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HW5 Solution   2020.3.3

[3.5] 1 Consider $x^2 \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

(1) Determine b_n

(2) For what values of x $\exists$ equality

Sol: (1) From (3.5.6), one has $\frac{x^2}{2} \sim \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3^3} + \frac{\sin\frac{5\pi x}{L}}{5^3} + \cdots\right)$

From (3.3.11) (3.3.12) $x = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$

$$\Rightarrow x^2 \sim L\left(\sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}\right) - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3^2} + \frac{\sin\frac{5\pi x}{L}}{5^3} + \cdots\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where } b_n = \begin{cases} \frac{2L}{n\pi} - \frac{8L^2}{n^3\pi^3} & n \text{ odd} \\ \frac{-2L^2}{n\pi} & n \text{ even} \end{cases}$$

(2) $0 < x < L$. otherwise, x^2 will be odd instead of being even.

□

2. Obtain the Fourier cos series of x^2.

Sol: Integrating the Fourier series of x, one gets

$$x^2 = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{3}$$

□

5. Show that $B_n = a_n / \frac{n\pi}{L}$

Sol: $B_n = \frac{1}{L}\int_{-L}^{L} G(x)\sin\frac{n\pi x}{L}\,dx = \frac{L}{n\pi L}\int_{-L}^{L}\left(F(x) - a_0(x+L)\right)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)$

$$= \frac{1}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L} f(x)\,dx = \frac{L}{n\pi}a_n$$

□

[3.4] 11

$$\begin{cases} \partial_t u = k\partial_x^2 u + g(x) \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Pf: Let $u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L}$

From initial data, one has $f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\frac{n\pi x}{L}$

$$\left(\Rightarrow B_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)$$

Take ∂_x^2

$$\partial_x^2 u \sim \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 B_n(t) \sin\frac{n\pi x}{L}$$

Since $\partial_t u$ is piecewise smooth, one has $\partial_t u = \sum_{n=1}^{\infty} B_n'(t) \sin\frac{n\pi x}{L}$

Now, the Fourier series of g is $\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$, $A_n = \frac{2}{L}\int_0^L g(x)\sin n\pi x\,dx$

plug these into heat eq. $\Rightarrow$ $B_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 B_n(t) + A_n$

$$\Rightarrow B_n(t) = B_n(0)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \frac{A_n}{k\left(\frac{n\pi}{L}\right)^2}$$

$$\text{So } u(x,t) = \sum_{n=1}^{\infty}\left(B_n(0)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \frac{A_n}{k\left(\frac{n\pi}{L}\right)^2}\right)\sin\frac{n\pi x}{L}$$

□

[8.2] 2.(d) Reduce ~~Solve~~:

$$\begin{cases} \partial_t u = k\partial_x^2 u + Q(x,t) \\ u(0,x) = f(x) \\ u(0,t) = 0,\quad \partial_x u(L,t) + h(u(L,t) - B(t)) = 0 \end{cases}$$

Sol: Let $V(x,t) = u(x,t) - B(t)\,\frac{h(x-L)x}{L}$

then $\partial_t V = \partial_x^2 u + \bar{Q}(x,t)$ for some $\bar{Q}$ ~~where $\bar{Q} = Q - \partial_t r$~~ ~~$r = \frac{h\,B(t)\,x}{1+hL}$~~

$$\begin{cases} V(0,t) = u(0,t) - 0 = 0 \\ \frac{\partial V}{\partial x}(L,t) + hV(L,t) = \partial_x u(L,t) + h u(L,t) - hB(t) - h\frac{(x-L)x}{L}B(t)\Big|_{x=L} \\ \qquad = 0 \end{cases}$$

□

[8.3] 6. Solve $\begin{cases} \partial_t u = \partial_x^2 u + e^{-2t}\sin 5x \\ u(0,t)=1 \\ u(\pi,t)=u(x,0)=0 \end{cases}$

Sol: Similarly as in 8.2.2, we set $v(x,t)=u(x,t)-(1-\frac{x}{\pi})$

so $\begin{cases} \partial_t v = \partial_x^2 v + e^{-2t}\sin 5x \\ v(0,t)=v(\pi,t)=0 \\ v(x,0)=-1+\frac{x}{\pi} \end{cases}$

By separating variables we can assume $v(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin nx$.

Plug this into heat eq:

$$\Rightarrow \sum_{n\geq 1} (a_n'(t)+n^2 a_n)\sin nx = e^{-2t}\sin 5x, \quad \forall t; x.$$

so $a_n'(t)+n^2 a_n = \begin{cases} 0 & n\neq 5 \\ e^{-2t} & n=5 \end{cases}$

$$a_n(0) = -\frac{2}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\sin nx\, dx$$

$$a_n(t) = \begin{cases} a_n(0)\, e^{-n^2 t} & n\neq 5 \\ \left(\frac{1}{23}(e^{23t}-1)+a_5(0)\right)e^{-25t} \end{cases}$$

□