Diffusion theory

- Chapter 7
  - Markov / strong Markov property
  - Generator / characteristic operator (Dynkin formula).
- Chapter 8
  - Kolmogorov Backward / forward equ.
  - Feynman-Kac formula
  - The martingale problem
  - Random time change
  - Change of measure (Girsanov Theorem)

1. Kolmogorov's backward / forward equ.

Let X_t be an Ito diffusion w/ generator A. i.e. $dX_t = b(X_t)dt + \sigma(X_t)dB_t$; Lipschitz

$Af = b\cdot\nabla f + \frac{1}{2}(\sigma\sigma)^T : \mathrm{Hess}(f)$, $\forall f \in C_0^2$

Recall Dynkin's formula:

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right]. \quad \text{for } \tau: \text{stopping time}, \ \mathbb{E}^x[\tau] < \infty$$

$\overset{\tau = t.}{\Longrightarrow}$ $u(t,x) \leftarrow \mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x\left[\int_0^t Af(X_s)ds\right]$

$\Rightarrow$ $\boxed{\frac{d}{dt}\mathbb{E}^x[f(X_t)] = \mathbb{E}^x[Af(X_t)].}$ $\overset{?}{=}$ $A\,\mathbb{E}^x[f(X_t)] \Rightarrow \partial_t u = Au$

Theorem 8.1.1 (Kolmogorov's backward equ.) Let $f \in C_0^2(\mathbb{R}^n)$, $u(t,x) = \mathbb{E}^x[f(X_t)]$

(a). Then $u(t,\cdot) \in \mathrm{Dom}(A)$ $\forall t$, $\begin{cases} \partial_t u = Au, & t>0, x\in\mathbb{R}^n \\ u(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases}$

(b) If $w(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ is bdd & satisfying $\begin{cases} \partial_t w = Aw \\ w(0,x) = f(x) \end{cases}$, then $w = \mathbb{E}^x[f(X_t)]$.

Proof: a) $Au(t,x) = \lim_{r\downarrow 0} \frac{\mathbb{E}^x[u(t,X_r)] - u(t,x)}{r} = \frac{1}{r}\mathbb{E}^x\left[\underbrace{\mathbb{E}^{X_r}[f(X_t)]}_{=\,\mathbb{E}^x[f(X_{t+r})|\mathcal{F}_r]} - f(X_t)\right]$ Markov.

$= \lim_{r\downarrow 0} \frac{1}{r}\mathbb{E}^x[f(X_{t+r}) - f(X_t)]$ $\qquad \mathbb{E}^x[f(X_t)] = \mathbb{E}^x[\mathbb{E}^x[f(X_t)|\mathcal{F}_r]]$

$= \lim_{r\downarrow 0} \frac{1}{r}[u(t+r,x) - u(t,x)]$

$= \partial_t u.$ #

Fix (s,x)

b) $\begin{cases} \tilde{A}w = -\partial_t w + Aw = 0 \\ w(0,x) = f(x) \end{cases}$ $\quad \tilde{A}: \quad Y_t = \begin{pmatrix} s-t \\ X_t^{0,x} \end{pmatrix} \quad dY_t = \begin{pmatrix} -dt \\ dX_t \end{pmatrix} = \begin{pmatrix} -1 \\ b(X_t) \end{pmatrix}dt + \begin{pmatrix} 0 \\ \sigma(X_t) \end{pmatrix}dB_t$

By Dynkin's formula, $\mathbb{E}^{s,x}[w(Y_{t\wedge\tau_R})] = w(s,x) + \mathbb{E}^{s,x}\left[\int_0^{t\wedge\tau_R} \tilde{A}w(Y_r)dr\right] = w(s,x)$

$R\uparrow\infty \downarrow$ $\quad \hookrightarrow \tau_R = \inf\{t>0: |X_t| \geq R\}.$

$\mathbb{E}^{s,x}[w(Y_t)]$, $\quad \forall t \geq 0.$

Choosing $t = s$:

