

[Higham 01]: elementary intro. on simulation

[KP92]: our reference

Numerical Approximation of SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t; \quad X_0 = x_0 \quad (1)$$

- Euler-Maruyama; Milstein; \triangleright Ito-Taylor expansion
- Strong / weak App.; order of convergence.
- Stability

1. Time discrete Approximations

$$X_t = x_0 + \int_0^t a(X_s) ds + \int_0^t b(X_s) dW_s; \quad t \in [0, T].$$

consider discrete times: $0 = t_0 < t_1 < \dots < t_N = T, \quad t_i = i \frac{T}{N}$.

$$\begin{aligned} \text{EM: } Y_{n+1} &= Y_n + a(Y_n) \Delta t + b(Y_n) \Delta W_n; \quad Y_0 = x_0; \quad n = 0, 1, \dots, N-1. \\ &\downarrow \\ &= W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t); \quad \text{i.i.d.} \end{aligned}$$

$$\text{Milstein: } Y_{n+1} = Y_n + a(Y_n) \Delta t + b(Y_n) \Delta W_n + \frac{1}{2} b b' (\Delta W_n)^2 - \Delta t).$$

- ($\Delta W_n \sim \sqrt{\Delta t} \mathcal{N}(0, 1)$ i.i.d., generated by a pseudo-random number generator.)

Question:

- ① How do we get them? What is a "valid" scheme?
- ② order of convergence?
- ③ Higher-order schemes?
- ④ High-D version?
- ⑤ stability?

2. Ito-Taylor expansion (Wagner-Platen formula) (similar to numerical ODE)

Apply Ito's formula [chain rule for $f \in C^2$: $df(X_t) = f'(X_t)[a(X_t)dt + b(X_t)dW_t] + \frac{1}{2}f''(X_t)b^2dt$]

$$X_t = x_0 + \int_0^t a(X_s) ds + \int_0^t b(X_s) dW_s$$

$$= x_0 + \int_0^t a(x_0) ds + \int_0^t b(x_0) dW_s + \int_0^t [a(X_s) - a(x_0)] ds + \int_0^t [b(X_s) - b(x_0)] dW_s$$

$t \downarrow 0$

$$= x_0 + \underbrace{a(x_0)t}_{O(t)} + \underbrace{b(x_0)(W_t - 0)}_{O(\sqrt{t})} + \left(\begin{array}{l} \text{ } \\ \text{ } \end{array} \right) \quad a, b \in C_b^2$$

$$a(X_s) - a(x_0) = \int_0^s a'(X_r) [a(X_r)dr + b(X_r)dW_r] + \frac{1}{2}a''b^2dr$$

$$\int_0^t [a(X_s) - a(x_0)] ds = \int_0^t \int_0^s \left[\underbrace{a'a + \frac{1}{2}a''b^2}_f dr + \underbrace{a'b}_g dW_r \right] ds$$

$$= [a'a + \frac{1}{2}a''b^2](x_0) \frac{1}{2}t^2 + [a'b](x_0) \int_0^t (W_s - W_0) ds + \int_0^t \int_0^s \left[\underbrace{f(X_r) - f(x_0)}_h dr + \underbrace{g(X_r)g(X_s)}_i dW_r \right] ds$$

$$\int_0^t [b(X_s) - b(x_0)] ds = \int_0^t \int_0^s \left[\underbrace{b'a + \frac{1}{2}b''b^2}_f dr + \underbrace{b'b}_g dW_r \right] ds \quad O(t^{3/2}) \quad O(t^{3/2})$$

$$= [b'a + \frac{1}{2}b''b^2](x_0) \int_0^t s dW_s + [b'b](x_0) \int_0^t \int_0^s dW_r dW_s + \int_0^t \int_0^s \left[\underbrace{f(X_r)}_h dr + \underbrace{g(X_r)g(X_s)}_i dW_r \right] dW_s$$

$$\quad O(t^{3/2}) \quad O(t) \quad O(t) \quad \downarrow O(t^{3/2})$$

EM: $X_t = x_0 + a(x_0)t + b(x_0)(W_t - W_0) + o(\sqrt{t})$

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W_{t_{n+1}} - W_{t_n}) + o(\sqrt{\Delta t})$$

