Undergrad PDE

HW 7

Junyan Zhang

[5.3] 3. Consider the non-S-L eq: $\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + (\lambda\beta(x)+\gamma(x))\phi = 0$ ①

Multiply $H(x)$. Determine $H(x)$ s.t. ① becomes a standard S-L eq.

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \lambda(\sigma(x)+q(x))\phi = 0 \quad \cdots ②$$

Given α, β, γ, what are p, q, σ?

pf: ② $\Rightarrow p(x)\frac{d^2\phi}{dx^2} + p'(x)\frac{d\phi}{dx} + (\lambda\sigma(x)+q(x))\phi(x) = 0$

① $\times H(x) \Rightarrow \quad H(x)\phi''(x) + H(x)\alpha(x)\phi'(x) + H(x)(\lambda\beta(x)+\gamma(x))\phi(x) = 0$

It follows that $p = H \quad \sigma = H\beta$

$p' = H\alpha. \quad q = H\gamma.$

Suppose $H \neq 0$. then $p = e^{\int \alpha(x)dx}$. $H = p$

$\sigma = p\cdot\beta. \quad q = p\cdot\gamma.$ $\square$

4. Consider $\partial_t u = k\partial_x^2 u - V_0\partial_x u$.

(1). Show that the spatial ODE obtained by Separation of variables is not a S-L eq.

pf: Let $u = X(x)T(t) \Rightarrow XT' = kX''T - V_0X'T$

$$\Rightarrow \frac{kX'' - V_0X'}{X} = \lambda = \frac{T'}{T}$$

$\Rightarrow kX''(x) - V_0X'(x) + \lambda X(x) = 0$

is S-L $\Leftrightarrow$. $p(x) = k. \quad p'(x) = -V_0$. impossible $\blacksquare$

(c) Solve IBVP with $\partial_x u(0,t) = 0 = \partial_x u(L,t)$, $u(x,0) = f(x)$.

$T'(t) = -\lambda T(t) \Rightarrow T(t) = Ce^{-\lambda t}$

$kX''(x) - V_0 X'(x) + \lambda X(x) = 0$

$\{ X'(0) = X'(L) = 0$ $\Rightarrow X(x) = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}$

$$\alpha_{1,2} = \frac{V_0 \pm \sqrt{V_0^2 - 4k\lambda}}{2k} \quad x$$

~~We suppose~~

$X^0(x)$ is not trivial iff $V_0^2 - 4\lambda k < 0$

(otherwise $\alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow C_1 = C_2 = 0$).

$$\Rightarrow X_n(x) = e^{\frac{V_0}{2k}x}\left(C_1 \cos\left(\frac{\sqrt{-V_0^2+4k\lambda}}{2k}x\right) + C_2 \sin\left(\frac{\sqrt{4\lambda k - V_0^2}}{2k}x\right)\right).$$

plug in $X'(0) = X'(L) = 0 \Rightarrow$

- $\frac{\sqrt{4\lambda k - V_0^2}}{2k} L = n\pi$. $n = 1, 2, \ldots$

$$C_1 = -\frac{\sqrt{4\lambda k - V_0^2}}{V_0} C_2 = -\frac{2kn\pi}{LV_0} C_2.$$

$$\Rightarrow \lambda_n = \frac{\frac{4k^2n^2\pi^2}{L^2} + V_0^2}{4k^2}. \qquad n = 1, 2, \ldots$$

So $u(x,t) = \sum_{n\geq 1} e^{\frac{V_0}{2k}x - \lambda_n t} a_n \left(-\frac{2kn\pi}{LV_0}\cos\frac{n\pi x}{L} + \sin\frac{n\pi x}{L}\right)$

$u(x,0) = f(x) \Rightarrow$ $a_n = \frac{2}{L}\int_0^L f(x) e^{-V_0 x/2k} \sin\frac{n\pi x}{L}\, dx.$

$\square$

5. For S-L $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$

Verify

(1) $\exists$ infinitely many eigenvalues with smallest but no largest.

Pf: $\phi(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

$$\left.\begin{array}{l}\phi'(0)=0\\ \phi'(L)=0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(C_1 - C_2)\sqrt{-\lambda} = 0\\ C_1\sqrt{-\lambda}\,(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) = 0\end{array}\right. \Rightarrow \begin{array}{l}\lambda > 0.\\ C_1 = C_2\\ \sin\sqrt{\lambda}L = 0\end{array}$$

$\therefore \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n \geq 1$ ✓

(2). The nth eigenfunction has n-1 zeros

$$\frac{d^2\phi_n}{dx^2} + \lambda_n\phi_n = 0 \Rightarrow \phi_n = \cos\frac{(n-1)\pi x}{L}$$

$$\Rightarrow \phi_n(x) = 0 \iff x = \underbrace{\frac{L}{2(n-1)} = \frac{(1+2\times 0)L}{2(n-1)}, \ldots, \frac{(1+2(n-2))L}{2(n-1)}}_{n-1 \text{ many}}$$ ✓

(3) follows from the orthogonality $\wedge$ completeness and the of $\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots\}$ ✓

(4) Rayleigh quotient. $\lambda = \frac{-\phi\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int \phi^2 dx}$

$\lambda \leq 0$ requires $\phi(0)\frac{d\phi}{dx}(0) - \phi(L)\frac{d\phi}{dx}(L) + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \leq 0$

$\phi'(0) = \phi'(L) = 0 \Rightarrow \int_0^L \left(\frac{d\phi}{dx}\right)^2 \leq 0 \Rightarrow \phi \equiv 0$

$\therefore$ no negative eigenvalues

$\square$

8. Show ~~that~~ $\lambda \geq 0$ for the eigenvalue problem.

$$\phi''(x) + (\lambda - x^2)\phi = 0$$
$$\phi'(0) = \phi'(1) = 0$$

Is $\lambda = 0$ an eigenvalue?