$w(s,x) = \mathbb{E}^{s,x}[w(Y_s)] = \mathbb{E}\left[w\left(\begin{pmatrix} 0 \\ X_s^{0,x} \end{pmatrix}\right)\right] = \mathbb{E}[f(X_s^{0,x})] = u(s,x).$ #

Rmk: semigroup: $u(t,\cdot) = Q_t f = \mathbb{E}^x[f(X_t)]$; $\partial_t u = \frac{d}{dt}Q_t f = Q_t(Af) = A(Q_t f)$

$\Rightarrow Q_t = e^{At}$ (Need further explanation, b.c. A is unbounded)

Exe 8.3 (Kolmogorov's forward Equ.) (Fokker-Planck Equ.)

Assume that X_t^x has a density $p_t(x,\cdot)$: $\mathbb{E}^x[f(X_t)] = \int f(y)\, p_t(x,y)\, dy$, $f \in C_0^2$

If $p_t(x,\cdot)$ is smooth, then $\partial_t p_t(x,y) = A_y^* p_t(x,y)$, $\forall x, y$.

$$A_y^* \phi = -\sum_i \partial_{y_i}(b_i \phi) + \sum_{ij} \partial_{y_i y_j}(a_{ij}\phi), \quad \forall \phi \in C^2.$$

Proof: By Dynkin's formula: $\mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x\left[\int_0^t Af(X_s)\, ds\right]$

$$\int_{\mathbb{R}^n} f(y)\, p_t(x,y)\, dy = f(x) + \int_{\mathbb{R}^n} \int_0^t Af(y)\, p_s(x,y)\, ds\, dy.$$

$\frac{d}{dt}$: $\langle f, \partial_t p_t \rangle_{L^2(\mathbb{R}^n)} = \langle Af, p_t \rangle = \langle f, A^* p_t \rangle$. #

How to solve the PDE? Does it have an equilibrium and what is it? Au = f? How is it related to invariant measure?

The Resolvent $(\alpha I - A)^{-1}$

Def: (Resolvent Operator) $R_\alpha g(x) = \mathbb{E}^x\left[\int_0^\infty e^{-\alpha t} g(X_t)\, dt\right]$, $\forall g \in C_b(\mathbb{R}^n)$

(formally: $\int_0^\infty e^{-\alpha t} e^{At} g\, dt = \int_0^\infty e^{-(\alpha I - A)t} dt\, g = (\alpha I - A)^{-1} g$)

Lemma (Feller cts) Let $g \in C_b(\mathbb{R}^n)$. Then $u(t,x) = \mathbb{E}^x[g(X_t)]$ is cts.

$\Rightarrow$ $R_\alpha g$ is bdd & cts.

Theorem 8.1.5 (a) If $f \in C_0^2(\mathbb{R}^n)$, then $R_\alpha(\alpha - A)f = f$, $\forall \alpha > 0$.

(b) If $g \in C_b(\mathbb{R}^n)$, then $R_\alpha g \in \mathrm{Dom}(A)$ and $(\alpha - A) R_\alpha g = g$, $\forall \alpha > 0$.

Proof, (a) by Dynkin's formula, $R_\alpha(\alpha - A)f = \alpha R_\alpha f - R_\alpha A f = \alpha \int_0^\infty e^{-\alpha t}\mathbb{E}^x[f(X_t)]\, dt - \int_0^\infty e^{-\alpha t} \mathbb{E}^x[Af(X_t)]\, dt$

$$= -e^{-\alpha t} u(t,x)\Big|_0^\infty + \underbrace{\int_0^\infty e^{-\alpha t}\partial_t u(t,x)\, dt - \int_0^\infty e^{-\alpha t} Au\, dt}_{=0}$$

$$= f(x).$$

b). Need $\partial_t \mathbb{E}^x[R_\alpha g(X_t)] = A[.]$:

