sigma_{jk}(x)\partial_{x_j}, \quad k = 1, \cdots, m$$

Define the Lie algebras (with Lie bracket $[A_j, A_i] = A_jA_i - A_iA_j$)

$$\mathcal{A}_0 = \mathrm{Lie}(A_1, \cdots, A_m)$$

$$\mathcal{A}_1 = \mathrm{Lie}([A_0, A_1], \cdots, [A_0, A_m])$$

$$\vdots$$

$$\mathcal{A}_k = \mathrm{Lie}([A_0, X], X \in \mathcal{A}_{k-1})$$

and the vectors $\mathcal{H} = \mathrm{Lie}(\mathcal{A}_0, \mathcal{A}_1, \cdots)$.

Hormander's condition: $\mathrm{span}\{H(x) : H \in \mathcal{H}\} = T_x\mathbb{R}^n = \mathbb{R}^n$.

Hormander's Theorem: Assume that b & σ are smooth w/ bdd derivative of all orders, and Hormander's condition holds. Then X_t has a smooth transition density $p(t,\cdot,\cdot)$.

§6.2 Hypocoercivity.

To prove convergence of $\rho(t,\cdot\cdot)$, we consider $h(t,\cdot\cdot) = \rho(t,\cdot,\cdot)/\rho_\beta(\cdot,\cdot)$

Then, from $\partial_t \rho = L^* \rho = (-p\cdot\nabla_q + \nabla_q V\cdot\nabla_p)\rho + \gamma(\nabla_p\cdot(p\rho) + \beta^{-1}\Delta_p \rho)$; $L^*\rho_\beta = 0$

we get $\boxed{\partial_t h = L_{kin} h} = \underbrace{(-p\nabla_q + \nabla_q V\cdot\nabla_p)}_{-B} h + \gamma\underbrace{(-p\cdot\nabla_p + \beta^{-1}\Delta_p)}_{S} h$

- Almost the generator $L = \underbrace{p\cdot\nabla_q - \nabla_q V\cdot\nabla_p}_{B} + \gamma\underbrace{(-p\cdot\nabla_p + \beta^{-1}\Delta_p)}_{S}$
- If we write $\rho(t,q,p) = \hat{h}(t,q,-p)\,\rho_\beta(q,p)$, then $\partial_t \hat{h} = L\hat{h}$.

- NOT reversible
  (h represents reversed DS.)
  ← reverse momentum

The theory of hypercoercivity applies to evolution equation of the form

$$\partial_t h + \underbrace{(A^*A - B)}_{=\mathcal{L}} h = 0 \quad ; \qquad B^* = -B$$

Let $\mathcal{H}$ be a Hilbert space, $D(A)\cap D(B) \subseteq \mathcal{H}$. Then, $\forall h \in D(A)\cap D(B)$

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 = -\|Ah\|^2 + \langle Bh, h\rangle = -\|Ah\|^2 \qquad \langle Bh,h\rangle = \langle h,-Bh\rangle = 0$$

- If $-\langle \mathcal{L}h, h\rangle = \|Ah\|^2 \geq \lambda\|h\|^2$, w/ $\mathcal{L} = -(A^*A-B)$, we get $\|h\|^2 \leq e^{-2\lambda t}\|h(0,\cdot)\|^2$

However, we only have $-\langle L_{kin} f, f\rangle_{L^2(\rho_\beta)} \geq 0$.

<u>Def 6.2</u> (Coercivity) Let J be an unbdd operator on a Hilbert space H and let $\ker(J) = \{f\in H;\ Jf=0\}$.

Assume that there exists another Hilbert space $\tilde{H} \subseteq \ker(J)^\perp$, with inner product $\langle\cdot,\cdot\rangle_{\tilde{H}}$.

Then, the operator J is said to be λ-coercive on $\tilde{H}$ if

$$\langle Jh, h\rangle_{\tilde{H}} \geq \lambda\|h\|^2_{\tilde{H}}, \quad \forall\ h\in \ker(J)^\perp \cap D(J).$$

- $\Leftrightarrow$ $\|e^{-Jt}h\|_{\tilde{H}} \leq e^{-\lambda t}\|h\|_{\tilde{H}}$, $\forall t\geq 0$ (Prop. 6.4)

<u>Definition 6.3</u> (Hypercoercivity) Assume further that J generates a continuous semigroup. Then

J is λ-hypercoercive on $\tilde{H}$ iff $\exists$ a constant $K>0$ s.t.

$$\|e^{-Jt}h\|_{\tilde{H}} \leq K e^{-\lambda t}\|h\|_{\tilde{H}}, \quad \forall h\in\tilde{H},\ t\geq 0.$$

<u>Remark</u>: the constant K makes the hypercoercivity <u>invariant under a change of equivalent norms</u>.

Thm 6.3 (Exponential convergence). Assume that

(1) V satisfies the Poincare Ineq. w/ constant λ;

(2) $\exists\, C>0$ s.t. $|\partial_q^2 V| \le C(1+|\partial_q V|)$

Then, the density $\rho(t,p,q)$ converges exponentially in time; $\forall\, h_0 \in H^1(\rho_\beta)$

$$\left\| e^{-tL_{\mathrm{Lan}}} h_0 - \int h_0 \rho_\beta \right\|_{H^1(\rho_\beta)} \le C e^{-\lambda t} \|h_0\|_{H^1(\rho_\beta)}$$

Here: • $H^1(\rho_\beta) = \{ h \in L^2(\rho_\beta) : \partial_p h, \partial_q h \in L^2(\rho_\beta) \}$.

• condition (1): $\liminf_{|x|\to\infty} \left(\frac{1}{2} |V'(x)|^2 - V''(x) \right) = \infty$

or Bakry-Emery criterion $V''(x) \ge \lambda$