PDE Undergrad
HW2 Solution

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[2.2]4 (1) $L(u_p + C_1u_1 + C_2u_2)$

$\overset{L \text{ linear}}{=} Lu_p + C_1Lu_1 + C_2Lu_2 = f+0+0 = f$

(2) $\underbrace{V = u_{p_1} + u_{p_2}}$. $\Rightarrow LV = L(u_{p_1} + u_{p_2}) = f_1 + f_2$. □

[2.3] Consider $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$. Determine the eigenvalue λ and corresponding eigen functions if ϕ satisfies the following bdry conditions:

(c). $\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$

Sol: Case 1: $\lambda = 0$ $\Rightarrow \frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = A + Bx \Rightarrow \frac{d\phi}{dx} = B$

since $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0 \Rightarrow$ ~~$\begin{cases} A=0 \\ A+LB=0 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=0 \end{cases} \Rightarrow \phi \equiv 0$~~ $\cdot B=0$ $\phi = A$ (const)

thus 0 is ~~not~~ an eigenvalue, ϕ = const

Case 2: $\lambda < 0$ $\Rightarrow \phi(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$.

$\Rightarrow \frac{d\phi}{dx} = \sqrt{-\lambda}\,(C_1 e^{\sqrt{-\lambda}x} - C_2 e^{-\sqrt{-\lambda}x})$.

$\frac{d\phi}{dx}(0) = 0 \Rightarrow \sqrt{-\lambda}\,(C_1 - C_2) = 0 \Rightarrow C_1 = C_2$

$\frac{d\phi}{dx}(L) = 0 \Rightarrow \sqrt{-\lambda}\,(C_1 e^{\sqrt{-\lambda}L} - C_2 e^{-\sqrt{-\lambda}L}) = 0 \Rightarrow \sqrt{-\lambda}\,C_1(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) = 0$

$\Rightarrow e^{\sqrt{-\lambda}L} = e^{-\sqrt{-\lambda}L}$ (impossible)

or $C_1 = 0 = C_2$ (makes $\phi \equiv 0$).

so $\lambda < 0$ is impossible

Case 3: $\lambda > 0$ $\Rightarrow \phi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$\frac{d\phi}{dx} = \sqrt{\lambda}\,(-C_1\sin(\sqrt{\lambda}x) + C_2\cos(\sqrt{\lambda}x))$.

$\begin{cases} \frac{d\phi}{dx}(0) = 0 \\ \frac{d\phi}{dx}(L) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ -C_1\sin(\sqrt{\lambda}L) = 0 \end{cases} \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$ is the eigenvalue $n \in \mathbb{Z}_+$

and the corresponding $\phi(x) = C_1\cos\left(\frac{n\pi}{L}x\right)$ □

[2.3]3. Consider $\begin{cases} \partial_t u = k\partial_x^2 u \\ u(0,t) = 0 = u(L,t). \end{cases}$

(b) If $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$, then solve the IVP.

Sol: Use separation of variables: $u(x,t) = X(x)\cdot T(t)$

Then the equation becomes $T'X = k\,X''T$

$$\Rightarrow \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

$\uparrow$ we assume

$\Rightarrow$ get 2 ODEs

$$\begin{cases} ① \; T' + k\lambda T = 0 \\ ② \; X'' + \lambda X = 0 \end{cases}$$

• Solve ②

Mimicing the proof in 2.3.2, one can get the eigenvalues of ② is $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n\in\mathbb{Z}_+$. (thus $\lambda_n > 0$).

and $\underline{X_n(x) = B_n\sin\left(\frac{n\pi x}{L}\right)}$ for some $B_n \neq 0$.

• Solve ① $\Rightarrow T_n(t) = e^{-k\lambda_n t}$.

Therefore one can ~~be~~ re-write the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\,T_n(t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right).$$

Inserting u into the initial-bdry conditions $\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,L) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\frac{3\pi x}{L} \end{cases}$

one can get $B_1 = 3 \quad B_3 = -1 \quad B_n = 0$ for $n \neq 1, 3$

$$\Rightarrow u(x,t) = 3\sin\frac{\pi x}{L}\, e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\frac{3\pi x}{L}\, e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

$\square$

[2.3] 5. Evaluate $I = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$. (Orthognoality of Trigonometric Functions)

only consider nonnegative m, n.

Sol: Let $y = \frac{x}{L}$ $\Rightarrow I = L \int_0^1 \sin n\pi y \sin m\pi y \, dy$

~~1) m=n~~. $\sin n\pi y \sin m\pi y = \frac{1}{2}\left(\cos((n-m)\pi y) - \underbrace{\cos((n+m)\pi y)}\right)$

1°) $m=n$. $I = \frac{L}{2}\int_0^1 \cos((n-m)\pi y)\, dy - 0$. $\int_0^1 \cdots = 0$

$= \frac{L}{2}$.

2°) $n \neq m$. then $\int_0^1 \cos(n-m)\pi y \, dy = 0$

So $I = \begin{cases} 0 & n \neq m \text{ or } n=m=0 \\ \frac{L}{2} & n = m \neq 0 \end{cases}$

$\Rightarrow I = 0$

Similarly one can prove $\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n=m\neq 0 \\ L & n=m=0 \end{cases}$ $\square$

$\int_0^L \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0$

[2.4] 1 (b) Solve $\begin{cases} \partial_t u = k\partial_x^2 u. & 0<x<L. \; t>0. \\ \partial_x u(0,t) = \partial_x u(L,t) = 0 & t>0 \\ u(x,0) = 6 + 4\cos\frac{3\pi x}{L}. \end{cases}$

Sol: Use separation of variables as in [2.3] 3:

let $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$.

$u(x,0) = 6 + 4\cos\frac{3\pi x}{L} \overset{\text{let } t=0}{\Rightarrow}$ $A_0 = 6$ $\Rightarrow u(x) = 6 + 4\cos\frac{3\pi x}{L} \cdot e^{-\left(\frac{3\pi}{L}\right)^2 kt}$

$A_3 = 4$

$A_n = 0$ if $n \neq 0, 3$

$\square$