$$\mathbb{E}^x[R_\alpha g(X_t)] = \mathbb{E}^x\left[\mathbb{E}^{X_t}\left[\int_0^\infty e^{-\alpha s} g(X_s)\, ds\right]\right] \overset{\text{Markov}}{=} \mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty e^{-\alpha s} g(X_{t+s})\, ds \,\Big|\, \mathcal{F}_t\right]\right]$$

$$= \mathbb{E}^x\left[\int_0^\infty e^{-\alpha s} g(X_{t+s})\, ds\right] = \int_0^\infty e^{-\alpha s}\, \mathbb{E}^x[g(X_{t+s})]\, ds \qquad \mathbb{E}^x[g(X_{t+s})] = \frac{d}{ds}\int_t^{t+s} \mathbb{E}^x[g(X_v)]\, dv$$

$$\overset{IBP}{=} \alpha \int_0^\infty e^{-\alpha s} \int_t^{t+s} \mathbb{E}^x[g(X_v)]\, dv\, ds$$

$\frac{d}{dt}$ $\downarrow$

$$A R_\alpha g = \frac{d}{dt}\mathbb{E}^x[R_\alpha g(X_t)]\Big|_{t=0} = \alpha \int_0^\infty e^{-\alpha s}\, \mathbb{E}^x[g(X_{t+s})] - \mathbb{E}^x[g(X_t)]\Big|_{t=0} ds = \alpha R_\alpha g - g.$$ #

§8.2 The Feynman-Kac formula

Theorem 8.2.1 $f\in C_0^2$, $q\in C(\mathbb{R}^n)$ lower bdd. Then

(a) $v(t,x)=\mathbb{E}^x[\exp(-\int_0^t q(X_s)ds)f(X_t)]$ solves $\begin{cases}\partial_t v = Av-qv, & t>0,\ x\in\mathbb{R}^n\\ v(0,x)=f(x). & x\in\mathbb{R}^n\end{cases}$

(b) If $w(t,x)\in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ is bdd on $K\times\mathbb{R}^n$ for each $K\subseteq\mathbb{R}$ cpt, and w solve the equ. then $w=v$.

Proof: (a) Let $Y_t=f(X_t)$, $Z_t=e^{-\int_0^t q(X_s)ds}$. $dY_t=Af(X_t)dt+\nabla f\cdot\sigma(X_t)dB_t$

$dZ_t=-Z_t\,q(X_t)dt$

$\Rightarrow\quad d(Y_tZ_t)=Y_t dZ_t+Z_t dY_t$, an Ito diffusion.

$\Rightarrow\quad v(t,x)=\mathbb{E}^x[Y_tZ_t]$ is C^1 in t

Then, $\frac{1}{r}\left(\mathbb{E}^x[v(t,X_r)]-v(t,x)\right)=\frac{1}{r}\left(\mathbb{E}^x[\mathbb{E}^{X_r}(Z_tf(X_t))]-\mathbb{E}^x[Z_tf(X_t)]\right)$

$\downarrow r\downarrow 0$

Av

$=\frac{1}{r}\left(\mathbb{E}^x[\mathbb{E}^x[f(X_{t+r})\exp(-\int_0^t q(X_{s+r})ds)|\mathcal{F}_r]-\mathbb{E}^x[Z_tf(X_t)|\mathcal{F}_r]]\right)$

$=\frac{1}{r}\mathbb{E}^x[Z_{t+r}\exp\int_0^r q(X_s)ds\, f(X_{t+r})-Z_tf(X_t)]$

$=\frac{1}{r}\mathbb{E}^x[Z_{t+r}f(X_{t+r})-f(X_t)Z_t]+\frac{1}{r}\mathbb{E}^x[\underbrace{(e^{\int_0^r q(X_s)ds}-1)Z_{t+r}f(X_{t+r})}_{\text{bdd.}}]$

