Undergrad PDE HW8 Junyan Zhang

[5.5] 1. S-L eigen-value problem is called self-adjoint if $p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$

Since then $\int_a^b u\,L(v) - v\,L(u)\,dx = 0$ $\forall u, v$ satisfying the bdry condition.

Verify the self-adjointness.

~~(a) $\phi(0) = \phi(L) = 0$~~

(c). $\frac{d\phi}{dx}(0) - h\phi(0) = 0$. $\frac{d\phi}{dx}(L) = 0$.

then: $$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L = p(L)\,u(L)\overset{0}{\overset{\shortparallel}{\frac{dv}{dx}(L)}} - p(L)\,v(L)\overset{0}{\overset{\shortparallel}{\frac{du}{dx}(L)}}$$

$$- p(0)\,u(0)\underset{hv(0)}{\underset{\shortparallel}{\frac{dv}{dx}(0)}} + p(0)\,v(0)\underset{hu(0)}{\underset{\shortparallel}{\frac{du}{dx}(0)}}$$

$$= -hp(0)u(0)v(0) + h\,p(0)u(0)v(0)$$

$$= 0.$$

(e). $\phi(a) = \phi(b)$. $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$. $p(a) = p(b)$

From the computation above. $u(b) = u(a)$. $p(b) = p(a)$

then the first term cancels with the third term.

Similarly the second term cancels with the fourth term.

(f) $\phi(L) = 0$. $p(0) = 0$. $\phi(0)$ bdd. $p(x)\frac{d\phi}{dx} = 0$.

$\phi(L) = 0 \Rightarrow$ The first two terms are 0.

~~$-\phi$~~ $-p(0)\,u(0)\frac{dv}{dx}(0) + p(0)v(0)\,\frac{du}{dx}(0)$

$$= -u(0)\lim_{x\to 0} p(x)\frac{dv}{dx}(x) + v(0)\lim_{x\to 0} p(x)\frac{du}{dx}(x)$$

$$= 0.$$

□

8. Let $L = \frac{d^4}{dx^4}$.

(1) Show that $uL(v) - vL(u)$ is exact

(2) $\int_0^1 uL(v) - vL(u)$

(3) $= 0$ if u, v satisfy $\phi(0) = \phi(1) = 0$

$\frac{d\phi}{dx}(0) = 0$

$\frac{d^2\phi}{dx^2}(1) = 0$

(4). Give another example of bdry conditions s.t. $\int_0^1 uLv - vLu = 0$

(5). $\frac{d^4\phi}{dx^2} + \lambda e^x \phi = 0$. subject to (3). Show that eigenfunctions are orthogonal. and find the weight.

Sol: (1) $uL_v - vL_u = uV_{xxxx} - Vu_{xxxx}$

$= uV_{xxxx} + U_xV_{xxx}$

$- U_xV_{xxx} - U_{xx}V_{xx}$

$+ U_{xx}V_{xx} - U_{xxx}V_x$

$+ U_{xxx}V_x - U_{xxxx}V$

$= \frac{d}{dx}\left(uV_{xxx} - U_xV_{xx} + U_{xx}V_x - U_{xxx}V\right)$

$\Rightarrow$ exact.

(2). By fundamental Thm of Calculus we have

$$\int_0^1 uLv - vLu = \left(uV_{xxx} - U_xV_{xx} + U_{xx}V_x - U_{xxx}V\right)\Big|_{x=0}^{x=1}.$$

(3). Under these boundary conditions, each term in (2) is zero.

(4). $\phi(0) = \phi(1) = 0$. $\phi'(1) = 0$. $\phi''(0) = 0$.

(5) Suppose λ_n's are all distinct eigenvalue of L, the corresponding eigenfunctions of the are $\{\psi_n\}$ then $\sigma(x) = e^x$ is the weight.

$$\int_0^1 \psi_m \underbrace{(-\lambda_m e^x \psi_n)}_{L\psi_n} - \psi_n \underbrace{(-\lambda_m e^x \psi_m)}_{L\psi_m} \, dx$$

$$= (\lambda_m - \lambda_n) \int_0^1 \psi_n \psi_m \underbrace{e^x}_{\sigma(x)} \, dx = 0 \quad \forall m \neq n$$

Since $\lambda_m \neq \lambda_n$, we know ψ_n, ψ_m are orthogonal w.r.t $\sigma(x) = e^x$

$\square$

[5.6] 1. Use Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of.

(1). $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ $\phi(1) = 0$

Sol.

Let $u = x^2 - 1$. then $\frac{du}{dx}(0) = 0$.

$$\lambda_1 \leq \frac{-pu\frac{du}{dx}\Big|_0^1 + \int_0^1 \left(p\left(\frac{du}{dx}\right)^2 - qu^2\right) dx}{\int_0^1 u^2 \underbrace{\sigma(x)}_{=1} \, dx}$$

where $p(x) = 1$, $\sigma(x) = 1$
$q(x) = -x^2$

Compute RHS.

$\Rightarrow \quad \lambda_1 \leq \frac{37}{14}$

2. Consider $\begin{cases} \dfrac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \\ \phi'(0) = 0 \quad \phi'(1) = 0 \end{cases}$ Show that $\lambda > 0$

pf: This is a Sturm-Liouville type eq: with $p(x) = 1$ $q(x) = -x^2$, $\sigma(x) = 1$

By Rayleigh quotient, (by boundary condition)

$$\lambda = \frac{-\underbrace{\phi(x)\,\phi'(x)\Big|_0^1}_{=0} + \int_0^1 \underbrace{(\phi'(x))^2 + x^2\phi^2}_{>0}\,dx}{\underbrace{\int_0^1 \phi^2(x)\,dx}_{>0}}$$

> 0 for all $\phi \neq 0$.

$\square$