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n$$

Melstein: $X_{t_{n+1}} = \dots + [b'b](X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^s dW_r dW_s + o(\Delta t)$

$$\int_0^t \int_0^s dW_r dW_s = \int_0^t (W_s - W_0) dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [b'b](Y_n) \frac{1}{2}(\Delta W_n^2 - \Delta t)$$

IT1.5 $X_t = x_0 + a(x_0)t + b(x_0)(W_t - W_0) + [b'b](x_0) \int_0^t \int_0^s dW_r dW_s + [b'a + \frac{1}{2}b''b^2](x_0) \int_0^t s dW_s$
 $+ [a'b](x_0) \int_0^t (W_s - W_0) ds + [b'b']b(x_0) \int_0^t \int_0^s \int_0^r dW_u dW_r dW_s + o(t^{3/2}) + [a'a + \frac{1}{2}a''b^2](x_0) \frac{1}{2}t^2$

Attention: relation between multiple integrals: $I_1, I_{11}, I_{1r}, I_{111}, I_{01}$

$$I_{11} = \frac{1}{2}(I_1^2 - t)$$

$$\mathbb{E}[I_{10}] = 0; \quad \mathbb{E}[I_{1r}^2] = \frac{1}{3}t^3 \quad \int_0^t \int_0^t \mathbb{E}[W_s W_r] ds dr$$

$$I_{111} = \frac{1}{3!}(I_1^3 - 3tI_1)$$

$$\mathbb{E}[I_{10}I_{1r}] = \frac{1}{2}t^2$$

$$\int_0^t \int_0^s \int_0^r ds dr ds$$

$$I_{01} = tI_1 - I_{10}$$

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds$$

$$\mathbb{E}[\int_0^t W_s ds W_t] = \int_0^t s ds = \frac{1}{2}t^2$$

$$\Delta Z = I_{10} = \int_0^t W_s ds$$

$$\begin{aligned}
Y_{n+1} &= Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [bb'](Y_n) \frac{1}{2}(\Delta W_n^2 - \Delta t) \\
&\quad + [b'a + \frac{1}{2}b''b'](Y_n)(\Delta t \Delta W_n - \Delta Z) + [a'b'](Y_n)\Delta Z \\
&\quad + [(b'b)'](Y_n) \frac{1}{6}(\Delta W_n^3 - 3\Delta t \Delta W_n) + [da + \frac{1}{2}a''b^2](Y_n) \frac{1}{2}\Delta t^2 \\
\Delta W_n &= W_{t_{n+1}} - W_{t_n} = \sqrt{\Delta t} \xi_n, \quad \Delta Z_n = \frac{1}{2}(\Delta t)^{3/2} (\xi_n + \frac{1}{3}\eta_n), \quad \xi_n, \eta_n \sim N(0,1), \text{ i.i.d.}
\end{aligned}$$

• When $b' = 0$, i.e. constant diffusion coefficient,

EM = Milstein.

$$\text{IT(1.5): } Y_{n+1} = Y_0 + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [a'b'](Y_n)\Delta Z_n + [a'a + \frac{1}{2}a''b](Y_n) \frac{1}{2}\Delta t^2.$$

High-D: $x \in \mathbb{R}^n$.

$$\text{EM } \vee \quad \underline{Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n}$$

Milstein: Derive from Ito's chain rule, becoming complicated.

- In practice, hybrid schemes are often used, eg. RK4 + EM for additive noise
 - The drift is nonlinear (local Lipschitz) and dominates the dynamics (stability \star). } Splitting methods for Hamiltonian eqn.
 - The order is limited by the lowest order part.