$\xrightarrow{r\downarrow 0}\ \partial_t v+qv$ ✓

(b). Note that $\begin{cases}\tilde{A}w(t,x)=-\partial_t w+Aw-qw=0, & \forall t>0,\ x\in\mathbb{R}^n\\ w(0,x)=f(x) & \forall x\in\mathbb{R}^n.\end{cases}$

Let $H_t=(s-t,\ X_t^{0,x},\ Z_t)$ $Z_t=z+\int_0^t q(X_s)ds$

Then H_t is an Ito diffusion w/ generator $A_H\phi(s,x,z)=-\partial_s\phi+A\phi+q(x)\partial_z\phi$, $\phi\in C_0^2$

For $\phi=e^{-z}w(s,x)$: $A_H\phi=e^{-z}[-\partial_s w_s+Aw-qw]=0$

By Dynkin's formula w/ ϕ and $\tau_R=\inf\{t>0: |H_t|\geq R\}$:

$$\mathbb{E}^{s,x,z}[\phi(H_{t\wedge\tau_R})]=\phi(s,x,z)+\underbrace{\mathbb{E}^{s,x,z}[\int_0^{t\wedge\tau_R}A_H\phi(H_r)dr]}_{=0}$$

$\Rightarrow\quad w(s,x)=\phi(s,x,0)=\mathbb{E}^{s,x,0}[\phi(H_{t\wedge\tau_R})]=\mathbb{E}^x[\exp(-\int_0^{t\wedge\tau_R}q(X_r)dr)\,w(s-t\wedge\tau_R,X_{t\wedge\tau_R})]$

$\xrightarrow{R\to\infty}\mathbb{E}^x[\exp(-\int_0^t q(X_r)dr)w(s-t,X_t)]$ (w. bdd)

Let $t=s$,

$w(s,x)=\mathbb{E}^x[\exp(-\int_0^s q(X_r)dr)\,w(0,X_s^{0,x})]=v(s,x)$.   #

What is the diffusion with the A-qI being a generator?

§ 8.3 The Martingale problem.

- Strook & Varadhan (1979) : weak solution to SDE & martingale

(*) $dX_t = b(X_t)dt + \sigma(X_t)dB_t$; $X_0 = x$. $\Leftrightarrow$ generator $A = b\cdot\nabla + \frac{1}{2}\sigma\sigma^T : \text{Hess}$

$$f(X_t) = \underset{\sim}{f(x)} + \int_0^t Af(X_s)ds + \underbrace{\int_0^t \nabla f^T \cdot \sigma(X_s)dB_s} \longrightarrow \text{an mG. } \mathcal{F}_t^B \text{ or } \mathcal{F}_t^X$$

$$\Rightarrow M_t = f(X_t) - \int_0^t Af(X_s)ds = f(x) + \int_0^t \nabla f^T \cdot \sigma(X_s)dB_s \text{ is a mG wrt } \mathcal{F}_t^X, \quad \forall f \in C_0^2(\mathbb{R}^n)$$

That is, the weak sdu. $\{X_t\}$ to (*) makes $M_t(f)$ be a mG wrt $\mathcal{F}_t^X$ $\forall f \in C_0^2$.

$\downarrow$ Law of X_t $\qquad\qquad\qquad\qquad \searrow$

a probab. meas. $\tilde{Q}^x$ on $((\mathbb{R}^n)^{[0,\infty)}, \mathcal{B}, \tilde{Q}^x)$ $\longrightarrow$ M_t is a $\tilde{Q}^x$-mG, $\forall f \in C_0^2$.

$\swarrow$

Def 8.3.2 (Martingale problem) Let $L = b\cdot\nabla + a : \nabla^2 = \sum_i b_i \partial_{x_i} + \sum_{ij} a_{ij} \partial_{x_i x_j}$

- b_i, a_{ij} locally bdd, Borel meas.
- semi-elliptic: $a(x) \geq 0, \forall x$

We say that a probability meas. $\tilde{\mathbb{P}}^x$ (on the path space $((\mathbb{R}^n)^{[0,\infty)}, \mathcal{B})$) solves the martingale problem for L of the process

$$M_t = f(\omega_t) - \int_0^t Lf(\omega_r)dr; \quad M_0 = f(x)$$

is a $\tilde{\mathbb{P}}^x$ martingale w.r.t. $\mathcal{B}_t$ for all $f \in C_0^2(\mathbb{R}^n)$

- The mG problem is well-posed if $\exists!\ \mathbb{P}^x$.