Pf: $\lambda = \dfrac{\int_0^1 (\phi')^2 + x^2\phi^2\,dx}{\int_0^1 \phi^2\,dx} \geq 0$

$p = 1,\ q = -x^2,\ \sigma = 1$

$\lambda = 0 \Rightarrow \phi \equiv 0 \Rightarrow$ ~~ϕ~~ not □

9. $x^2\phi'' + x\phi' + \lambda\phi = 0 \qquad \phi(1) = 0 \quad \phi(b) = 0$

(1). Multiplying $\frac{1}{x}$ $\Rightarrow$ $x\phi'' + \phi' + \frac{\lambda}{x}\phi = 0$

$\Rightarrow \frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \frac{\lambda\phi}{x} = 0$ with $p(x) = x$
(S-L eq) $\quad q(x) = 0$, $\sigma(x) = \frac{1}{x}$

(2). $\lambda = \dfrac{\int_1^b x(\phi'(x))^2}{\int_1^b \frac{\phi^2(x)}{x}dx} \geq 0$

(3) Let. $\phi(x) = x^m$ $\Rightarrow$ $m(m-1)x^m + mx^m + \lambda x^m = 0$

$\Rightarrow m = \pm\sqrt{-\lambda} = \pm i\lambda$

Thus $\phi(x) = A_1\cos(\sqrt{\lambda}\ln x) + A_2\sin(\sqrt{\lambda}\ln x)$

$\phi(1) = 0 \Rightarrow C_1 = 0$. $\quad \phi(b) = 0 \Rightarrow \lambda\ln b = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{\ln b}\right)^2$. $n \geq 1$

If $\lambda = 0$. then $\phi(x) \equiv 0 \Rightarrow 0$ is not an eigenvalue

(4) Eigenfunctions $\varphi_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$. $n = 1, 2, \ldots$

let the weight $\sigma(x) = \frac{1}{x}$ Then $\forall m \neq n$ $\int_1^b \sigma(x)\varphi_m(x)\varphi_n(x)dx \overset{t=\ln x}{=} 0$

(5). $\varphi_n = 0 \Leftrightarrow \frac{n\pi}{\ln b} = \ln x = k\pi$ $\quad k = 0, 1, 2, \ldots$ $1 < x < b \Leftrightarrow x = b^{k/n}$. so $\underbrace{k = 1, 2, \ldots, n-1}_{\text{are possible eigenvalues}}$ □

[5.4]5. Consider $\rho \partial_t^2 u = T_0 \partial_x^2 u + \alpha u$ $\alpha(x) < 0 < \rho(x)$

T_0 const

$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f \quad \partial_t u(x,0) = g \end{cases}$$

Solve the IBVP assuming the eigen function are known

Sol: By separating variables, let $u(x) = X(x) T(t)$

$$\Rightarrow \begin{cases} T''(t) + \lambda T(t) = 0 \Rightarrow T(t) = C_1 \cos(\sqrt{\lambda} t) + C_2 \sin(\sqrt{\lambda} t) \\ \left.\begin{array}{l} T_0 \phi''(x) + \alpha \phi(x) + \lambda \rho \phi(x) = 0 \\ \phi(0) = \phi(L) = 0 \end{array}\right\} \text{S-L equ.} \end{cases}$$

By Rayleigh quotient, $\lambda_n = \dfrac{\int_0^L \left(T_0 (\phi'(x))^2 - \alpha(x) \phi^2\right) dx}{\int_0^L \phi^2(x) \rho(x) dx}$

So, $u(x,t) = \sum_{n \geq 1} \phi_n(x) \cdot T(t) = \sum_{n \geq 1} \left(a_n \cos(\sqrt{\lambda_n} t) + b_n \sin(\sqrt{\lambda_n} t)\right) \phi_n(x)$

$f(x) = u(x,0)$ ~~$a_n =$~~ where $a_n = \dfrac{\int_0^L f(x) \phi_n(x) \rho(x) dx}{\int_0^L \phi_n^2(x) \rho(x) dx}$

$= \sum a_n \phi_n(x)$

$g(x) = \partial_t u(x,0) = \sum_{n \geq 1} b_n \sqrt{\lambda_n} \phi_n(x)$, $b_n = \dfrac{1}{\sqrt{\lambda_n}} \dfrac{\int_0^L g(x) \phi_n(x) \rho(x) dx}{\int_0^L \phi_n^2(x) \rho(x) dx}$

□