3. Strong/weak order of convergence

$$X_T \leftrightarrow Y_N^\delta \quad \delta = \frac{T}{N}$$

strong order ν : $\mathbb{E} |X_T - Y_N^\delta| \leq C \delta^\nu$ for all $\delta \in (0, \delta_0)$.

weak order β : $|\mathbb{E}[g(X_T) - g(Y_N^\delta)]| \leq C_g \delta^\beta$ - - - - , $\forall g \in C^{2+2\beta}$.

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

(Thm 10.2.2) EM has strong order of convergence $\nu = 0.5$. if

1) $\mathbb{E}[|X_0|^2] < \infty$; $\mathbb{E}[|X_0 - Y_0^\delta|^2] \leq K_1 \delta$

2) Global Lipschitz & linear growth a.b.

3) (Holder in time) $|a(s, x) - a(t, x)| + |b(s, x) - b(t, x)| \leq K(1+|x|)|t-s|^\frac{1}{2}$.

(Thm 14.1.5) It has weak order $\beta = 1$ if g is smooth & $bb^T \geq \lambda I$ for some $\lambda > 0$.

(Thm 10.3.5) Multstein has strong order $\nu = 1$. if 1) and (2-3) for a, b and derivatives.
weak order $\beta = 2$.

About weak approximation, when only moments are of interest, can replace ΔW_n by $\Delta \hat{W}_n$

EM: $Y_{n+1} = Y_n + a\Delta t + b\Delta \hat{W}_n$, $P(\Delta \hat{W}_n = \pm \sqrt{\Delta t}) = \frac{1}{2}$

IT-2.0 weak: $Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [bb](Y_n)(\frac{1}{2}\Delta W_n^2 - \Delta t)$
 $+ [b'a + \frac{1}{2}b''b](Y_n)(\Delta t \Delta W_n - \Delta Z) + [a'b](Y_n)\Delta Z$
 $+ \frac{1}{6}(\Delta W_n^3 - 3\Delta t \Delta W_n) + [da + \frac{1}{2}a''b](Y_n)\frac{1}{2}\Delta t^2$

$$\Delta W_n = W_{t_{n+1}} - W_{t_n} = \sqrt{\Delta t} \xi_n, \quad \Delta Z_n = \frac{1}{2}(\Delta t)^{\frac{3}{2}}(\xi_n + \frac{1}{3}\xi_n^3) \approx \frac{1}{2}(\Delta t)^{\frac{3}{2}}\xi_n$$

Order test: How to do a numerical order test?

Note: $W_n^\delta = W_{t_{n+1}} - W_{t_n}$, $W_n^{k\delta} = W_{t_{(n+1)k\delta}} - W_{t_{nk\delta}} = \sum_{j=1}^{(n+1)k} (W_{t_{j\delta}} - W_{t_{(j-1)\delta}})$

ΔZ_n ? Need to consider: $\Delta Z_n = \int_{t_n}^{t_{n+1}} W_s ds$

4. Stability.

- convergence alone is no guarantee of "stability", particularly stiff DEs.
- Propagation of initial errors and roundoff errors.

4.1. Stability of the process/solution

Recall: ODE $x'(t) = \lambda x(t)$; $\Rightarrow x(t) = x(0)e^{\lambda t}$; 0 is a steady state. it is Stable if $\lambda < 0$.

SDE ?

$$\left\{ \begin{array}{l} \text{Mean-square stable: } \lim_{t \rightarrow \infty} \mathbb{E}[X_t^2] = 0 \\ \text{Asymptotic stable: } \lim_{t \rightarrow \infty} X_t = 0 \text{ w.p. 1} \end{array} \right. \Downarrow$$

or $\text{Re}(\lambda) < 0$
 $\lim_{t \rightarrow \infty} x(t) = 0, \forall x(0)$

Example 1. $dX_t = \lambda X_t dt + \mu X_t dW_t$; $\Rightarrow X_t = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W_t}$; 0 is a steady state.