Strook & Varadhan : weak sdu $\Leftrightarrow$ $\exists\ \tilde{\mathbb{P}}^x$ for the mG. problem.

weak sdu. $\underline{+\text{Markov}}$ $\Leftrightarrow$ $\underline{\text{well-posed}}$ mG problem.

Q1: when the weak sdu. is NOT Markov?

Q2: when the measure is NOT unique?

§8.4 Ito process $\overset{?}{\Rightarrow}$ Ito diffusion

Ito process: (P44) $\quad dX_t = u(t,\omega)dt + v(t,\omega)dB_t$

Ito diffusion (P114) $\quad dX_t = b(X_t)dt + \sigma(X_t)dB_t \qquad (*)$

- If X_t is an Ito process, then $\varphi(X_t)$ is also an Ito process.

  - - - - - Ito diffusion, - - - - - - - - - - . Will $\varphi(X_t)$ be an Ito diffusion?

  NOT in general. yes if ↓

<u>Example</u> 8.4.1 (The Bessel process) $n \geq 2$

$$R_t = |B(t)| = \left(\sum_{i=1}^n B_i(t)^2\right)^{\frac{1}{2}} \Rightarrow dR_t = \sum_{i=1}^n \frac{B_i\, dB_i}{R_t} + \frac{n-1}{2R}dt$$

$g(x) = |x| = (|x|^2)^{\frac{1}{2}}$

$\nabla g(x) = \frac{1}{2}(|x|^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{|x|}$

$\partial_i \partial_j g = |x|^{-1} \cdot \delta_{ij} - \frac{x_j x_i}{|x|^3}$

$\Delta g = |x|^{-1} \cdot n - \frac{|x|^2}{|x|^3} = |x|^{-1}(n-1)$

- This is NOT an SDE in the form of $(*)$.
- But if $\quad Y_t = \sum_i \int_0^t \frac{B_i\, dB_i}{|R_t|} \sim \hat{B}_t$ a Bm.

then $\quad dR_t = \frac{n-1}{2R_t}dt + d\hat{B}_t$, a diffusion

with generator. $\quad Af(x) = \frac{1}{2}f'' + \frac{n-1}{2x}f'$.

(weak uniqueness needs Lipschitz)

yes. $\quad V = \frac{B}{|B|}, \quad VV^T = \frac{1}{|B|^2}BB^T = 1$

Theorem 8.4.2. An Ito process $\quad dY_t = V\, dB_t;\ Y_0 = 0$ w/ $V(t,\omega) \in \mathcal{V}^{n\times m}$

↑ coincides in law w/ an n-D Bm iff $\quad VV^T(t,\omega) = I_n$, a.a. $(t,\omega) \in dt \times dP$.

Theorem 8.4.3. Let $\quad dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $b \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n\times m}$, $X_0 = x$

$$dY_t = u(t,\omega)dt + v(t,\omega)dB_t, \quad u \in \mathbb{R}^n, \quad v \in \mathbb{R}^{n\times m}, \quad Y_0 = x.$$

Then $\quad X_t \simeq Y_t$ iff $\quad \mathbb{E}[u(t,\cdot)|\underbrace{\mathcal{N}_t}_{\mathcal{N}_t^Y}] = b(Y_t^x)(\omega); \quad vv^T(t,\omega) = \sigma\sigma^T(Y_t^x)(\omega)$ a.a. $dt \times dP$

