PDE Undergrad
HW3 Solution

Junyan Zhang
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[2.5] 1 Solve Laplacian equ. in a rectangle $[0, L] \times [0, H]$ with the bdry condition

(d) $u(0,y) = g(y)$. $u(L,y) = 0$
$\partial_y u(x,0) = 0$ $u(x,H) = 0$

Sol: Let $u(x,y) = X(x)Y(y)$

$$\Delta u = \partial_x^2 u + \partial_y^2 u = 0 \Rightarrow \cancel{\frac{X''(x)}{X(x)}} \; X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\Downarrow$$

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = \lambda$$

Boundary conditions: $X(0)\,Y(y) = g(y)$
$X(L) = 0$
$Y'(0) = 0$
$Y(H) = 0$

Therefore one can suppose $X(x) = A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x}$
$Y(y) = B_1 \sin(\sqrt{\lambda}y) + B_2 \cos(\sqrt{\lambda}y)$

Combining the boundary conditions, one can get:

$$\left\{\begin{array}{l} Y'(0) = B_1\sqrt{\lambda} = 0 \Rightarrow B_1 = 0 \Rightarrow Y(y) = B_2\cos(\sqrt{\lambda}y) \\ Y(H) = B_2\cos(\sqrt{\lambda}H) = 0 \Rightarrow \sqrt{\lambda} = \frac{(n+\frac{1}{2})\pi}{H} \quad (n \geq 0) \end{array}\right\} \Rightarrow Y_n(y) = B_n \cdot \cos\left(\frac{(n+\frac{1}{2})\pi}{H}y\right)$$

$$X(L) = 0 \Rightarrow X_n(x) = \cancel{A_n \sinh}\; A_n \cdot sh\left(\frac{(n+\frac{1}{2})\pi}{H}(x-L)\right) \qquad sh\, t = \frac{e^t - e^{-t}}{2}$$

So:
$$u(x,y) = \sum_{n=0}^{\infty} \cancel{A_n}\, \underbrace{C_n}_{A_n \cdot B_n} sh\left(\frac{(n+\frac{1}{2})\pi}{H}(x-L)\right)\cos\left(\frac{(n+\frac{1}{2})\pi}{H}y\right)$$

$$X(0)Y(y) = g(y) \Rightarrow C_n = \frac{2}{H\sinh\left(-\frac{(n+\frac{1}{2})\pi L}{H}\right)}\int_0^H g(y)\cos\left(\frac{(n+\frac{1}{2})\pi y}{\cancel{2}H}\right)dy$$

$\square$

5 (b). Solve Laplacian eq. inside the $\frac{1}{4}$-circle of radius 1 $(0 \le \theta \le \frac{\pi}{2})$, $0 \le r \le 1$

subject to. $\partial_\theta u(r,0) = 0$ $\quad \partial_\theta u(r, \frac{\pi}{2}) = 0$ $\quad u(1,\theta) = f(\theta)$.

Sol: Let $u(r,\theta) = \Theta(\theta) R(r)$.

Then one can get: $\frac{r}{R(r)} (rR')' = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$

① $\Rightarrow$ $\Theta(\theta) = C_1 \sin(\sqrt{\lambda}\theta) + C_2 \cos(\sqrt{\lambda}\theta)$

$\partial_\theta u(r,0) = \partial_\theta u(r, \frac{\pi}{2}) = 0 \Rightarrow \Theta'(0) = \Theta'(\frac{\pi}{2}) = 0$

$$\left.\begin{aligned} \Rightarrow\ & C_1\sqrt{\lambda} = 0 \\ & -C_2\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot\tfrac{\pi}{2}) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 0 \\ \lambda &= 4n^2 \end{aligned}$$

$\Rightarrow \Theta(\theta) = C_2 \cos(2n\theta)$

② $\frac{r}{R(r)} (rR')' = \lambda$

$\Rightarrow rR' + r^2 R'' = 4n^2 R$. one can solve this ODE by

$$R_n(r) = \begin{cases} \tilde{C}_1 r^{2n} + \tilde{C}_2 r^{-2n} & \\ \tilde{C}_1 + \tilde{C}_2 \ln r & n = 0 \end{cases}$$

Since R_n is bounded $\Rightarrow$ $\tilde{C}_2 = 0$

$R_n(r) = \tilde{C}_1 r^{2n}$ $\quad n \ge 0$.

$\Rightarrow u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{2n} \cos 2n\theta$.

③ $u(1,\theta) = f(\theta) \Rightarrow \sum_{n=0}^{\infty} A_n \cos(2n\theta) = f(\theta)$

$$\Rightarrow A_n = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta & n = 0 \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos 2n\theta\, d\theta & n > 0 \end{cases}$$

$\square$

10. ~~Let~~

Proof. Let u_1, u_2 be two solutions to $\begin{cases} \Delta u = g & \text{in } U \\ u = f & \text{on } \partial U \end{cases}$

$$V := u_1 - u_2 \Rightarrow \begin{cases} \Delta V = 0 & \text{in } U \\ V\big|_{\partial U} = 0 \end{cases}$$

By maximum principle we have $V = 0$ $\square$

[3.2] Sketch the Fourier series of $f(x)$ in $[-L, L]$ and determine the Fourier coefficients.

(d). $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0. \end{cases}$

Sol: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{L}{4}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

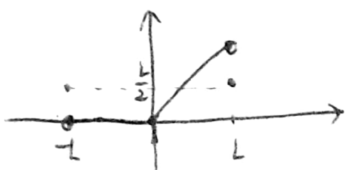
$$= \frac{1}{L}\int_{0}^{L} x\cos\frac{n\pi x}{L}\,dx = \begin{cases} -\frac{2L}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad \left(\text{or say } \frac{L}{n^2\pi^2}\left((-1)^n - 1\right)\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}(-1)^n = (-1)^{n-1}\frac{L}{n\pi}$$

so $f(x) = \frac{L}{4} + \sum_{n=1}^{\infty} \frac{L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

for $-L < x < L$

Cannot attain endpoint



$f(\pm L) = \frac{0 + L}{2} = \frac{L}{2}$

since f is piecewise C^1.

(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

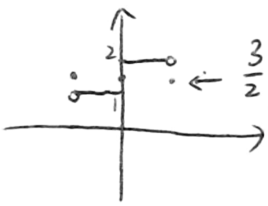
Similarly as in (d), one has

$$a_0 = \frac{3}{2}$$

$$a_n = 0 \qquad n \geq 1.$$

$$b_n = \frac{1-(-1)^n}{n\pi}.$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \qquad \underline{\underline{-L<x<L}}$$



$\square$.