$$\left\{ \begin{array}{l} \mathbb{E}[X_t^2] = \mathbb{E}[X_0^2 e^{2(\lambda - \frac{1}{2}\mu^2)t + 2\mu W_t}] = \mathbb{E}[X_0^2] e^{2(\lambda - \frac{1}{2}\mu^2)t + 2\mu^2 t} \searrow \text{if } \lambda + \frac{1}{2}\mu^2 < 0. \\ X_t \rightarrow 0 \text{ a.s.} \end{array} \right. \text{if } \lambda - \frac{1}{2}\mu^2 < 0$$

Stability of $\{X_n^t\}$

$$\left\{ \begin{array}{l} \text{Mean-square stable: } \lim_{j \rightarrow \infty} \mathbb{E}[X_j^2] = 0 \\ \text{Asymptotic stable: } \lim_{j \rightarrow \infty} X_j = 0 \text{ a.s.} \end{array} \right.$$

EM of Example 1: $Y_{j+1} = (1 + \lambda \Delta t) Y_j + \mu Y_j \Delta W_j$

⊙ M.S.: $\mathbb{E}[Y_{j+1}^2] = (1 + \lambda \Delta t)^2 \mathbb{E}[Y_j^2] + \mu \Delta t \mathbb{E}[Y_j^2]$
 $= [(1 + \lambda \Delta t)^2 + \mu \Delta t] \mathbb{E}[Y_j^2] \searrow 0, \text{ if } |h \cdot| < 1$

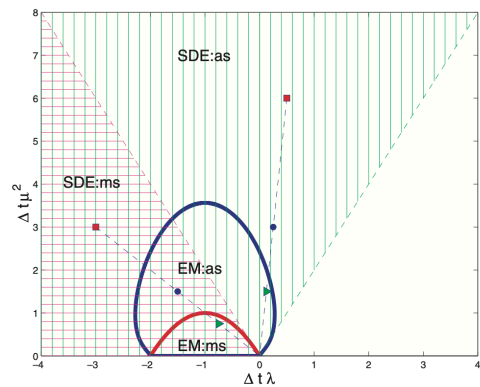
⊙ A.S.: $Y_{j+1} = (1 + \lambda \Delta t + \mu \Delta W_j) Y_j = \prod_{i=1}^{j+1} (1 + \lambda \Delta t + \mu \Delta W_i) Y_0 \xrightarrow{j \rightarrow \infty} 0$

$\Leftrightarrow \prod_{i=1}^{j+1} |1 + \lambda \Delta t + \mu \Delta W_i| \rightarrow 0 \Leftrightarrow \exp\left(\sum_{i=1}^{j+1} \log |1 + \lambda \Delta t + \mu \Delta W_i|\right) \rightarrow 0$

$= \exp\left(\sum_{i=1}^{j+1} (Z_i - \mathbb{E}Z) + (j+1)\mathbb{E}Z\right)$

$\approx S_{j+1}$: Iterated law of Log: $\lim_{j \rightarrow \infty} \frac{S_j}{\sqrt{j}} = \sigma_Z$

Fig. 6 Mean-square and asymptotic stability regions.



stability regions

$Z = \log |1 + \lambda \Delta t + \mu \sqrt{\Delta t} N(0,1)|$
 $\Leftrightarrow \mathbb{E}Z < 0$

Numerical Stability regions:

- Δt , parameter (λ, μ) or drift & diffusion
- The larger $|\lambda|$, the smaller Δt that we can tolerate.

Q: $dX_t = \lambda X_t dt + dW_t$, NO steady-state. How to define "stability"?

① steady state \rightarrow stationary soln. $\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} E[|X_n^{tot}|^2] = \lim_{t \rightarrow \infty} E[X_t^2]$

② Numerical stability. KP 92; §9.8

The appr. (Y_n^δ) of (Y) is stochastically numerically stable if $\exists \Delta_0$ s.t., $\forall \delta \in (0, \Delta_0)$, $\forall \epsilon > 0$,

$$\lim_{|Y_0^\delta - Y_0| \rightarrow 0} \sup_{0 \leq n \delta \leq T} P(|Y_n^\delta - Y_{n\delta}| > \epsilon) = 0. \quad (1)$$

----- Asymptotically numerically stable if -----

$$\lim_{|Y_0^\delta - Y_0| \rightarrow 0} \lim_{T \rightarrow \infty} P\left(\sup_{0 \leq n \delta \leq T} |Y_n^\delta - Y_{n\delta}| > \epsilon\right) = 0. \quad (2)$$

Remark 1. Proof of (1) usually by controlling $E[|Y_n^\delta - Y_{n\delta}|^2]$ & Markov inequality.