<u>Proof</u>: $(\Leftarrow)$ Let $A = b\cdot\nabla + \frac{1}{2}(\sigma\sigma^T) : \text{Hess}.$ be the generator of X_t. $\forall f \in C_0^2$, define

$$Hf(t,\omega) := \sum_i u_i\, \partial_i f(Y_t) + \frac{1}{2}\sum_{ij} (vv^T)_{ij} \cdot \partial_{x_i x_j} f(Y_t).$$

Then, <u>Ito formula</u> → $\quad f(Y_s) \overset{s>t}{=} f(Y_t) + \int_t^s Hf(r,\omega)dr + \int_t^s \nabla f^T \cdot v\, dB_r$

$$\mathbb{E}[f(Y_s)\,|\,N_t] = f(Y_t) + \int_t^s \underbrace{\mathbb{E}[Hf(r,\omega)\,|\,N_t]}\,dr + 0$$

$$= \mathbb{E}[\underbrace{\mathbb{E}[Hf(r,\omega)\,|\,N_r]}_{Af(Y_r)}\,|\,N_t] \overset{(*)}{=} \mathbb{E}[Af(Y_r)\,|\,N_t]$$

$$\downarrow \qquad = f(Y_t) + \mathbb{E}\Big[\int_t^s Af(Y_r)\,dr\,\Big|\,N_t\Big]$$

Let $M_t = f(Y_t) - \int_0^t Af(Y_r)\,dr$,

$$\Rightarrow \mathbb{E}[M_s\,|\,N_t] = \mathbb{E}\Big[f(Y_s) - \int_0^s Af(Y_r)\,dr\,\Big|\,N_t\Big] = f(Y_t) + \mathbb{E}\Big[\int_t^s Af(Y_r)\,dr\,\Big|\,N_t\Big] - \mathbb{E}\Big[\int_0^s Af(Y_r)\,dr\,\Big|\,N_t\Big]$$

$$= f(Y_t) - \int_0^t Af(Y_r)\,dr = M_t.$$

By uniqueness of the solu. to the mG. problem, we conclude that $X_t \simeq Y_t$.

√√ $(\Rightarrow)$. Assume that $X_t \simeq Y_t$. Let $f \in C_0^2$. By Ito formula,

$$(1)\quad \lim_{h\downarrow 0} \frac{1}{h}\Big(\mathbb{E}^x[f(Y_{t+h})\,|\,N_t] - f(Y_t)\Big) = \lim_{h\downarrow 0} \int_t^{t+h} \mathbb{E}[Hf(s,\omega)\,|\,N_t]\,ds$$

$$= \mathbb{E}^x[Hf(t,\omega)\,|\,N_t] = \sum_i \mathbb{E}^x[u_i(t,\omega)\,\partial_i f(Y_t)\,|\,N_t] + \sum_{ij} \mathbb{E}^x[(VV^T)_{ij}(t,\omega)\,\partial_{ij} f(Y_t)\,|\,N_t]$$

$$(2)\quad = \sum_i \mathbb{E}^x[u_i(t,\omega)\,|\,N_t]\,\partial_i f(Y_t) + \sum_{ij} \mathbb{E}^x[VV^T(t,\omega)\,|\,N_t]\,\partial_{ij} f(Y_t).$$

On the other hand, since $X_t \simeq Y_t$ & X_t is a Markov process, so is Y_t. Hence,

$$(1) = \lim_{h\downarrow 0} \frac{1}{h}\Big(\mathbb{E}^{Y_t}[f(Y_h)] - \mathbb{E}^{Y_t}[f(Y_0)]\Big) = \mathbb{E}^{Y_t}[Hf(0,\omega)]$$

$$= \sum_i \mathbb{E}^{Y_t}[u_i(0,\omega)\,\partial_i f(Y_0)] + \frac{1}{2}\sum_{ij} \mathbb{E}^{Y_t}[(VV^T)_{ij}(0,\omega)\,\partial_{ij} f(Y_0)]$$

$$(3)\quad = \sum_i \mathbb{E}^{Y_t}[u_i(0,\omega)]\,\partial_i f(Y_t) + \frac{1}{2}\sum_{ij} \mathbb{E}^{Y_t}[(VV^T)_{ij}(0,\omega)]\,\partial_{ij} f(Y_t)$$

$\mathbb{E}^{Y_t}[F(\omega)g(Y_s)] = \mathbb{E}[F(\omega)g(Y_s)\,|\,Y_{s=0} = Y_t]$

Thus $(2) = (3)$, $\forall f \in C_0^2$. Take $f(x) = x_i$ (use stopping time & C_0^2 approximation)

$$\mathbb{E}^x[u_i(t,\omega)\,|\,N_t] = \mathbb{E}^{Y_t}[u_i(0,\omega)] \quad a.a. \quad = b_i(Y_t^x) \qquad √√$$

Take $f(x) = 2x_i x_j \Rightarrow \mathbb{E}^x[(VV^T)_{ij}(t,\omega)\,|\,N_t] = \mathbb{E}^{Y_t}[(VV^T)_{ij}(0,\omega)]$, a.a. $= (\sigma\sigma^T)_{ij}(Y_t^x)$.

Also, since $X_t \simeq Y_t$, their generator are the same, i.e.

$\Downarrow$ △△ ? ?

$VV^T(t,\omega) = (\sigma\sigma^T)(Y_t^x)$

Lemma 8.4.4 √√ Let $dY_t = u(t,\omega)dt + v(t,\omega)dB_t$, $Y_0 = x$. as above

There $\exists$ an N_t adapted process $W(t,\omega)$ st. $VV^T(t,\omega) = W(t,\omega)$.

Proof: By Ito's formula, $g(x) = x_i x_j$: $Y_iY_j(t,\omega) = x_i x_j + \int_0^t [Y_i\,dY_j(s) + Y_j\,dY_i(s)] + \int_0^t (VV^T)_{ij}(s,\omega)\,ds$

$$\Rightarrow \int_0^t (VV^T)_{ij}(s,\omega)\,ds = H_{ij}(t,\omega) = Y_iY_j(t,\omega) - x_i x_j - \int_0^t (\quad), \quad N_t \text{ adapted}.$$

$$\Rightarrow (VV^T)_{ij}(t,\omega) = \lim_{r\downarrow 0} \frac{1}{r}\Big(H_{ij}(t+r,\omega) - H_{ij}(t,\omega)\Big) \in N_t. \qquad \#$$

△ Not on $U(t,\cdot)$ or $V(t,\cdot)$: $\downarrow$

<u>Rmk</u> 1. $u(t,\cdot)$ may NOT be $\mathcal{N}_t$ meas.. E.g.

$$dY_t = B_t^1 \, dt + dB_t^2 \quad ,$$

§8.5 Random time change

Let $c(t,\omega) \geq 0$ be an $\mathcal{F}_t$-adapted process

$$\beta_t := \beta(t,\omega) = \int_0^t c(s,\omega)\,ds$$

- a (random) time change w/ time change rate $c(t,\omega)$
- β_t is NON-decreasing

$$\alpha_t = \inf\{s : \beta_s > t\}$$

- α_t is a right-inverse of β_t,

$$\beta(\alpha(t,\omega),\omega) = t, \qquad \forall \omega.\ \forall t,$$

- α_t is right-continuous.
- If $c > 0$, then
  - $t \mapsto \beta_t$ is strictly increasing
  - $t - \alpha_t$ is continuous
  - α_t is a left-inverse of β_t:

$$\alpha(\beta(t,\omega),\omega) = t, \quad \forall t$$

- $\omega \mapsto \alpha(t,\omega)$ is an $\{\mathcal{F}_t\}$-stopping time. (all the above are deterministic, until here.)

Question: In Theorem 8.4.3. X_t Ito diffusion, Y_t Ito process $X_t \simeq Y_t$.

When does there exist a time change β_t st $Y_{\alpha_t} \simeq X_t$?