Proof of (2) usually by Doob's mg. inequality.

Remark 2. In practice, often interested in approximating the invariant measure; large $\delta \rightarrow$ faster

Example: $dX_t = \lambda X_t dt + dW_t$ $\lambda \in \mathbb{R}$, $X_0 \in \mathbb{R}$

EM $Y_{n+1}^\delta = Y_n^\delta + \lambda Y_n^\delta \cdot \delta + W_n^\delta = (1 + \lambda\delta) Y_n^\delta + W_n^\delta$

• if $\lambda > 0$, then blowup, i.e. $|Y_{n+1}^\delta| \xrightarrow{n \rightarrow \infty} \infty$, b.c. $Y_{n+1}^\delta = (1 + \lambda\delta)^{n+1} Y_0^\delta + \sum_{j=0}^n (1 + \lambda\delta)^j W_j^\delta$
 $\approx O((1 + \lambda\delta)^n)$

• If $\lambda < 0$ $\left\{ \begin{array}{l} \text{if } |1 + \lambda\delta| > 1, \text{ blowup} \\ \text{if } |1 + \lambda\delta| < 1, \text{ then bounded.} \end{array} \right.$

$$|1 + \lambda\delta| < 1 \Leftrightarrow -2 < \lambda\delta < 0 \Leftrightarrow 0 < \delta < -2/\lambda$$

Implicit Euler: $Y_{n+1}^\delta = Y_n^\delta + \lambda Y_{n+1}^\delta \delta + W_n^\delta \Leftrightarrow Y_{n+1}^\delta = (1 - \lambda\delta)^{-1} (Y_n^\delta + W_n^\delta)$

Thus, stable if $|1 - \lambda\delta| > 1$, $\rightarrow \lambda < 0$, $\delta > 0$ arbitrary!

Invariant measure. $X_\infty \sim N(0, \frac{1}{2\lambda})$ $\xleftarrow{\lambda < 0}$ $X_t = e^{\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dB_s$

EM: $Y_\infty^\delta \sim N(0, \frac{\delta}{1 - (1 + \lambda\delta)^2}) \rightarrow \frac{-1}{2\lambda - \lambda^2\delta}$

IE: $Y_\infty^\delta \sim N(0, \frac{(1 - \lambda\delta)^{-2} \delta}{(1 - (1 - \lambda\delta)^2)} \rightarrow \frac{1}{(1 - \lambda\delta)^2 - 1} = \frac{1}{2\lambda - \lambda^2\delta}$

$\left\{ \begin{array}{l} \text{b.c. } Z_{n+1} = \alpha Z_n + \beta W_n \Rightarrow Z_\infty \sim N(0, \sigma^2) \\ \sigma^2 = E[Z_{n+1}^2] = \alpha^2 E[Z_n^2] + \beta^2 \Rightarrow \sigma^2 = \frac{\beta^2}{1 - \alpha^2} \end{array} \right.$

$$\sigma^2 = E[Z_{n+1}^2] = \alpha^2 E[Z_n^2] + \beta^2 \Rightarrow \sigma^2 = \frac{\beta^2}{1 - \alpha^2}$$

Remark 2: Implicit Euler schemes: $Y_{n+1}^{\delta} = Y_n^{\delta} + a(t_{n+1}, Y_{n+1}^{\delta})\delta + b(t_n, Y_n^{\delta})\Delta W_n$
 or $\dots \dots \dots b(t_{n+1}, Y_{n+1}^{\delta})\Delta W_n$

Reading:

- KP 92: §6.3; §9.8, §12.5
- Mattingly + Hegham + Stuart 02: Implicit schemes for local Lipschitz SDEs.